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Combinatorially Derived Properties of Young Tableaux

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by

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Abstract

We examine properties of Young tableaux of shape λ and weight μ or of shape $\{\lambda_{(i)}\}$, a sequence of partitions. First we use combinatorial arguments to re-derive results about individual tableaux from Behrenstein and Zelevinskii regarding Kostka numbers and from Gates, Goldman, and Vinroot regarding when the weight μ on a tableau of shape λ is the unique weight with $K_{\lambda\mu} = 1$. Second we generalize these results to sequences of tableaux. Specifically we show under what conditions is $K_{\{\lambda_{(i)}\}\mu} = 1$ for a sequence of partitions $\{\lambda_{(i)}\}$ and weight μ and when is there a unique weight μ for a sequence of partitions with $K_{\{\lambda_{(i)}\}\mu} = 1$.

Contents

Chapter 1

Introduction

We begin Chapter 2 by introducing the concept of Kostka numbers through the multiplicities of tableaux with shape and weight defined by two partitions. Carl Kostka first formalized the notion of Kostka numbers in 1882 [6]. Given partitions λ and μ of a positive integer n, the Kostka number $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and weight μ. For a sequence of partitions $\{\lambda_{(i)}\}$ and a weight μ, the Kostka number $K_{\{\lambda_{(i)}\}\mu}$ is the number of sequences of semistandard Young tableaux with shapes $\{\lambda_{(i)}\}$ and weight μ. Motivated by the importance of Kostka numbers and their generalizations in the representation theory of Lie algebras and the finite groups of Lie type, we examine how Behrenstein and Zelevinskii characterized precisely when $K_{\lambda\mu} = 1$ [1] and then we prove when $K_{\{\lambda_{(i)}\}\mu} = 1$. We then use both results to examine the sequences of partitions $\{\lambda_{(i)}\}$ for which $K_{\{\lambda_{(i)}\}\mu} = 1$ if and only if $\mu = \tilde{\lambda}$ where $\tilde{\lambda}$ is the annealed partition.

After discussing some preliminary properties of individual and sequences of tableaux in Chapter 2, we revisit some results from Berenshtein and Zelevinskii [1] in Chapter 3. We will develop a combinatorial proof for one of their theorems that exposes a notation error in their paper [1, Theorem 1.5]. In Chapters 3 and 4, we explore the number of partitions μ such that $K_{\lambda\mu} = 1$ and $K_{\{\lambda_{(i)}\}\mu} = 1$ respectively. Specifically when is there only one weight μ such that $K_{\lambda\mu} = 1$ or $K_{\{\lambda_{(i)}\}\mu} = 1$. Chapter 3 concludes with a result from Gates, Goldman, and Vinroot [3]: when is there only one μ such that $K_{\lambda\mu} = 1$. Chapter 4 delves into proving when $K_{\{\lambda_{(i)}\}\mu} = 1$ and on which sequences of partitions $\{\lambda_{(i)}\}$ is the weight μ such that $K_{\{\lambda_{(i)}\}\mu} = 1$ unique.

Kostka numbers play an important role in many aspects of representation theory and other related disciplines. In 1908, Carl Kostka computed Kostka numbers for the purpose of generating a change of bases for the symmetric function up to 11 dimensions [7]. The results shown here can be directly applied into factorization of Kostka polynomials [5]. The Kostka number for partitions λ and μ, $K_{\lambda\mu}$

is also equal to the number of times the Specht module S^λ for the symmetry group is a composition factor in the permutation module M^μ [5]. The symmetry group can be a powerful tool for physical chemists in computing equilibrium positions or vibrational modes of complex molecules [4]. Furthermore Kostka numbers and their variants have other applications in the representation theory of Lie algebras [8], Weyl groups [9], and tensor product decompositions [2], with Kostka multiplicities appearing in the decomposition of various representations. Kostka multiplicity of one guarantees that a particular representation will inherit certain properties of other representations. Finally, the Kostka multiplicities of sequences of tableaux with depths are another way to represent degenerate Gelfand-Graev characters [10]. We conclude by suggesting novel approaches for future work with sequences of tableaux with depths.

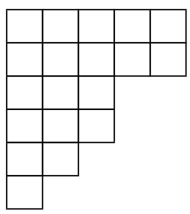
Chapter 2

Background

2.1 Partitions

A *partition* λ of a non-negative integer n is a tuple of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^{r} \lambda_i = n$. The *size* of λ is n, written $|\lambda| = n$. The numbers λ_i are the *parts* of the partition and the *length* of λ is the number of positive parts, written $l(\lambda)$. We consider two partitions to be identical if they have equivalent sequences of positive parts. Another notation we will use is $\lambda = (k_1^{m_1}, k_2^{m_2}, \ldots, k_s^{m_s})$ where $\lambda_i = k_j$ for all $\sum_{r=1}^{j-1} m_r < i \leq \sum_{r=1}^{j} m_r$. For example $\lambda = (12, 9, 9, 9, 9, 5, 4, 2, 2, 2, 1, 1, 1, 1, 1, 1)$ would be $\lambda = (12^1, 9^4, 5^1, 4^1, 2^3, 1^6)$.

Given a partition λ the *Young diagram* for λ is an array of left-justified rows of boxes, where the ith row from the top has λ_i boxes. For example the Young diagram for $\lambda = (5^2, 3^2, 2^1, 1^1)$ is



The *dominance partial order* on partitions of some fixed non-negative integer n, written $\trianglerighteq$, is defined as follows. If both λ and μ are partitions of n then $\lambda \trianglerighteq \mu$ if for all $m \geq 1$, $\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} \mu_i$ holds. For example if $\lambda = (3, 2, 1)$, $\mu = (2, 2, 2)$, and $\nu = (3, 1, 1, 1)$ then $\lambda \trianglerighteq \mu$, $\lambda \trianglerighteq \nu$, and ν and μ are incomparable.

2.2 Sequences of Partitions

A *sequence of partitions* $\{\lambda_{(i)}\}$ is a tuple of individual partitions, written

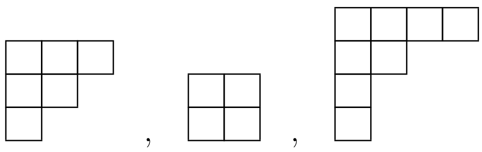
$$\{\lambda_{(i)}\} = \{{}_1\lambda,\ {}_2\lambda, \dots,\ {}_s\lambda\}$$

Each ${}_i\lambda$ represents an individual partition with its own parts:
${}_i\lambda = ({}_i\lambda_1,\ {}_i\lambda_2, \dots,\ {}_i\lambda_r)$. Thus a sequence of partitions $\{\lambda_{(i)}\}$ can alternatively be expressed as follows.

$$\{\lambda_{(i)}\} = \{({}_1\lambda_1,\ {}_1\lambda_2, \dots,\ {}_1\lambda_r), ({}_2\lambda_1,\ {}_2\lambda_2, \dots,\ {}_2\lambda_r), \dots, ({}_s\lambda_1,\ {}_s\lambda_2, \dots,\ {}_s\lambda_r)\}$$

Some parts ${}_i\lambda_j$ may be zero so the different partitions ${}_i\lambda$ in the sequence can have different lengths. For instance, ${}_1\lambda$ could be a partition of length three, while ${}_2\lambda$ could be a partition of length 17. In this case, ${}_1\lambda_i$ would be zero for at least $4 \le i \le 17$. Thus we see that $l({}_i\lambda)$ is not necessarily equal to $l({}_k\lambda)$.

Given a sequence of partitions $\{\lambda_{(i)}\}$ the Young diagram is the ordered sequence of Young diagrams for each individual partition. For example the Young diagram of $\{\lambda_{(i)}\} = ((3,2,1),(2,2),(4,2,1,1))$ is



Conceptually, sequences of partitions can be considered as partitions of the parts of some other partition of a positive integer. From the example above, $\{\lambda_{(i)}\} = ((3,2,1,0),(2,2,0,0),(4,2,1,1))$ is a sequence of partitions for the integer $18 = \sum_{i=1}^{3}\sum_{j=1}^{4} {}_i\lambda_j$, where $6+4+8$ is the partition of 18, ${}_1\lambda$ is a partition of 6, ${}_2\lambda$ is a partition of 4, and ${}_3\lambda$ is a partition of 8.

We can consider an analog to the dominance partial order of individual partitions for a sequence of partitions in relation to a weight as follows. Let $\{\lambda_{(i)}\}$ be a sequence of partitions and μ be a weight, where $\sum_{i=1}^{s}\sum_{j=1}^{r} {}_i\lambda_j = |\{\lambda_{(i)}\}| = |\mu| = n$ for some positive integer n. Then the analog to $\lambda \trianglerighteq \mu$ is if for all $m \ge 1$ we have $\sum_{i=1}^{s}\sum_{j=1}^{m} {}_i\lambda_j \ge \sum_{j=1}^{m} \mu_j$. For example, we note that for the sequence of partitions $\{\lambda_{(i)}\} = \{(4,2,1,1),(4,3)\}$ and weight $\mu = (5,5,5)$ that $\sum_{i=1}^{2}\sum_{j=1}^{3} {}_i\lambda_j = 14 < 15 =$

$\sum_{j=1}^{3} \mu_j$. This implies that the sequence of partitions $\{\lambda_{(i)}\} = \{(4,2,1,1),(4,3)\}$ does not satisfy the conditions on the weight $\mu = (5,5,5)$.

2.3 Tableaux

A *semistandard Young tableau* of shape λ and weight μ is a filling of a Young diagram for the partition λ with positive integers whereby integers from left to right across a row are non-decreasing and from top to bottom down a column are increasing. Define the tuple $\mu = (\mu_1, \mu_2, \dots)$ to be the frequency with which each individual integer occurs in the tableau. That is μ_i represents the number of i's that occur in boxes of the Young diagram. It has been shown that the number of semi-standard Young tableaux with shape λ and weight μ is invariant upon permutation of the parts of μ. Thus we can consider all weights μ to be partitions since any other tuple can be re-ordered such that $\mu_1 \geq \mu_2 \geq \dots$ to create a partition.

For example we can consider a semi-standard Young tableau of shape $\lambda = (3,3,2)$ and weight $\mu = (3,2,2,1)$.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

and a filling that is not a semi-standard Young tableau due to the decreasing values in the third row and the non-increasing values in the first column

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline 4 & 3 \\ \cline{1-2}\end{array}$$

The number of possible tableaux of shape λ and weight μ is defined as the Kostka number, written $K_{\lambda\mu}$. From the example above we observe that $K_{\lambda\mu} = 2$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$$

The Kostka number for a sequence of partitions $\{\lambda_{(i)}\}$ and weight μ is defined as the number of distinct sequences of semi-standard Young tableaux. For example consider the sequence of partitions $\{\lambda_{(i)}\} = ((2,1),(2,2,2),(2,1))$ with weight

$\mu = (6,3,3)$ where $K_{\{\lambda_{(i)}\}\mu} = 2$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

Chapter 3

Individual Tableaux

Our first result is a combinatorial proof for a theorem by Berenshtein and Zelevinskii that characterizes when $K_{\lambda\mu} = 1$ [1, Theorem 1.5]. To do so however we will need to show a few preliminary results. First it is known that for any two partitions λ and μ of a positive integer n that $K_{\lambda\mu} \neq 0$ if and only if $\lambda \trianglerighteq \mu$. Second we will prove the following lemma, regarding partitions λ with restricted lengths.

Lemma 3.0.1. *If $\lambda \trianglerighteq \mu$, for some partitions λ and μ, where $l(\lambda) \geq l(\mu) - 1$, and $\lambda_1 = \lambda_2 = \cdots = \lambda_{l(\mu)-1}$ then $K_{\lambda\mu} = 1$.*

Proof. Let the partitions λ and μ be defined as above. Since $\lambda \trianglerighteq \mu$ then $K_{\lambda\mu} \geq 1$. Next we assume that $l(\lambda) > l(\mu)$. Then with this assumption, we find that $\sum_{i=1}^{l(\mu)} \lambda_i < \sum_{i=1}^{l(\mu)} \mu_i$, which implies that $\lambda \ntrianglerighteq \mu$. This is a contradiction and therefore $l(\mu) - 1 \leq l(\lambda) \leq l(\mu)$ so we can consider partitions λ of two different lengths as our only cases.

Case 1 $l(\lambda) = l(\mu) - 1$: Since any filling of a column must be strictly increasing and there is exactly one more distinct value that can fill a box than the total number of boxes in the column, we observe that any column filling can equivalently be symbolized by the unique numerical value that does not appear in any of its boxes. Thus each column filling is missing exactly one numerical value from the weight. For example consider $\lambda = (5^4)$ and $\mu = (4^5)$ and above it the sequence of entries not occurring in each column filling.

5	**4**	**3**	**2**	**1**
1	1	1	1	2
2	2	2	3	3
3	3	4	4	4
4	5	5	5	5

If we simply know the length of the columns and the sequence of numbers not appearing in the column filling, then we can reconstruct the exact shape and weight of the tableau. For example if γ_i is the number of columns that lack an entry with i then $\mu = (l(\lambda) - \gamma_1, l(\lambda) - \gamma_2, \ldots, l(\lambda) - \gamma_{l(\mu)})$. Furthermore we see that under these constraints, a column missing μ_i would have each row λ_j filled by μ_j for $j < i$ and λ_j filled by μ_{j+1} for all $j \geq i$. Due to the non-decreasing across rows requirement, these columns have a defined ordering. Columns missing the least-valued weight must be the rightmost columns since they have fewer rows λ_j filled with entries from μ_j, and more rows λ_j filled with entries from μ_{j+1}. Given the strict ordering, there can be only one possible filling and arrangement of these columns and so $K_{\lambda\mu} = 1$.

Case 2 $l(\lambda) = l(\mu)$: Then there are $\lambda_{l(\lambda)}$ columns with $l(\lambda)$ rows and $l(\lambda)$ distinct numerical values that can fill them. Thus the leftmost $\lambda_{l(\lambda)}$ columns must be filled in incremental order down each column and we can then independently consider the $\lambda_{l(\lambda)-1} - \lambda_{l(\lambda)}$ rightmost columns and the remaining, unused numerical entries. The remaining boxes and remainder of the weight meet the conditions for Case 1, since the number of distinct valued entries is equal to one greater than the number of available rows to fill, and so we find that there is only one filling of the Young diagram for λ that results in a semi-standard Young tableau. Thus $K_{\lambda\mu} = 1$. $\square$

This will constitute one of the primary cases for the proof of Berenshtein and Zelevinskii's Theorem. We also note that the requirement on the length of both partitions is what makes the result important. With columns of length smaller than $\mu - 1$ we see that there is no guaranteed ordering of columns. For instance there is no intuitively defined ordering for cases where $l(\mu) = l(\lambda) + 2$. In the example below we look at $l(\mu) = 5$ and $l(\lambda) = 3$. There is no possible way to arrange all four columns in a semi-standard Young tableau. Additionally distinct column fillings could swap entries and result in a completely distinct total filling of all of the columns. In the example below, the two leftmost columns can swap an entry of 3 for an entry of 5. However when $l(\mu) = l(\lambda) + 1$ then any switching would result in just a re-ordering of the same quantities of each type of column filling.

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\;,\quad \begin{array}{|c|}\hline 1\\\hline 4\\\hline 5\\\hline\end{array}\;,\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array}\;,\quad \begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline\end{array}$$

While Behrenstein and Zelevinskii used Lie algebra and Weyl groups to derive precisely when $K_{\lambda\mu} = 1$, they made a notational mistake in their paper. Specifically when they decompose the partitions λ and μ into sub-partitions, their notation implies an overlap of the last part of one sub-partition with the first part of the next sub-partition [1]. We will see that this is impossible when we go through a combinatorial proof of the conditions.

Theorem 3.0.2. *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *and* $\mu = (\mu_1, \ldots, \mu_l)$ *where some* λ_i *or* μ_i *may equal 0 (so there is some choice in* l*). Then* $K_{\lambda\mu} = 1$ *if and only if there exists a choice of indices* $0 = i_0 < i_1 < \cdots < i_k = l$ *such that for* $j = 1, \ldots, k$ *the sub-partitions* $\lambda^j = (\lambda_{i_{j-1}+1}, \lambda_{i_{j-1}+2}, \ldots, \lambda_{i_j})$ *and* $\mu^j = (\mu_{i_{j-1}+1}, \mu_{i_{j-1}+2}, \ldots, \mu_{i_j})$ *satisfy the following:*

1. $|\lambda^j| = |\mu^j|$ *and* $\lambda^j \trianglerighteq \mu^j$*, and*
2. *either* $\lambda_{i_{j-1}+1} = \lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j-2} = \lambda_{i_j-1}$ *or*
   $\lambda_{i_{j-1}+2} = \lambda_{i_{j-1}+3} = \cdots = \lambda_{i_j-1} = \lambda_{i_j}$

Proof. Assume for some partitions λ and μ of an integer n that there exist a set of indices that satisfy the above conditions. Thus for $j = 1, \ldots, k$ we note that $\displaystyle\sum_{r=1}^{i_j} \lambda_r = \sum_{r=1}^{i_j} \mu_r$ and for any $i_{j-1}+1 \leq s \leq i_j$, $\displaystyle\sum_{r=i_{j-1}+1}^{s} \lambda_r \geq \sum_{r=i_{j-1}+1}^{s} \mu_r$ and therefore $\displaystyle\sum_{r=1}^{s} \lambda_r \geq \sum_{r=1}^{s} \mu_r$ for any $1 \leq s \leq l$ or equivalently $\lambda \trianglerighteq \mu$. Since $\lambda \trianglerighteq \mu$ then $K_{\lambda\mu} \neq 0$ so there exists at least one semi-standard Young tableaux of shape λ and weight μ.

We know that any entry in the λ_r row must be $s \geq r$ (since down a column the weights must strictly increase). Thus for any i_j, no entry of value $s \leq i_j$ can fill a space in any row λ_r for $r > i_j$. From the above equality we know that the rows λ_1 through λ_{i_j} can be completely filled with entries of value 1 through i_j. Since none of these weights can appear in lower rows then no entries from μ_s for $s > i_j$ can fill any space in rows λ_1 through λ_{i_j}.

Thus we can separate any of the semi-standard Young tableaux into sub-tableaux of shape λ^j and weight μ^j. Since none of the weights from any of these sub-tableaux can have entries in another sub-tableau then we can consider each sub-tableau independently. Furthermore any tableau of shape λ and weight μ must be some combination of distinct sub-tableaux of shape λ^j and weight μ^j. Since $\lambda^j \trianglerighteq \mu^j$ then $K_{\lambda^j\mu^j} \geq 1$ so we can calculate $K_{\lambda\mu} = \displaystyle\prod_{j=1}^{k} K_{\lambda^j\mu^j}$.

Case 1: For sub-tableaux of shape $\lambda_{i_{j-1}+1} = \cdots = \lambda_{i_j-1}$ we know from Lemma 3.0.1 that $K_{\lambda^j\mu^j} = 1$, since $l(\lambda^j) = l(\mu^j)$, and $\lambda^j \trianglerighteq \mu^j$.

Case 2: For sub-tableaux of shape $\lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j}$ we know that the left-most λ_{i_j} columns must be filled in strictly increasing order by the weights $\mu_{i_{j-1}+1}$ through μ_{i_j} since there are the same number of distinct valued entries as rows. The remaining $\lambda_{i_{j-1}+1} - \lambda_{i_j}$ boxes from the row $\lambda_{i_{j-1}+1}$ must be filled in non-decreasing order. For example consider $\lambda = (6^1, 3^4)$ and $\mu = (5^1, 4^1, 3^3)$

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{1-3} \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{1-3} \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{1-3} \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & \ & \ & \ \\ \hline 2 & 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-3} 3 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} 4 & 4 & 4 & \multicolumn{3}{c}{} \\ \cline{1-3} 5 & 5 & 5 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-3} 3 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} 4 & 4 & 4 & \multicolumn{3}{c}{} \\ \cline{1-3} 5 & 5 & 5 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$$

There is only one way to fill a single row in non-decreasing order so there is only one way that the weights from μ^j can fill λ^j. Thus $K_{\lambda^j \mu^j} = 1$ for all λ^j and μ^j where λ^j is of the shape $\lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j}$.

For every sub-tableau we have shown that the Kostka number is one. Thus there is only one distinct combination of sub-tableaux so there is only one distinct tableau of shape λ and weight μ. Therefore we conclude that $K_{\lambda\mu} = 1$.

Next we will show by induction that for any two partitions λ and μ of n where $K_{\lambda\mu} = 1$ that there exists a set of indices i_j and sub-partitions λ^j and μ^j that satisfy the above conditions. Define a block as a subset of the parts of λ that are all of the same size (ie. $\lambda_\alpha = \lambda_{\alpha+1} = \cdots = \lambda_{\alpha+\beta}$). We know that since $K_{\lambda\mu} = 1$ then $\lambda \trianglerighteq \mu$ and so if we split up the partitions λ and μ into sub-partitions of equal size and length (where some of the final rows of λ can have zero size) then we must have that $\lambda^j \trianglerighteq \mu^j$ since $K_{\lambda^j \mu^j} = 1 \neq 0$.

Let λ only have one distinct block of length m. If the length of μ is less than or equal to $m+1$ we can choose the entire partitions as sub-partitions and the conditions will be satisfied. If the length of μ is greater than $m+1$ we will show that $\displaystyle\sum_{r=1}^{m-1} \lambda_r = \sum_{r=1}^{m-1} \mu_r$ and therefore our choice of indices becomes $i_0 = 0,\ i_1 = m-1,\ i_k = \ell(\mu)$.

Case 1: An entry with value m does not appear in the row λ_m. Then we can fill the leftmost column with the smallest values (ie. a value from μ_m fills the leftmost column of row λ_m), and then fill the remaining $\lambda_1 - 1$ columns by using the smallest valued number from the weight going from left to right across rows and then sequentially down the rows. This is an alternate Young semi-standard tableau, which is a contradiction since $K_{\lambda\mu} = 1$. Thus a number of value m must appear in the row λ_m.

Case 2: An entry with value m appears in the row λ_m. Then there must exist an entry of value $m + \alpha$ for some $\alpha \geq 0$ in some row other than λ_m (otherwise we have $\displaystyle\sum_{r=1}^{m-1} \lambda_r = \sum_{r=1}^{m-1} \mu_r$ and are done). Thus there must exist some entry $m + \alpha$ in the rightmost column of the row λ_{m-1} since down a column, weights must strictly increase.

If $\alpha = 0$ then we can switch this value with the leftmost entry that is $m+1$ in the row λ_m to generate another Young semi-standard tableau (since the bottom right entry must be the largest value and there are at least $m+2$ distinct values). This is a contradiction since $K_{\lambda\mu} = 1$.

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 3 \\ \hline 3 & 3 & 4 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 4 \\ \hline 3 & 3 & 3 & 5 \\ \hline \end{array}$$

If $\alpha > 0$ then we can consider the leftmost entry in row λ_{m-1} as β for $\beta > m$. If we switch that with the rightmost entry in row λ_m that is less in value, we have another Young semi-standard tableau, which is a contradiction.

$$\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & 4 & 5 \\ \hline 2 & 2 & 3 & 3 & 4 & 5 & 6 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & 3 & 5 \\ \hline 2 & 2 & 3 & 4 & 4 & 5 & 6 \\ \hline \end{array}$$

Thus it must be the case that $\displaystyle\sum_{r=1}^{m-1} \lambda_r = \sum_{r=1}^{m-1} \mu_r$ so we can choose our indices $i_0 = 0$, $i_1 = m-1$, and $i_2 = \ell(\mu)$ where the sub-partitions λ^j and μ^j satisfy the conditions for $j = 1, 2$. Thus any partition λ with only one distinct block and $K_{\lambda\mu}$ must have at least one set of indices and sub-partitions that obey the above conditions.

Now assume that for all partitions λ and μ, where λ is composed of α distinct blocks and $K_{\lambda\mu} = 1$, then there exists a choice of indices i_j that split the partitions into sub-partitions λ^j and μ^j that satisfy the conditions. Now let λ and μ be partitions composed of $\alpha + 1$ distinct blocks and $K_{\lambda\mu} = 1$. Let the length of the first distinct block be m so $\lambda_1 = \cdots = \lambda_m$. Then we will show that either $\displaystyle\sum_{r=1}^{m} \lambda_r = \sum_{r=1}^{m} \mu_r$ or $\displaystyle\sum_{r=1}^{m+1} \lambda_r = \sum_{r=1}^{m+1} \mu_r$.

Case 1: An entry of value $m+1$ does not appear in the row λ_{m+1}. Then we can consider an alternate filling whereby the leftmost column is filled with the least possible entry in each row. This would put an entry of $m+1$ in the row λ_{m+1}. The remaining values can fill the remaining spaces in λ by filling the least value from left to right across rows and then down to the next row. This will be an alternative, Young semi-standard tableau, which is a contradiction. Thus an entry of value $m+1$ must appear in the row λ_{m+1}.

Case 2: An entry of value $m+1$ does appear in the row λ_{m+1}. Since no entries of value less than β can fill any row below the row $\lambda_{\beta+1}$ and there exists a Young semi-standard tableau, we can conclude the following two conditional statements. If no numbers of value greater than m appear in the row λ_m then we must have

$\sum_{r=1}^{m} \lambda_r = \sum_{r=1}^{m} \mu_r$. If no numbers of value greater than $m+1$ appear in the row λ_{m+1} then we must have $\sum_{r=1}^{m+1} \lambda_r = \sum_{r=1}^{m+1} \mu_r$. Thus we will only consider cases where there exist entries in row λ_m that are strictly greater than m and entries in row λ_{m+1} that are strictly greater than $m+1$.

Define the rightmost entry in the row λ_m of a tableau as x. Switch the leftmost entry in λ_m of same value as x with the rightmost entry in λ_{m+1} that has value strictly less than x. Consider the example below.

$$\begin{array}{|c|c|c|c|c|c|}\hline & & & 2 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline & & & 2 & 3 & \mathrm{x} \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline & & & 2 & 2 & \mathrm{x} \\ \hline 2 & 3 & 3 \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline & & & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} \end{array}$$

This alternate filling creates another Young semi-standard tableau of shape λ and weight μ, which is a contradiction since $K_{\lambda\mu} = 1$. Thus none of these cases are possible and all possible cases result in either $\sum_{r=1}^{m} \lambda_r = \sum_{r=1}^{m} \mu_r$ or $\sum_{r=1}^{m+1} \lambda_r = \sum_{r=1}^{m+1} \mu_r$.

Now we consider the partitions λ^* and μ^* by removing either the first block of λ or the first block and the first row of the second block of λ (depending on which of the above conditions is met). The partitions λ^* and μ^* have α distinct blocks and $K_{\lambda^*\mu^*} = 1$ so by our assumptions there exist a set of indices $0 = \tilde{i}_0 < \tilde{i}_1 < \cdots < \tilde{i}_{\tilde{k}} = \ell(\tilde{\mu})$ and sub-partitions $\lambda^{*j} = (\lambda^*_{i_{j-1}+1}, \lambda^*_{i_{j-1}+2}, \ldots, \lambda^*_{i_j})$ and $\mu^{*j} = (\mu^*_{i_{j-1}+1}, \mu^*_{i_{j-1}+2}, \ldots, \mu^*_{i_j})$ that satisfy the above conditions. Now if we simply create a new set of indices $0 = i_0 < i_1 = \tilde{i}_0 + m$ or $+m+1 < \cdots < i_k = \tilde{i}_{\tilde{k}} + m$ or $+m+1 = \ell(\mu)$ (depending on whether either $\sum_{r=1}^{m} \lambda_r = \sum_{r=1}^{m} \mu_r$ or $\sum_{r=1}^{m+1} \lambda_r = \sum_{r=1}^{m+1} \mu_r$). The resulting sub-partitions λ^j and μ^j obey the above conditions and so the statement holds for all partitions where λ and μ where λ has $\alpha + 1$ distinct blocks. Thus by induction, the statement holds for all partitions with any number of distinct blocks.

We have therefore shown that for any two partitions λ and μ of an integer n that $K_{\lambda\mu} = 1$ if and only if there exists a set of indices and sub-partitions that satisfy the above detailed conditions. $\square$

As shown above in the proof, the sub-partitions must be completely separate and distinct. When Behrenstein and Zelevinskii defined sub-partitions $\lambda^j = (\lambda_{i_{j-1}}, \lambda_{i_{j-1}+1}, \ldots, \lambda_{i_j})$ and $\mu^j = (\mu_{i_{j-1}}, \mu_{i_{j-1}+1}, \ldots, \mu_{i_j})$, they wrongly included the

$\lambda_{i_{j-1}}$ and $\mu_{i_{j-1}}$ terms. This error we can overlook certain pairs of partitions that have a Kostka multiplicity of one. For example, we consider $\lambda = (5, 3, 2)$ and $\mu = (4, 4, 1, 1)$. Since λ has all distinct parts then for $K_{\lambda\mu} = 1$ we must find sub-partitions. We know $\lambda_1 \neq \mu_1$ so then $\lambda^1 = (5, 3)$. With $\mu^1 = (4, 4)$ we find that $\lambda^1 \trianglerighteq \mu^1$. Now if we define λ^2 and μ^2 according to Behrenstein and Zelevinskii, we have $\lambda^2 = (3, 2)$ and $\mu^2 = (3, 1)$. This however leaves $\lambda^3 = (0)$ and $\mu^3 = (1)$, which means $\lambda^3 \ntrianglerighteq \mu^3$. According to their definitions this implies that $K_{\lambda\mu} \neq 1$. However we can see from our combinatorial proof and with non-overlapping indices that $K_{\lambda\mu} = 1$. The unique semi-standard Young tableaux is depicted below.

1	1	1	1	2
2	2	2		
3	4			

Theorem 3.0.2 will also help us in defining properties on sequences of partitions $\{\lambda_{(i)}\}$ and weights μ such that $K_{\{\lambda_{(i)}\}\mu} = 1$. Through a reductionist approach, we will anneal the individual partitions $\lambda_i \in \{\lambda_{(i)}\}$ and use properties of that annealed partition to determine information about the sequence.

With our revised theorem from Behrenstein and Zelevinskii, we can also look for partitions λ where the only weight μ with Kostka multiplicity of one, $K_{\lambda\mu} = 1$, is λ itself: $\mu = \lambda$. We know that for any partition λ with weight λ we can decompose both into sub-partitions, each with length one. In this way all the conditions of Theorem 3.0.2 are met so $K_{\lambda\lambda} = 1$. With this knowledge we can look for the number of weights μ such that $K_{\lambda\mu} = 1$, defined as $J(\lambda)$. As we just described, we know that $J(\lambda) \geq 1$ for all λ and so we aim to find λ where there are no other weights besides λ itself. In other words, what λ have $J(\lambda) = 1$?

Corollary 3.0.3. *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ then $J(\lambda) = 1$ if and only if $\lambda_i - \lambda_{i+1} \leq 1$ for all $1 \leq i < l$ and $\lambda_l \leq 1$.*

Proof. First consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ where for some i, $\lambda_i - \lambda_{i+1} > 1$. Then we can define a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ where $\mu_i = \lambda_i - 1$, $\mu_{i+1} = \lambda_{i+1} + 1$, and $\mu_j = \lambda_j$ for all other j. Choosing indices such that each sub-tableau is only one row, except for the rows λ_i and λ_{i+1}, which is a sub-tableau of two rows, we find that these sub-tableaux meet the conditions from Theorem 3.0.2 and so $K_{\lambda\mu} = 1$ and thus $J(\lambda) \geq 2$ since both $K_{\lambda\lambda}$ and $K_{\lambda\mu} = 1$.

Next we will again consider some partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Now let $\lambda_i - \lambda_{i+1} \leq 1$ for all $1 \leq i < l$. Any partition μ such that $K_{\lambda\mu} = 1$ must satisfy the conditions of Theorem 3.0.2. Thus we will examine a set of sub-tableaux that can be created from the conditions.

Case 1: In the sub-tableau of shape $\lambda^j = (\lambda_{i_{j-1}+1}, \lambda_{i_{j-1}+2}, \ldots, \lambda_{i_j})$ and weight $\mu^j = (\mu_{i_{j-1}+1}, \mu_{i_{j-1}+2}, \ldots, \mu_{i_j})$, we have $\lambda_{i_{j-1}+1} = \lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j-1} = \lambda_{i_j} + 1$.

If for some r, $\mu_r > \lambda_r$ then $\mu_s > \lambda_r$ for all $s \leq r$ and if $r \neq i_j$ then $\lambda_r = \lambda_{r-1} = \cdots = \lambda_{i_{j-1}+1}$ so we see $\sum_{k=0}^{i_{j-1}+1} \mu_{r-k} > \sum_{k=0}^{i_{j-1}+1} \lambda_{r-k}$. Furthermore if $r = i_j$ then $\sum_{k=i_{j-1}+1}^{i_j} \mu_k = \mu_{i_j} + \sum_{k=i_{j-1}+1}^{i_j-1} \lambda_k > \sum_{k=i_{j-1}+1}^{i_j} \lambda_k$. This is a contradiction in both cases so $\mu_r \leq \lambda_r$ for all r.

If for some r, $\mu_r < \lambda_r$ then $\mu_s < \lambda_r$ for all $s \geq r$. Thus $\sum_{k=r}^{i_j} \mu_k < \sum_{k=r}^{i_j} \lambda_k$ since $\lambda_k \geq \lambda_r - 1 \geq \mu_s$ for all $s \geq r$ and all k and for at least one s there must be $\lambda_s > \mu_s$. Since $|\mu^j| = |\lambda^j|$ then a number must fill a row with a value less than the corresponding row number, which will create a contradiction, because somewhere above it in the same column will be a non-increasing pair of values but by definition it is a semi-standard Young tableau. Thus $\mu_r \geq \lambda_r$ for all r. In conclusion $\mu_r = \lambda_r$. Since this was for any arbitrary r then $\mu = \lambda$ so for any μ such that $K_{\lambda\mu} = 1$ it must be that $\mu^j = \lambda^j$ for all sub-tableaux of this shape.

Case 2: In the sub-tableau of shape $\lambda^j = (\lambda_{i_{j-1}+1}, \lambda_{i_{j-1}+2}, \ldots, \lambda_{i_j})$ and weight $\mu^j = (\mu_{i_{j-1}+1}, \mu_{i_{j-1}+2}, \ldots, \mu_{i_j})$, we have $\lambda_{i_{j-1}+1} - 1 = \lambda_{i_{j-1}+2} = \cdots = \lambda_{i_j-1} = \lambda_{i_j}$. Since there are an equal number of distinct values that can fill boxes as rows and μ is a partition, we know that $\mu_{i_{j-1}+1} \geq \mu_{i_{j-1}+2} \geq \cdots \geq \mu_{i_j} \geq \lambda_{i_j}$. Since $K_{\lambda\mu} = 1$ we can deduce that $\lambda_{i_{j-1}+1} \geq \mu_{i_{j-1}+1} \geq \mu_{i_{j-1}+2} \geq \cdots \geq \mu_{i_j} \geq \lambda_{i_j} = \lambda_{i_{j-1}+1} - 1$ since μ^j is a partition whose smallest, positive part must be at greater than or equal to the size of the smallest part of λ^j (otherwise $\sum_{k=i_{j-1}+1}^{i_j-1} \lambda_k < \sum_{k=i_{j-1}+1}^{i_j-1} \mu_k$). If $\mu_{i_{j-1}+1} = \lambda_{i_j}$ then $\mu_{i_{j-1}+1} = \mu_{i_{j-1}+2} = \cdots = \mu_{i_j} = \lambda_{i_j} = \lambda_{i_j-1} = \cdots = \lambda_{i_{j-1}+2} = \lambda_{i_{j-1}+1} - 1$ and so $\sum_{k=i_{j-1}+1}^{i_j} \lambda_k \neq \sum_{k=i_{j-1}+1}^{i_j} \mu_k$, which is a contradiction. Thus $\mu_{i_{j-1}+1} = \lambda_{i_{j-1}+1}$ and so it must be true that $\mu^j = \lambda^j$ since the remaining unfilled columns all have the same number of rows as distinct valued entries. Thus $\mu^j = \lambda^j$ for all sub-tableaux of this shape.

Since $\mu^j = \lambda^j$ for all possible sub-tableaux then $\mu = \lambda$ so any μ such that $K_{\lambda\mu} = 1$ must be equal to λ. Thus $J(\lambda) = 1$. $\square$

Chapter 4

Sequences of Tableaux

4.1 The Annealed Partition

In analyzing sequences of partitions, we often will find it useful to consider combining them into one partition. While this process loses some of the information such as what columns are in which part of the sequence, it helps simplify some of the logic and provide more intuition towards further proofs. As such we will define the combined partition as the annealed partition and the process of linking them as annealing.

The idea of the annealed partition $\tilde{\lambda}$ of a sequence of partitions $\{\lambda_{(i)}\}$ is to combine all of the partitions ${}_i\lambda$ from $\{\lambda_{(i)}\}$. Recursively the i^{th} partition from the sequence $\{\lambda_{(i)}\}$ will add each column to the right of any columns of greater or equal length in $\tilde{\lambda}$ and to the left of any columns of lesser length. Thus in the final $\tilde{\lambda}$ for each set of columns of equal length the leftmost columns will have originated from partitions earlier in the sequence $\{\lambda_{(i)}\}$ than partitions that donated the rightmost columns. Formally the annealed partition $\tilde{\lambda}$ of a sequence of partitions $\{\lambda_{(i)}\}$ is the partition $\tilde{\lambda} = (\tilde{\lambda_1}, \tilde{\lambda_2}, \ldots, \tilde{\lambda_l})$ where $\tilde{\lambda_i} = \sum_{{}_j\lambda \in \{\lambda_{(i)}\}} {}_j\lambda_i$.

We can consider a small example where the top box will be filled simply to represent the origin of each column in the annealed partition.

$$\begin{ytableau} 1 & 2 & 3 \\ \; & \; & \; \\ \; \end{ytableau}\ , \quad \begin{ytableau} 4 & 5 \\ \; & \; \end{ytableau}\ , \quad \begin{ytableau} 6 & 7 \\ \; \\ \; \end{ytableau} \quad \Rightarrow \quad \begin{ytableau} 1 & 6 & 2 & 3 & 4 & 5 & 7 \\ \; & \; & \; & \; & \; & \; \\ \; & \; \end{ytableau}$$

This leads to a result that will help us define conditions on sequences of partitions such that the Kostka number is one.

Lemma 4.1.1. *If a sequence of partitions $\{\lambda_{(i)}\}$ with weight μ has a Kostka number* *of one,* $K_{\{\lambda_{(i)}\}\mu} = 1$ *then the Kostka number of the annealed partition* $\tilde{\lambda}$ *with the same weight must be one:* $K_{\tilde{\lambda}\mu} = 1$.

Proof. Assume $K_{\tilde{\lambda}\mu} \neq 1$. If $K_{\tilde{\lambda}\mu} = 0$ then $\tilde{\lambda} \not\trianglerighteq \mu$, which means that $\sum_{j=1}^{l} \tilde{\lambda}_j < \sum_{j=1}^{l} \mu_j$ for some l. Since $\sum_{j=1}^{l} \tilde{\lambda}_j = \sum_{j=1}^{l} \sum_{i=1}^{m} {}_i\lambda_j$ for all l then $\sum_{i=1}^{m} \sum_{j=1}^{l} {}_i\lambda_j < \sum_{j=1}^{l} \mu_j$ so $K_{\{\lambda_{(i)}\}\mu} = 0$.

If $K_{\tilde{\lambda}\mu} \neq 0$ then there exist at least two distinct, semi-standard Young tableaux of shape $\tilde{\lambda}$ and weight μ. If we return the filled, annealed tableau back into a sequence of tableaux, keeping the entries in their corresponding columns, then we will have two distinct sequences of semi-standard Young tableaux. Since the columns must have all entries increasing down each column in the annealed tableau then they will be so in the sequence of tableaux. Furthermore in each column in each tableau in the sequence, the column to the right of it originated from a column to the right of it in the annealed partition and so along each row must be non-decreasing values. Thus $K_{\{\lambda_{(i)}\}\mu} \neq 0$. □

This result can be very powerful in deriving properties of arbitrarily long and complex sequences of partitions $\{\lambda_{(i)}\}$. We can quickly use Theorem 3.0.2 to discern if a sequence of partitions with a weight does not have a Kostka multiplicity of one, simply by using Lemma 4.1.1. While this will not guarantee that the sequence of partitions and weight have a unique filling, it will easily pull out many sequences of partitions with weights that have more than one filling.

By condensing sequences of partitions to an annealed partition we lose information about which columns came from where, but we gain the simplicity of an individual tableau. This in turn helps us define more restrictive conditions on sequences of partitions and weights such that there exists only one valid sequence of tableaux. In the following lemma, we see just how the importance of returning columns back from the annealed partition can result in multiple distinct sequences of tableaux.

Lemma 4.1.2. *If* $K_{\{\lambda_{(i)}\}\mu} = 1$ *for some sequence of partitions* $\{\lambda_{(i)}\}$ *and weight* μ *then in the unique sequence of semi-standard Young tableaux of shape* $\{\lambda_{(i)}\}$ *and weight* μ*, all columns of the same length that exist in at least two distinct tableaux in the sequence must be filled identically.*

Proof. For some sequence of partitions $\{\lambda_{(i)}\}$ and weight μ let $K_{\{\lambda_{(i)}\}\mu} = 1$ and assume there are at least two columns of the same length in ${}_{i_1}\lambda$ and ${}_{i_2}\lambda$ that have distinct fillings in the unique sequence of semistandard Young tableaux. From

Lemma 4.1.1 we know that $K_{\tilde{\lambda}\mu} = 1$ and so we can consider the sub-tableaux containing the rows of these columns where there are distinct fillings. From the proof of Lemma 3.0.1 we know that these columns can be defined by the one value they lack and that the columns have a defined ordering, which is to say that given any set of columns they can be put into an order to make a valid tableau.

Thus if we let α_1 be the column from ${}_{i_1}\lambda$ and α_2 be the column from ${}_{i_2}\lambda$ with distinct fillings then we can consider a filling of the sequence of partitions with ${}_{i_2}\lambda$ using the filling for α_1 and ${}_{i_1}\lambda$ using the filling for α_2. From the annealed partition, we know that these columns can properly be placed between columns of different lengths and from the proof of Lemma 3.0.1 we know that these columns can be re-ordered in a way to make a valid tableau.

This filing and the filling using simply the exact same filling as the annealed tableau are distinct semi-standard sequences of Young tableaux. Thus $K_{\{\lambda_{(i)}\}\mu} \geq 2$, which is a contradiction. Therefore all columns of the same length that exist in at least two distinct partitions ${}_{i_1}\lambda$ and ${}_{i_2}\lambda$ must be filled identically in the unique sequence of semi-standard Young tableaux. $\square$

As suggested before we see how the process of annealing loses the origin of columns of equal lengths. This in turn occludes the ability to intuit how many possible Young semi-standard tableaux there are of shape $\{\lambda_{(i)}\}$ and weight μ. Lemma 4.1.2 pieces back some of the conditions required on $\{\lambda_{(i)}\}$ and μ for there to be a unique filling. We will see later on how this fits into a rigorous equivalency on the shapes of $\{\lambda_{(i)}\}$ and μ such that $K_{\{\lambda_{(i)}\}\mu} = 1$.

4.2 Kostka Numbers of Sequences of Partitions

The first question we pose when considering the shape and weight of a tableau or sequence of tableaux is does there exist at least one valid filling? Earlier we introduced the concept of dominance partial order, and an analog for comparing sequences of partitions with a weight. We will now revisit this with its application for Kostka numbers.

Lemma 4.2.1. *The Kostka number of a sequence of partitions $\{\lambda_{(i)}\}$ with weight μ is nonzero if and only if $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{l} {}_i\lambda_j \geq \sum_{j=1}^{l}\mu_j$ for all l.*

Proof. Assume that $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{l} {}_i\lambda_j < \sum_{j=1}^{l}\mu_j$ for some l. Then there must be some entry k for $k \leq l$ in a row ${}_i\lambda_j$ where $j > l$ since the total number of distinct entries of value less than or equal to l are greater than the number of available spaces in the first l rows of all m partitions in $\{\lambda_{(i)}\}$. Thus an entry of value k appears in a column that cannot be strictly increasing down the column since there are more than k boxes with at most k distinct possible entries that can fill boxes above λ_k. Thus $K_{\{\lambda_{(i)}\}\mu} = 0$.

Assume that $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{l} {}_i\lambda_j \geq \sum_{j=1}^{l}\mu_j$ for all l. Then we can fill ${}_n\lambda_1$ with entries of value 1 for some n where $\displaystyle\sum_{i=1}^{n-1} {}_i\lambda_1 < \mu_1 \leq \sum_{i=1}^{n} {}_i\lambda_1$. We can keep filling with incrementally larger values from the weight μ, step-wise filling ${}_i\lambda_j$ for all i before incrementing j and filling the next row. Given the condition, each number that newly fills a box will have strictly lesser values above it and numbers of equal or greater value to its right. Thus this filling method produces a sequence of semi-standard Young tableaux. □

From Lemma 4.2.1 we see that for any set of the first l rows from all the partitions ${}_i\lambda \in \{\lambda_{(i)}\}$ we must have more boxes than numbers less than or equal to l. This must be true for all positive l. Otherwise no matter how we fill the sequence there will be at least one column that disobeys the strictly increasing order from top to bottom. For example, we consider here $\{\lambda_{(i)}\} = \{(4,3,2),(2,1,1,1),(3,3,3),(2)\}$ and $\mu = (10,9,4,2)$. If we only fill in the Young diagrams with the first part of the weight, we see that there are only 8 positions left in the first two rows for the second part of the weight, but $\mu_2 = 9$.

$$\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 \\ \, & \, \end{ytableau}\ ,\quad \begin{ytableau} 1 & 1 \\ 2 \\ \, \\ \, \end{ytableau}\ ,\quad \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \, & \, & \, \end{ytableau}\ ,\quad \begin{ytableau} 1 & 2 \end{ytableau}$$

Thus any filling will not be a sequence os semi-standard Young tableaux. We also note that any other filling of the first two rows would produce a similar contradiction. Below is one example this.

$$\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 3 \end{ytableau}\ ,\quad \begin{ytableau} 1 & 1 \\ 2 \\ 3 \\ 4 \end{ytableau}\ ,\quad \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \end{ytableau}\ ,\quad \begin{ytableau} 1 & 2 \end{ytableau}$$

This leads us to the next problem: a generalization of the Berenshtein and Zelevinskii result for sequences of partitions [1]. As such we will draw on many of the results already proved to show how some conditions on the shapes of partitions in a sequence $\{\lambda_{(i)}\}$ and weight μ guarantees one and only one sequence of semi-standard Young tableaux.

Theorem 4.2.2. *For a sequence of partitions* $\{\lambda_{(i)}\} = \{{}_1\lambda,\ {}_2\lambda, \ldots,\ {}_m\lambda\}$, *where each* ${}_i\lambda = ({}_i\lambda_1,\ {}_i\lambda_2, \ldots,\ {}_i\lambda_l)$ *and some* ${}_i\lambda_j$ *may be zero, and weight* $\mu = \{\mu_1, \mu_2, \ldots, \mu_l\}$, *where some* μ_i *may be zero, the Kostka number is one,* $K_{\{\lambda_{(i)}\}\mu} = 1$, *if and only if there exist a choice of indices* $0 = j_0 < j_1 < \cdots < j_k = l$ *such that for* $p = 1, 2, \ldots, k$ *there exist sequences of sub-partitions* $\{\lambda^p\} = \{{}_1\lambda^p,\ {}_2\lambda^p, \ldots,\ {}_m\lambda^p\}$ *(where each* ${}_i\lambda^p = \{{}_i\lambda_{j_{p-1}+1},\ {}_i\lambda_{j_{p-1}+2}, \ldots,\ {}_i\lambda_{j_p}\}$*) and sub-partitions* $\mu^p = \{\mu_{j_{p-1}+1}, \mu_{j_{p-1}+2}, \ldots, \mu_{j_p}\}$ *with the following properties.*

1. $|\{\lambda^p\}| = \sum_{i=1}^{m} |{}_i\lambda^p| = |\mu^p|$ *and* $\sum_{i=1}^{m} \sum_{q=j_{p-1}+1}^{l} {}_i\lambda_q \geq \sum_{q=j_{p-1}+1}^{l} \mu_q$, *where* l *is any integer such that* $j_{p-1} + 1 \leq l \leq j_p$.

2. *For only one* ${}_i\lambda^p \in \{\lambda^p\}$ *either* ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_{p-1}+2} = \cdots = {}_i\lambda_{j_p-1}$ *or* ${}_i\lambda_{j_{p-1}+2} = {}_i\lambda_{j_{p-1}+3} = \cdots = {}_i\lambda_{j_p-1} = {}_i\lambda_{j_p}$. *For all other* ${}_i\lambda^p \in \{\lambda^p\}$ *then* ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_{p-1}+2} = \cdots = {}_i\lambda_{j_p}$.

Proof. Assume that $K_{\{\lambda_{(i)}\}\mu} = 1$. Therefore there exists a unique semi-standard Young tableau with the shape of the annealed partition $\tilde{\lambda}$ and weight μ: $K_{\tilde{\lambda}\mu} = 1$. From the annealed partition $\tilde{\lambda}$ we know that we can split μ and $\tilde{\lambda}$ or each ${}_i\lambda$ into a set of sub-partitions such that $|\tilde{\lambda}^p| = \sum_{i=1}^{m} |{}_i\lambda^p| = |\mu^p|$ for ${}_i\lambda^p = \{{}_i\lambda_{j_{p-1}+1},\ {}_i\lambda_{j_{p-1}+2}, \ldots,\ {}_i\lambda_{j_p}\}$ and $\mu^p = \{\mu_{j_{p-1}+1}, \mu_{j_{p-1}+2}, \ldots, \mu_{j_p}\}$. Furthermore since $K_{\{\lambda_{(i)}\}\mu} > 0$ then we know that $\sum_{i=1}^{m}\sum_{q=1}^{l} {}_i\lambda_q \geq \sum_{q=1}^{l}\mu_q$ for all l. Thus $\sum_{i=1}^{m} \sum_{q=j_{p-1}+1}^{l} {}_i\lambda_q \geq \sum_{q=j_{p-1}+1}^{l} \mu_q$ for all $j_{p-1} + 1 \leq l \leq j_p$ since $\sum_{i=1}^{m} \sum_{q=j_{p-1}+1}^{j_p} {}_i\lambda_q = \sum_{q=j_{p-1}+1}^{j_p} \mu_q$ for all p.

Next we consider some possible cases for the shape of each $\tilde{\lambda}^p$. Since $K_{\tilde{\lambda}\mu} = 1$ then we know that all ${}_i\lambda^p \in \{\lambda^p\}$ have shape either ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_{p-1}+2} = \cdots = {}_i\lambda_{j_p-1}$ or ${}_i\lambda_{j_{p-1}+2} = {}_i\lambda_{j_{p-1}+3} = \cdots = {}_i\lambda_{j_p-1} = {}_i\lambda_{j_p}$ (some ${}_i\lambda$ may have ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_p}$). Furthermore from Lemma 4.1.2 we know that columns of the same length that are from at least two distinct ${}_i\lambda \in \lambda$ must be filled identically. If for all but at most one ${}_i\lambda^p \in \{\lambda^p\}$, we have ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_{p-1}+2} = \cdots = {}_i\lambda_{j_p}$, and that at most one ${}_i\lambda^p$ has shape ${}_i\lambda_{j_{p-1}+1} = {}_i\lambda_{j_{p-1}+2} = \cdots = {}_i\lambda_{j_p-1}$ or ${}_i\lambda_{j_{p-1}+2} = {}_i\lambda_{j_{p-1}+3} = \cdots = {}_i\lambda_{j_p-1} = {}_i\lambda_{j_p}$. Then the conditions are satisfied for this sequence of sub-partitions. If more than one $i^p_\lambda \in \{\lambda^p\}$ has ${}_i\lambda_{j_{p-1}+1} \neq {}_i\lambda_{j_p}$ then we can consider the only two possible cases separately.

Case 1: If ${}_i\lambda_{j_{p-1}+1} \neq {}_i\lambda_{j_{p-1}+2}$ for at least two ${}_i\lambda \in \{\lambda_{(i)}\}$ then we know that all columns of length $j_{p-1}+1$ must be filled identically in the unique sequence of semi-standard Young tableaux of shape $\{\lambda_{(i)}\}$ and weight μ. Thus only one distinct value can fill the rightmost ${}_i\lambda_{j_{p-1}+1} - {}_i\lambda_{j_{p-1}+2}$ columns of row ${}_i\lambda_{j_{p-1}+1}$. Furthermore we know that the leftmost ${}_i\lambda_{j_{p-1}+2}$ columns of each ${}_i\lambda^p$ has the same number of rows as the length of μ^p. Thus each row ${}_i\lambda_q \in {}_i\lambda^p$ must be filled entirely with entries of value q since otherwise there is a column with values that do not strictly increase down the length of the column. Since μ is a partition then $\mu_{j_{p-1}+l} \not> \mu_{j_{p-1}+1}$ for any $l > 1$. Therefore the only possibility is that each ${}_i\lambda^p$ with entries from μ^p, which requires $\mu_{j_{p-1}+1} = \sum_{i=1}^{m} {}_i\lambda_{j_{p-1}+1}$.

Now we can consider splitting this sequence of sub-partitions and the sub-partition of the weight into two. Define $\{\lambda^{p+}\} = \{({}_1\lambda_{j_{p-1}+1}), ({}_2\lambda_{j_{p-1}+1}), \ldots, ({}_m\lambda_{j_{p-1}+1})\}$, $\{\lambda^{p-}\} = \{({}_i\lambda_{j_{p-1}+2}, {}_i\lambda_{j_{p-1}+3}, \ldots, {}_i\lambda_{j_p})|$ for all $1 \leq i \leq m\}$, $\mu^{p+} = \{\mu_{j_{p-1}+1}\}$ and $\mu^{p-} = \{\mu_{j_{p-1}+2}, \mu_{j_{p-1}+3}, \ldots, \mu_{j_p}\}$. The sequences of sub-partitions $\{\lambda^{p+}\}$ and $\{\lambda^{p-}\}$ and corresponding weightsμ^{p+} and μ^{p-} satisfy the above conditions.

Case 2: If ${}_i\lambda_{j_p-1} \neq {}_i\lambda_{j_p}$ for at least two ${}_i\lambda \in \{\lambda_{(i)}\}$ then we know that in the unique sequence of semi-standard Young tableaux of shape $\{\lambda_{(i)}\}$ and weight μ all columns of length $j_p - 1$ must be filled identically. Thus only one distinct value can fill the rightmost ${}_i\lambda_{j_p-1} - {}_i\lambda_{j_p}$ columns of each row ${}_i\lambda_{j_p-l}$ for $1 \leq l \leq j_p - j_{p-1} - 1$. Furthermore we know that the leftmost ${}_i\lambda_{j_{p-1}+2}$ columns of each ${}_i\lambda^p$ has the same number of rows as there are distinct valued entries from μ^p. Thus each row ${}_i\lambda_q \in_i \lambda^p$ must be filled entirely with entries of value q since otherwise there is a column with entries that do not strictly increase down the length of the column. Since μ is a partition then $\mu_{j_p} \not> \mu_{j_{p-1}+l}$ for any $1 \leq l \leq j_p - j_{p-1}$, and so entries of value j_p cannot fill any row besides ${}_i\lambda_{j_p}$ since otherwise, according to Lemma 4.1.2, it would have to fill the entirety of that row in each ${}_i\lambda \in \{\lambda_{(i)}\}$ and that would imply there exists a $\mu_{j_{p-1}+l} < \mu_{j_p}$. Therefore we can consider splitting this sequence of partitions and weight into two. Define $\{\lambda^{p+}\} = \{({}_i\lambda_{j_{p-1}+1}, {}_i\lambda_{j_{p-1}+2}, \ldots, {}_i\lambda_{j_p-1})|$ for all $1 \leq i \leq m\}$, $\{\lambda^{p-}\} = \{({}_1\lambda_{j_p}), ({}_2\lambda_{j_p}), \ldots, ({}_m\lambda_{j_p})\}$, $\mu^{p+} = \{\mu_{i_{p-1}+1}, \mu_{i_{p-1}+2}, \ldots, \mu_{i_p-1}\}$, and $\mu^{p-} = \{\mu_{i_p}\}$. The sequences of sub-partitions $\{\lambda^{p+}\}$ and $\{\lambda^{p-}\}$ and corresponding weights μ^{p+} and μ^{p-} satisfy the above conditions.

Thus all sequences of sub-partitions $\{\lambda^p\}$ and weights μ^p either satisfy the above conditions or can be split into other sequences of sub-partitions $\{\lambda^{p+}\}$ and $\{\lambda^{p-}\}$ and weights μ^{p+} and μ^{p-}, which satisfy the conditions. For any sequence of partitions $\{\lambda_{(i)}\}$ and weight μ with the property $K_{\{\lambda_{(i)}\}\mu} = 1$ we can create this set of sequences of sub-partitions and sub-partitions of weights that satisfy the above conditions.

Next assume that for some sequence of partitions $\{\lambda_{(i)}\}$ and weight μ that the above conditions hold. Since $\displaystyle\sum_{i=1}^{m}\sum_{q=j_{p-1}+1}^{l} {}_i\lambda_q \geq \sum_{q=j_{p-1}+1}^{l} \mu_q$ for all p and l then we know that $\displaystyle\sum_{i=1}^{m}\sum_{q=1}^{l} {}_i\lambda_q \geq \sum_{q=1}^{l}\mu_q$ for all l. Thus from Lemma 4.2.1 we know that $K_{\{\lambda_{(i)}\}\mu} > 0$.

We will show by induction that boxes in the Young diagram for $\{\lambda^p\}$ must only be filled by values from μ^p for all p. Since columns must be filled with entries in strictly increasing order, we know that any entry of value less than or equal to j_1 cannot fill a row ${}_i\lambda_{j_1+q}$ for any $q > 0$ for any ${}_i\lambda \in \{\lambda_{(i)}\}$ (otherwise there would be somewhere in that column with an entry beneath another entry of equal or greater value). Thus all entries of value less than or equal to j_1 must fill spaces in $\{\lambda^1\}$. Since $|\{\lambda^1\}| = |\mu^1|$ then these entries completely fill all rows of ${}_i\lambda^1$ for every ${}_i\lambda \in \{\lambda^1\}$, and so no numerical entry from μ^j can fill any space in $\{\lambda^1\}$ for $j \neq 1$. Next assume for some p that all $\{\lambda^{p-q}\}$ are filled only by entries from μ^{p-q} for any $0 < q < p$. We know that values from μ^p cannot fill any space in any $\{\lambda^{p+r}\}$ for any $r > 0$ (otherwise there would exist a column with a number of lesser or equal value below another value). Furthermore all the spaces in every $\{\lambda^{p-q}\}$ are filled completely by values from μ^{p-q} so no number from μ^p can fill any other space besides one in λ^p. Since $|\{\lambda^p\}| = |\mu^p|$ then all the spaces in each ${}_i\lambda^p \in \{\lambda^p\}$ must be filled only with values from μ^p. Thus by induction values from μ^p must completely and only fill spaces in $\{\lambda^p\}$.

Let $\gamma_p = j_p - j_{p-1} - 1$ be the number of rows in each ${}_i\lambda^p \in \{\lambda^p\}$ and the length of μ^p. Since we know from above that there exists at least one sequence of Young semi-standard tableaux of shape $\{\lambda_{(i)}\}$ and weight μ, then the leftmost ${}_i\lambda_{j_{p-1}+1} - {}_i\lambda_{j_p}$ columns of each ${}_i\lambda^p$ must be filled such that row ${}_i\lambda_{j_{p-1}+q}$ is filled only with values $j_{p-1}+q$. Otherwise a number will be in a column beneath an entry of equal or greater value. If there is a ${}_\eta\lambda^p \in \{\lambda^p\}$ such that ${}_\eta\lambda_{j_{p-1}+1} \neq {}_\eta\lambda_{j_p}$ then we can define $\displaystyle\tilde{\mu_q} = \mu_q - \sum_{r=j_{p-1}+1}^{\eta-1} {}_r\lambda_q - \sum_{j=\eta+1}^{m} {}_j\lambda_q$. Using subtraction on both sides from $\displaystyle\sum_{i=1}^{m}\sum_{q=j_{p-1}+1}^{l} {}_i\lambda_q \geq \sum_{q=j_{p-1}+1}^{l}\mu_q$, we find that $\displaystyle\sum_{q=j_{p-1}+1}^{l} {}_\eta\lambda_q \geq \sum_{q=j_{p-1}+1}^{l}\tilde{\mu_q}$ for all $l \geq j_{p-1}+1$. Furthermore $|{}_\eta\lambda^p| = |\tilde{\mu^p}|$ and either ${}_\eta\lambda_{j_{p-1}+1} = {}_\eta\lambda_{j_{p-1}+2} = \cdots = {}_\eta\lambda_{j_p-1}$ or ${}_\eta\lambda_{j_{p-1}+2} = {}_\eta\lambda_{j_{p-1}+3} = \cdots = {}_\eta\lambda_{j_p}$. By these conditions on a single sub-partition ${}_\eta\lambda^p$ and $\tilde{\mu^p}$ we know that $K_{{}_\eta\lambda^p\tilde{\mu^p}} = 1$ from Theorem 3.0.2. Thus there is only one way to fill the spaces of each ${}_i\lambda^p \in \{\lambda^p\}$ with values from μ^p. Since this is true for all ${}_\eta\lambda \in \{\lambda^p\}$ for any arbitrary $\{\lambda^p\}$, we can therefore conclude that

$K_{\{\lambda_{(i)}\}\mu} = 1.$ □

4.3 Unique Weights with Kostka Number of One

Gates, Goldman, and Vinroot showed conditions on an individual partition that guarantees that only one weight can fill it in a unique way (ie. only one weight μ for the partition λ has $K_{\lambda\mu} = 1$) [3]. Drawing on Theorem 4.2.2 we will generalize their result for sequences of partitions $\{\lambda_{(i)}\}$. Below we will show when $J(\{\lambda_{(i)}\}) = 1$, which is to say when is $\tilde{\lambda}$ the only weight on $\{\lambda_{(i)}\}$ with a Kostka number of one?

Corollary 4.3.1. *For a sequence of partitions*

$$\{\lambda_{(i)}\} = \{({}_1\lambda_1,\ {}_1\lambda_2, \ldots,\ {}_1\lambda_l), ({}_2\lambda_1,\ {}_2\lambda_2, \ldots,\ {}_2\lambda_l), \ldots, ({}_m\lambda_1,\ {}_m\lambda_2, \ldots,\ {}_m\lambda_l)\}$$

then $J(\{\lambda_{(i)}\}) = 1$ *if and only if for every* j *either* $\sum_{i=1}^{m} {}_i\lambda_j - \sum_{i=1}^{m} {}_i\lambda_{j+1} \leq 1$ *or there exist at least two* ${}_i\lambda \in \{\lambda_{(i)}\}$ *such that* ${}_i\lambda_j - {}_i\lambda_{j+1} \geq 1$.

Proof. For any sequence of partitions $\{\lambda_{(i)}\}$ we know that the partition $\mu = (\sum_{i=1}^{m} {}_i\lambda_1, \sum_{i=1}^{m} {}_i\lambda_2, \ldots, \sum_{i=1}^{m} {}_i\lambda_l)$ has the property $K_{\{\lambda_{(i)}\}\mu} = 1$, since we can separate the sequence of tableaux into sequences of sub-tableaux with shape consisting of one row and weight of length one. According to Theorem 4.2.2 this guarantees that $K_{\{\lambda_{(i)}\}\mu} = 1$. Thus $J(\{\lambda_{(i)}\}) \geq 1$ for all $\{\lambda_{(i)}\}$.

Assume for the sequence of partitions $\{\lambda_{(i)}\}$ that there exists some r and j such that ${}_r\lambda_j - {}_r\lambda_{j+1} = \sum_{i=1}^{m} {}_i\lambda_j - \sum_{i=1}^{m} {}_i\lambda_{j+1} > 1$. Then we can define $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ where $\mu_s = \sum_{i=1}^{m} {}_i\lambda_s$ for all $s \neq r$ or $r+1$. We also define $\mu_r = -1 + \sum_{i=1}^{m} {}_i\lambda_r$ and $\mu_{r+1} = 1 + \sum_{i=1}^{m} {}_i\lambda_{r+1}$. Then there exists sequences of sub-tableaux where all but one sequence have tableaux of length one: $\{\lambda^j\} = \{({}_1\lambda_j), ({}_2\lambda_j), \ldots, ({}_m\lambda_j)\}$ for $j \neq r$ or $r+1$ and $\{\lambda^r\} = \{({}_1\lambda_r,\ {}_1\lambda_{r+1}), ({}_2\lambda_r,\ {}_2\lambda_{r+1}), \ldots, ({}_m\lambda_r,\ {}_m\lambda_{r+1})\}$. These sequences of sub-tableaux are all completely filled, each $\{\lambda^j\} \trianglerighteq \mu^j$, and the shape of each sub-tableaux is rows of at most two different sizes, so according to Theorem 4.2.2 $K_{\{\lambda_{(i)}\}\mu} = 1$. From above we know there is at least one other weight on $\{\lambda_{(i)}\}$ such that the Kostka number is one so $J(\lambda) \geq 2$.

Next assume for the sequence of partitions $\{\lambda_{(i)}\}$ that for every j either $\sum_{i=1}^{m} {}_i\lambda_j - \sum_{i=1}^{m} {}_i\lambda_{j+1} \leq 1$ or there exist at least two ${}_i\lambda \in \{\lambda_{(i)}\}$ such that ${}_i\lambda_j - {}_i\lambda_{j+1} \geq 1$. Now consider any $\mu = (\mu_1, \mu_2, \ldots, \mu_{l(\mu)})$ such that $K_{\{\lambda_{(i)}\}\mu} = 1$.

By definition, for any j such that both ${}_i\lambda_j \neq {}_i\lambda_{j+1}$ and ${}_k\lambda_j \neq {}_k\lambda_{j+1}$ for $k \neq i$ then these two rows cannot be a part of any sequence of sub-tableaux that satisfy the conditions of Theorem 4.2.2. Thus we will examine only the possible sequences of sub-tableaux that are guaranteed by Theorem 4.2.2. For any sequence of sub-partitions $\{\lambda^j\}$ where for any ${}_i\lambda \in \{\lambda^j\}$ then ${}_i\lambda = ({}_i\lambda_{k_{j-1}+1},\ {}_i\lambda_{k_{j-1}+2}, \ldots,\ {}_i\lambda_{k_j})$. Let ${}_r\lambda \in \{\lambda^j\}$ be the only sub-partition where ${}_r\lambda_{j_k-1}$ can be a different size than ${}_r\lambda_{k_j}$.

Since all ${}_i\lambda \in \{\lambda^j\}$ except ${}_r\lambda^j$ have all parts of equal size and the length of each ${}_i\lambda^j$ is either equal to 0 or the length of μ^j, then any μ^j that satisfies Theorem 4.2.2 must fill each row completely with the values such that ${}_i\lambda_p$ is filled only with entries of value p. Thus we are left with only the values that fill the sub-partition ${}_r\lambda$ and that ${}_r\lambda_p - {}_r\lambda_{p+1} \leq 1$ for all p so according to Corollary 3.0.3 then $J({}_r\lambda^j) = 1$. Since we know that for $K_{\{\lambda_{(i)}\}\mu} = 1$ then $K_{\{\lambda^j\}\mu^j} = 1$ and thus there can only be one semi-standard Young tableau of shape ${}_r\lambda$ and weight generated by the remaining unused values of μ. However we have shown that this implies that the weight must be equal to ${}_r\lambda$, which is the shape of the sub-tableau. Thus the only possible μ^j such that $K_{\{\lambda^j\}\mu^j} = 1$ must be $\mu^j = (\sum_{i=1}^{m} {}_i\lambda_{k_{j-1}+1}, \sum_{i=1}^{m} {}_i\lambda_{k_{j-1}+2}, \ldots, \sum_{i=1}^{m} {}_i\lambda_{k_j})$. Therefore there is only one possible μ such that $K_{\{\lambda_{(i)}\}\mu} = 1$ and so $J(\{\lambda_{(i)}\}) = 1$. $\square$

Chapter 5

Conclusion

5.1 Summary

We began our investigation by considering properties of two partitions that equate to characteristics about their Kostka multiplicity. From Behrenstein and Zelevinskii [1] we found a typo in one of their results about when $K_{\lambda\mu} = 1$ for two partitions λ and μ. By using combinatorial arguments, we found the correct conditions. We extended these arguments to revisit a result from Gates, Goldman and Vinroot [3] about which partitions λ have only one possible weight with a Kostka multiplicity of one, which is to say when is $J(\lambda) = 1$.

After individual partitions, we generalized our results to sequences of partitions $\{\lambda_{(i)}\}$. The annealed partition $\tilde{\lambda}$ enabled us to use the results about individual partitions to understand sequences of partitions. With that tool, we found both when a sequence of partitions $\{\lambda_{(i)}\}$ and weight μ have a Kostka number of one and when $J(\{\lambda_{(i)}\}) = 1$.

Results about partitions and Kostka numbers play a role in representation theory and have deep implications in symmetry groups. Symmetry groups in turn have been crucial in simplifying computational simulations of vibrational states of complex molecules. Vibrational states of inorganic molecules are particularly important in understanding the electrical properties of potential semiconductor components. In organic molecules, different vibrational states or steady conformations can guide understanding of biological molecules and reactions.

5.2 Future Work

The next question is to define the conditions on sequences of partitions with depth and a given weight such that the Kostka multiplicity would be one. A sequence of partitions with depth assigns an integer value to each partition in the sequence.

From a Young diagram perspective, each partition would be considered as a set of cubes where one surface is the same as a normal Young diagram, but each cube has the same number of cubes behind it as the integer valued depth. We can consider our results for sequences of partitions to all have a depth of one. Any filling of the partition with weights must fill all boxes at all the depths with the same weight. Thus if the top, left box is filled with a weight 1 then all the boxes behind it must be filled with a one.

We note that it is highly non-trivial to find even when a sequence of partitions with depth and a given weight have at least one valid filling. Furthermore we can also re-formulate this partition theory question into a graph theory problem. The Kostka number can be considered the number of distinct, balanced sub-graphs of a graph defined by vertices representing a part of the weight and one possible sum of depths to equal that part of the weight. The directed, weighted edges between vertices are defined with value equal to a depth used in the partition of the part of the weight of the source vertex and directed to any vertex with a greater value than the value of the depth.

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