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Relaxation of Planar Graphs With dΔ≥2 and No 4-Cycles

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Abstract

First, let a graph be a set of vertices (points) and a set of edges (lines) connecting these vertices. Further, let a planar graph be a graph that can be represented on the plane without any edges crossing. Define a (c_1, c_2, c_k)-coloring of graph G as $\psi : G \to \{1, 2, k\}$ such that for every i, $1 \leq i \leq k, G[V_i]$ has maximum degree at most c_i, where $G[V_i]$ represents the subgraph induced by the vertices colored with i. The Four Color Theorem by Appel and Haken (1973) states that all planar graphs are 4-colorable. The Bordeaux Conjecture (2003) postulates that planar graphs with no 5-cycles and without intersecting triangles is 3-colorable. Liu, Li and Yu (2014) proved that planar graphs without intersecting triangles and 5-cycles is (2,0,0) colorable. We prove that all planar graphs without 4-cycles and no less than two edges between triangles is also (2,0,0) colorable.

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1 Introduction

Graph theory is the study of mathematical structures composed of pairwise relationships between points (called vertices). These relationships are represented by lines (called edges) between vertices. A *planar graph* is a graph that can be drawn on a plane without edges crossing. Vertices that are connected with an edge are *adjacent*, or *neighbors*. If there are n neighbors of vertex v, then we say that the vertex has a degree of n, or $deg(x) = d(x) = n$. An m-cycle is a sequence of m linked edges where the first and last vertex is the same, and no edge is used more than once. If a vertex is on the path of a cycle, then the vertex is *incident* to the cycle.

Consider some cycle, g. Let $int(g)$ be the vertices inside, or internal to, a cycle g, and $ext(g)$ be the vertices outside of g. Let a *separating cycle* be a cycle where $int(g) \neq 0$ and $ext(g) \neq 0$. Note that in a *nonseparating cycle* (also called a face), either $int(g) = 0$ or $ext(g) = 0$. An n-face f has a degree of n, or $deg(f) = d(f) = n$. Faces are also described by the degrees of their incident vertices. For example, in Figure 1, 3-face xyz is a $(3, 3, 4)$ face and 5-face $abcde$ is a $(3, 3, 4, 3, 5)$-face.

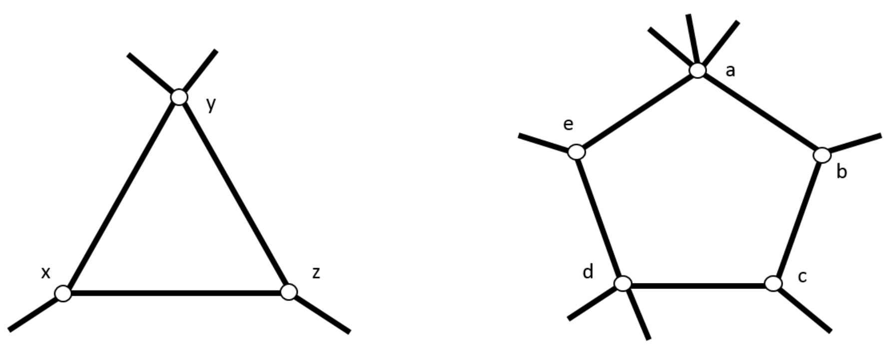

Figure 1: A $(3, 3, 4)$-face and a $(3, 3, 4, 3, 5)$-face

A *pendant* face to vertex v is a face with an incident 3-vertex x, where x is adjacent to v. The 3-vertex is called a *dependent* vertex of v. In Figure 2, m is the dependant vertex of n, and face mop is a pendant 3-face of n.

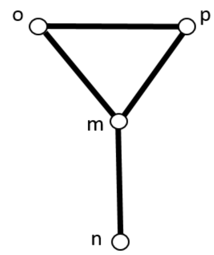

Figure 2: A pendant

Graph coloring is a process where vertices in a graph are assigned colors, represented by numbers. In a properly colored graph, each vertex v is assigned a color distinct from all vertices adjacent to v. If x is the minimum number of colors used to obtain a proper coloring of a graph, then we say the graph is "x-colorable". The smaller the x, the more difficult it is to have a proper coloring of a graph.

The Four Color Theorem, proved in 1973 by Appel and Haken (see [1]-[2]), states that all planar graphs are at most 4-colorable. The 3-colorability of planar graphs is NP-complete. Since 1973, mathematicians have been working to determine the necessary and sufficient structures for which planar graphs are 3-colorable. Grözch [5] proved that planar graphs with no 3-cycles are 3-colorable. Steinberg [9] proposed the following conjecture:

Conjecture 1.1 *Steinberg: Planar graphs without 4-cycles or 5-cycles are 3-colorable.*

Havel [6] proposed that there might be an m such that the distance between 3-cycles in a planar graph G is at least m, then G is 3-colorable. A conjecture by Borodin and Raspaud [4] combines Havel and Steinberg for a proposed set of conditions for 3-colorability in planar graphs.

Conjecture 1.2 *The Bordeaux (2003) Conjecture: Planar graphs with no 5-cycles and without intersecting triangles are 3-colorable.*

Both conjectures are NP-complete, but some progress has been made with relaxed colorings and similar conditions.

In a relaxed coloring, a vertex can have some number of adjacent vertices colored similarly to itself. More exactly, (c_1, c_2, c_k)-coloring of graph G as $\psi : G \to \{1, 2, k\}$ such that for every i, $1 \leq i \leq k, G[V_i]$ has maximum degree at most c_i, where $G[V_i]$ represents the subgraph induced by the vertices colored with i. That is, a vertex colored with i may be adjacent to c_i vertices also colored with i. A nicely colored vertex is colored with i and has a maximum of $c_i - 1$ adjacent vertices also colored with i, where $c_i \geq 1$.

Owen Hill et al. [7] proved that planar graphs without cycles of length 4 or 5 are $(3, 0, 0)$ colorable. Xu [10] proved that planar graphs without adjacent triangles or 5-cycles is $(1, 1, 1)$-colorable. Note that $adjacent\ faces$ have an incident vertex or vertices in common. Yang and Yerger [11] proved that planar graphs without intersecting triangles or 4-cycles is both $(2, 1, 0)$ colorable and $(4, 0, 0)$-colorable. Borodin and Glebov [3] proved that planar graphs without 5-cycles and where 3-cycles are not closwer than two edges apart is 3-colorable. Liu, Li and Yu [8] proved that planar graphs without intersecting triangles and 5-cycles is (2,0,0) colorable. In a similar vein, we will prove that all planar graphs without 4-cycles and no less than two edges between triangles is also (2,0,0) colorable.

2 Preliminaries and Definitions

Before we can begin with the proof, some notation and terminology must be explained.

Definition Let $d_\triangle$ be the distance between 3-cycles on some graph H, measured by the number of edges between these cycles.

Definition Let $\mathcal{G}$ be the set of planar graphs where there are no 4-cycles and $d_\triangle \geq 2$. Further, let $G \in \mathcal{G}$ be the graph in $\mathcal{G}$ such that $|G| \leq M$ for $M \in \mathcal{G}$.

Definition Let ϕ be a $(2, 0, 0)$ coloring of some induced subgraph H of the minimum counterexample $G \in \mathcal{G}$.

Definition We say that coloring ϕ_H on $H \subseteq G$ can be *superextended* to G if some $(2,0,0)$-coloring ϕ_H can be extended to G where $\phi_H(v) \neq \phi_H(u)$ for $v \in H$ and $u \in G \backslash H$.

We wish to prove that:

Theorem 2.1 *Every triangle, 5-cycle and 6-cycle of graph $G \in \mathcal{G}$ is superextendable.*

Assuming Theorem 2.1, we can prove Corollary 2.2. This proof is modeled after Theorem 1 in in a paper by Xu [10].

Corollary 2.2 *Every graph in $\mathcal{G}$ is $(2,0,0)$-colorable.*

Proof Let G be a graph in $\mathcal{G}$. If G has no triangles, then G is 3-colorable by Grözch's theorem, and so it is also $(2,0,0)$-colorable. If G contains a triangle, 5-cycle or 6-cycle, then any $(2,0,0)$-coloring of the cycle can be superextended to G by Theorem 2.1. $\square$

To prove Theorem 2.1, we are studying a graph $G \in \mathcal{G}$ that is a minmum counterexample to the theorem, containing a cycle C that is a 3, 5 or 6-cycle where a $(2,0,0)$ coloring ϕ of some subgraph H of G cannot be superextended to G. Let $|C| = r$.

Definition Let a "bad x-cycle" for $x \in \{3,5,6\}$ be a cycle f for which a $(2,0,0)$ coloring of $G \backslash int(f)$ cannot be extended to G. All bad 3-cycles and bad 5-cycles can be found in Figure 3.

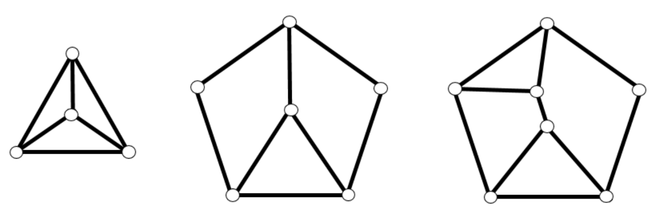

Figure 3: Bad 3-cycles and bad 5-cycles

Lemma 2.3 *There are no bad 3-cycles or 5-cycles in G.*

Proof We know that there can be no bad 3-cycles, else $d_\triangle < 2$, or there would be a 4-cycle in G, contradicting the construction of G. So, we consider 5-cycles.

Let $deg(f) = 5$. Suppose $int(f)$ is one 3-vertex. Then, either $d_\triangle < 2$, or there is a 4-cycle, contradicting the construction of G.

In a second scenario, $int(f)$ contains exactly two adjacent 3-vertices. However, then there would again be a 4-cycle in G, or $d_\triangle < 2$. $\square$

Before we can study separating cycles, we consider vertices with very small degrees.

Lemma 2.4 *There are no 2^- vertices in $G \backslash K$, where K is the outer cycle of G.*

Proof Assume to the contrary that a 2^--vertex v is in $G' = G\backslash K$. Note that $d_\triangle \geq 2$ and $\sigma(G') < \sigma(G)$. Because G is the minimum counterexample to Lemma 2.1, we know that $G'\backslash\{v\}$ is $(2,0,0)$-colorable. We can reintroduce v and properly color it. So, G is $(2,0,0)$, but this is a contradiction. $\square$

The following lemma is similar to Lemma 1 in B. Xu's [10] paper.

Lemma 2.5 *Minimum counterexample $G \in \mathcal{G}$ does not contain separating triangles, 5-cycles or 6-cycles.*

Proof Suppose $|C| = 3, 5$

If C is a separating 3- or 5-cycle, then C is superextendable in both $G\backslash ext(C)$ and $G\backslash int(C)$, and hence is superextendable in G, which contradicts the choice of C. So, we may assume, without loss of generality, that C is the boundary of the outer face of f_o of G. Recall that for any cycle C' of G we still use C' to represent the vertex set of C'.

Let $C' \neq C$ be a triangle or 5-cycle. By Lemma 2.3 that there are no bad 3 or 5-cycles in G. So, $int(C') = \emptyset$, which means that then C is superextendable in $G\backslash int(C')$ and C' is superextendable in $G\backslash ext(C')$, and hence C is superextendable in G. So, we have a contradiction. This means that there are no separating 3-cycles or 5-cycles in G.

To prove that there are no separating 6-cycles in G, we must prove that there are no bad 6-cycles in G.

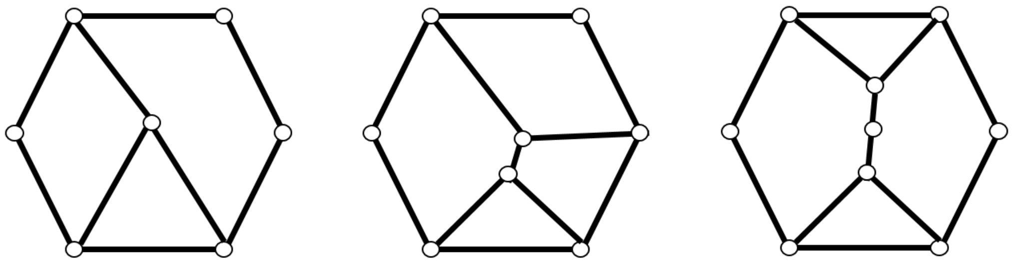

Figure 4: Bad 6-cycles

Consider 6-cycle g, where $int(g)$ is one 3-vertex. Such a 6-cycle would have a 4-cycle, contradicting the construction of G.

Second, consider a 6-cycle where $int(g)$ has two adjacent 3-vertices. By assumption, $d_{\triangle} \geq 2$, and there are no 4-cycles. So, there must be a third vertex in $int(g)$ that is adjacent to both of the 3-vertices. Because it is an internal vertex, by Lemma 2.4, it must have a degree of 3^{+}. However, this means that there are two 3-faces and two 5-faces in g. We just proved that both 3-cycles and 5-cycles are not separating, so this bad 6-cycle is not in G.

Thus, there are no bad 6-cycles in G. Similar to the above argument for separating 3-cycles and 5-cycles, since there are no bad 6-cycles in G, there are no separating 6-cycles in G. $\square$

Now we know that C is the outer boundary cycle of G. Let vertex $v \in G \backslash C$ be an *internal* vertex. The following lemma was inspired by Xu's [10] paper.

Lemma 2.6 *C, the outer boundary cycle of G, is chordless, and for $x, y \in C$, with $xy \notin E(C)$, $N(x) \cap N(y) \subseteq C$*

Proof The conclusion is trivial if $r = 3$. So, we suppose that $r = 5$ or $r = 6$.

If C has a chord, then it separates the 5-cycle into a 3-cycle and a 4-cycle, contradicting the construction of G. If $|C| = 6$, then the graph would be separated into a 4-cycle and a 5-cycle, contradicting the construction of G, or a 3-cycle and a 5-cycle. In the second case, one of these would have to be a separating cycle, contradicting Lemma 2.5.

We now consider x, y such that $xy \notin E(C)$. Suppose that there exists $v \in N(x) \cap N(y)$ where $v \notin E(C)$. Then, the inside of C would be comprised of a 5-cycle and a 4-cycle, which, again, is impossible.

So, we can say that C is chordless, and for $x, y \in C$, with $xy \notin E(C)$, $N(x) \cap N(y) \subseteq C$.

□

3 Reducible Configurations

We now consider the structure of G itself, and narrow down the possible structures in G. The following lemma is a foundational one for our paper and is also Lemma 2.3 in the paper by C. Yang and C. Yerger [11].

Lemma 3.1 *Suppose the three neighbors v_1, v_2, v_3 of an internal 3-vertex v of $G \backslash C$ are pre-colored in a (d_1, d_2, d_3)-coloring of $G \backslash v$. Assume that the neighbors of v are also internal. If v_i is colored by i, $d(v_i) \geq d_i + 3$ for $1 \leq i \leq 3$.*

Proof If vertices v_1, v_2, v_3 do not all use different colors in a (d_1, d_2, d_3)-coloring of $G \backslash v$, v can be properly colored. Therefore there would exist a (d_1, d_2, d_3)-coloring of G that superextends the coloring, contradicting G as a minimum counterexample. So we can assume WOLOG that v_i is colored by i for $1 \leq i \leq 3$. Suppose to the contrary that $d(v_i) \leq d_i + 2$ for some i. From Lemma 2.4, $d(v_i) \geq 3$, thus $d_i \geq 1$. We consider two cases.

Case 1: If v_i is nicely colored by i, v can be colored by i, a contradiction.

Case 2: Suppose v_i is not nicely colored by i. Then, v_i has at least d_i neighbors colored by i. Since $d(v_i) \leq d_i + 2$ by assumption and v_i has an uncolored neighbor v, the neighbors of v_i are not colored by all three colors. That is, v_i can be properly recolored and so v can be colored by i. Then, G is $(2, 0, 0)$-colorable, a contradiction. □

We can conclude the following lemma from Lemma 3.1:

Lemma 3.2 *Every internal 3-vertex in G with three internal neighbors is adjacent to at least one* 5^+*-vertex.*

Lemma 3.3 *If a vertex* $v \in G \backslash C$ *has exactly two colored neighbors* ($v_1, v_2 \in G \backslash C$) *where* v_1 *and* v_2 *are adjacent to each other, then* v *can be recolored to 1 unless the neighbors are* 6^+ *and* 3^+*, or* 5^+ *and* 4^+*. Further, if the two neighbors are not adjacent to each other,* v *can be recolored to 1 unless one vertex has degree* 5^+*.*

Proof Consider a vertex $v \in G \backslash C$ with two colored neighbors, also in $G \backslash C$, and are adjacent to each other. Call these neighbors v_1 and v_2. Assume to the contrary that vertex $v \in G \backslash C$ cannot be recolored to 1, and $deg(v_1) = 3$ and $deg(v_2) \leq 5$.

If v is colored WOLOG with 2, then at least one outer neighbor must be colored with 1. Note that if v_1 is the only neighbor colored with 1, then it is either nicely colored or can be recolored, so that v can be colored with 1, a contradiction.

So, we assume that v_2 is colored with 1. Vertex v_2 must have two neighbors colored with 1, and neighbors colored each with 2 and 3, so that v_2 cannot be recolored. Else, v can be recolored.

Suppose v_1 is colored with 1. If v_1 is not properly colored, it can be properly recolored, and so v_2 is nicely or properly colored with 1. Then, v can be colored with 1. So, v_1 must be colored with 3. This means that the outer neighbors of v_2 (the neighbors that are not v or v_1) are colored with 1, 1, and 2.

Note that v_1 must also have an outer neighbor colored with 1, so that v_1 and v_2 cannot exchange colors, making v colorable with 1. However, then v_1 can be recolored to 2, v_2 can be recolored to 3, and v can be colored with 1. So, either v_1 must have a second outer neighbor colored with 2, so that v_2 is not recolorable, or v_1 must have a fourth outer neighbor, colored with 3, so that v_1 cannot be recolored after v_2 is colored. However, this means either $deg(v_1) \geq 6$ or $deg(v_2) \geq 4$, a contradiction.

To prove the further part, we assume that v has exactly two nonadjacent colored neighbors, v_1 and v_2. Further, we assume that v cannot be colored with 1 and $deg(v_1), deg(v_2) \leq 4$. If v cannot be colored with 1, then one of its outer neighbors must be colored with 1. Suppose $\phi(v_1) = 1$. Then, v_1 must have two neighbors colored with 1, else v can be colored with 1. Further, v_1 must have a neighbor colored with 2, and another colored with 3. So, $deg(v_1) \geq 5$, a contradiction. $\square$

Based on Lemma 3.3, we introduce a new definition.

Definition A 3-vertex v is *special* if it is adjacent to two 4^- vertices and all three vertices are incident to a 5-face. Note that by Lemma 3.1, v is also the dependent to a 5^+ vertex.

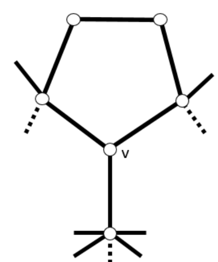

Figure 5: Special vertex v

Lemma 3.4 *Given some vertex $v \in G \backslash C$, where m is the total number of adjacent internal special vertices, and n is the total number of internal 3-vertices adjacent to v and incident to $(3, 3, 5^-)$ and $(3, 4, 4)$ faces, $m + n \leq deg(v) - 2$.*

Proof Suppose to the contrary, there exists some $v \in G \backslash C$ where $m + n > deg(v) - 2$. That is, let $m + n = deg(v) - 1$.

Then, we consider $G' = G \backslash \{v\}$. We know that $G' \subseteq \mathcal{G}$, so G' is $(2, 0, 0)$ colorable. We attempt to extend this coloring to G. By Lemma 3.3, we can recolor all adjacent "special" vertices to 1. This means all the neighbors of v are colored with 1, save one. We can properly color v distinct from the final vertex, and so G is $(2, 0, 0)$-colorable, a contradiction. $\square$

3.1 5-Faces

Lemma 3.5 *Consider a 5-face in G with one 4- or 5-vertex, u, and four 3-vertices. The two 3-vertices adjacent to u must be adjacent to a second 5^+ vertex.*

Proof Consider a $(3,3,3,3,5)$ or $(3,3,3,3,4)$ face, f, in G, where the 4- or 5- vertex incident to f is called u. Assume to the contrary that a 3-vertex on f and adjacent to u (call it v) is the dependent vertex of a 4^- vertex. Delete all the 3 vertices, and nicely recolor both u and the outer adjacent vertex of v. Properly color the 3-vertices in order, leaving v for last. Note that it has one properly colored neighbor and two nicely colored neighbors. So, it can be colored. □

As a corollary to Lemma 3.5, if u is a 4-vertex or a 5-vertex, u cannot be incident to two $(3,3,3,3,d(u))$ faces that share an edge that is also incident to u.

Consult the following figure for Lemmas 3.6- 3.8.

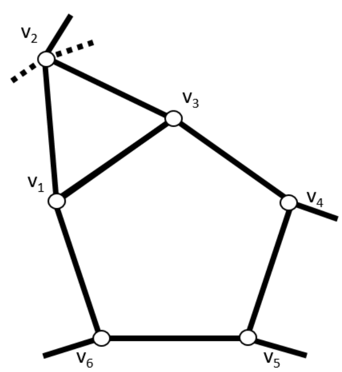

Figure 6: 6-cycle with chord

Lemma 3.6 *Given an internal 6-cycle $v_1 ... v_6$ in $G \backslash C$ with chord $\{v_1, v_3\}$, where $deg(v_1) = deg(v_3) = 3$, $deg(v_2) \leq 5$, then $deg(v_5) \geq 4$.*

Proof Assume to the contrary that this structure is internal in G. Let the vertices be labeled $v_1 ... v_6$, with $deg(v_1) = deg(v_3) = 3$, $deg(v_2) \leq 5$, and $deg(v_5) = 3$.

Note that by Lemma 3.10, $deg(v_4) = deg(v_6) \geq 5$.

Consider $H = G[v_6, v_4]\backslash\{v_1, v_2\}$. Call the identified vertex v'. Note that there can be no path of length 3 from v_6 to v_4 in G that does not go through v_1, v_2, else there would be a separating 5-cycle in G. Further, since v_1, v_3 are triangular in G, neither v_4 nor v_6 can be triangular. So, $d_\triangle \geq 2$ in H. Finally, there can be no path of length 4 from v_4 to v_6, else there would be a separating 6 cycle in G. Finally, since $\sigma(H) < \sigma(G)$, H must be $(2,0,0)$-colorable. Call the coloring ϕ_H.

We can extend this $(2,0,0)$ coloring of H to G. Let $\phi_G(x) = \phi_H(x)$ for $x \in G\backslash\{v'\}$ Nicely recolor v_2, and properly color v_5. Note that at least one of v_4, v_6 is nicely colored. WOLOG, suppose v_4 is nicely or properly colored. Properly color v_1, and then color v_3. Then, G is $(2,0,0)$ colorable, a contradiction.

So, a 5-face in G cannot be incident to two or fewer nonadjacent vertices with degree 4^+ if neither of the 4^+ vertices are triangular, and the 5-face shares an edge with a $(3,3,5^-)$ face. $\square$

Lemma 3.7 *Given a 6-cycle $v_1 ... v_6$ in $G\backslash C$, with chord $\{v_1, v_3\}$, and $deg(v_6) = 5$, $deg(v_3) = 4$, and $deg(v_1) = deg(v_2) = 3$, at least one of v_4, v_5 has degree 4^+*

Proof Assume to the contrary that such a 6-cycle is in $G\backslash C$, where $deg(v_4) = deg(v_5) = 3$. We consider $H = G\backslash\{v_i : i \neq 6\}$. We know that $\sigma(H) < \sigma(G)$, so $H \in \mathcal{G}$, and H is $(2,0,0)$ colorable. We extend the coloring to G by nicely recoloring v_6, then properly coloring v_5, v_4, v_3, v_2 in that order. Then, v_1 can be colored, and G has a $(2,0,0)$ coloring, a contradiction. So, either $deg(v_4) \geq 4$, or $deg(v_5) \geq 4$ $\square$

Lemma 3.8 *Given an internal 6-cycle $v_1 ... v_6$ with chord $\{v_1, v_3\}$, where $deg(v_1) = deg(v_3) = 3$, $deg(v_2) \leq 5$ and one of v_4, v_6 has degree 5, with two outer pendant $(3,3,5^-)$ faces, $(3,4,4)$ faces, and/or adjacent special 3-vertices, then the 6-cycle is not in G.*

Proof Suppose to the contrary that such a 6-cycle is in $G\backslash C$. WOLOG, we suppose that v_4 is the 5-vertex with two outer pendant $(3,3,5^-)$ faces, $(3,4,4)$ faces, or adjacent special 3-vertices. Then, we consider $H = G[v_4, v_6]\backslash\{v_1, v_3\}$. Let the newly identified vertex be called v'.

We know that no new 3-cycles were created in H, else there would be a separating 5-cycle in G, contradicting Lemma 2.5. By the same lemma, no 4-cycles are created with this identification. Finally, since v_4, v_6 are each one edge away from a 3-face in G, the identification of these two vertices does not decrease the distance between 3-faces any less than 2 edges. So, $d_\triangle \geq 2$ in H, and H has no 4-cycles. So, $H \subseteq \mathcal{G}$, and since $\sigma(H) < \sigma(G)$, we know that there exists some $(2,0,0)$ coloring of H, called ϕ_H. We now attempt to extend the coloring to G. Call the coloring of G ϕ_G. Let $\phi_G(x) = \phi_H(x)$ for $x \in H \backslash \{v'\}$, and let $\phi_G(v_4) = \phi_G(v_6) = \phi_H(v') = \alpha$. Nicely recolor v_2.

If WOLOG $\alpha = 2$, then we can properly color v_2, and color v_1. Then, ϕ_G is a $(2,0,0)$ coloring of G, a contradiction.

If $\alpha = 1$, and $\phi_G(v_5) = 2$ or 3, or $\phi_G(v_5) = 1$ and has two or fewer neighbors colored with 1, then at least one of v_4, v_6 is nicely colored with 1. WOLOG, suppose v_4 is nicely colored. Then, we properly color v_1, and color v_3. So, ϕ_G is a $(2,0,0)$ coloring of G, a contradiction.

If $\alpha = 1$ and $\phi_G(v_5) = 1$, and v_5 has three neighbors colored with 1, then we remove the color of v_4, and recolor the two pendant special 3-vertices or 3-faces with 1 by Lemma 3.3, and then properly color v_4. Then, we properly color v_1 and v_2 and color v_3. Then, G is $(2,0,0)$ colorable, a contradiction. So, a 6-cycle $v_1 ... v_6$ with chord $\{v_1, v_3\}$, where $deg(v_1) = deg(v_2) = deg(v_3) = 3$, and one of v_4, v_6 has degree 5, with two outer pendant $(3,3,5^-)$ faces, $(3,4,4)$ faces, and/or adjacent special 3-vertices is not in G. $\square$

3.2 3-Faces

Lemma 3.9 *Given a $(4,*,*)$-face in $G \backslash C$, if the outer adjacent vertices of the 4-vertex are all internal and have degree 4^-, then the 3-face must be $(4, 4^+, 5^+)$.*

Proof Assume to the contrary that that there exists a $(3,4,5^-)$ face, with the outer vertices of the 4-vertex are internal and have degree 4^-. Call the 4-vertex incident to the

3-face v, and the 3-vertex incident to the 3-face z. We know that $G\backslash\{v,z\}$ is $(2,0,0)$ colorable. We can extend the coloring to G as follows: nicely recolor the 5-vertex and the outer vertices of v, then properly color z. Then, v can be colored. $\square$

The following figure applies to Lemmas 3.10- 3.12:

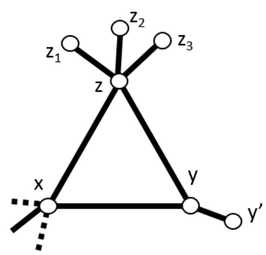

Figure 7: $(3, 5^-, 5)$-face

Lemma 3.10 *Given a* $(3, 3, 5^-)$ *face* $f \in G\backslash C$*, where all vertices adjacent to* f *are also* $G\backslash C$*, the outer neighbors of the 3-vertices incident to* f *must have degree* 5^+*.*

Proof Consider a 3-face $f = xyz$ in $G\backslash C$, where $deg(x) \leq 5$, and $deg(y) = deg(z) = 3$, and all vertices incident to x, y and z are also internal vertices (Figure 7). Assume to the contrary that the outer neighbors of one of the 3-vertices (WOLOG say y) has degree 4^-. Call this outer neighbor of y by y'. Consider $H = G\backslash\{y, z\}$. Because $H \subseteq G$, we know that there exists a superextension ϕ_H on H of ϕ. So, H is $(2,0,0)$ colorable. If x, y' and are not nicely colored in H, then they can be properly recolored.

We extend the coloring to G by properly coloring z, and then coloring y, obtaining the desired contradiction. $\square$

Lemma 3.11 *If a* $(3, 5^-, 5)$ *face,* $f = xyz$ *is in* $G\backslash C$*, with* $deg(x) \leq 5$ *and* $deg(y) = 3$*, with outer neighbors of* z *called* $z_1, z_2, z_3 \in G\backslash C$ *in clockwise order, then for at least one* $a \in \{z_1, z_3\}$*, either* $deg(a) \geq 4$*, or* a *is not "special"–that is, at least one of the other two neighbors of* a *have degree* 5^+*.*

Proof Let $f = xyz$, $deg(x) \leq 5$ and $deg(y) = 3$. Assume to the contrary that all outer vertices of z, labeled z_1, z_2, z_3 are special. Assume that x is "next to" z_1–i.e. that they are incident to the same 5-face.

For the first part, we let $S = \{z\}$, $M = \{x, z_2\}$, $N = \{z_1, z_3\}$.

Consider $H = G[M]\backslash S$. Call the newly identified vertex v'. Note that no new 3-cycles are created with this identification, else there would be a separating 5-cycle in G, contradicting Lemma 2.5. Further, since the vertices in S are incident to a 3-face in G, we know that $d_\triangle \geq 2$ in H. We also know that there can be no new 4-cycles in G, else there would be a separating 6-cycle in G, contradicting Lemma 2.5. So, $H \in \mathcal{G}$, and since $\sigma(H) < \sigma(G)$, we know that there exists a $(2, 0, 0)$ coloring, say ϕ_H of H.

We attempt to extend this coloring to G. Let $\phi_G(x) = \phi_H(x)$ for $x \in H\backslash\{v'\}$, and for $a \in M, \phi_G(a) = \phi_H(v') = \alpha$. Properly (re)color y and the vertices in N. Vertex z can be colored unless $\alpha = 1$, and one of the vertices in N is also colored with 1. Note that y cannot be colored with 1, else it can be properly recolored, and then z can be colored. So, one of the vertices in N is colored with 1, and the other is colored with either 2 or 3. WOLOG, suppose $\phi_G(z_1) = 2$. Then, z_1 can be recolored to 1 by Lemma 3.3, and z can be colored with 2. Thus, G is $(2, 0, 0)$ colorable, a contradiction. $\square$

Lemma 3.12 *If a $(3, 5^-, 5)$ face, $f = xyz$ is in $G\backslash C$, with $deg(x) \leq 5$ and $deg(y) = 3$, with outer neighbors of z called $z_1, z_2, z_3 \in G\backslash C$ in clockwise order, then z_2 is nonspecial.*

Proof Let $f = xyz$, $deg(x) \leq 5$ and $deg(y) = 3$. Assume to the contrary that both z_2 is special.

Consider $H = G[x, z_3]\backslash\{y, z\}$. Call the newly identified vertex v'. Note that no new 3-cycles are created with this identification, else there would be a separating 5-cycle in G, contradicting Lemma 2.5. Further, since y is incident to a 3-face in G, $d_\triangle \geq 2$ in H. Finally, no new 4-cycles are in H, else there would be a separating 6-cycle in G, contradicting Lemma 2.5. So, $H \in \mathcal{G}$, and since $\sigma(H) < \sigma(G)$, we know that there exists a $(2, 0, 0)$ coloring, say ϕ_H of H.

We attempt to extend this coloring to G. Let $\phi_G(x) = \phi_H(x)$ for $x \in H\backslash\{v'\}$. Let $\phi_G(z_1) = \phi_G(z_3) = \phi_H(v') = \alpha$. Recolor z_2 to 1 by Lemma 3.3 and properly color y.

If WOLOG $\alpha = 2$, then z can be colored, extending the coloring ϕ_H to G, a contradiction. So, $\alpha = 1$.

Then, WOLOG z_1 must be colored with 2 and y must be colored with 3.

Note that x must be at least nicely colored with 1, else x can be properly recolored and z can be colored with 1, extending the coloring to G, a contradiction. So, suppose x is nicely colored with 1. Then, either y can be recolored to 1 or 2, depending on the coloring of its outer vertex. Then, z can be colored, a contradiction. So, z_2 must be nonspecial. □

The following figure will assist in studying Lemma 3.13:

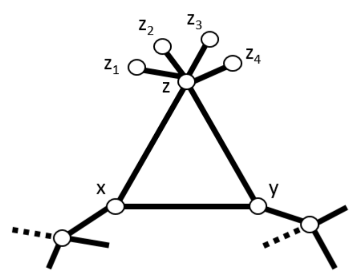

Figure 8: $(3, 3, 6)$-face

Lemma 3.13 *Consider a* $(3, 3, 6)$ *face,* $f = xyz$ *is in* $G\backslash C$*, with deg*$(z) = 6$*. Let all outer vertices of the 3-face be in* $G\backslash C$ *also. If at least one of the outer vertices of* x *and* y *has degree* 4^-*, then none of the outer neighbors of* z*, called* $z_i : i = 1, 2, 3, 4$*, are special. Also, if both of the outer vertices of* x *and* y *have degree* 5^+*, then no more than two* z_i *are special.*

Proof For the first part, suppose to the contrary that a $(3, 3, 6)$ face in G only has three or fewer "not special" 3-faces–that is, assume that at least one of $z_i : i = 1, 2, 3, 4$ is special. For the "also" part, assume to the contrary that z can have three of the adjacent z_i special.

By symmetry, we consider z_1 and z_2. Let $H_1 = G[x, z_2, z_4]\backslash\{z, y\}$ or $H_2 = G[y, z_1, z_3]\backslash\{z, x\}$.

If z_1 is special, then consider H_1. If z_2 is special, then consider H_2. For H_1, let $S = \{x, z_2, z_4\}$, and for H_2, let $S = \{y, z_1, z_3\}$. Finally, let the triangular vertex not in S be called n. Call the triangular vertex in S, m.

WOLOG consider H_1. Call the identified vertex v'. Note that no new 3-cycles are created with this identification, else there would be a separating 5-cycle in G, contradicting Lemma 2.5. Further, since v was incident to a 3-face, we know that $d_{\triangle} \geq 2$. Note that the only path with length 4 between the vertices is between z_4 and x, and the deletion of y separates this path, so a 4-cycle is not constructed. Also, no new 4-cycles are in H_1, else there would be a separating 6-cycle in G. So, $H_1 \in \mathcal{G}$, and since $\sigma(H_1) < \sigma(G)$, we know that there exists a superextension ϕ_{H_1} of H_1, where ϕ_{H_1} is a $(2, 0, 0)$ coloring of H_1.

We will extend the coloring to G with coloring ϕ_G.

Let $\phi_G(a) = \phi_{H_1}(a)$ for $a \in H_1 \backslash \{v'\}$, and for $b \in S$, $\phi_G(b) = \phi_{H_1}(v') = \alpha$. Suppose we are just doing the first part.

Recolor z_1 to 1 by Lemma 3.3. Then, properly color y.

Suppse $\alpha = 2$ (WOLOG). If the other z_i is improperly colored with 1, then z can be colored with 3 unless n is colored with 3. We now have two possibilities. Either the outer vertex of n has degree 4^-, or m has degree 4^-. If the outer vertex of n has degree 4^-, then that outer vertex can be nicely recolored, and n can be colored with 1, and z can be colored. If the outer vertex of n is 5^+, then the outer vertex of m has degree 4^-. If n cannot be colored with 1, then m can be colored with 1, and n can be colored with 2. Then, z can be colored with 3. So, $\alpha = 1$.

Then, if z cannot be colored, n is uniquely colored with (2 or 3), then it can be recolored to (3 or 2), else 1, and z can be colored. So, the 6-vertex of a $(3, 3, 6)$ face, where at least one of the outer vertices of a triangular 3-vertex has degree 4^-, does not have special outer neighbors.

To prove the "also" part, begin with the coloring of H_1 mapped onto G, with the removed vertices still uncolored. If $\alpha = 1$, then v can be properly colored.

WOLOG, suppose $\alpha = 2$. Then, either v can be colored with 3, else 1, or one or both of the special vertices has two neighbors colored with 1. Then, those special vertices can be recolored to 2, and v can be nicely colored with 1. So, if x and y have outer vertices with degree 5^+, z must have no more than two adjacent special vertices. $\square$

4 Discharge Procedure

Finally, we will solve the theorem through a discharge procedure.

Let V, E, F denote the total number of vertices, edges and faces in G, respectively. Let vertex $v \in G$ have an initial charge of $\mu(v) = 2d(v) - 6$, and face $f \in G$ be $\mu(f) = d(f) - 6$. Finally, recall that $2E = \sum_{f \in F} d(f) = \sum_{u \in V} d(u)$, and by Euler's Theorem, $V - E + F = 2$.

So, $\sum_{u \in V} \mu(u) + \sum_{f \in F} \mu(f)$
$= \sum_{u \in V} (2d(u) - 6) + \sum_{f \in F} (d(f) - 6)$
$= 2 \sum_{u \in V} d(u) - 6 \sum_{u \in V} 1 + \sum_{f \in F} d(f) - 6 \sum_{f \in F} 1$
$= 2(2E) - 6(V) + (2E) - 6F$
$= 6(-V + E - F) = -12$

So, $\sum_{u \in V} \mu(u) + \sum_{f \in F} \mu(f) = -12 = \sum_{u \in V} \mu(u) + \sum_{f \in F \setminus C} \mu(f) + d(C) + 6 = 0$

Let $\mu^*(u)$ denote the charge of any $u \in V(G)$ after the discharge procedure, and $\mu^*(f)$ for any $f \in F(G)$ denotes the charge of a given face after the discharge procedure. We will prove that $\sum_{u \in V} \mu^*(u) + \sum_{f \in F} \mu^*(f) > 0$, contradicting Euler's Theorem and proving that the minimum counterexample, G, does not exist. Then, Theorem 2.1 is proven.

Let F_C be the faces incident to C, and F^* be the faces in G that are not in F_C. Similarly, let V_C be the vertices incident to C, and V^* be the set of vertices in G that are not in V_C.

Let an x-face with one vertex in V_C be F'_x, and an x-face with two or more vertices in V_C be F''_x, for $x \in \{3, 5\}$. Note that by Lemma 2.6, there cannot be a 3-face or 5-face where all vertices of the 3-face are on C.

For vertices in V^*:

R1. Given an internal 4-vertex:

(a) Give $1\frac{2}{5}$ to incident$(3,4,5^-)$-face in F^*, and 1 to incident $(4^+,4^+,4^+)$ faces and incident $(3,4,6^+)$ faces in F^*

(b) Give $\frac{2-a}{b}$ to each of the b incident 5-faces, where a is the amount of charge given to each 3-face

R2. For 3-faces in F^*, let internal triangular 5-vertices give $1\frac{2}{5}$ to incident $(3,3,5)$ faces, $1\frac{3}{5}$ to incident $(3,4,5)$ faces, $1\frac{1}{2}$ to incident $(3,5,5)$ faces, and 1 to both incident $(3,5,6^+)$ faces and to incident $(4^+,4^+,5)$ faces with no incident 6^+ vertices

R3. Given an internal 5^+ vertex:

(a) give $\frac{1-(a+b)}{m}$ to internal 5-faces, where a is the charge from the special 3-vertices that are incident to the 5-face, b is the charge from any 4-vertices that are incident to to the 5-face, and m is the number of 5^+ vertices incident to the 5-face.

(b) 1 to pendant $(3,3,3)$ faces and incident $(4^+,4^+5^+)$ faces, $\frac{4}{5}$ to pendant $(3,3,4)$ faces and pendant $(3,3,5)$ faces, and finally $\frac{1}{5}$ to pendant $(3,4,4)$ faces.

(c) $\frac{1}{5}$ to pendant 5-faces where the dependent vertex is special

R4. Let internal triangular 6^+ vertices give 3 to incident $(3,3,6^+)$-face in F^* 2 to incident $(3,4^+,6^+)$ faces in F^*

For vertices in V_C:

First, recall that C has a degree of $12+(deg(C)-6)=6+deg(C)$.

R5. Let C give

(a) $\frac{12}{5}$ to triangular 4-vertices in V_C that are incident to one F_3' and two F_5''

(b) 2 to all other triangular 3^+ vertices in V_C, 2-vertices, and nontriangular 3-vertices with a pendant $(3,3,5^-)$ or $(3,4,4)$ face and two incident F_5" faces.

(c) $\frac{6}{5}$ to all other vertices in V_C.

R6. After the charge of C has been distributed, let 3^+ vertices in V_C give

(a) $6-x$ to F'_x for $x \in \{3,5\}$ and $\frac{6-x}{2}$ to F''_x for $x \in \{3,5\}$

(b) 1 to pendant $(3,3,5^-)$ and $(3,4,4)$ faces and $\frac{1}{5}$ to pendant 5-faces where the dependent vertex is special and in V^*

4.1 Discharge Procedure For Faces

Discharge for Outer Cycle

First, we consider the outer cycle, C. Note that the degree of C is either 3, 5 or 6. Recall that C starts out with a charge of $deg(C)+6$.

Suppose $deg(C) = 3$. Then, by definition of G, V_C cannot contain any triangular vertices, or vertices with pendant 3-faces. V_C can contain up to two 2-vertices. By R5.b and R5.c, we give 2 to the 2-vertices and $\frac{6}{5}$ to the nontriangular 3^+ vertex with no pendants. So, we have $deg(C)+6-(2(2)+2) > 0$

Suppose $deg(C) = 5$. Then, V_C can contain a maximum of one triangular 4-vertices with one incident F'_3 face and two incident F''_5 faces. The triangular 4-vertices get $\frac{12}{5}$ from C by R5.a. All other vertices receive up to 2 from C. So, we have $deg(C)+6-(\frac{12}{5}+2(4)) > 0$

Finally, suppose $deg(C) = 6$. Then V_C can contain a maximum of two triangular 4-vertices with one incident F'_3 face and two incident F''_5 faces. By R5.a, the triangular 4-vertices get $\frac{12}{5}$ each from C. Up to two other vertices on C can be 2-vertices. The other two vertices must be 3^+ vertices, and if they are 3-vertices, they cannot have a pendant 3-face, else the 4-vertices would not have two incident F''_5 faces. So, each of the remaining two vertices get $\frac{6}{5}$ from C by R5.c. So, we have $deg(C)+6-(\frac{12}{5}(2)+2(2)+\frac{6}{5}(2)) > 0$

So, for all possible degrees of C, the charge of the outer face is greater than 0. If we can show that that the charge of all vertices and faces is nonnegative after the discharge procedure, then we will have proven that $\sum_{u \in V} \mu(u) + \sum_{f \in F \setminus C} \mu(f) + d(C) + 6 > 0$, obtaining the desired contradiction.

Discharge for Internal Faces

Let $f \in G \setminus C$ be some face.

Suppose $d(f) = 3$.

First, suppose $f \in F_C$.

If $f \in F_3'$, then f gets 3 from the vertex incident to f and in V_C by R6.a. So, $d(f) - 6 + 3 = 0$

If $f \in F_3''$, then f gets $\frac{3}{2}$ from each 3^+ vertex incident to f and in V_C by R6.a. So, $d(f) - 6 + (\frac{3}{2}(2)) = 0$

Next, suppose $f \in F^*$.

By R3.b, $(3,3,3)$ face gets 1 from each outer neighbor, since by Lemma 3.1, each outer neighbor has degree 5^+, or is on C and has the necessary charge to give 1 to the $(3,3,3)$ face. So $d(f) - 6 + (1(3)) = 0$

If f is a $(3,3,4)$-face, f gets $1\frac{2}{5}$ from the incident 4-vertex by R1.a, and at least $\frac{4}{5}$ from the outer neighbors of the 3-vertices by R3.b. So, $d(f) - 6 + (1\frac{2}{5}(1) + \frac{4}{5}(2)) = 0$

By R2. and R3.b, $(3,3,5)$ faces get $1\frac{2}{5}$ from the 5-vertex and at least $\frac{4}{5}$ from the two outer neighbors. So, $d(f) - 6 - (1\frac{2}{5} + \frac{4}{5}(2)) = 0$

If f is a $(3,4,4)$-face, f gets $1\frac{2}{5}$ from each incident 4-vertex by R1.a, and $\frac{1}{5}$ from the outer neighbor of the 3-vertex by R3.c. So, $d(f) - 6 + (1\frac{2}{5}(2) + \frac{1}{5}(1)) = 0$

If f is a $(3,4,5)$-face, f gets $1\frac{2}{5}$ from 4-vertex by R1.a and $1\frac{3}{5}$ from 5-vertex by R2.. So,$d(f) - 6 + (1\frac{2}{5}(1) + 1\frac{3}{5}(1)) = 0$

If f is a $(3,5,5)$-face, f gets $1\frac{1}{2}$ from each incident 5-vertex, by R2.. So, $d(f) - 6 + (1\frac{1}{2}(2)) = 0$

If f is a $(3,3,6^+)$-face, f gets 3 from the 6^+-vertex by R4.. So, we have $d(f)-6+(3)=0$

If f is a $(3,4^+,6)$ face, f gets 2 from the 6-vertex by R4. and 1 from the 4^+ vertex by R1.a, R2. and R4.. So, we have $d(f)-6+(2+1)=0$

If f is a $(4^+,4^+,4^+)$ face, then by R1. and, f gets 1 from each incident vertex by R3.b. So, $d(f)-6+(1(3))=0$.

Suppose $d(f)=5$.

First, we consider the faces in F_C:

If f is F_5', then by R6.a, f gets 1 from the vertex that is in V_C and $d(f)-6+1=0$

If f is F_5'', then by R6.a, f gets $\frac{1}{2}$ from two 3^+. So, $d(f)-6+(\frac{1}{2}(2))=0$

Next, we consider the faces in F^*:

If f is a $(4^-,4^-,4^-,4^-,4)$-face, by R1.b and R3.c, f gets $\frac{1}{5}$ from each outer 5^+ vertex adjacent to a special 3-vertex, and from any vertices in V_C that are adjacent to a special 3-vertex. Further, f gets $\frac{1}{5}$ or more from each 4-vertex by R1.b. So, $d(f)-6+(\frac{1}{5}(5))=0$, at least.

If f is a $(5^+,*,*,*,*,*)$-face, f gets at least $\frac{2}{5}$ from the four other vertices by R1., R3.a and R3.c. Finally, by R3.a, the 5^+ vertex gives the necessary charge for f to have final charge of 0.

4.2 Discharge Procedure for Vertices

Discharge for Vertices in V_C

First, we consider the vertices in V_C.

3^- **Vertices in** V_C If $d(v)=2$, v gets 2 from C by R5.b. We have $2d(v)-6+2=0$.

Next, suppose that $d(v)=3$. If v is triangular, or has a pendant $(3,3,5^-)$ or $(3,4,4)$-face with two incident F_5" faces, then it gets 2 from C by R5.b. Else, by R5.c, v gets $\frac{6}{5}$ from C.

1. If v is triangular, then it is incident to a F_3'' face. It can also be incident to a F_5'' face. By R6.a, v gives $\frac{3}{2}$ to the F_3'' and $\frac{1}{2}$ to the F_5'' face. We have $2d(v)-6+2-(\frac{3}{2}+\frac{1}{2})=0$

2. Suppose v has a pendant $(3,3,5^-)$ or $(3,4,4)$-face and is incident to two F_5'' faces. By R6.a, v gives $\frac{1}{2}$ to the F_5'' faces, and by R6.b. We have $2d(v)-6+2-(1+\frac{1}{2}(2))=0$

3. If v does not have a pendant 3-face and is not triangular, then v gets $\frac{6}{5}$ from C by R5.c. Further, v is incident to two F_5'' faces and adjacent to a special vertex. By R6.a, vertex v gives $\frac{1}{2}$ to each F_5'' face and $\frac{1}{5}$ to the adjacent special vertex by R6.a. We have $2d(v) - 6 + \frac{7}{5} - (\frac{1}{2}(2)) > 0$

4-Vertices in V_C Next, suppose $deg(v) = 4$.

Triangular 4-Vertices in V_C If v is incident to a F_3' face and two F_5'' faces, then v gets $\frac{12}{5}$ from C by R5.a. If v is incident to a F_3' face but incident to less than two F_5'' faces, or is incident to a F_3'' face, v gets 2 by R5.b.

1. We consider the first scenario. Vertex v gives 3 to F_3' by R6.a, $\frac{1}{2}$ to each F_5'' face by R6.a, and $\frac{1}{5}$ to any adjacent special 3-vertices not in V_C by R6.a. We have $2d(v) - 6 + \frac{12}{5} - (3 + \frac{1}{2}(2) + \frac{1}{5}(2)) = 0$

2. If v is incident to a F_3'' face, then it can also be incident to a F_5' face and a F_5' face . Vertex v gives 1 to F_5' by R6.a, $\frac{1}{2}$ to the F_5'' face by R6.a, and finally $\frac{3}{2}$ to the F_3'' face. Finally, v gives $\frac{1}{5}$ to any adjacent special 3-vertices not in V_C by R6.a. We have $2d(v) - 6 + 2 - (\frac{3}{2} + \frac{1}{2} + 1 + \frac{1}{5}) > 0$

Nontriangular 4-Vertices in V_C Finally, if v is nontriangular, then it gets $\frac{6}{5}$ from C by R5.c. Further, it is incident to three 5-faces: two F_5'' and one F_5'. Vertex v gives 1 to the F_5' face, $\frac{1}{2}$ to each F_5'' face by R6.a, and $\frac{1}{5}$ to any adjacent special 3-vertices not in V_C by R6.a. We have $2d(v) - 6 + \frac{6}{5} - (1 + \frac{1}{2}(2) + \frac{1}{5}(2)) > 0$

5^+ **Vertices in V_C** Next, we consider if $deg(v) \geq 5$.

Triangular 5^+ Vertices in V_C Suppose v is triangular. Note that v gets 2 from C by R5.b. Either v is incident to a F_3' face, or a F_3'' face.

If v is incident to a F_3' face, then v can also be incident to two F_5'' faces, and $(d(v)-4)$ F_5' faces. Further, v can be adjacent to $d(v)-2$ special 3-faces.Vertex v gives $\frac{3}{2}$ to the F_3'' face, 1 to the F_5' face, and $\frac{1}{2}$ to the F_5'' face by R6.a. We have $2d(v)-6+2-(3+\frac{1}{2}(2)+(d(v)-4)(1)+(d(v)-2)(\frac{1}{5}))>0$

If v is incident to a F_3'' face, then v can also be incident to one F_5'' face, and $(d(v)-3)$ F_5' faces. Finally, v can also be adjacent to a max of $d(v)-3$ special 3-faces. We have $2d(v)-6+2-(\frac{3}{2}+\frac{1}{2}+(d(v)-3)(1)+(d(v)-3)\frac{1}{5})>0$

Nontriangular 5^+ Vertices in V_C If v is nontriangular, v can be incident to two F_5'' faces, and $(d(v)-3)F_5'$ faces, as well as $\lceil\frac{d(v)-2}{2}\rceil$ pendant $(3,3,5^-)$ or $(3,4,4)$ faces, and $\lfloor\frac{d(v)-2)}{2}\rfloor$ adjacent special 3-faces. By R6.a, v gives $\frac{1}{2}$ to the F_5" faces and 1 to the F_5' faces. By R6.b, v gives 1 to each pendant $(3,3,5^-)$ or $(3,4,4)$ face, and $\frac{1}{5}$ to each adjacent special vertex. We have $2d(v)-6+\frac{6}{5}-(\frac{1}{2}(2)+(d(v)-3)+\lceil\frac{d(v)-2}{2}\rceil+(\lfloor\frac{d(v)-2)}{2}\rfloor)\frac{1}{5})\geq 0$ Note that for each added pendant $(3,3,5^-)$ or $(3,4,4)$ face greater than $\lceil\frac{d(v)-2}{2}\rceil$, there are two fewer incident F_5' faces and one fewer adjacent special vertex. Since $2+\frac{1}{5}>1$, v can be adjacent to any number of pendant 3-faces, up to $d(v)-2$ pendants, giving 1 to each. So, the sum of the charges of the vertices in V_C is nonnegative. We now consider internal vertices of G.

Discharge for Internal Vertices

4^- **Vertices** By Lemma 2.4, we know that if v is an internal vertex, then $d(v)\geq 3$. If $deg(v)=3$, then $2d(v)-6=0$. So, we consider $d(v)\geq 4$.

Suppose $d(v)=4$. We implement R1..

By construction, v never gives more than its initial charge to its incident faces, and it gives nothing to pendant faces. So, the charge of all 4-vertices after the discharge procedure is 0.

5^+ Vertices First, we will make some calculations that apply to 5^+ vertices. Let v be a 5^+ vertex.

Suppose v has one pendant $(3,3,5^-)$ face. By R3.b, v gives up to 1 to the pendant 3-face. Then, at least one vertex adjacent to v has a degree of 4^+ by Lemma 3.6. By R3.a, v gives up to $\frac{2}{5}$ to two incident 5-faces, and gives $\frac{3}{5}$ to the other $(d(v)-2)$ incident 5-faces. Note that by Lemma 3.4, v can be adjacent to up to $(d(v)-3)$ special 3-vertices. By R3.c, v gives $\frac{1}{5}$ to those special 3-vertices. Then, $2d(v)-6-(1+\frac{3}{5}(d(v)-2)+\frac{2}{5}(2)+\frac{1}{5}(d(v)-3)) \geq 0$ for $deg(v) \geq 5$.

Suppose v has n pendant $(3,4,4)$ faces and/or adjacent special vertices, where $0 \leq n \leq (d(v)-2)$. By R3.b, v gives $\frac{1}{5}$ to the pendant 3-face. Note that by Lemma 3.4, v can be adjacent to up to $(d(v)-3)$ special 3-vertices if there is a pendant $(3,4,4)$ face, and up to $d(v)-(n+2)$ special vertices. By R3.c, v gives $\frac{1}{5}$ to those special 3-vertices. By R3.a, v gives up to $\frac{3}{5}$ to $d(v)$ incident 5-faces. We have $2d(v)-6-(\frac{1}{5}(d(v)-2)+\frac{3}{5}(d(v))) \geq 0$

Suppose v has two pendant $(3,3,3)$ faces. Then, by R3.b, v gives 1 to the pendant $(3,3,3)$ faces. By Lemma 3.6, v has at least two adjacent 4^+ vertices, and so by R3.a, v gives $\frac{2}{5}$ to at least four incident 5-faces, and up to $\frac{3}{5}$ to all other incident 5-faces. Finally, by R3.b, v gives $\frac{1}{5}$ to any adjacent special 3-vertices. Note that if the vertices are separate, then there is at minimum five incident 5-faces with two adjacent 4^+ vertices.

If the pendant 3-faces are next to each other: $2d(v)-6-(2+\frac{2}{5}(4)+\frac{3}{5}(d(v)-5)+\frac{1}{5}(d(v)-4)) > 0$

If the pendant 3-faces are separated by one or more vertices: $2d(v)-6-(2+\frac{2}{5}(5)+\frac{3}{5}(d(v)-5)+\frac{1}{5}(d(v)-5)) > 0$

Suppose v has one pendant $(3,3,3)$ face, and one $(3,3,5)$, $(3,3,4)$, $(3,4,4)$ face. v gives 1 to the pendant $(3,3,3)$ face, and up to $\frac{4}{5}$ to the second pendant 3-face. By Lemma 3.6, v has at least one adjacent 4^+ vertices, two if the pendant 3-faces are not next to each other. By R3.a, v gives $\frac{2}{5}$ to at least four incident 5-faces, and up to $\frac{3}{5}$ to all other incident 5-faces. Finally, by R3.b, v gives $\frac{1}{5}$ to any adjacent special 3-vertices. If they are next to each other: $2d(v) - 6 - (1 + \frac{4}{5} + \frac{2}{5}(2) + \frac{3}{5}(d(v) - 3) + \frac{1}{5}) > 0$

If they are separate: $2d(v) - 6 - (1 + \frac{4}{5} + \frac{2}{5}(4) + \frac{3}{5}(d(v) - 4) + \frac{1}{5}(d(v) - 4)) > 0$

Suppose v has two pendant $(3, 3, 4)$, $(3, 3, 5)$ or $(3, 4, 4)$ faces. Vertex v gives up to $\frac{4}{5}$ to each pendant 3-face by R3.b. If the pendant 3-faces are next to each other, then at least two incident 5-faces must have two incident 5^+ vertices by Lemma 3.4 and Lemma 3.3, or two incident 4^+ vertices that are adjacent. Vertex v gives up to $\frac{1}{2}$ to these incident 5-faces, and $\frac{3}{5}$ to the other incident 5-faces. Note that by Lemma 3.6, if the faces are separate and at least one face is not a $(3, 4, 4)$ face, then one of the vertices adjacent to v must be a 4^+ vertex, and one incident 5-face must have two adjacent 4^+ vertices or two 5^+ vertices. If they are next to each other: $2d(v) - 6 - (\frac{4}{5}(2) + \frac{1}{2}(2) + \frac{3}{5}(d(v) - 3) + \frac{1}{5}(d(v) - 4) \geq 0$ If they are separate and at least one pendant is not a $(3, 4, 4)$ face: $2d(v) - 6 - (\frac{4}{5}(2) + \frac{2}{5}(2) + \frac{3}{5}(d(v) - 3) + \frac{1}{2} + \frac{1}{5}(d(v) - 4) \geq 0$

5 Vertices Suppose $deg(v) = 5$.

Triangular 5-Vertices Suppose a 5-face is incident to a $(5, *, *)$-face and to four 5-faces. Label these 5-faces clockwise from the 3-face f_1, f_2, f_3 and f_4. Note that vertices x, z and z_1 are incident to f_1, and vertices z_1, z_2 and z are incident to f_2, and so on. (See Figure 7).

If v is incident to a $(3, 3, 5)$ face and four 5-faces, then we know by Lemma 3.10 that f_1 and f_4 have at least two 5^+ vertices.By R3.a, v gives a max of $\frac{1}{2}$ to these 5-faces. By Lemma 3.11 and Lemma 3.12, v has one adjacent special vertex. By R3.c, v gives $\frac{1}{5}$ to this vertex. Finally, by R2., the 5-vertex gives $1\frac{2}{5}$ to the incident 3-face. We have $2d(v) - 6 - (1\frac{2}{5} + \frac{1}{2}(2) + \frac{3}{5}(2) + \frac{1}{5}) > 0$

Suppose v is incident to a $(3, 4, 5)$ face. Let f_1 be incident to the triangular 4-vertex. By Lemma 3.11 and Lemma 3.12, v can be incident to no more than two special vertices.

By R3.c, v gives $\frac{1}{5}$ to these vertices. Note that by Lemma 3.5 and Lemma 3.9, f_4 has two 4^+ vertices, and the non-v vertices in this face give at least $\frac{1}{2}$ to f_4. So, v gives up to $\frac{1}{2}$ to the face by R3.a. Also by R3.a, v gives $\frac{2}{5}$ to f_1. One of f_2, f_3 has two 5^+ vertices, and so v gives $\frac{1}{2}$ to this incident 5-face, and up to $\frac{3}{5}$ to the other 5-face by R3.a. Finally, by R2., v gives $1\frac{3}{5}$ to the incident 5-face. We have $2d(v) - 6 - (1\frac{3}{5} + \frac{2}{5} + \frac{1}{2}(2) + \frac{3}{5} + \frac{1}{5}(2)) = 0$

Suppose v is incident to a $(3, 5, 5)$ face and four 5-faces. By R3.a, v gives $\frac{3}{5}$ to the faces that have only one incident 4^+ vertex, and gives up to $\frac{1}{2}$ to the faces that have two 4^+ vertices. Note that by the same rule, f_1 only gets $\frac{2}{5}$ from the 5-face. Further, by R2., v gives $1\frac{1}{2}$ to the 3-face, and by R3.c, gives $\frac{1}{5}$ to the adjacent special vertex. Note that v has no more than one adjacent special vertex by Lemma 3.11 and Lemma 3.12. We have $2d(v) - 6 - (1\frac{1}{2} + \frac{2}{5} + \frac{3}{5}(3) + \frac{1}{5}) > 0$. Suppose v is incident to a $(3, 5, 6^+)$ face and four 5-faces. By R3.a, v gives $\frac{3}{5}$ to the faces that have only one incident 4^+ vertex, and $\frac{2}{5}$ to f_1. By R2., v gives 1 to the incident 3-face. Finally, by R3.c, v gives $\frac{1}{5}$ to up to three adjacent special vertices. We have $2d(v) - 6 - (1 + \frac{2}{5} + \frac{3}{5}(3) + \frac{1}{5}(3)) > 0$

Suppose v is incident to a $(4^+, 4^+, 5)$ face and four 5-faces. Note that there can be a max of three adjacent special vertices by Lemma 3.4, and by R3.c, v gives $\frac{1}{5}$ to each of them. Finally, v gives 1 to the incident 3-face by R2.. We have $2d(v) - 6 - (1 + \frac{1}{5}(3) + \frac{3}{5}(2) + \frac{2}{5}(2)) > 0$.

Nontriangular 5-Vertices We now suppose that v is nontriangular.

We suppose that v has three pendant $(3, 3, 5^-)$ or $(3, 4, 4)$ faces. By Lemma 3.4, v is not adjacent to any special vertices.

By R3.b, v gives up to 1 to pendant 3-faces, and up to $\frac{3}{5}$ to each incident 5-face by R3.a.

If they are all pendant $(3, 3, 3)$ faces, then by Lemma 3.8, v can only be incident to one 5-face. By R3.b, v gives 1 to the pendant 3-faces and by R3.a, v gives up to $\frac{3}{5}$ to the incident 5 face. We have $2d(v) - 6 - (1(3) + \frac{3}{5}) > 0$.

If there are two pendant $(3, 3, 3)$ faces, and one pendant $(3, 4, 4)$, $(3, 3, 4)$ or $(3, 3, 5)$

face, then v can have a maximum of two incident 5-faces by Lemma 3.8. We have $2d(v) - 6 - (1(2) + \frac{4}{5} + \frac{3}{5}(2)) = 0$

If there is one pendant $(3,3,3)$ face and two pendant $(3,3,4)$ or $(3,3,5)$ faces, then there is a maximum of two incident 5-faces by Lemma 3.8. By R3.a, v gives up to $\frac{3}{5}$ to each incident 5-face. By R3.b, vertex v gives 1 to the pendant $(3,3,3)$ face, and up to $\frac{4}{5}$ to the other two pendant 3-faces. We have $2d(v) - 6 - (1 + \frac{4}{5}(2) + \frac{3}{5}(2)) = 0$

If there is one pendant $(3,3,3)$ face, one pendant $(3,4,4)$ face, and one pendant $(3,4,4)$, $(3,3,4)$ or $(3,3,5)$ face, then by Lemma 3.8, v is incident to a maximum of three 5-faces. By R3.a, v gives up to $\frac{3}{5}$ to each incident 5-face. By R3.b, vertex v gives 1 to the pendant $(3,3,3)$ face, and up to $\frac{4}{5}$ to the other two pendant 3-face. We have $2d(v) - 6 - (1 + \frac{4}{5} + \frac{1}{5} + \frac{3}{5}(3)) > 0$

If there are three pendant $(3,3,4)$ or $(3,3,5)$ faces, then by Lemma 3.8, v is incident to up to three 5-faces. By Lemma 3.7, the faces incident to v either have two 5^+ vertices, or have a nontriangular 4^+ vertex incident to them. By R3.a, v gives up to $\frac{1}{2}$ to these incident 5 faces. We have $2d(v) - 6 - (\frac{4}{5}(3) + \frac{1}{2}(3)) > 0$

If there are two pendant $(3,3,4)$ or $(3,3,5)$ faces, with exactly one pendant $(3,4,4)$ face, there can be a maximum of four incident 5 faces. Note that by Lemma 3.7, at least two incident 5 faces either has two incident 5^+ vertices, or has a nontriangular 4^+ vertex incident to it. By R3.a, v gives up to $\frac{1}{2}$ to these incident 5-faces, and up to $\frac{3}{5}$ to the other incident 5 faces. We have $2d(v) - 6 - (\frac{4}{5}(2) + \frac{1}{5} + \frac{3}{5}(2) + \frac{1}{2}(2)) = 0$

If there are two or more pendant $(3,4,4)$ faces and one pendant $(3,3,4)$ or $(3,3,5)$ or $(3,4,4)$ face, v gives $\frac{1}{5}$ to the pendant $(3,4,4)$ faces and up to $\frac{4}{5}$ to the third pendant face by R3.b We have $2d(v) - 6 - (\frac{4}{5} + \frac{1}{5}(2) + \frac{3}{5}(4)) > 0$

6 Vertices

Triangular 6 Vertices Suppose $deg(v) = 6$.

If v is incident to a $(3,3,6)$ face and five 5-faces, we know that either the 3-face is pendant to at least one 4^{-}-vertex, or the 3-face is pendant to two 5^{+}-vertices.

If the 3-face is pendant to at least one 4^{-} vertex, then by Lemma 3.13, v is adjacent to no more than two special vertices, and these special vertices are triangular. By R4., v gives 3 to the incident 3-face, and v can give up to $\frac{1}{2}$ to each incident 5-face by R3.a. We have $2d(v) - 6 - (3 + \frac{1}{2}(5) + \frac{1}{5}(2)) > 0$

If the 3-face is pendant to two 5^{+} vertices, then by Lemma 3.13, v can be incident to up to two special vertices. By R3.c, v gives $\frac{1}{5}$ to these special vertices. By R3.a, v gives up to $\frac{1}{2}$ to two adjacent 5-faces, or $\frac{1}{3}$ to one and $frac35$ to the other. Since $\frac{1}{3} + \frac{3}{5} < \frac{1}{2}(2)$, we will assume that four incident 5-faces have two 5^{+} vertices. We have $2d(v) - 6 - (3 + \frac{3}{5} + \frac{1}{2}(4) + \frac{1}{5}(2)) > 0$ If v is incident to a $(3, 4^{+}, 6)$ face, by R3.c, v gives $\frac{1}{5}$ to up to four adjacent special vertices. By R3.a, v gives up to $\frac{3}{5}$ to all incident 5-faces, and by R4., v gives 2 to the incident 3-face. We have $2d(v) - 6 - (2 + \frac{1}{5}(4) + \frac{3}{5}(5)) > 0$

Nontriangular 6-Vertices Next, suppose v is nontriangular. First, suppose v has three pendant $(3,3,5^{-})$ or $(3,4,4)$ faces.

If v has three pendant $(3,3,3)$ faces, then by Lemma 3.6, v is incident to at least four 5-faces with two incident adjacent 4^{+} vertices. By R3.a, v gives up to $\frac{2}{5}$ to these incident 5 faces, and gives up to $\frac{3}{5}$ to the other two incident 5 faces. By R3.b, v gives 1 to the pendant $(3,3,3)$ faces, and by R3.c $\frac{1}{5}$ to one adjacent special vertex. We have $2d(v) - 6 - (1(3) + \frac{2}{5}(4) + \frac{3}{5}(2) + \frac{1}{5}) = 0$

If v has two pendant $(3,3,3)$ faces and one pendant $(3,3,4), (3,3,5)$ or $(3,4,4)$ face, by R3.b, v gives 1 to the pendant $(3,3,3)$ faces and up to $\frac{4}{5}$ to the other pendant. By Lemma 3.6, vertex v is incident to six 5-faces with two adjacent 4^{+} vertices if the pendants are all separated, four 5-faces with two adjacent 4^{+} vertices if two pendants are next to each other, and two 5-faces with two adjacent 4^{+} vertices if all three pendants are next to each other. By R3.a, v gives $\frac{2}{5}$ to these 5-faces, and $\frac{3}{5}$ to all other incident 5-faces.

Finally, by R3.c $\frac{1}{5}$ to one adjacent special vertex.

Separated pendants: $2d(v) - 6 - (1(2) + \frac{4}{5} + \frac{2}{5}(6) + \frac{1}{5}) > 0$

Two pendants next to each other: $2d(v) - 6 - (1(2) + \frac{4}{5} + \frac{2}{5}(4) + \frac{3}{5} + \frac{1}{5}) > 0$

All pendants next to each other: $2d(v) - 6 - (1(2) + \frac{4}{5} + \frac{2}{5}(2) + \frac{3}{5}(2) + \frac{1}{5}) > 0$

If v has one pendant $(3, 3, 3)$ face and two pendant $(3, 3, 4), (3, 3, 5)$ or $(3, 4, 4)$ faces, by R3.b, v gives 1 to the pendant $(3, 3, 3)$ faces and up to $\frac{4}{5}$ to the other two pendants. If they are all separate, then v is incident to four 5-faces with two adjacent 4^+ vertices, and two other 5-faces. If two pendants are next to each other, v is incident to two 5-faces with two adjacent 4^+ vertices, and three other 5-faces. Finally, if they are all next to each other, v is incident to four 5-faces. By R3.a, v gives $\frac{2}{5}$ to these 5-faces, and $\frac{3}{5}$ to all other incident 5-faces. Finally, by R3.c $\frac{1}{5}$ to one adjacent special vertex.

Separated pendants: $2d(v) - 6 - (1 + \frac{4}{5}(2) + \frac{2}{5}(4) + \frac{3}{5}(2) + \frac{1}{5}) > 0$

Two pendants next to each other: $2d(v) - 6 - (1 + \frac{4}{5}(2) + \frac{2}{5}(2) + \frac{3}{5}(3) + \frac{1}{5}) > 0$

All pendants next to each other: $2d(v) - 6 - (1 + \frac{4}{5}(2) + \frac{3}{5}(4) + \frac{1}{5}) > 0$

If v has three pendant $(3, 3, 4)$ or $(3, 3, 5)$ faces, then by R3.b, v gives $\frac{4}{5}$ to each pendant. If they are all separate, then by Lemma 3.6, at least four incident 5 faces have two adjacent 4^+ vertices. By R3.a, v gives up to $\frac{2}{5}$ to those 5-faces, and up to $\frac{3}{5}$ to all other incident 5 faces. Finally, by R3.c $\frac{1}{5}$ to one adjacent special vertex. We have $2d(v) - 6 - (\frac{4}{5}(3) + \frac{2}{5}(4) + \frac{3}{5}(2) + \frac{1}{5}) > 0$

If v has one pendant $(3, 4, 4)$ face and two $(3, 3, 4)$, $(3, 3, 5)$ or $(3, 4, 4)$ faces, by R3.b, v gives $\frac{1}{5}$ to the pendant $(3, 4, 4)$ face and up to $\frac{4}{5}$ to the other two pendants. v gives up to $\frac{3}{5}$ to each incident 5-face by R3.a and $\frac{1}{5}$ to an adjacent special vertex by R3.c. We have $2d(v) - 6 - (\frac{4}{5}(2) + \frac{1}{5} + \frac{3}{5}(6) + \frac{1}{5}) > 0$

If v has three pendant $(3, 3, 4)$, $(3, 3, 5)$ or $(3, 4, 4)$ faces, by R3.b, v gives $\frac{4}{5}$ to each pendant. By R3.a, v gives up to $\frac{3}{5}$ to each incident 5 face, and by R3.c, v gives $\frac{1}{5}$ to one special vertex. There can be up to five incident 5-faces. We have $2d(v) - 6 - (\frac{4}{5}(3) + \frac{3}{5}(5) + \frac{1}{5}) > 0$

Suppose v has four pendant 3-faces. By Lemma 3.4, v is not adjacent to any special vertices.

Suppose v is adjacent to three pendant $(3,3,3)$ faces, and a fourth pendant $(3,3,5^-)$ or $(3,4,4)$ face. By R3.b, v gives 1 to each pendant 3 face. Note that by Lemma 3.6, v is adjacent to at least two 5-faces that have two adjacent 4^+ vertices. By R3.a, v gives up to $\frac{2}{5}$ to these faces, and up to $\frac{3}{5}$ to the other incident 5 faces. Note that by construction, v can have a maximum of four incident 5-vertices. We have $2d(v) - 6 - (1(4) + \frac{2}{5}(2) + \frac{3}{5}(2)) = 0$

Suppose v is adjacent to two or fewer $(3,3,3)$ faces, and the other pendants are $(3,3,4)$, $(3,3,5)$, $(3,4,4)$ faces. By R3.b, v gives 1 to the pendant $(3,3,3)$ face and up to $\frac{4}{5}$ to the other pendant 3 faces. By construction, v can have a maximum of four incident 5-vertices. By R3.a, v gives up to $\frac{3}{5}$ to the incident 5 faces. We have $2d(v)-6-(1(2)+\frac{4}{5}(2)+\frac{3}{5}(4)) = 0$

7^+ vertices

Triangular 7^+ Vertices Suppose $deg(v) \geq 7$.

If v is incident to a$(3^+, 3^+, 7^+)$ face, then v gives 3 to the incident 3-face by R4. Note that v can be incident to up to $d(v) - 1$ 5-faces, and gives up to $\frac{3}{5}$ to each one by R3.a. Finally, v can have $d(v) - 2$ adjacent special 3-vertices, and by R3.c, gives $\frac{1}{5}$ to each one. So, $2d(v) - 6 - (3 + \frac{1}{5}(d(v) - 2) + \frac{3}{5}(d(v) - 1) \geq 0$.

Nontriangular 7^+ Vertices Suppose v is nontriangular. If we can show that $deg(v)$ is non-negative with $\lfloor \frac{d(v)}{2} \rfloor$ pendant $(3,3,5^-)$ or $(3,3,4)$ faces and $\lceil \frac{d(v)}{2} \rceil$ pendant $(3,3,5^-)$ or $(3,3,4)$ faces, giving the maximum charge to all incident 5-faces, then v can give the proper charge to any number of possible pendant 3-faces and possible incident 5-faces without having a net negative charge.

Suppose v has $\lfloor \frac{d(v)}{2} \rfloor$ pendant $(3,3,5^-)$ or $(3,3,4)$ faces. By Lemma 3.6, v is adjacent to up to $(d(v) - (\lfloor \frac{d(v)}{2} \rfloor + 2))$ special 3-vertices. By R3.b, v gives up to 1 to each pendant 3-face, and by R3.c, up to $\frac{1}{5}$ to the special 3-vertices. Finally, by R3.a, v gives $\frac{3}{5}$ to each incident 5-face. We have $2d(v) - 6 - (\lfloor \frac{d(v)}{2} \rfloor + \frac{3}{5}(d(v)) + \frac{1}{5}(d(v) - (\lfloor \frac{d(v)}{2} \rfloor + 2)) > 0$.

Suppose v has $\lceil \frac{d(v)}{2} \rceil$ pendant $(3, 3, 5^-)$ or $(3, 3, 4)$ faces. By Lemma 3.6, v is adjacent to up to $(d(v) - (\lceil \frac{d(v)}{2} \rceil + 2))$ special 3-vertices. By R3.b, v gives up to 1 to each pendant 3-face, and by R3.c, up to $\frac{1}{5}$ to the special 3-vertices. Finally, by R3.a, v gives $\frac{3}{5}$ to each incident 5-face. We have $2d(v) - 6 - (\lceil \frac{d(v)}{2} \rceil + \frac{3}{5}(d(v)) + \frac{1}{5}(d(v) - (\lceil \frac{d(v)}{2} \rceil + 2)) > 0$.

So, after the discharge procedure, all vertices and faces have a non-negative charge. That is, $\sum\limits_{u \in V} \mu^*(u) + \sum\limits_{f \in F} \mu^*(f) > 0$. The proof is complete. $\square$

5 Future Research

With the successful completion of this proof, I propose further study of graphs in $\mathcal{G}$, to determine if all graphs in $\mathcal{G}$ are $(1, 0, 0)$ colorable, or even properly 3-colorable. Such research would provide more insight and another set of constraints for which planar graphs are 3-colorable, furthering research into 3-colorability of planar graphs. This research may also shed light on Steinberg's Conjecture [9] and the Bordeaux Conjecture [4].

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Bibliography

[1] K. Appel, W. Haken, *Every planar graph is four colorable. Part I. Discharging*, Illinois J. Math, (1977), 429–490.

[2] K. Appel, W. Haken. *Every planar graph is four colorable. Part II. Reducibility*, Illinois J. Math, (1977), 491–567.

[3] O.V. Borodin, A. Glebov, *Planar Graphs with Neither 5-Cycles Nor Close 3-Cycles Are 3-Colorable*, J. Graph Theory, (2010), 1-31.

[4] O.V. Borodin, A. Raspaud, *A sufficient condition for planar graphs to be 3-colorable*, J. Combinatorial Theory, Series B, (2003), 17–27.

[5] H. Grözch, *Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreis-freie Netze auf der Kugel.* (German), Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg, Math-Nat. (1958/1959) 109–120.

[6] I. Havel, *On a Conjecture of B. Grünbaum*, J. Combinatorial Theory, (1969), 184-186.

[7] O. Hill, D. Smith, Y. Wang, L. Xu, G. Yu, *Planar Graphs Without Cycles of Length 4 or 5 are* $(3, 0, 0)$ *colorable*, submitted.

[8] R. Liu, X. Li, G. Yu, *A relaxation of the Bordeaux Conjecture*, European Journal of Combinatorics, to appear.

[9] R. Steinberg, *The state of the three color problem*, Quo Vadis, Graph Theory?, Ann. Discrete Math (1993), 211–248.

[10] B. Xu, *On (3,1)*-Coloring of Plane Graphs* SIAM J. Discrete Math, (2008), 205–220.

[11] C. Yang, C. Yerger, *The Bordeaux 3-color Conjecture and Near-Coloring*, (2013), status unknown.