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On the Non-Symmetric Spectra of Certain Graphs

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ABSTRACT

The study of eigenvalue list multiplicities of matrices with certain graphs has appeared in volumes for symmetric real matrices. Very interesting properties, such as interlacing, equivalent geometric and algebraic multiplicities of eigenvalues, and “Parter-Weiner-Etc. Theory” drive the study of symmetric real matrices. We diverge from this and analyze non-symmetric real matrices and ask if we can attain more possible eigenvalue list multiplicities. We fully describe the possible algebraic list multiplicities for matrices with graphs P_n, S_n, K_n, and $K_n - K_m$.

Contents

Chapter 1

Background

1.1 Graphs and Matrices

We begin by giving details of graph theory and matrix theory upon which we will build our proofs. We aim to give background details which are sufficient for a reader who is somewhat familiar with matrix theory. No prior knowledge of graph theory should be necessary to understand the proofs in this text. However, if one wishes to consult further background material, reference [7] gives quite an in depth discussion of matrix theory and [13] serves the same purpose for graph theory.

1.1.1 Graphs

A *graph* is a set of *vertices*, $V(G)$, and a set of *edges*, $E(G)$, such that an edge represents a connection between exactly two vertices. A *simple graph* is a graph for which the edges have no orientation, distinct vertices share at most one edge, and no vertex is both endpoints of an edge. For the remainder of the text we deal only with simple graphs and simplify the language to "graph".

For vertices $u, v \in V(G)$, a *path* from u to v is a sequence of distinct vertices $(v_1, v_2, \cdots, v_n)$ with $v_1 = u$, $v_n = v$, and $v_i v_{i+1} \in E(G)$ for each $1 \leq i \leq n-1$. A graph is *connected* if for any two vertices $u, v \in V(G)$, a path from u to v exists in G. If a graph is not *connected* then we say it is *disconnected*. A *subgraph* H of a graph G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is an *induced subgraph* of G if for vertices $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$. Maximal connected induced subgraphs of G are called *connected components* of G. A *cut edge*, or *bridge*, is an edge $uv \in E(G)$ such that the removal of uv from G increases the number of connected components.

An edge $e \in E(G)$ is *incident* with the vertices $u, v \in V(G)$ if u and v are the endpoints of e. The *degree* of a vertex $d_G(v)$ is the number of edges incident with v in G. A set of vertices $S \subseteq V(G)$ is an *independent set* if for any $u, v \in S$, $uv \notin E(G)$. A *tree* is a connected graph with $n-1$ edges. Trees are extremely important as they are minimally connected graphs on n vertices, that is, the removal of an edge results in a disconnected graph. The *path* on n vertices P_n is the tree with maximum vertex degree 2. The *complete bipartite graph*, $K_{m,n}$ is the graph obtained by taking two independent sets of vertices X and Y with size m and n, respectively, and for any $x \in X$ and $y \in Y$, letting an edge exist between x and y. The *star* on n vertices is defined $S_n = K_{1,n-1}$. The *complete graph* on n vertices K_n is the graph with n vertices where any two distinct vertices share an edge.

1.1.2 Matrices

We denote the set of n-by-n matrices over real numbers $\mathbb{R}$ by $M_n(\mathbb{R})$. Given a matrix $A \in M_n(\mathbb{R})$, it is a well known fact that A has n *eigenvalues*, which are also the n roots of the polynomial $p_A(t) = det(tI - A)$. We call $p_A(t)$ the *characteristic polynomial* of the matrix A. The *spectrum* of A, denoted $\sigma(A)$, is the set of eigenvalues of A. For $\lambda \in \sigma(A)$, the *algebraic multiplicity* of λ, denoted $m_A(\lambda)$, is the multiplicity of λ as a root of $p_A(t)$. For $\lambda \in \sigma(A)$, the *geometric multiplicity* of λ is $n - rank(\lambda I - A)$.

If A is an $n \times n$ matrix, then for a set $\alpha \subseteq \{1, 2, \cdots, n\}$, $A(\alpha)$ is the $(n-|\alpha|)\times(n-|\alpha|)$ matrix obtained by deleting the rows and columns indexed by the set α. The property of *interlacing* for a real symmetric matrix A states that if $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\sigma(A(i)) = \{\mu_1, \mu_2, \cdots, \mu_{n-1}\}$ where $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$, then $\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \mu_{n-1} \le \lambda_n$. This has interesting implications on the possible algebraic multiplicities of $(n-1)-$by$-(n-1)$ principal submatrices of real symmetric matrices which may not apply to non-symmetric matrices.

1.1.3 A Matrix and its Graph

Given a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$, the graph G of the matrix A is constructed by letting $V(G) = \{1, 2, \cdots, n\}$ and letting $ij \in E(G)$ if and only if $i \neq j$ and $a_{ij} \neq 0$. It is important to note that the diagonal entries of A have no effect on the graph of A. Much research has appeared regarding the set $\mathcal{S}(G)$ of real symmetric matrices with graph G. We relax the condition of symmetry and consider the set $\mathcal{R}(G)$ of real matrices with graph G. As $\mathcal{S}(G) \subseteq \mathcal{R}(G)$, theorems regarding $\mathcal{S}(G)$ serve as a bound for the flexibility obtained by using $\mathcal{R}(G)$. Let $\mathcal{L}_{\mathcal{S}}(G)$ and $\mathcal{L}_{\mathcal{R}}(G)$ be the sets of all possible lists of multiplicities of the eigenvalues of matrices in $\mathcal{S}(G)$ and $\mathcal{R}(G)$, respectively. We say that the set $\mathcal{L}_{\mathcal{S}}(G)$ (or $\mathcal{L}_{\mathcal{R}}(G)$) is full if it contains all partitions of $|V(G)|$.

We may observe a few relationships between graphs and matrices. A simple example is that an irreducible tridiagonal $n \times n$ matrix has graph P_n. Matrices A and B are *similar* if there exists an invertible matrix S such that $A = S^{-1}BS$. In this case, we are performing a *similarity* on B. For matrices A and B, the direct sum of A and B is $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Since we are dealing with matrices with simple graphs, we define a matrix to be *irreducible* if its graph is connected. If a symmetric matrix A is reducible, then there exist matrices A_1 and A_2 such that A is permutationally similar to $A_1 \oplus A_2$. This implies that a matrix with a disconnected graph G is similar to a matrix which is the direct sum of irreducible matrices whose graphs are the connected components of G. Since $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, the spectrum of a graph is determined entirely by the spectra of its connected components, so we primarily study matrices with connected graphs.

1.1.4 Sub-direct Sums, Trees, and the Bridge Formula

Given a matrix A with graph G such that G has a cut edge e with endpoints v_1 and v_2, call H_1 and H_2 the connected components of $G - e$ which contain v_1 and v_2, respectively. Furthermore, for an induced subgraph H of G, call A_{H_i} the principal sub matrix of A which has graph H_i. We can relax the definition of the *bridge formula* shown in [10] for calculating the characteristic polynomial of A to cover non-symmetric matrices as well. The formula will be

$$p_A(t) = p_{A_{H_1}}(t)p_{A_{H_2}}(t) - (a_{v_1v_2})p_{A_{H_1-v_1}}(t)p_{A_{H_2-v_2}}(t).$$

Given real, square matrices A and B, if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where both A_{22} and B_{11} are k-by-k matrices, the *k-sub-direct sum* of A and B as defined in [4] is $A \oplus_k B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix}$. Let T be a tree and $e \in E(T)$ be incident with vertices u and v and let T_u and T_v be the connected components of $T - e$ containing u and v, respectively. Then one may represent matrices in $\mathcal{R}(T)$ as the 2-sub-direct sums of matrices in $\mathcal{R}(T_u)$ and $\mathcal{R}(T_v)$.

The combination of these two concepts helps us realize that for a tree T, the understanding of the spectra of matrices in $\mathcal{R}(T)$ can be progressed quickly by fully understanding not only the possible spectra of trees, but also the possible spectra of the subtrees of trees with a fixed spectrum.

1.1.5 Standard Form of Matrices in $\mathcal{R}(T)$

Given a tree T and a matrix $A \in \mathcal{R}(T)$, one can see that off-diagonal values are recorded in minors of A as pairs. That is, for $i \neq j$, the a_{ij} entry will always be recorded in the product $a_{ij}a_{ji}$. Because of this, we suggest a standard form for matrices in $\mathcal{R}(T)$ where we consider a matrix to have only $1's$ and $0's$ above the main diagonal. By the flexibility of non-symmetry in $\mathcal{R}(T)$, the set of matrices in this form does not restrict the spectra of matrices in $\mathcal{R}(T)$. When speaking of matrices with trees as a graph, we will consider them to be of this form. For example, the following matrices will have the same characteristic polynomials.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 2 & 2 & -3 \\ 0 & 6 & 0 & 0 \\ 0 & -2 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 1 \\ 0 & 12 & 0 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

1.2 Introduction

It is a well known fact that for Hermitian matrices, the geometric multiplicity of an eigenvalue is equal to the algebraic multiplicity of that eigenvalue. For this, the study of eigenvalue structure of symmetric, real matrices can be conducted through either the study of algebraic or geometric multiplicities, though the study of geometric multiplicity seems more difficult. However, sometimes this is not the case. As we will see later, one may show using geometric multiplicity that the only multiplicity list for matrices in $\mathcal{S}(P_n)$ is $1, 1, \cdots 1$. The reference [10] gives a quite detailed explanation of possible algebraic multiplicity lists of various graphs utilizing "Parter-Weiner-Etc. Theory." This theory takes advantage of the interlacing property of Hermitian matrices to describe possible list multiplicities of submatrices of a given matrix.

In particular, the study of maximum multiplicity of matrices with prescribed graphs commands interest. The maximum multiplicity of matrices in $\mathcal{S}(T)$ is investigated in [8] and matrices whose graph contains exactly one cycle is shown in [5]. When the possible multiplicity lists are known, it is natural to extend the question to describe what possible spectra are attainable. We see this in [6] as the possible list multiplicities for $\mathcal{S}(P_n)$ are known, however, Gray and Wilson show that it is possible that for any strict interlacing sequence $\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n$, there exists a matrix $A \in \mathcal{S}(P_n)$ such that $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and $\sigma(A(1)) = \{\mu_1, \mu_2, \cdots, \mu_{n-1}\}$. We make the next natural extension of the study of multiplicity lists by allowing matrices to be non-symmetric.

One may ask a few general questions about the possible multiplicity lists with respect to $\mathcal{S}(G)$ and $\mathcal{R}(G)$. For example, since $\mathcal{S}(G) \subseteq \mathcal{R}(G)$, is $\mathcal{L}_{\mathcal{S}}(G)$ a proper subset of $\mathcal{L}_{\mathcal{R}}(G)$? We believe in the case of trees that is true.

Conjecture 1. *For a tree T, the set $\mathcal{L}_{\mathcal{R}}(T)$ is full.*

Furthermore, we may ask questions about the geometric multiplicity lists of matrices in $\mathcal{R}(G)$. For geometric multiplicities, we have come to the following conjecture.

Conjecture 2. *For a graph G, the maximum geometric multiplicity of matrices in $\mathcal{S}(G)$ is the same as the maximum geometric multiplicity of matrices in $\mathcal{R}(G)$.*

Note that this conjecture does not entirely classify the possible geometric multiplicity lists of matrices in $\mathcal{R}(G)$, but simply restricts the maximum multiplicity of any such list.

Chapter 2

The Star

We begin with the example of the star on n vertices. It was shown in [9] that the set $\mathcal{L}_{\mathcal{S}}(S_n)$ consists of lists of the form $a_1, a_2, \cdots, a_k$ if and only if $\sum_{i=1}^{k} a_i = n$ and $a_i > 1$ implies that $1 < i < k$ and $a_{i-1} = a_{i+1} = 1$. Though this set is fairly large, we aim to encompass all eigenvalue list multiplicities by relaxing the condition of symmetry. We see that not only do we attain all eigenvalue list multiplicities for matrices in $\mathcal{R}(S_n)$, but a much stronger statement regarding the possible spectra of matrices in $\mathcal{R}(S_n)$.

Theorem 1. *For any real, monic polynomial $q(t)$ of degree n, there exists a matrix $A \in \mathcal{R}(S_n)$ such that $p_A(t) = q(t)$. As a result, $\mathcal{L}_{\mathcal{R}}(S_n)$ is full.*

Proof. Consider a real, monic polynomial $q(t)$ of degree n and let M be the matrix

$$M = \begin{bmatrix} d_1 & 1 & 1 & \cdots & 1 \\ a_2 & d_2 & 0 & \cdots & 0 \\ a_3 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & d_n \end{bmatrix}$$

with $a_i \in \mathbb{R} - \{0\}$ for $2 \leq i \leq n$, so $M \in \mathcal{R}(S_n)$. Now we let $p_M(t) = t^n - c_{n-1}t^{n-1} + c_{n-2}t^{n-2} - \cdots \pm c_0$ and see that with careful choices of the a and d variables in M, we may achieve any real characteristic polynomial for M.

Let

$$S_k(\{x_1, x_2, \cdots, x_n\}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (\prod_{j=1}^{k} x_j)$$

be the k^{th} elementary symmetric function on $x_1, x_2, \cdots x_n$. For notation's sake, let us call $D_i = S_{n-i}(\{d_1, d_2, \cdots, d_n\})$.

Then we may see that $c_i = D_i - \sum_{j=2}^{n} a_j(S_{n-i-2}(\{d_2, d_3, \cdots, d_n\} - \{d_j\}))$ for $0 \le i \le n-3$ and when $i = n-2$ the latter term is replaced by $\sum_{j=2}^{n} a_j$ and when $i = n-1$, the latter term is replaced by 0. Now as the d_i are fixed, we may rearrange this to get $\tilde{c}_i = D_i - c_i = \sum_{j=2}^{n} a_j(S_{n-i-2}(\{d_2, d_3, \cdots d_n\} - \{d_j\}))$ for $0 \le i \le n-3$ and $\tilde{c}_{n-2} = S_{n-i-2}(\{d_2, d_3, \cdots d_n\} - \{d_j\}) - c_{n-2} = \sum_{i=2}^{n} a_i$. From here we may apply a linear transformation to get the following equation

$$N \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \tilde{c}_{n-2} \\ \tilde{c}_{n-3} \\ \vdots \\ \tilde{c}_0 \end{bmatrix}$$

where $N = (n_{ij})$ and $n_{ij} = 1$ when $i = 1$ and $n_{ij} = S_{i-1}(\{d_2, d_3, \cdots, d_n\} - \{d_{j+1}\})$. Therefore, it suffices to prove that N is invertible to show that we may achieve a matrix M with $p_M(t) = q(t)$. Consider the matrix

$$A = (a_{ij}) = \left[(-1)^{j+1} \frac{d_{i+1}^{n-(j+1)}}{\prod_{k=2;k\neq i+1}^{n}(d_{i+1} - d_k)} \right]_{ij}.$$

We may test to see that $A = N^{-1}$. Let the row vectors of A be denoted $\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n$ and the column vectors of n be denoted $\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_n$. Then, one may verify that

$$\hat{a}_i \hat{n}_j = \frac{d_i^{n-2} + \sum_{k=3}^{n} (-1)^k d_i^{n-k}(S_{k-2}(\{d_2, d_3, \cdots, d_n\} - \{d_j\}))}{\prod_{k=2;k\neq i}^{n}(d_i - d_k)}.$$

We may see that when $i = j$, the numerator is a summation of products of all possible choices of $(n-2)$ elements of $\{-d_2, \cdots, -d_{i-1}, d_i, -d_{i+1}, \cdots, -d_n\}$ where only d_i may have multiplicity higher than 1, then separated by the number of factors equal to d_i. Another way of representing the sum of all possible products of $(n-2)$ elements of $\{-d_2, \cdots, -d_{i-1}, d_i, -d_{i+1}, \cdots, -d_n\}$ where only d_i has a multiplicity greater than 1 is $\prod_{k=2;k\neq i}^{n}(d_i - d_k)$. Therefore when $i = j$, we get $\hat{a}_i \hat{n}_j = 1$. Now, when $i \neq j$, for some $k \in \mathbb{N}$ where $k \geq 3$, we have

$$d_i^{n-k}(S_{k-2}(\{d_2, d_3, \cdots, d_n\} - \{d_j\})) =$$

$$= d_i^{n-k+1}(S_{k-3}(\{d_2, d_3, \cdots, d_n\} - \{d_j\})) + d_i^{n-k}(S_{k-2}(\{d_2, d_3, \cdots, d_n\} - \{d_i, d_j\}))$$

we may use this fact to recursively show that the numerator of $\hat{a}_i \hat{n}_j = 0$. Therefore $A = N^{-1}$, and we may use A to give values for the a_i. Finally, we must discuss what to do when there are some 0 values for various

a_i variables. One can see that in our equation

$$\begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = A \begin{bmatrix} \tilde{c}_{n-2} \\ \tilde{c}_{n-3} \\ \vdots \\ \tilde{c}_0 \end{bmatrix}$$

gives values

$$a_i = \frac{\sum_{j=0}^{n-2}(-1)^{j+1} d_i^j (D_j - c_j)}{\prod_{k=2,k\neq i}^{n}(d_i - d_k)}$$

and thus each a_i acts as a polynomial which is rational in the d_i and thus not identically zero. From [2] we know that a finite vector of polynomials over an infinite field, none of which are identically zero, has a totally non-zero solution. Furthermore, we may find a solution for which $d_i \neq d_j$ when $i \neq j$, which is necessary for the existence of A. Such a solution gives us a and d values which we may place in the matrix M to have $p_M(t) = q(t)$. Since any polynomial with all real roots is real, it follows that $\mathcal{L}_{\mathcal{R}}(S_n)$ is full.

□

This shows that for some real polynomials, it must be that all matrices in $\mathcal{R}(S_n)$ with that characteristic polynomial must have a negative entry below the main diagonal given our standard form for $\mathcal{R}(S_n)$. Furthermore, since $\mathcal{L}_{\mathcal{R}}(S_n)$ is full, $\mathcal{L}_{\mathcal{S}}(S_n)$ is a proper subset of $\mathcal{L}_{\mathcal{R}}(S_n)$ and some such polynomials have all real roots.

Although we may attain a larger set of eigenvalue list multiplicities for matrices in $\mathcal{R}(S_n)$ than for matrices in $\mathcal{S}(S_n)$, one may ask if the same distinction applies to geometric multiplicities. We note that the maximum geometric multiplicity of matrices in $\mathcal{R}(S_n)$ is the same as for matrices in $\mathcal{S}(S_n)$, that is $n - 2$. One may easily check this as for a matrix $A \in \mathcal{R}(S_n)$ and any $\lambda \in \mathbb{C}$, the matrix $A - \lambda I$ has full first row and column excluding possibly the $1,1$ entry. Therefore $\text{rank}(A - \lambda I) \geq 2$ and the maximum geometric multiplicity is $n - 2$.

Returning to algebraic multiplicities, it is important to note that the proof of Theorem 1 only provides an implicit solution for matrices of this form and that a clever manipulation of the process described could yield an explicit solution. The importance of finding explicit solutions lies in the possibility of finding solutions for which we may choose an entry, providing a base for possible induction proofs as we will see later.

It would be convenient to utilize this method of proof when analyzing different trees, however, given the matrix M in the previous proof, one may see that principal minors of M are linear in terms of the a variables. As we progress to more complex graphs, principal minors of matrices corresponding to these graphs no longer are linear in terms of the off-diagonal entries. This makes it difficult to prove that these matrices are invertible and even more difficult to find such an explicit inverse. Thus as we move on to different graphs we utilize different methods.

Chapter 3

The Path

Recall that given an eigenvalue λ of an $n \times n$ matrix A, its geometric multiplicity is $n - rank(A - \lambda I)$. It is a well known fact that real symmetric tridiagonal matrices have maximum geometric multiplicity 1, and that for real symmetric matrices, the algebraic multiplicity is equal to the geometric multiplicity. Therefore, as shown in [10] the set $\mathcal{L}_{\mathcal{S}}(P_n)$ consists of only the multiplicity list $1, 1, \cdots, 1$. We are able to show that $\mathcal{L}_{\mathcal{S}}(P_n)$ is full, and thus the extension of $\mathcal{S}(P_n)$ to non-symmetric matrices drastically changes the set of attainable spectra.

A matrix is *non-derogatory* if its maximum geometric multiplicity is 1. A matrix in *Schwarz form* is an $n \times n$ real, irreducible tridiagonal matrix with entries as follows

$$W = \begin{bmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 \\ w_n & 0 & -1 & \ddots & & \vdots \\ 0 & w_{n-1} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & w_3 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & w_2 & w_1 \end{bmatrix}.$$

Theorem 2 (Schwarz)**.** *Let W be a matrix in the Schwarz form with $w_i \neq 0$ for $1 \leq i \leq n$. Then the number of positive entries in the sequence $w_1, w_1w_2, \cdots, w_1w_2\cdots w_n$ is equal to the number of eigenvalues lying in the open right half-plane.*

The following proposition given by [3] utilizes Theorem 2 to describe the Schwarz form of non-derogatory matrices with eigenvalues with positive real part. We use it to show that we may attain any spectrum for matrices in $\mathcal{R}(P_n)$ while also showing that the matrices are certainly non-symmetric by our construction.

In [3], it is explained that a matrix A is similar to a matrix in the Schwarz form if the Routh scheme of $p_A(t)$ has a non-vanishing first column which occurs when all eigenvalues lie in the open right half-plane. A matrix A is *symmetric in modulus* if the matrix obtained by taking entrywise absolute values of A is symmetric. Elsner and Hershkowitz utilize the Schwarz form to make a series of claims about irreducible, symmetric in modulus matrices in $\mathcal{R}(P_n)$. The following proposition found in [3] describes the structure of matrices attainable in Schwarz form given a spectrum of eigenvalues with positive real part.

Proposition 1. *Given a real, monic polynomial $q(t)$ which has roots in the open right (left) half-plane, there exists a symmetric in module tridiagonal matrix A with positive super diagonal, negative subdiagonal, and all zeros along the diagonal, excluding $a_{nn} > 0$ such that $p_A(t) = q(t)$.*

We extend their method and show that we may attain any possible spectrum in $\mathcal{R}(P_n)$ by applying a transformation, utilizing the proposition, and translating the eigenvalues back to their original position.

Theorem 3. *For any real, monic polynomial $q(t)$ of degree n, there exists a matrix $A \in \mathcal{R}(P_n)$ with $p_A(t) = q(t)$. As a result, $\mathcal{L}_{\mathcal{R}}(P_n)$ is full.*

Proof. Let $q(t)$ be such a polynomial with roots $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ where $m = \max_{1 \le i \le n}\{|Re(\lambda_i)|\}$. Then for some $\epsilon \in \mathbb{R}^+$ consider the collection $(m+\epsilon)+\Lambda$. By the proposition of [3], there exists a matrix $A \in \mathcal{R}(P_n)$ with $\sigma(A) = (m+\epsilon)+\Lambda$. As a shift by the identity matrix doesn't affect the graph of a matrix, it follows that $A-(m+\epsilon)I \in M(P_n)$ and $\sigma(A-(m+\epsilon)I) = \Lambda$, so $p_{(A-(m+\epsilon)I)}(t) = q(t)$. □

It is interesting to note that this construction gives an explicit method of constructing a matrix $A \in \mathcal{R}(P_n)$ with $q(t)$ as its characteristic polynomial, and further, when we transform A into the standard form for trees, we will always have a negative subdiagonal so A will never be symmetric. Therefore, for any real, monic polynomial $q(t)$ we may always find a matrix $A \in (\mathcal{R}(P_n) - \mathcal{S}(P_n))$ with $p_A(t) = q(t)$.

Though the set $\mathcal{L}_{\mathcal{R}}(P_n)$ is much larger than $\mathcal{L}_{\mathcal{S}}(P_n)$, we can see that the lists of geometric multiplicities of $\mathcal{R}(P_n)$ and $\mathcal{S}(P_n)$ are the same. This comes from the fact that the maximum geometric multiplicity of a matrix in $\mathcal{R}(P_n)$ is 1. This follows the similar argument seen in [10] in which given a matrix $A \in \mathcal{R}(P_n)$ and any $\lambda \in \mathbb{C}$, we delete the first row and last column of $A - \lambda I$, resulting in an upper triangular matrix with non-zero diagonal. Thus the resulting matrix is a rank $n-1$ submatrix of $A - \lambda I$, so $\text{rank}(A - \lambda I) \ge n-1$ implying that the maximum geometric multiplicity of A is 1.

Since the sets of geometric multiplicities of $\mathcal{S}(P_n)$ and $\mathcal{R}(P_n)$ are the same, we focus on the possible algebraic multiplicities of $\mathcal{R}(P_n)$. Recall our proposition that given a tree T and matrix $A \in \mathcal{R}(T)$, understanding the possible spectra of subtrees of T given a fixed spectrum for A is important to further our understanding of possible spectra of general trees. Now given a matrix $A \in \mathcal{R}(P_n)$, we investigate the possible spectra of $A(1)$. The following restriction on the spectrum of $A(1)$ holds.

Theorem 4. *For any real, irreducible tridiagonal matrix A, $(p_A(t), p_{A(1)}(t)) = 1$.*

Proof. We recall that any real, irreducible tridiagonal matrix A is assumed to be in the form

$$A = \begin{bmatrix} a_1 & 1 & 0 & \cdots \\ b_1 & a_2 & 1 & \cdots \\ 0 & b_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

as the super and sub diagonal entries appear within principal minors as pairs. See that for a 2×2 matrix $A = \begin{bmatrix} a_1 & 1 \\ b_1 & a_2 \end{bmatrix}$ if $\lambda \in \sigma(A)$ it must be that $det(A - \lambda I) = (a_1 - \lambda)(a_2 - \lambda) - b_1 = 0$. This implies that if $(p_A(t), p_A(1)) \neq 1$ then $b_1 = 0$, a contradiction. Now proceed by induction and since we know that $p_A(t) = (t - a_1)p_{A(1)}(t) - b_1 p_{A(1,2)}(t)$, if it were that $(p_A(t), p_{A(1)}(t)) \neq 1$, then we would also have $(p_{A(1)}(t), p_{A(1,2)}(t)) \neq 1$ or $b_1 = 0$, a contradiction. $\square$

Observing the two previous theorems, we note that given any monic, real, degree n polynomial $q(t)$, we may find a matrix $A \in \mathcal{R}(P_n)$ with $p_A(t) = q(t)$ and that we must have $(p_A(t), p_{A(1)}(t)) = 1$. Now we ask, given real, monic polynomials $q(t)$ and $r(t)$ where $deg(q(t)) = deg(r(t)) + 1 = n$, what are the exact restrictions for which we may find a matrix $A \in \mathcal{R}(P_n)$ for which $p_A(t) = q(t)$ and $p_{A(1)}(t) = r(t)$. For such a matrix A, the more flexibility we have with the spectrum of $A(1)$ results in extra flexibility with the a_1 entry. This could be of use in extending our knowledge of spectra on more complex trees as we see in section on the spectra of matrices in $\mathcal{R}(K_n - K_m)$. We end the discussion of the path with the following conjecture.

Conjecture 3. *Let $q(t)$ and $r(t)$ be real, monic polynomials such that $deg(q(t)) = deg(r(t)) + 1 = n$ and $(q(t), r(t)) = 1$. Then there is $A \in \mathcal{R}(P_n)$ such that $p_A(t) = q(t)$ and $p_{A(1)}(t) = r(t)$, subject to a set of recursive restrictions on the coefficients of $q(t)$ and $r(t)$. Furthermore, the a_1 and b_2 entries are entirely determined by the desired spectra.*

It seems to me this set of recursive relations is a finite set totally determined by the coefficients of $q(t)$ and $r(t)$, leaving a proof which would allow us to inductively choose a matrix A with $p_A(t) = q(t)$, $p_{A(1)}(t) = r(t)$ and having a_1 and b_2 be of our choice.

Chapter 4

Graphs with Few Edges Missing

The study of eigenvalue list multiplicities in the symmetric case has been extended to complete graphs with few edges missing. This includes K_n, K_n with few independent edges missing, and $K_n - P_3$. We further extend our knowledge to include possible spectra of matrices in both $\mathcal{S}(K_n - K_m)$ and $\mathcal{R}(K_n - K_m)$ for $m \leq \lfloor \frac{n}{2} \rfloor$. Our method of proof for various eigenvalue list multiplicities consists of an explicit construction of a real diagonal matrix with the given multiplicity list and performing 2-by-2 similarities which act on a principal submatrix of the original matrix. A useful property to observe is that symmetry is invariant under orthogonal similarity.

We use 2-by-2 similarities to act on n-by-n matrices by letting an n-by-n similarity matrix be the matrix obtained from replacing a 2-by-2 principal minor of I_n with the desired 2-by-2 similarity matrix. Thus when we say we are performing an (i,j) transform, we are performing a similarity by the matrix W, where $W(i,j) = I_{n-2}$ and $W[i,j]$ is a 2-by-2 invertible matrix so $W = I_{i-1} \oplus \tilde{W} \oplus I_{n-j}$ where $\tilde{W}$ is of the form

$$\tilde{W} = \begin{bmatrix} * & 0 & \cdots & 0 & * \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ * & 0 & \cdots & 0 & * \end{bmatrix}.$$

The method of using these similarity matrices is both used and described in detail in [11].

4.1 A Note on K_n

We see that in [11], $\mathcal{L}_{\mathcal{S}}(K_n)$ consists of all multiplicity lists containing at least 2 distinct eigenvalues. Furthermore, we may obtain any spectra satisfying the multiplicity list constraint. We utilize the Schwarz form method of finding a matrix in $\mathcal{R}(P_n)$ to extend $\mathcal{L}_{\mathcal{S}}(K_n)$ to be full in the non-symmetric case.

Corollary 5. *For any real, monic polynomial $q(t)$ of degree n, there exists a matrix $A \in \mathcal{R}(K_n)$ such that $p_A(t) = q(t)$. As a result, $\mathcal{L}_{\mathcal{R}}(K_n)$ is full.*

Proof. We know that we may find a matrix $A \in \mathcal{R}(P_n)$ satisfying $p_A(t) = q(t)$ by the process shown in Theorem 3. By this process, the $(n-1)^{th}$ and n^{th} diagonal entries of A will be distinct, allowing us to perform $(i, i-1)$ transforms for $i = n, n-1, n-2, \cdots, 2$. This will yield a full matrix with characteristic polynomial $q(t)$. □

We will see this become useful in section 4.3 which describes the spectra of matrices in $\mathcal{R}(K_n - K_m)$.

4.2 Spectra of Matrices in $\mathcal{S}(K_n - K_m)$

In [11], it was shown that for a graph G which is K_n minus an edge, the set $\mathcal{L}_{\mathcal{S}}(G)$ contains all multiplicity lists with maximum multiplicity $n-2$. We seek to generalize this statement, both by extending to graphs $K_n - K_m$ and by extending to the set $\mathcal{R}(K_n - K_m)$. It is first necessary to describe the effects of orthogonal similarities on a prescribed symmetric matrix. The following proposition comes from [11].

Proposition 2. *The matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(\mathbb{R})$ is transformed to a full symmetric matrix $\begin{bmatrix} a' & c \\ c & b' \end{bmatrix}$ with numbers a' and b' properly in between a and b, subject to $a' + b' = a + b$, by orthogonal similarity via U if and only if U is full and $a \neq b$. If $a = b$, the similarity returns the same diagonal matrix.*

This proposition also shows how we may utilize similar diagonal values to some manipulate off-diagonal entries while maintaining zeros as certain off-diagonal entries. We use this fact to prove the following theorem.

Theorem 6. *Let G be the graph $G = K_n - K_m$ where $m \leq \lfloor \frac{n}{2} \rfloor$, the set $\mathcal{L}_{\mathcal{S}}(G)$ contains all multiplicity lists with maximum multiplicity $n-2$.*

Proof. Let $m_1, m_2, \cdots, m_k$ be a multiplicity list with maximum multiplicity m_k and let eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ have $m(\lambda_i) = m_i$ for $1 \leq i \leq k$. We have two cases. When $m_k \geq m-1$, then we consider the multiplicity list $m_1, m_2, \cdots, m_{k-1}, (m_k - m + 1)$ and $m(\lambda_k) = m_k - m + 1$. We construct a diagonal matrix D with the eigenvalues described above, letting the first and second diagonal entries differ, letting λ_k be neither the minimum nor maximum eigenvalue, and letting the $(n-k+1)^{th}$ entry be different from λ_k. Then utilizing the proposition, we may fill D by performing $(i, i+1)$ transforms in such a way that the last diagonal entry is λ_k. Call the resulting matrix $\tilde{D}$. Then for $n-k \leq i \leq n-1$ we apply $(i, i+1)$ transforms to the matrix $\tilde{D} \oplus \lambda_k I_{k-1}$, resulting in a matrix which lies in $\mathcal{S}(G)$ and has eigenvalue multiplicity list $m_1, m_2, \cdots, m_k$.

When $m_k < m-1$,we consider the diagonal matrix D given by letting the first $n-k+1$ diagonal entries be the first $n-k+1$ of the λ_i and letting the last $k-1$ diagonal entries be the last $k-1$ of the λ_i in reverse order. Assuming $n-m+1 = m_1 + m_2 + \cdots + m_{j-1} + (m_j - t)$ where $0 \leq i \leq m_j - 1$, D will take the form

$$D = \begin{bmatrix} \lambda_1 I_{m_1} & & & & & \\ & \ddots & & & & \\ & & \lambda_j I_{m_j - t} & & & \\ & & & \lambda_k I_{m_k} & & \\ & & & & \ddots & \\ & & & & & \lambda_j I_t \end{bmatrix}.$$

Now we let $\lambda_1 < \lambda_2 < \cdots < \lambda_{j-1} < \lambda_k < \lambda_{k-1} < \cdots \lambda_j$. For $1 \leq i \leq m-1$, we perform $(n-k+1-i, n-k+i)$ transforms such that the $(n-k+i)^{th}$ entry is λ_k after the transformation. The resulting matrix, call it $\tilde{D}$ is of the form

$$\tilde{D} = \left[\begin{array}{ccc|cccc} * & & & & & & \\ & \ddots & & & & & \\ & & * & & & & \\ \hline & & & * & & & * \\ & & & & \ddots & \iddots & \\ & & & & \iddots & \ddots & \\ & & & * & & & * \end{array}\right]$$

with the last $k-1$ entries the same. Now it is important that we have the first two entries different in this matrix, so if $m(\lambda_1) \geq 2$, do the same process, only switching the second occurrence of λ_1 with the first occurrence of λ_2 in D. Then for $1 \leq i \leq n-1$, we perform $(i, i+1)$ transforms to $\tilde{D}$ in such a way that after the $(n-k-1, n-k)^{th}$ transform, the $(n-k)^{th}$ entry is λ_k. This yields a matrix in $\mathcal{S}(G)$ which is similar to D and thus has the eigenvalue list multiplicity $m_1, m_2, \cdots, m_k$. $\square$

4.3 Spectra of Matrices in $\mathcal{R}(K_n - K_m)$

We again ask if non-symmetric matrices with a given graph have a larger set of possible eigenvalue list multiplicities than symmetric matrices with that graph. We begin with a concrete method of describing the possible eigenvalue list multiplicities for matrices in $\mathcal{R}(K_n - K_m)$. Afterwords, we show an alternative proof which relies on the validity of the conjecture proposed in the previous chapter. This proof shows the importance of having freedom to fix entries for a matrix with a given graph and characteristic polynomial. The following proposition facilitates attaining a matrix with desired diagonals by similarity given a diagonal matrix.

Proposition 3. *If a and b are distinct real numbers, then the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(\mathbb{R})$, can be transformed to a full matrix $\begin{bmatrix} a' & c \\ d & b' \end{bmatrix}$ with a' and b' free of choice, provided $a' + b' = a + b$ by similarity.*

This serves as a relaxation of the proposition given in 4.1 as it no longer requires orthogonal similarity and the preservation of symmetry.

Theorem 7. *Let G be the graph $G = K_n - K_m$ where $m \leq \lfloor \frac{n}{2} \rfloor$ and let $q(t)$ be any monic, degree n polynomial with real roots. If $q(t)$ has at least two distinct roots, then there exists a matrix $A \in \mathcal{R}(G)$ such that $p_A(t) = q(t)$. As a result, the set $\mathcal{L}_{\mathcal{R}}(G)$ contains all multiplicity lists excluding n itself.*

Proof. Let $q(t)$ be a degree n monic polynomial with real roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ where λ is the root with highest multiplicity. We give the λ_i an ordering, σ such that $\lambda_{\sigma(1)} + \lambda_{\sigma(n-m+1)} \neq \lambda_{\sigma(2)} + \lambda_{\sigma(n-m+2)}$, for $1 \leq i \leq m-1$, either $\lambda_{\sigma(i)} \neq \lambda_{\sigma(n-m+1+i)}$ or $\lambda_{\sigma(n-k+m+i)} = \lambda$, and finally, $\lambda_{\sigma(n-m)} \neq \lambda$. The only case in which we may not do this is with the multiplicity list $n-1, 1$, which we will discuss later. Let D be the diagonal matrix

$$D = \begin{bmatrix} \lambda_{\sigma(1)} & & & \\ & \lambda_{\sigma(2)} & & \\ & & \ddots & \\ & & & \lambda_{\sigma(n)} \end{bmatrix}$$

for $1 \leq i \leq k-1$, perform $(i, n-m+1+i)$ transforms which leave the $(n-m+1+i)^{th}$ diagonal entry equal to λ. After this series of transformations we will have a matrix $\tilde{D}$ with totally non-zero $(n-m)^{th}$ super and subdiagonal and all other super and subdiagonals with 0 entries so $\tilde{D}$ will be of the form

$$\tilde{D} = \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \\ \tilde{D}_3 & \lambda I_{m-1} \end{bmatrix} \quad \text{where } \tilde{D}_2 \text{ and } \tilde{D}_3 \text{ have non-zero main diagonal entries and } \tilde{D}_1 \text{ is diagonal.}$$

The reordering of the eigenvalues ensures that the first and second diagonal entries are different after this sequence of transforms which is necessary to fill out the upper left portion of the matrix $\tilde{D}$. Then for $1 \leq i \leq n-m+1$, we may perform $(i, i+1)$ transforms to fill $\tilde{D}_1$ and portions of $\tilde{D}_2$ and $\tilde{D}_3$. Furthermore, utilizing proposition 4, we may do this in such a way that the $(n-m)^{th}$ entry equals λ after the transforms. Now, for $1 \leq i \leq m-1$, we perform $(n-m, n-m+1)$ transforms which fills $\tilde{D}_2$ and $\tilde{D}_3$ while preserving λI_{m-1}. The resulting matrix is a matrix in $\mathcal{R}(G)$ which is similar to D. Since the characteristic polynomial of a matrix is similarity invariant, we are done and may move on to the case when the multiplicity list of 1 is $n-1, 1$.

In this case, let the distinct roots of q be λ_1 and λ_2 where $m(\lambda_1) = n-1$. Then let

$$D = \begin{bmatrix} \lambda_2 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_1 \end{bmatrix}$$

and perform a $(1,2)$ transform, then a $(1, n-m)$ transform. Then proceed with $(i, i+1)$ transforms for $1 \leq i \leq n-1$ as described above, forcing the $(n-m-1, n-m)$ transform to give λ_1 as the $(n-m)^{th}$ diagonal entry. $\square$

We may now use Corollary 5 to extend $\mathcal{L}_{\mathcal{R}}(K_n - K_m)$ to be full.

Corollary 8. *The list $\mathcal{L}_{\mathcal{R}}(K_n - K_m)$ is full.*

Proof. Let $q(t) = (t-\lambda)^n$ where $\lambda \in \mathbb{R}$. Then see that we may achieve a matrix $A \in \mathcal{R}(K_(n-m))$ with $p_A(t) = (t-\lambda)^{n-m}$. Furthermore, by using proposition 3 and the process described in Corollary 5, we may perform the $(n-m, n-m-1)$ transform in such a way that we may allow the $(n-m)^{th}$ entry of A to be λ. Then for $n-m \leq i \leq n-1$, we perform $(i, i+1)$ transforms on $A \oplus \lambda I_m$ resulting in a matrix with graph $K_n - K_m$ and having characteristic polynomial $(t-\lambda)^n$. $\square$

4.3.1 An Alternate Proof

Recall the conjecture from Chapter 3 which states that given monic, real polynomials $q(t)$ and $r(t)$, where $deg(q(t)) = deg(r(t)) + 1 = n$ and $(q(t), r(t)) = 1$, then there exists a matrix $A \in \mathcal{R}(P_n)$ such that $p_A(t) = q(t)$ and $p_{A(1)}(t) = r(t)$ exactly when the coefficients of the polynomials q and r satisfy a certain set of recursive relations. It seems this set of relations would allow us to find a matrix $A \in \mathcal{R}(P_n)$ with a_{11} a real number of our choice. Thus we proceed to shorten the proof the previous theorem using this conjecture.

Proof. Let $q(t)$ be a degree n polynomial with real roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ with λ being the root with maximum multiplicity. If $m(\lambda) \geq m - 1$, then we may find a matrix $A \in \mathcal{R}(P_{n-m+1})$ with

$$p_A(t) = \frac{q(t)}{(t-\lambda)^{m-1}}$$

and a_{11} entry not equal to λ. Then we take the matrix $\tilde{A} = \lambda I_{m-1} \oplus A$ and for $1 \leq i \leq n-1$, we may perform $(i, i+1)$ transforms to arrive at a matrix in $\mathcal{R}(K_n - K_m)$ which is similar to the matrix $\tilde{A}$ and thus has characteristic polynomial $q(t)$. With $m(\lambda) < m - 1$, we use the method described above. $\square$

This proof shows that in some cases, we may simplify the proofs of existence of matrices with certain characteristic polynomials. This, however, relies on the knowledge of possible spectra of submatrices of a matrix with given spectrum.

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