# The Dynamics of Long-Term Mass Transport in Estuaries

John M. Hamrick
*Virginia Institute of Marine Science*

# The Dynamics of Long-Term Mass Transport in Estuaries

**John M. Hamrick**

Virginia Institute of Marine Science
The College of William and Mary
Gloucester Point, VA 23062

**ABSTRACT**

The dynamics of long-term mass transport in estuaries having weakly nonlinear long wave dynamics are investigated. Low pass filtered long-term mass transport equations are derived for conditions of weak and strong vertical stratification. The dynamics of the residual mass transport velocity, or lowest order approximation to the Lagrangian residual velocity, are investigated by perturbation analyses of the hydrodynamics equations. For weak vertical stratification conditions, analytical results showing the influence of topography, the earth's rotation, and channel curvature on the distribution of the residual mass transport velocity field are presented. The calculation of the residual mass transport velocity field from field current meter measurements and numerical hydrodynamic model output is also discussed.

## I. Introduction

It is now recognized that the management of estuarine resources necessitates the ability to predict and understand the transport and transformation of dissolved and suspended biogeochemical constituents over time periods on the order of several years. From the modeling point of view, the disparity between short time scale transport processes, associated with tides, for example, and the desired long-term evaluation of water quality, makes the integration of mass transport equations at the short time steps necessary to resolve short time scale transport processes unattractive. Likewise, from the point of view of achieving an understanding of the pathways of long-term mass transport in estuaries, a formulation and solution of the mass transport problem, which filters out short term advective and diffusive transport variations, while effectively retaining the correlations or interactions of short term variations is desirable. Both of these considerations have led to the formulation of time-averaged or filtered mass transport equations.

Early attempts at formulating time-averaged or filtered long-term mass transport equations were primarily ad hoc in approach, and resulted in physically unjustifiable temporal subscale transport parameterization. Typically in these equations, the time-averaged concentration field is advectively transported by some type of time-averaged velocity field, with the nonzero averaged subscale advective transport completely ignored or unjustifiably parameterized as diffusive or dispersive transport depending upon the spatial dimensions of the equations. A recent example of this approach, starting from the three-dimensional advection-diffusion equation in conservative form using a stretched vertical coordinate, used the Eulerian residual transport velocity field obtained from simple tidal cycle time averaging as the long-term advective transport velocity, (Hydroqual, 1987). Calibration to field data was relied upon to determine the magnitude of diffusion coefficients necessary to represent temporal subscale transport processes. An alternate approach to the formulation of time-averaged or filtered long-term mass transport equations has been the utilization of perturbation analyses of the three-dimensional advection-diffusion equation for conditions under which the instantaneous advective transport field is dominated by a velocity field resulting

Coastal and Estuarine Studies, Vol. 38
R. T. Cheng (Ed.)
Residual Currents and Long-term Transport

from weakly nonlinear long wave motions, associated with tidal and/or atmospheric forcings. Using this approach, Feng et al. (1986) derived a two-dimensional in the horizontal, steady, time averaged mass transport equation from the unsteady depth averaged advection-dispersion equation. Depending upon the scaling of the horizontal shear dispersion terms, two equations were obtained. In an equation for strong shear dispersion transport, the advective transport field is the lowest order barotropic Lagrangian residual or residual mass transport velocity presented by Longuet-Higgins (1969). In an equation for weak shear dispersion transport, an additional next order correction, termed the Lagrangian drift velocity, is made to the advective transport field. Subsequently, Hamrick (1986a) derived an unsteady form of the equation of Feng et al., appropriate for strong shear dispersion from the three-dimensional advective-diffusion equation, and provided a rigorous definition of vertical stratification conditions for which the equation is appropriate.

A three-dimensional, unsteady, expected value, long-term mass transport equation, appropriate for conditions of strong vertical stratification, was derived using the perturbation approach, by Hamrick (1987), with a similar steady time averaged equation presented by Feng (1987). In both of these equations, the advective transport field is the three-dimensional lowest order Lagrangian residual or residual mass transport velocity, and is consistent with the three-dimensional Eulerian-Lagrangian transformation presented by Longuet-Higgins (1969). The parameterization of vertical diffusive transport in Hamrick's (1987) and Feng's (1987) equations use respectively the expected value and time averaged values of the instantaneous vertical turbulent diffusion coefficient, while Hamrick's equation also includes horizontal turbulent diffusion parameterized in a similar fashion.

Along a complementary line of investigation, beginning with Longuet-Higgins (1969), numerous researchers have recognized that the Lagrangian residual velocity field, and in particular its lowest order approximation termed the residual mass transport velocity field, is the appropriate representation of long-term advective mass transport in estuaries and coastal seas having weakly nonlinear long wave dynamics. The ability to predict and understand the long-term transport pathways in estuaries now becomes strongly dependent upon knowledge of the dynamics of the residual mass transport velocity which is composed of the Eulerian residual velocity and the Stokes drift velocity. Until the last decade, dynamic theories of residual circulation in estuaries have been dominated by the two-dimensional vertical plane density driven Eulerian residual circulation theory formalized by Hansen and Rattray (1965) and most recently extended by Hamrick (1979) and Oey (1984), and to a lesser extent the two-dimensional horizontal plane density driven Eulerian residual circulation theory proposed by Fischer (1972) and extended by Imberger (1976), Hamrick (1979) and Smith (1980).

Over the last decade, the scope of study of residual circulation dynamics has expanded to consider Eulerian residual circulation induced by nonlinear interaction of long wave velocity and surface elevation fields, as well as the Stokes drift velocity field associated with the long wave motion. For two-dimensional motion in the vertical plane, Ianniello (1977, 1979) presented analytical solutions for the Eulerian residual velocity field, the Stokes drift velocity and the residual mass transport velocity for homogeneous density and a single constituent tidal forcing. Subsequently, Najarian et al. (1984) reported the results of a numerical study of the two dimensional geometry considered by Ianniello. Two-dimensional in the horizontal plane, tide induced Eulerian and Lagrangian residual circulation in estuaries has primarily been studied with the aid of numerical hydrodynamic models, with recent studies including Cheng and Casulli (1982), Oey et al. (1985), Feng et al. (1986), and Smith and Cheng (1987). In contrast, an extensive body of analytical results for two-dimensional in the horizontal plane Eulerian and Lagrangian residual circulation, associated with topographic influences on the nonlinear interaction of long wave velocity and surface elevation fields in coastal seas, exist with recent reviews given by Zimmerman (1981) and Robinson (1983).

The purpose of the present paper is to present some results on the dynamics of long-term mass transport in estuaries. The paper is organized in the following manner. The hydrodynamic and mass transport equations in a scaled form consistent with weakly nonlinear long wave dynamics and suitable for perturbation analysis are presented. This is followed by a discussion of the relationship between temporal filtering and the multiple time scale perturbation expansion to be used in the long-term mass transport analysis. Some linearization considerations that are useful in subsequent analytical treatments and in understanding the scaling of the vertical turbulent viscosity and diffusion coefficients are also presented. Filtered long-term mass transport equations for weak and strong vertical stratification conditions are next presented as are the two-dimensional barotropic and the three-dimensional residual mass transport velocity fields. The paper concludes with an analysis of the dynamics of the residual mass transport velocity field, including the presentation of some analytical results for the two-dimensional barotropic field and a discussion of the general features of the three dimensional field. Methodologies for numerically determining the three-dimensional mass transport velocity and analyzing field current meter data are also presented.

## II. Hydrodynamic and Mass Transport Equations

The scaled hydrodynamic and mass transport equations in a horizontal curvilinear orthogonal and vertically stretched coordinate system are:

$$\partial_t u_\alpha + F\left(\frac{u_\beta}{m_\beta}\partial_\beta u_\alpha + \frac{w}{h}\partial_z u_\alpha\right) - F_R f e_{\alpha\beta} u_\beta - \frac{F}{m_\alpha m_\beta}(u_\beta \partial_\alpha m_\beta - u_\alpha \partial_\beta m_\alpha) u_\beta$$
$$- \frac{1}{h^2}\partial_z(N_v \partial_z u_\alpha) + \frac{1}{F m_\alpha}\partial_\alpha\left[g(h-h_o) + g\beta \frac{F^2}{F_D^2} h \int_z^1 s\,dz\right]$$
$$- g\beta \frac{F^2}{F_D^2}\frac{s}{m_\alpha}\partial_\alpha (h_o - zh) + 0\,(h_s\lambda_s) = 0 \qquad (1)$$

$$\partial_t h + \frac{F}{m}\partial_\beta\left(\frac{m}{m_\beta} h\, u_\beta\right) + \partial_z w = 0 \qquad (2)$$

$$\partial_t s + F\left(\frac{u_\beta}{m_\beta}\partial_\beta s + \frac{w}{h}\partial_z s\right) - \frac{F^\epsilon}{h^2}\partial_z(K_v \partial_z s) + 0\,(h_s\lambda_s) = 0 \qquad (3)$$

with $m_\beta$ being the square root of the diagonal components of the metric tensor, m the square root of the metric tensor determinant, $u_\alpha$ the components of the horizontal velocity vector, w the physical vertical velocity in the stretched vertical coordinate, $0 \le z \le 1$, h the total water column depth, $h_o$ the still water column depth, s the salinity, $\beta$ the volumetric expansion coefficient, and $N_v$ and $K_v$ vertical turbulent diffusion coefficients for momentum and mass. The scaling parameters are: F the long wave Froude number or ratio of free surface displacement, $h-h_o$, scale to still water depth scale, $F_R$ the rotational Froude number, and $F_D$ the densimetric long wave Froude number. It is noted that Eqs.(1-3) assume dimensionless form by setting f, g and $\beta$ to unity, while dimensional forms are obtained by setting the three scaling parameters to unity.

Horizontal velocity has been scaled by FC, with C being the shallow water wave speed, $(gh)^{1/2}$. Time and horizontal distance have been scaled by the long wave motion frequency, $\omega_s$, and wave number $\lambda_s$, with $C = \omega_s/\lambda_s$. The vertical turbulent viscosity, $N_v$, has been scaled by $\omega_s h_s^2$, thus including frictional effects at lowest order. The vertical turbulent diffusivity, $K_v$, has been scaled by $F^\epsilon \omega_s h_s^2$. The exponent, $\epsilon$, is used to adjust the scaling to accommodate different vertical stratification conditions with a value of zero for weak vertical stratification and of two for strong vertical stratification. The rational for this variable scaling will be discussed further in a following section. The omitted terms of order $h_s \lambda_s$ in Eqs.(1) and (3) are the horizontal diffusive momentum and mass fluxes, assuming that the scale of the horizontal diffusion coefficients does not exceed $\omega_s h_s/\lambda_s$, a conservatively large value.

## III. Scaling, Filtering and Linearization Considerations

### III.1 Multiple time scaling and filtering

For weakly nonlinear long wave dynamics, the long wave Froude number, F, is much less than unity, and may be taken as the small parameter in a regular perturbation analysis of Eqs.(1-3). It is expedient to introduce a multiple time scale expansion of the time derivative,

$$\partial_t = \partial_{t_o} + F^2 \partial_{t_2} \tag{4}$$

which serves to isolate short-term variations in the fast time variable, $t_o$, from long-term variations in the slow time variable, $t_2$. Correspondingly, the time scale associated with long-term variations is $F^{-2}\omega_s^{-1}$, corresponding to weeks or months for long-term variability as opposed to hours or days for short term variability.

Previous derivations of long-term mass transport equations have utilized simple time averaging over the period of the long wave motion for single frequency motion (Feng et al. 1986), time averaging over the synodic period for deterministic multiple period long wave motion (Hamrick, 1987), and ensemble averaging for a continuous spectrum of mixed deterministic and random long wave motion, (Hamrick, 1987). A more general approach for the analysis of continuous time variation and discrete time variation, characteristic of both field data and numerical model output, is to employ a filtering operation rather than a time or ensemble average operation.

The multiple time scale expansion of the time derivative, Eq.(4), applied to the salinity, for example, is equivalent to the time domain convolution operation

$$\Delta^* s = (\Delta_o + F^2 \Delta_2)^* s \tag{5}$$

or the frequency domain multiplication operation

$$\hat{\Delta}\,\hat{s} = (\hat{\Delta}_o + F^2 \hat{\Delta}_2)\hat{s} \tag{6}$$

where, $*$, denotes convolution and, $\hat{}$, denotes the Fourier transform. The Fourier transforms of the derivative operators may be defined as

$$\hat{\Delta} = i\omega;\ 0 \le |\omega| \le \infty \tag{7a}$$

$$\hat{\Delta}_o = \begin{cases} 0;\ 0 \le |\omega| \le \infty \\ i\omega;\ \omega_p < |\omega| \le \infty \end{cases} \tag{7b}$$

$$\hat{\Delta}_2 = \begin{cases} i\omega;\ 0 \le |\omega| \le \omega_p \\ 0;\ \omega_p < |\omega| \le \infty \end{cases} \tag{7c}$$

with $\omega_p$ defining the intermediate frequency scale satisfying, $F^2\omega_s < \omega_p < \omega_s$. A low pass filter operator is now defined in the time domain by

$$\langle s \rangle = \phi^* s \tag{8}$$

with the Fourier transformation of the filtering operation defined theoretically by the ideal frequency domain filter response function

$$\hat{\phi} = \begin{cases} 1;\ 0 \le |\omega| \le \omega_p \\ 0;\ \omega_p \le |\omega| \le \infty \end{cases} \tag{9}$$

where $\omega_p$ is now identified as the upper limit of the filter pass band. When the filter operation is applied to the time derivative, the result is

$$\langle \partial_t s \rangle = F^2 \partial_{t_2} \langle s \rangle \tag{10}$$

representing the slow or long-term variation of low pass filtered salinity. For application to digital processing of discrete data, the ideal filter, Eq.(9) is replaced with a filter tapered about the frequency $\omega_p$.

The choice of $\omega_p$, the upper limit of the filter is related to the dynamics of the situation of interest. For weakly nonlinear wave motion, the lowest order motion at order $F^0$ should not pass through the filter. The next order motions at order $F^1$ represent nonlinear corrections including higher harmonics and higher frequency compounds and steady and lower frequency compounds. It is these steady and lower frequency compound components and the slowly varying density and atmospheric forced motions, also presumed of order $F^1$, that are the residual circulation and must pass through the low pass filter, thus determining $\omega_p$.

### III.2 Vertical diffusivity scaling and linearization

In this section linearizations of the vertical diffusive fluxes of momentum and mass are presented, which are useful in subsequent analytical solution procedures and provide insight into the scaling of the momentum and mass diffusion coefficients. Beginning with the vertical momentum flux, the vertical boundary conditions are

$$\frac{N_v}{h} \partial_z u_\alpha = \tau_{b\alpha} = c_b |u_\beta| u_\alpha ; \quad z = z_b \tag{11a}$$

$$\frac{N_v}{h} \partial_z u_\alpha = \tau_{s\alpha} ; \quad z = 1 \tag{11b}$$

$$c_b^{1/2} = \frac{\kappa}{\ln\left(\frac{z_b + z_o}{z_o}\right)} , \tag{12}$$

where $\tau_{b\alpha}$ and $\tau_{s\alpha}$ are the bottom and free surface kinematic stresses respectively and $c_b$ is the bottom friction coefficient, which depends on $z_b$, the height above the bottom boundary at which the boundary condition is applied, $z_o$, the roughness height, and the von Karman constant, $\kappa$. It is noted that as $z_b$ approaches zero, the friction coefficient $c_b$ approaches infinity and the bottom boundary condition becomes the no slip condition. For stratified turbulent estuarine flow a simple, but realistic, relationship for the vertical turbulent viscosity or momentum diffusivity is

$$\frac{N_v}{h} = S_N \left(\frac{\Lambda}{h}\right)^2 |\partial_z u_\beta| , \tag{13}$$

where $S_N$ is a Richardson number dependent stability function such as that given by Mellor and Yamada (1982), and $\Lambda$ is the mixing length.

The bottom boundary condition, Eq.(11) and the form of $N_v$ given by Eq.(13) are strongly nonlinear, in both temporal and spatial dependence. For analytical solution of Eq.(1), a constant momentum diffusivity is defined by

$$\frac{N_v}{h} = q_N \tag{14}$$

where $q_N$ has dimensions of velocity. This allows a linearized bottom boundary condition

$$q_N \partial_z u_\alpha = \tau_{b\alpha} = \gamma q_N u_\alpha , \qquad \text{at } z = 0 \tag{15}$$

$$\frac{1}{\gamma} = (z_m + z_o) \ln\left(\frac{z_m + z_o}{z_o}\right), \tag{16}$$

where $\gamma$ is a slip coefficient chosen such that the constant diffusivity velocity profile matches the logarithmic profile at a height $z_m$. Requiring that the depth integrated, low pass filtered, turbulent energy production by vertical shear be equivalent between the nonlinear and linearized formulations gives

$$q_N = \frac{\langle c_b |u_\beta(z_b)|^3 + \int_{z_b}^{1} S_N \left(\frac{\Lambda}{h}\right)^2 |\partial_z u_\beta|^3 \, dz \rangle}{\langle \gamma |u_\beta(0)|^2 + \int_0^1 |\partial_z u_\beta|^2 \, dz \rangle} \tag{17}$$

allowing the magnitude of $q_N$ to be determined such that bulk energy conservation is insured.

The preceeding results can be used to support the scaling of the vertical turbulent diffusivity, $N_v$. In essence the scaling of $N_v$ is chosen consistent with the scaling of the vertical momentum flux whose scaling is in turn set by the bottom boundary stress. This is immediately evident from Eqs.(14) and (15), which indicate that $N_v$ scales as

$$N_{vs} = q_{Ns} h_s = \frac{c_b}{\gamma} u_s = \frac{c_b}{\gamma} FC \tag{18}$$

in the limit of no vertical shear. At tidal forcing frequencies and for representative matching and roughness heights the scale given by Eq.(18) is consistent with the scale $\omega_s h_s^2$.

For the vertical salinity flux, the boundary conditions are

$$\frac{K_v}{h} \partial_z s = 0 \ ; \ z = 0, 1 \tag{19}$$

where the vertical turbulent mass diffusivity, $K_v$, is given by

$$\frac{K_v}{h} = S_K (\frac{\Lambda}{h})^2 \ |\partial_z u_\beta| \tag{20}$$

with $S_K$ being a Richardson number dependent stability function. For analytical solution of Eq.(3), a constant mass diffusivity is defined by

$$\frac{K_v}{h} = q_K \tag{21}$$

where $q_K$ has units of velocity. Requiring that the depth integrated, low pass filtered, turbulent energy production by the vertical buoyancy or salinity flux be equivalent between the nonlinear and linearized formulations gives

$$q_K = \frac{< \int_o^1 S_k (\frac{\Lambda}{h})^2 \ |\partial_z u_\beta| (-\partial_z s) dz >}{<s(1) - s(0)>}, \tag{22}$$

which allows the magnitude of $q_K$ to be determined such that bulk energy conservation is insured.

The results represented by Eq.(22) can now be used to justify the variable scale, $F^\epsilon \omega_s h_s^2$, introduced for the vertical turbulent mass diffusivity, $K_v$. For weak vertical stratification $\epsilon$ was chosen as unity with $K_v$ being scaled by $\omega_s h_s^2$, the same scale used for the vertical turbulent viscosity. The limiting case of weak vertical stratification is vertical homogeniety for which Eq.(22) fails to define $q_N$. Thus, for weak vertical stratification the scale for $K_v$ can be chosen consistent with the scale for $N_v$ based on similarity of mass and momentum diffusion in homogeneous flow. The limiting case of strong stratification is a two layer system with sufficient interfacial stability as to eliminate diffusive mass flux across the interface, with (22) giving $q_K$ equal to zero, however, a finite but very small scale for $K_v$ is necessary in the strong stratification case. To meet this condition, $\epsilon$ is chosen as two, giving a scale of $F^2 \omega_s h_s^2$, which serves to include vertical diffusive transport in the long-term mass transport equation for strong stratification.

## IV. Filtered Long-term Mass Transport Equations

### IV.1 Weak vertical stratification equation

For weak vertical stratification conditions, Eq.(3) in conservative form incorporating Eq.(4), is

$$(\partial_{t_o} + F^2 \partial_{t_2})(hs) + \frac{F}{m} \partial_\beta ( \frac{m}{m_\beta} h u_\beta s)$$
$$+ F \partial_z (ws) - \frac{1}{h} \partial_z (K_v \partial_z s) + O(h_s \lambda_s) = 0 \tag{23}$$

with the boundary conditions

$$\frac{K_v}{h} \partial_z s = 0 \ ; \ z = 0,\ 1. \tag{24}$$

The order $F^o$ approximation to Eq.(23) is

$$\partial_{to} s_o - h_o^{-2} \partial_z (K_v \partial_z s_o) = 0 \tag{25}$$

with the zero flux boundary conditions given by Eq.(24). For arbitrary temporal and spatial distribution of $K_v$ and an arbitrary initial condition, the solution for $s_o$ will exponentially decay on a time scale of $h_o^2/K_v$, asymptotically approaching its vertically averaged, $t_o$ time domain filtered value, $<\bar{s}_o>$, where, $\bar{}$, demotes the depth average over $0 \le z \le 1$. Without loss of generality $<\bar{s}_o>$ may be replaced by $<\bar{s}>$, the slowly or long-term varying filtered, depth averaged salinity for which a mass transport equation is sought.

The asymptotic order F approximation to Eq.(23) is, in nonconservative form,

$$\partial_{t_o} s_1 + \frac{u_{\beta o}}{m_\beta} \partial_\beta <\bar{s}> - \frac{1}{h_o^2} \partial_z (K_v \partial_z s_1) = 0 \tag{26}$$

subject to the no flux boundary condition, Eq.(24). Integrating Eq.(26) over the depth gives:

$$\partial_{t_o} \bar{s}_1 + \frac{\bar{u}_{\beta o}}{m_\beta} \partial_\beta <\bar{s}> = 0 \ , \tag{27}$$

which may be integrated to give

$$\bar{s}_1 = \bar{s}_1 \ (t_o = 0) - \int_o^{t_o} \bar{u}_{\beta o} dt_o \frac{1}{m_\beta} \partial_\beta <\bar{s}> = \frac{-\bar{\xi}_{\beta o}}{m_\beta} \partial_\beta <\bar{s}> \ , \tag{28}$$

where $\bar{\xi}_\beta$ is the barotropic displacement. For $\bar{s}_1$, given by Eq.(28) to be bounded in the $t_o$ time domain, $\bar{u}_{\beta o}$, the order $F^o$ barotropic or external mode horizontal velocity cannot pass the low pass filter, $<\bar{u}_{\beta o}> = 0$, which also results in $<\bar{s}_1> = 0$, consistent with the assumption $<\bar{s}_o> = <\bar{s}>$. This allows $\bar{u}_{\beta o}$ to be identified as the higher frequency barotropic or external velocity field associated with tidal and strong higher frequency atmospheric forcings. The salinity field, $\bar{s}_1$, simply represents the advection of the long-term varying salinity field, $<\bar{s}>$, by the short-term varying velocity field $\bar{u}_{\beta o}$.

Subtracting Eq.(27) from Eq.(26) gives

$$\partial_{t_o} s_1' + \frac{u'_{\beta o}}{m_\beta} \partial_\beta <\bar{s}> - \frac{1}{h_o^2} \partial_z (K_v \partial_z s_1') = 0 \tag{29}$$

the equation governing the order F vertical salinity variation or stratification. It is noted the vertical shear or internal mode velocity field, $u'_{\beta o}$, may include a long-term or slowly varying portion resulting in a slowly varying vertical stratification governed by

$$\frac{<u'_{\beta o}>}{m_\beta} \partial_\beta <\bar{s}> - \frac{1}{h_o^2} \partial_z <K_v \partial_z s_1'> = 0. \tag{30}$$

Hamrick (1986b), presented analytic solutions of Eqs.(29) and (30), for constant $K_v$.

The depth integrated, low pass filtered, order $F^2$ approximation to Eq.(23), in conservative form, is,

$$h_o \partial_{t_2} <\bar{s}> + \frac{1}{m} \partial_\alpha [\frac{m}{m_\alpha} h_o (<\bar{u}_{\alpha_1}><\bar{s}> + <h_1 \bar{u}_{\alpha o}><\bar{s}> + <\bar{u}_{\alpha o} \bar{s}_1> + <\overline{u'_{\alpha o} s_1'}>)] = 0. \tag{31}$$

The third divergence term in Eq.(31) may be rewritten using the solution for $s_1$, Eq.(28), to give

$$\partial_\alpha(\frac{m}{m_\alpha}h_o\langle\bar{u}_{\alpha o}\bar{s}_1\rangle) = \partial_\alpha[\frac{m}{m_\alpha m_\beta}e_{\alpha\beta}\partial_\beta(h_o\bar{A}_z)] \tag{32}$$

$$\bar{A}_z = \langle\bar{u}_{1o}\int_o^{t_o}\bar{u}_{2o}dt_o\rangle \tag{33}$$

where $h_o A_z$ is the vertical component of a vector potential. The fourth divergence term in Eq.(31) represents shear dispersion and may be written as

$$\langle\overline{u'_{\alpha o}s'_1}\rangle = -\frac{\langle D_{\alpha\beta}\rangle}{m_\beta}\partial_\beta\langle\bar{s}\rangle \tag{34}$$

with $\langle D_{\alpha\beta}\rangle$ being the low pass filtered generalization of the ensemble averaged shear dispersion coefficient tensor given by Hamrick (1986b).

It is useful to express Eq.(31) in expanded form using conventional notation, the results being

$$\begin{aligned} m_x m_y h_o \partial_{t_2}\langle s\rangle &+ \partial_x(m_y h_o \bar{u}_L\langle\bar{s}\rangle) + \partial_y(m_y h_o \bar{v}_L\langle\bar{s}\rangle) \\ &- \partial_x(\frac{m_y}{m_x}h_o\langle D_{xx}\rangle\partial_x\langle\bar{s}\rangle + h_o\langle D_{xy}\rangle\partial_y\langle\bar{s}\rangle) \\ &- \partial_y(h_o\langle D_{yx}\rangle\partial_x\langle s\rangle + \frac{m_x}{m_y}h_o\langle D_{yy}\rangle\partial_y\langle\bar{s}\rangle) = 0, \end{aligned} \tag{35}$$

the filtered long-term mass transport equation for weak vertical stratification. The advective transport field, $(u_L, v_L)$ is given by

$$\bar{u}_L = \langle\bar{u}_1\rangle + \langle\frac{h_1\bar{u}_o}{h_o}\rangle + \frac{1}{h_o m_y}\partial_y(h_o\bar{A}_z) \tag{36}$$

$$\bar{v}_L = \langle\bar{v}_1\rangle + \langle\frac{h_1\bar{v}_o}{h_o}\rangle - \frac{1}{h_o m_x}\partial_x(h_o\bar{A}_z) \tag{37}$$

$$\bar{A}_z = \langle\bar{u}_o\int_o^{t_o}\bar{v}_o dt_o\rangle, \tag{38}$$

which is identified as the lowest order approximation to the barotropic Lagrangian residual velocity (Longuet-Higgins, 1969), and may be appropriately termed the barotropic residual mass transport velocity. The residual mass transport velocity, $(\bar{u}_L,\bar{v}_L)$ is the sum of the Eulerian residual velocity, $(\langle\bar{u}_1\rangle,\langle\bar{v}_1\rangle)$, and the Stokes drift velocity. The Stokes drift velocity may be further divided into the wave transport velocity, $(\langle h_1\bar{u}_o\rangle/h_o, \langle h_1\bar{v}_o\rangle/h_o)$, since $h_1$ is the long wave surface displacement, and the vector potential transport velocity, the last terms in Eqs.(38) and (39).

The Eulerian residual velocity and the wave transport velocity may be combined to form the barotropic Eulerian residual transport velocity which satisfies the depth averaged, low pass filtered, order F approximation of the continuity equation, Eq.(2). In addition, the residual mass transport velocity satisfies the zero divergence condition

$$\partial_x(m_y h_o \bar{u}_L) + \partial_y(m_x h_o \bar{v}_L) = 0. \tag{39}$$

The weak stratification, long-term mass transport model is completed by noting that Eq.(30) may be used to determine the slowly or long-term varying vertical stratification, $\langle s_1\rangle$, after Eq.(35) has been used to determine the depth averaged, filtered salinity, $\langle\bar{s}\rangle$. Analytic solutions of Eq.(30) may be obtained using a constant vertical diffusion coefficient defined by Eqs.(21) and (22).

### IV.2 Strong vertical stratification equation

For strong vertical stratification conditions, Eq.(3) in conservative form incorporating Eq.(4), is

$$(\partial_{t_o} + F^2\partial_{t_2})(hs) + \frac{F}{m}\partial_\beta(\frac{m}{m_\beta} hu_\beta s)$$
$$+ F\partial_z(ws) - \frac{F^2}{h}\partial_z(K_v\partial_z s) + O(h_s\lambda_s) = 0 \tag{40}$$

with the boundary conditions given by Eq.(24). The order $F^o$ approximation to Eq.(40) has the solution, $s_o = \langle s_o\rangle$, and without loss of generality, $s_o = \langle s\rangle$, the low pass filtered salinity. The order F, nonconservative approximation to Eq.(40) is

$$\partial_{t_o} s_1 + \frac{u_{\beta o}}{m_\beta}\partial_\beta\langle s\rangle + \frac{w_o}{h_o}\partial_z\langle s\rangle = 0, \tag{41}$$

which may be integrated to give

$$s_1 = s_1(t_o=0) - \int_0^{t_o} u_{\beta o}dt_o \frac{1}{m_\beta}\partial_\beta\langle s\rangle - \int_0^{t_o} w_o dt_o \frac{1}{h_o}\partial_z\langle s\rangle$$

$$s_1 = s_1(t_o=0) - \xi_{\beta o}\frac{1}{m_\beta}\partial_\beta\langle s\rangle - \zeta_0\frac{1}{h_o}\partial_z\langle s\rangle \tag{42}$$

with $\xi_{\beta o}$ and $\zeta_o$ being the horizontal and vertical displacements. For $s_1$ to be bounded in the $t_o$ time domain, the order $F^o$ velocity field cannot pass the low pass filter. The order $F^o$ velocity field may be identified as the higher frequency three dimensional velocity field associated with tidal and strong higher frequency atmospheric forcings.

The low pass filtered, order $F^2$ approximation to Eq.(40), in conservative form is

$$h_o\partial_{t_2}\langle s\rangle + \frac{1}{m}\partial_\alpha[\frac{m}{m_\alpha}h_o(\langle u_1\rangle\langle s\rangle + \langle\frac{h_1u_o}{h_o}\rangle\langle s\rangle + \langle u_o s_1\rangle)]$$
$$+ \partial_z(\langle w_1\rangle\langle s\rangle + \langle w_o s_1\rangle) - \frac{1}{h_o}\partial_z(\langle K_v\rangle\partial_z\langle s\rangle) = 0. \tag{43}$$

The third horizontal divergence term, and the second vertical divergence term may be rewritten, using the solution for $s_1$ given by Eq.(42), to give

$$\partial_\alpha(\frac{m}{m_\alpha}h_o\langle u_o s_1\rangle) = \partial_\alpha[\frac{m}{m_\alpha m_\beta}e_{\alpha\beta}\partial_\beta(h_oB_z) - \frac{m}{m_\alpha}e_{\alpha\beta}\partial_zB_\beta] \tag{44}$$

$$\partial_z\langle w_o s_1\rangle = \partial_z[\frac{1}{m}e_{\alpha\beta}\partial_\alpha(m_\beta B_\beta)] \tag{45}$$

$$B_1 = \langle u_{20}\int_0^{t_o} w_o dt_o\rangle, \quad B_2 = \langle w_o\int_0^{t_o} u_{10}dt_o\rangle, \quad B_3 = \langle u_{10}\int_0^{t_o} u_{20}dt_o\rangle \tag{46}$$

where $B_\beta$ and $B_z$ are the horizontal and vertical components, respectively, of a vector potential.

Expressing Eq.(43) in expanded form gives

$$h_o\partial_{t_2}\langle s\rangle + \frac{1}{m_xm_y}\partial_x(m_yh_ou_L\langle s\rangle) + \frac{1}{m_xm_y}\partial_y(m_xh_ov_L\langle s\rangle)$$
$$+ \partial_z(w_L\langle s\rangle) - \frac{1}{h_o}\partial_z(\langle K_v\rangle\partial_z\langle s\rangle) = 0 \quad , \tag{47}$$

the filtered long-term mass transport equation for strong vertical stratification. The advective transport field is given by

$$u_L = \langle u_1\rangle + \langle\frac{h_1u_o}{h_o}\rangle + \frac{1}{h_om_y}\partial_y(h_oB_z) - \frac{1}{h_o}\partial_zB_y \tag{48a}$$

$$v_L = \langle v_1\rangle + \langle\frac{h_1v_o}{h_o}\rangle + \frac{1}{h_o}\partial_zB_x - \frac{1}{h_om_y}\partial_x(h_oB_z) \tag{48b}$$

$$w_L = \langle w_1\rangle + 0 + \frac{1}{m_xm_y}\partial_x(m_yB_y) - \frac{1}{m_xm_y}\partial_y(m_xB_x) \tag{48c}$$

$$B_x = \langle v_o \int_o^{t_o} w_o dt_o \rangle \tag{49a}$$

$$B_y = \langle w_o \int_o^{t_o} u_o dt_o \rangle \tag{49b}$$

$$B_z = \langle u_o \int_o^{t_o} v_o dt_o \rangle , \tag{49c}$$

which is identified as the lowest order approximation to the three-dimensional Lagrangian residual velocity, and may be appropriately termed the three-dimensional residual mass transport velocity. The residual mass transport velocity, $(u_L, v_L, w_L)$ is the sum of the Eulerian residual velocity, $(\langle u_1 \rangle, \langle v_1 \rangle, \langle w_1 \rangle)$, and the three-dimensional Stokes drift velocity. The Stokes drift velocity may be further divided into the horizontal wave transport velocity, $(\langle h_1 u_o \rangle / h_o, \langle h_1 v_o \rangle / h_o, 0)$ and the vector potential transport velocity. The vector potential transport velocity is composed of the terms in Eq.(48), which involve $(B_x, B_y, B_z)$, the components of the vector potential B and is equivalent to curl B since $h_o^z$ is equivalent to $m_z$.

The Eulerian residual velocity and the horizontal wave transport velocity may be combined to form the Eulerian residual transport velocity which satisfies the low pass filtered, order F approximation of the continuity equation, Eq.(2). In addition, the residual mass transport velocity satisfies the zero divergence condition

$$\partial_x (m_y h_o u_L) + \partial_y (m_x h_o v_L) + \partial_z w_L = 0 , \tag{50}$$

which allows the long-term mass transport equation, Eq.(47), to alternately be written in nonconservative form.

The parameterization of the vertical diffusive salinity transport in the long-term mass transport equation, Eq.(47), using the low-pass filtered vertical diffusivity requires some elaboration. If the lowest order and most significant contribution to the velocity field is a single dominant frequency tidal forcing, Eq.(20), indicates that the temporal structure of $K_v$ will involve steady and second harmonic of the dominant frequency components. For this simple forcing, the lowest order salinity will be steady in time, while Eq.(42), indicates that the next order salinity will be oscillatory at the dominant frequency. Thus, it is readily shown that for these temporal structures of the vertical diffusivity and salinity, the low pass filter of the vertical salinity flux gives the parameterization in the filtered long-term mass transport equation, Eq.(47).

## V. Dynamics of the Residual Mass Transport Velocity Field

A definitive analysis of the dynamics of the residual mass transport velocity field, for either weak or strong vertical stratification conditions, for a prototype estuary would require numerical integration of the hydrodynamic and mass transport equations and subsequent manipulation and filtering of the output. An alternative approach is to rigorously analyze the hydrodynamic equations using the perturbation techniques employed in deriving the two filtered long-term mass transport equations in the preceeding section, with an objective of gaining insight into the significant dynamic influences.

### V.1 Weak vertical stratification analysis

The major simplification to the hydrodynamic equations, Eqs.(1) and (2), for the condition of weak vertical stratification results from the vertical uniformity and short-term temporal independence of the order $F^o$ salinity. For weak vertical stratification conditions, the order $F^o$ approximations to the momentum and continuity equations, Eqs.(1) and (2), are:

$$\partial_{to} u_{\alpha o} + F_R f e_{\alpha\beta} u_{\beta o} - \frac{1}{h_o^2} \partial_z (N_v \partial_z u_{\alpha o}) + \frac{g}{m_\alpha} \partial_\alpha h_1 + g\beta \frac{F}{F_D^2} h_o \frac{(1-z)}{m_\alpha} \partial_\alpha \langle s \rangle = 0 \tag{51}$$

$$\partial_{t_0} h_1 + \frac{1}{m}\partial_\alpha\left(\frac{m}{m_\alpha} h_o u_{\alpha o}\right) + \partial_z w_o = 0 \tag{52}$$

where for the present, the order of $F/F_D^2$ is unspecified. Eqs.(51) and (52) may be integrated over the depth to give equations governing the external or barotropic mode. The presence of the slowly or long-term varying salinity gradient in the momentum equations is in conflict with the requirement that the order $F^o$ barotropic or external velocity field not pass the low pass filter. Since the order of $F/F_D^2$ is in general intermediate between $F^o$ and $F^1$, it is reasonable to rescale the barotropic portion of the horizontal salinity gradient forcing and the portion of the bottom boundary stress associated with it to order F. The resulting external equations are:

$$\partial_{to}\bar{u}_{\alpha o} + F_R f e_{\alpha\beta}\bar{u}_{\beta o} + g\frac{1}{m_\alpha}\partial_\alpha h_1 + \frac{\tau}{h_o}(\tau_{b\alpha o} - \tau_{s\alpha o}) = 0 \tag{52}$$

$$\partial_{t_o} h_1 + \frac{1}{m}\partial_\alpha\left(\frac{m}{m_\alpha} h_o \bar{u}_{\alpha o}\right) = 0 \ . \tag{54}$$

Before continuing with the analysis of the external equations, it is necessary to consider the internal or shear and baroclinic mode equations and the specification of the bottom boundary stress.

The internal or shear and baroclinic mode equations are obtained by subtracting the depth integrals of Eqs.(51) and (52) from the original equations, the results being

$$\partial_{to}u'_{\alpha o} + F_R f e_{\alpha\beta}u'_{\beta o} - \frac{1}{h_o^2}\partial_z(N_v\partial_z u'_{\alpha o})$$
$$+ \frac{1}{h_o}(\tau_{s\alpha o} - \tau_{b\alpha o}) + g\beta\frac{F}{F_D^2}\frac{1}{2}\frac{1}{h_o}\frac{(1-2z)}{m\alpha}\partial_\alpha\langle\bar{s}\rangle = 0 \tag{55}$$

$$\frac{1}{m}\partial_\alpha\left(\frac{m}{m_\alpha}h_o u'_{\alpha o}\right) + \partial_z w_o = 0. \tag{56}$$

The baroclinic portion of the horizontal salinity gradient forcing has been retained at this order such that the strong vertical shear associated with it might be incorporated into the shear dispersion transport in the long-term mass transport equation. Hamrick (1986b) has presented analytical solutions to Eq.(55) for a constant vertical turbulent viscosity, $N_v$, and with the horizontal velocity represented as

$$u'_{\alpha o} + \langle u'_{\alpha o}\rangle + \int_{\infty}^{\infty} u'_{\alpha o} e^{i\omega t_o} d\omega. \tag{57}$$

Using $N_v$ given by Eqs.(14) and (17), the solutions for no surface stress are:

$$\langle u'_{\alpha o}\rangle = -\delta_{\alpha\beta}(\Gamma_3 - r_3\Gamma_1 - r_4\Gamma_2)_{\sigma=0}\, g\frac{\beta h_o^2}{q_N}\frac{F}{F_D^2}\frac{1}{m_\beta}\partial_\beta\langle\bar{s}\rangle$$
$$- e_{\alpha\beta}(\Gamma_4 + r_4\Gamma_1 - r_3\Gamma_2)_{\sigma=0}\, g\frac{\beta h_o^2}{q_N}\frac{F}{F_D^2}\frac{1}{m_\beta}\partial_\beta\langle\bar{s}\rangle \tag{58}$$

$$U'_{\alpha o} = [-\delta_{\alpha\beta}(r_1\Gamma_1 + r_2\Gamma_2) - e_{\alpha\beta}(r_1\Gamma_2 - r_2\Gamma_1)]\bar{U}_{\beta o} \tag{59}$$

where the $\Gamma$'s are functions of z given by

$$\Gamma_1 = 2\sum_{n=1}^{\infty}\frac{(n^2\pi^2 + i\sigma)\cos(n\pi z)}{(n^2\pi^2 + i\sigma)^2 + \epsilon^2} \tag{60a}$$

$$\Gamma_2 = 2\epsilon\sum_{n=1}^{\infty}\frac{\cos(n\pi z)}{(n^2\pi^2 + i\sigma)^2 + \epsilon^2} \tag{60b}$$

$$\Gamma_3 = 2\sum_{n=1}^{\infty} \frac{[1-\cos(n\pi)]\cos(n\pi z)}{n^4\pi^4 + \epsilon^2} \tag{60c}$$

$$\Gamma_4 = 2\epsilon\sum_{n=1}^{\infty} \frac{[1-\cos(n\pi)]\cos(n\pi z)}{n^2\pi^2\ (n^4\pi^4 + \epsilon^2)} \tag{60d}$$

$$\sigma = \frac{\omega h_o}{q_N} = \frac{\omega h_o^2}{N_v} \tag{61}$$

$$\epsilon = F_R \frac{f h_o}{q_N} = F_R \frac{h_o^2}{N_v}\ . \tag{62}$$

The r coefficients are functions of $\sigma$, $\epsilon$ and the bottom slip parameter, $\gamma$. They are determined by requiring that $\tau_{b\alpha}$ satisfy Eq.(15), which in turn defines the Fourier transform of the bottom boundary stress as

$$T_{b\alpha o} = (\delta_{\alpha\beta} r_1 - e_{\alpha\beta} r_2) q_N \bar{U}_{\beta o} \tag{63}$$

where $\bar{U}_{\beta o}$ is the Fourier transform of the external or barotropic mode horizontal velocity.

The external or depth mean order $F^o$ hydrodynamic problem may now be analyzed by Fourier transforming Eqs.(53) and (54) and using Eq.(63), to give

$$[\delta_{\alpha\beta}(i\omega + r_1 \frac{q_N}{h_o}) - e_{\alpha\beta}(F_R f + r_2 \frac{q_N}{h_o})]\ \hat{U}_{\beta o} - \frac{g}{m_\alpha} \partial_\alpha H_1 = 0 \tag{64}$$

$$i\omega H_1 + \frac{1}{m} \partial_\alpha (\frac{m}{m_\alpha} h_o \bar{U}_{\alpha o}) = 0 \tag{65}$$

for the case of no surface stress. These equations may be combined into a single elliptic equation for the Fourier transform of the free surface displacement, $H_1$. The elliptic equation may be solved numerically for realistic bottom topography and shoreline geometry, however, analytical solutions of the two-dimensional problem with variable bottom topography are difficult, if not impossible to obtain. To gain some general insight into the role of topography, channel curvature and the earth's rotation in determining the external barotropic flow dynamics, and ultimately the residual mass transport velocity dynamics, Eqs.(64) and (65) can be solved by perturbation techniques for slight variations in topography, slight channel curvature and a narrow channel width resulting in a weak geostrophic effect.

Rescaling the lateral horizontal coordinate, $x_2 = y$, and the lateral velocity, $u_{20} = v_o$, by a small parameter $\delta$ such that the channel width is of order $\delta/\lambda_s$, and representing the topography and channel curvature by

$$m_x = 1 - \delta_c \frac{b}{R} \frac{y}{b} \tag{66}$$

$$m_y = 1 \tag{67}$$

$$h_o = h_{oo} + \delta_h h_{o1} \tag{68}$$

where y is zero along the channel centerline whose radius of curvature R varies with x along the channel. The spatially averaged still water depth is $h_{oo}$ while $h_{o1}$ represents a slight lateral, y, variation having zero mean when averaged across the constant channel width, b. The solutions of Eqs.(64) and (65) to lowest order in the perturbation parameters, in dimensional form are:

$$\bar{U}_o = \bar{U}_{oo} + (\frac{\nu}{i\omega+\nu}) \frac{h_{o1}}{h_{oo}} \bar{U}_{oo} + \frac{b}{R} \hat{U}_{oo} \frac{y}{b}$$
$$- (\frac{i\omega}{i\omega+\nu}) \frac{fb}{(gh_{oo})^{1/2}} \frac{H_{10}}{h_{oo}} (gh_{oo})^{1/2} \frac{y}{b} \tag{69}$$

$$\bar{V}_o = \left(\frac{i\omega+2\nu}{i\omega+\nu}\right) \frac{\omega b}{(gh_{oo})^{1/2}} \frac{H_{10}}{h_{oo}} (gh_{oo})^{1/2} \frac{1}{b} \int_{b/2}^{y} \frac{h_{01}}{h_{oo}}\, dy$$

$$+ \frac{i\omega b}{(gh_{oo})^{1/2}} \frac{b}{R} \frac{H_{10}}{h_{oo}} (gh_{oo})^{1/2} \left[\left(\frac{y}{b}\right)^2 - \frac{1}{4}\right] \qquad (70)$$

$$H_{11} = H_o - \frac{fb}{(gh_{oo})^{1/2}} \frac{\bar{U}_{oo}}{(gh_{oo})^{1/2}} h_{oo} \frac{y}{b} \qquad (71)$$

$$\nu = r_1 \frac{q_N}{h_{oo}} \qquad (72)$$

where $\bar{U}_{oo}$ and $H_{10}$ are the solutions of

$$(i\omega+\nu)\, \bar{U}_{oo} + g\partial_x H_{10} = 0 \qquad (73)$$

$$i\omega H_{10} + \partial_x (h_{oo} \bar{U}_{oo}) = 0 \qquad (74)$$

the equations for a constant depth, straight, nonrotating channel. Eqs.(69-71) capture the essential dynamic influences of lateral topography, channel curvature and the earth's rotation. As indicated by the solutions, the longitudinal velocity distribution across the channel is influenced by all three effects, while the lateral velocity results only from topography and curvature, and the lateral variation of the free surface displacement is associated only with the geostrophic effect.

To continue with the analysis, the order F hydrodynamic equations must be solved for the barotropic Eulerian residual horizontal velocity, $<u_{\alpha 1}>$. For shallow depth and strong turbulence as represented by a large value of $q_N \cong N_\nu/h$, the parameters $\sigma$ and $\epsilon$ defined by Eqs.(61) and (62) are small, indicating that the magnitude of the internal modes are small relative to the external, thus allowing the vertical integrals of quadratic products of the shear modes to be at lowest approximation neglected. Since the barotropic portion of the order $F/F_D^2$ horizontal salinity gradient forcing and its associated contribution to the bottom boundary stress have been rescaled to this order, it is reasonable to neglect higher order $F^2/F_D^2$ horizontal salinity gradient forcings. The resulting set of filtered equations for the order F barotropic Eulerian residual velocity are:

$$- F_R f e_{\alpha\beta} <\bar{u}_{\beta 1}> + <\frac{\bar{u}_{\beta o}}{m_\beta} \partial_\beta \bar{u}_{\alpha o}> - \frac{1}{m_\alpha m_\beta} <(\bar{u}_{\beta o} \partial_\alpha m_\beta + \bar{u}_{\alpha o} \partial_\beta m_\alpha) \bar{u}_{\beta o}>$$

$$- (\delta_{\alpha\beta} r_1 - e_{\alpha\beta} r_2) \frac{1}{h_o^2} q_N <h_1 \bar{u}_{\beta o}>$$

$$+ (\delta_{\alpha\beta} r_1 - e_{\alpha\beta} r_2) \frac{1}{h_o} q_N <\bar{u}_{\beta 1}> + \frac{g}{m_\alpha} \partial_\alpha <h_2>$$

$$+ (\delta_{\alpha\beta}(1-2r_3) + e_{\alpha\beta} r_2) g\beta \frac{1}{F_D^2} \frac{h_o}{2} \frac{1}{m_\beta} \partial_\beta <s> = 0, \qquad (75)$$

$$\frac{1}{m} \partial_\alpha \left[\frac{m}{m_\alpha} h_o \left(<\bar{U}_{\alpha 1}> + \frac{<h_1 u_{\alpha o}>}{h_o}\right)\right] = 0 . \qquad (76)$$

The most promising solution strategy for Eqs.(75) and (76) is to eliminate $<h_2>$ between the components of Eq.(75) forming a vorticity equation, and introduce the Eulerian residual transport stream function defined by

$$h_o <\bar{u}_{\alpha 1}> + <h_1 \bar{u}_{\alpha o}> = \frac{e_{\alpha\beta}}{m_\beta} \partial_\beta <\psi_1> \qquad (77)$$

The solution of the resulting equation for, $< \psi >$, readily allows c ... of the Eulerian residual transport velocity.

The equation for the Eulerian residual transport stream function was solved by perturbation methods for the narrow channel rescaling and the slight channel curvature and topography conditions specified by Eqs.(66-68). The longitudinal salinity gradient was also assumed independent of lateral position. Making use of the solutions of the order $F^o$ baratropic problem, Eqs.(69-71), the longitudinal, x, component of the Eulerian residual transport velocity is in dimensional form

$$\bar{u}_{ET} = \frac{-Q_f}{gh_{oo}}(1 + \frac{h_{01}}{h_{oo}} + \frac{y}{R}) - g\frac{bh_{oo}^2}{2\nu}(1-2r_3)\partial_x\langle s\rangle (\frac{h_{01}}{h_{oo}})$$

$$+ \frac{1}{2}\,\mathrm{Re}[\frac{i\omega}{(i\omega+\nu)}\frac{H_{10}\bar{U}_{oo}^*}{h_{oo}} - \frac{(5i\omega+2\nu)}{(i\omega+\nu)}\frac{H_1^*\bar{U}_{oo}}{h_{oo}}]\frac{h_{01}}{h_{oo}}$$

$$- \frac{fb}{2}\frac{|U_{oo}|^2}{gh_{oo}} + \frac{3\omega^2}{(\omega^2+\nu^2)}|\frac{H_{10}}{h_{oo}}|^2\frac{y}{b} \quad . \tag{78}$$

The above results show the important influences of topography, channel curvature and the earth's rotation on the Eulerian residual transport velocity. The fresh water river discharge and longitudinal salinity gradient driven portions are stronger in the seaward and landward directions, respectively in deeper regions of the lateral transect. Curvature serves to intensify the seaward river discharge toward the inner bank of a curved section. The geostrophic influence on the tidal rectification induced portion results in landward transport to the right side facing landward. The topographic influence of the tidal rectified portion is not immediately apparent.

The barotropic residual mass transport velocity, $u_L$, defined by Eqs.(36-38), may now be obtained by adding the vector potential transport velocity. The longitudinal vector potential transport velocity, evaluated using Eqs.(69) and (70) is

$$\frac{1}{h_o}\partial_y(h_o\bar{A}_z) = \frac{b}{R}\,\mathrm{Re}(\frac{\bar{U}_{oo}^*H_{10}}{h_{oo}})\frac{y}{b} + \frac{1}{2}\mathrm{Re}[\frac{(i\omega+2\nu)}{(i\omega+\nu)}\frac{\bar{U}_o^*H_{10}}{h_{oo}}]\frac{h_{01}}{h_{oo}} \quad . \tag{79}$$

Combining Eqs.(78) and (79) gives the longitudinal barotropic residual mass transport velocity,

$$\bar{u}_L = \frac{-Q_f}{bh_{oo}}(1 + \frac{h_{01}}{h_{oo}} + \frac{b}{R}\frac{y}{b}) - g\frac{\beta}{2\nu}h_{oo}^2(1-2r_3)\partial_x\langle s\rangle(\frac{h_{01}}{h_{oo}})$$

$$- \frac{3}{2}\mathrm{Re}(\frac{i\omega}{i\omega+\nu}\frac{H_1^*\bar{U}_{oo}}{h_{oo}})\frac{h_{01}}{h_{oo}} + \frac{b}{R}(\frac{\bar{U}_{oo}^*H_{10}}{h_{oo}})\frac{y}{b}$$

$$- \frac{fb}{2}\frac{|U_{oo}|^2}{gh_{oo}} + \frac{3\omega^2}{\omega^2+\nu^2}|\frac{H_{10}}{h_{oo}}|^2)\frac{y}{b} \tag{80}$$

in dimensional form. Eq.(79) indicates that the residual vector potential transport velocity is influenced by topography and curvature as would be expected from the results for the lateral order $F^o$ barotropic velocity, Eq.(70). Combining the residual vector potential transport velocity with the Eulerian residual transport velocity to form the residual mass transport velocity results in modifying the topographic influenced portion of tidal rectification component and adding a curvature influenced portion. For the topographic influenced portion in the limit of small friction, the real operator produces a negative quantity for a simple landward propagating wave, thus the transport velocity is seaward in shallower regions. The real operator in the curvature influenced portion also produced a negative quantity resulting in seaward transport toward the inner bank of a curving channel.

### V.2 Strong vertical stratification analysis

For the case of strong vertical stratification, the constraint that the order $F^o$ three-dimensional velocity not pass the low pass filter requires that the salinity gradient forcing terms in Eq.(1) not enter until order F, which is readily accomplished by specifying $F_D$ to be of order unity. The order $F^0$ approximations to Eqs.(1) and (2) are identical to Eq.(51), with the salinity gradient term absent and Eq.(52). The internal shear mode solutions to the horizontal momentum equations are given by Eq.(59), while for slight topography and curvature, and a narrow channel width the external or barotropic solutions, Eqs.(69-71), are also applicable. Thus, the condition of strong vertical stratification does not modify the order $F^0$ tidal driven long wave motion. For the functions, Γ, given by Eq.(60), very slowing varying with horizontal position, the Fourier transform of the vertical velocity is

$$W_o = \int_0^z [\ -i\omega(r_1\Gamma_1 + r_2\Gamma_2)H_1 + (r_1\Gamma_2 - r_2\Gamma_1)h_o\tilde{\Omega}_{zo}$$

$$+ (r_1\Gamma_2 - r_2\Gamma_1)(\frac{\tilde{V}_o}{m_x}\partial_x h_o - \frac{\tilde{U}_o}{m_y}\partial_y h_o)]dz \tag{81}$$

where $\tilde{\Omega}_{zo}$ is the Fourier transform of the order $F^0$ barotropic vorticity. Since the internal shear modes of the horizontal velocity exhibit cos(nπz) modal structure, the corresponding modes of the vertical velocity exhibit sin(nπz) structure, which satisfies the boundary conditions on $w_o$ in the stretched vertical coordinate exactly.

The analysis for strong vertical stratification could continue by presenting the order $F^1$ filtered approximations to Eqs.(1) and (2), which govern the three-dimensional Eulerian residual velocity field. However, such a lengthy exercise is beyond the scope of this work, and instead only the solution strategy will be briefly discussed, before moving to a qualitative discussion of the three-dimensional residual mass transport velocity field. Since the order $F^0$ internal solutions, Eqs.(59) and (81) are in a modal or spectral form with cosine and sine basis functions, the order F, filtered approximations to Eqs.(1) and (2) can be readily expanded in the same basis functions, for the three dimensional Eulerian residual transport velocity. The salinity would be expanded in the cosine functions as would the filtered long-term mass transport equation, Eq.(47). Semi-analytical solutions may then be possible for low order expansions while numerical solutions would be necessary for higher order expansions. For situations where, $F_D$, the densimetric long wave Froude number is less than unity, the salinity gradient driving force will likely be dominant in determining the vertical structure, while geostrophic, topographic and curvature influences will make significant contributions to the horizontal structure.

The general features of the three-dimensional vector potential transport velocity field may be briefly discussed at this point. The longitudinal component, from Eq.(48), is

$$u_{vp} = \frac{1}{h_o m_y}\partial_y(h_o B_z) - \frac{1}{h_o}\partial_z B_y \tag{82}$$

with $B_z$ and $B_y$ given by Eq.(49). Expressing the Fourier transforms of the order $F^0$ three-dimensional velocity field using cosine and sine basis modal forms allows $B_z$ and $B_y$ to be written as

$$B_z = < \int_{\infty}^{\infty} \bar{U}_0 e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{\bar{V}_0}{i\omega} e^{i\omega t_0} d\omega >$$

$$+ \sum_{n=1}^{\infty} < \int_{\infty}^{\infty} \bar{U}_0\ e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{\bar{V}_0^{(n)}}{i\omega}\ e^{i\omega t_0} d\omega$$

$$+ \int_{\infty}^{\infty} U_0^{(n)} e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{V_o}{i\omega} e^{i\omega t_0} d\omega > \cos(n\pi z) \tag{83}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} < \int_{\infty}^{\infty} U_0^{(m)} e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{V_0^{(n)}}{i\omega}\ e^{i\omega t_0} d\omega > \cos(m\pi z)\cos(n\pi z)$$

$$B_y = \sum_{n=1}^{\infty} < \int_{\infty}^{\infty} W_0^{(n)} e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{\bar{U}_0}{i\omega}\ e^{i\omega t_0} d\omega > \sin(n\pi z)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} < \int_{\infty}^{\infty} W_0^{(m)} e^{i\omega t_0} d\omega \int_{\infty}^{\infty} \frac{U_0^{(n)}}{i\omega}\ e^{i\omega t_0} d\omega > \sin(n\pi z)\cos(n\pi z). \tag{84}$$

It is readily seen that $B_z$ has a barotropic component equivalent to $A_z$ given by Eq.(38). The behavior of this barotropic component will likely dominate $B_z$ and the single series contribution will dominate the double series contribution. Thus a low order approximation of the portion of $u_{vp}$ associated with the lateral gradient of $B_z$ will be strongly influenced by topography and dominated by a barotropic component followed next by a two layer shear mode.

The vertical structure of $B_y$ will be dominated by the lowest mode of the single series which when differentiated with respect to z in forming its contribution to $u_{vp}$ in Eq.(82) will result in a two layer structure. Using $W_o$ given by Eq.(81), this two layer structure will likely have landward flow near the surface in opposition to the classical salinity gradient driven flow.

The four temporal representations of $B_z$ and $B_y$ in Eqs.(83) and (84) and a similar expression for $B_x$ suggest how the filtering operations necessary to obtain these quantities might be carried out. For current meter records or numerical hydrodynamic model output, the three-dimensional velocity time series are fast Fourier transformed and filtered in the frequency domain, which is equivalent to setting, for example $U_o$, equal to zero over the band, $|\omega| \leq \omega_p$. The time integration is also performed in the frequency domain by dividing the transformed quantities by $i\omega$. The multiplication of the two transform representations is best done after inverting each filtered and time integrated series back to the time domain. The final series is again fast Fourier transformed, filtered in the frequency domain and inverted to give slowly or long-term varying series for the vector potential components. The same procedure can be applied to determine the Eulerian residual and wave transport velocities. For values of quantities at discrete spatial points, local linear basis functions are the natural spatial representations, with spatial gradients determined by finite difference differentiation.

## VI. Summary and Conclusion

An approach for analyzing the dynamics of long-term mass transport in estuaries has been presented. The primary results are a pair of filtered long-term mass transport equations for conditions of weak and strong vertical stratification and weakly nonlinear long wave dynamics. The advective transport field in these equations has been shown to be the two and three-dimensional lowest order approximations to the Lagrangian residual velocity field, termed the residual mass transport velocity field. Analytical solutions were presented for the internal shear and baroclinic mode and external barotropic mode order $F^o$ velocity fields. These results were used to obtain an analytical solution for the weak stratification case, two-dimensional barotropic residual mass transport velocity showing the influences of topography, channel curvature and the earth's rotation. The dynamics of the three-dimensional residual mass transport velocity were discussed in a qualitative manner. A filtering procedure for analyzing current meter measurements and numerical hydrodynamic model output to determine the residual mass transport velocity was also presented.

**Acknowledgement**

Contribution number 1558 from the Virginia Institute of Marine Science, The College of William and Mary, Gloucester Point, VA.

## VII. References

Cheng, R. T., and V. Casulli, 1982: On Lagrangian residual currents with applications in South San Francisco Bay, Ca. Water Resour. Res., **18**, 1652-1662.

Feng, S., R. T. Cheng, and P. Xi, 1986: On tide induced Lagrangian residual current and residual transport. Part 2. Water Resour. Res., **22**, 1635-1646.

Feng S., 1987: A three-dimensional weakly nonlinear model of tide-induced Lagrangian residual current and mass transport, with an application to the Bohai Sea, in

J.C. Nihoul and P. M. Jamart, (eds). Three-Dimensional Models of Marine and Estuarine Dynamics. Elsevier, Amsterdam.

Fischer, H. B., 1972: Mass transport mechanisms in partially stratified estuaries. J. Fluid Mech., **53**, 671-687.

Hamrick, J. M., 1979: Salinity intrusion and gravitational circulation in partially stratified estuaries. PhD dissertation. University of California, Berkeley.

Hamrick, J. M., 1986a: Subtidal circulation and transport in estuaries. In Advancements in Aerodynamics, Fluid Mechanics and Hydraulics. American Society of Civil Engineers, New York.

Hamrick. J. M., 1986b: Long-term dispersion in unsteady skewed free surface flow. Estuarine. Coastal and Shelf Sci., **23**, 807-845.

Hamrick, J. M., 1987: Time averaged estuarine mass transport equations, in Proceeding of the National Conference on hydraulic Engineering, American Society if Civil Engineers, New York.

Hansen, D. V. and M. Rattray, Jr., 1965: Gravitational circulation in straits and estuaries. J. of Marine Res., **23**, 104-122.

Hydroqual, Inc., 1987: A steady state coupled hydrodynamic, water quality model of the eutrophication and anoxia process in Chesapeake Bay.

Ianniello, J. P., 1977: Tidally induced residual currents in estuaries of constant breadth and depth. J. Mar. Res., **35**, 755-786.

Ianniello, J. P., 1979: Tidally induced residual currents in estuaries of variable breadth and depth. J. Phys. Oceanogr., **9**, 962-974.

Imberger, J., 1976: Dynamics of a longitudinally stratified estuary, in Proceeding 15th Int. Conf. on Coastal Engr.

Longuet-Higgins, M. S., 1969: On the transport of mass by time-varying ocean currents. Deep-Sea Res., **16**, 431-447.

Mellor, G. L. and T. Yamada, 1982: Development of a turbulence closure model for geophysical fluid problems. Rev.Geophys., **20**, 851-875.

Najarian, T. O., D. -P. Wang, and P. -S. Huang, 1984: Lagrangian transport model for estuaries. J. Waterways. Port. Coastal and Ocean Engr., **110**, 321-333.

Oey, L. -Y., 1984: On steady salinity distribution and circulation in partially mixed and well mixed estuaries. J. Phys.Oceanogr., **14**, 629-645.

Oey, L. -Y., G. L. Mellor, and R. I. Hires, 1985: Tidal modeling of the Hudson-Raritan estuary. Estuarine, Coastal and Shelf Sci., **20**, 511-527.

Robinson, I. S., 1983: Tidally induced residual flows, in B. Johns (ed), Physical Oceanography of Coastal and Shelf Seas, Elsevier, Amsterdam.

Smith, L. H. and R. T. Cheng, 1987: Tidal and tidally averaged circulation characteristics of Suisun Bay, CA. Water Resour. Res., **23**, 143-155.

Smith, R., 1980: Buoyancy effects upon longitudinal dispersion in wide well-mixed estuaries. Philos. Trans, R. Soc. London, **A 296**, 467-496.

Zimmerman, J. T. F., 1981: Dynamics, diffusion and geomorphological significance of tidal residual eddies. Nature, **290**, 549-555.