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Abstract. An attraction-repulsion chemotaxis model with nonlinear chemotactic sensitivity functions and growth source is considered. The global-in-time existence and boundedness of solutions are proved under some conditions on the nonlinear sensitivity functions and growth source function. Our results improve the earlier ones for the linear sensitivity functions.

1. **Introduction.** In this paper, we consider the following parabolic-elliptic-elliptic attraction-repulsion chemotaxis model with nonlinear chemotactic sensitivity functions and a growth source function:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (\psi(u)\nabla v) + \xi \nabla \cdot (\phi(u)\nabla w) + f(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \gamma v + \delta u, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - \eta w + \rho u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = \dfrac{\partial w(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u(x,t)$ represents the density of cells at location $x \in \Omega$ and time t, $v(x,t)$ denotes the concentration of an attractive chemical signal and $w(x,t)$ is the concentration of a repulsive chemical signal; Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$; homogeneous Neumann boundary condition is imposed for u, v and w so that the system is a closed one; the smooth function $f(u)$ is the growth rate of the cells; positive parameters χ and ξ are the chemotactic coefficients, which measure the strength of the attraction and repulsion respectively; the mortality

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rates of v and w are γ and η respectively, and parameters δ and ρ are growth rates of the chemicals; and the chemotactic movements are cell density dependent which are indicated by nonlinear functions $\psi(u)$ and $\phi(u)$. Here the equations of v and w are assumed to be in steady state mode due to different reaction time scales. So (1.1) is a coupled system of a quasilinear parabolic equation and two linear elliptic equations.

Throughout the paper, we assume that functions $f(u)$, $\psi(u)$ and $\phi(u)$ satisfy the following hypotheses:

(H_1) The function $f(u) \in C^1([0,\infty))$, and $f(0) \geq 0$;
(H_2) The functions $\psi(u)$, $\phi(u) \in C^2([0,\infty))$, and $0 \leq \psi(u) \leq u^p$ with some $p > 0$ for all $u \geq 0$,

and one of the following hypotheses:

(H_3) The functions $f(u)$ and $\phi(u)$ satisfy $f(u) \leq a - b(1+u)^r$ and $0 \leq \phi(u) \leq (1+u)^q$ with some $a > 0$, $b > 0$, $r \geq 2$ and $q > 0$, for all $u \geq 0$; or
(H_4) The function $f(u)$ satisfies $f(u) \leq a - bu^r$ with some $a > 0$, $b > 0$, $r \geq 2$, for all $u \geq 0$; The function $\phi(u)$ satisfies $\phi(u) = u^q$, $q > 0$, $u \geq u_*$ with $u_* > 1$.

The studies of system (1.1) is motivated by recent extensive investigation of chemotaxis models arisen from biology. Chemotaxis is a chemosensitive movement of biological species which detects and responds to chemical substances in the environment. The first chemotaxis model was proposed by Keller and Segel [18], which describes the aggregation process of the slime mold formation in *Dictyostelium Discoidium*. In the Keller-Segel model, the cell movement is directed towards the increasing chemical signal concentration, which is called the attractive chemotaxis. There have been numerous results on the boundedness and blow-up of the solutions of Keller-Segel type models, and a remarkable characteristics of such models is that solution blow-up may occur in a finite time and whether the blow-up occurs or not is not only dependent on the initial data, but also the spatial dimension n and geometric shape of the spatial region Ω. It was known that when $n = 1$, all the solutions are globally bounded [29], and when $n \geq 2$, solution blow-up may happen [12, 13, 41]. Furthermore, under some additional assumptions, when $n \geq 2$, the global existence and boundedness of solutions was also obtained in [27, 39]. A recent survey of Keller-Segel type chemotaxis models can be found in [4].

There is another type chemotaxis model called repulsive chemotaxis, which indicates that cells move away from the increasing signal concentration and it also produces various interesting biological phenomena (see [10, 30, 43] for instance). There are only a few work concerning the repulsive chemotaxis systems. In [5], the global existence of smooth solutions and convergence to steady states based on a Lyapunov functional approach were obtained with $f(u) = u$ when $n = 2$, and when $n = 3, 4$, the global existence of weak solutions was obtained. In [33, 38], it was shown that under some assumptions, the classical solutions to the repulsion chemotaxis model are uniformly bounded in time and converge to the constant steady state as time goes to infinity.

Many biological processes may involve interactions between cells and a combination of attractive and repulsive signalling chemicals, and the corresponding attraction-repulsion chemotaxis model was proposed in [26, 30] to describes the aggregation process of microglia:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), & x \in \Omega,\ t > 0, \\ \tau_1 \dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \delta u, & x \in \Omega,\ t > 0, \\ \tau_2 \dfrac{\partial w}{\partial t} = \Delta w - \eta w + \rho u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = \dfrac{\partial w(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

For the case without growth term ($f(u) \equiv 0$), it was shown in in [34] that the solutions of (1.2) are globally bounded for the full parabolic case of $\tau_1 = \tau_2 = 1$ and $n = 2$ if the repulsion prevails over the attraction and the initial mass is small. In [15, 24], when the repulsion dominates over the attraction, the global existence of classical solutions of (1.2) when $n = 2$ for any nonnegative initial data was proved, and the global existence of a weak solution when $n = 3$ was also obtained in [15]. On the other hand, when $0 \le f(u) \le a - bu^r$ for all $u \ge 0$ ($a, b > 0$, $r \ge 1$), the global existence and uniform boundedness of the classical solution of (1.2) when $\tau_1 = \tau_2 = 1$ and $n = 1, 2$ were proved in [20, 22].

Since the chemicals diffuse much faster than cells, then the cases that $\tau_1 = \tau_2 = 0$ (parabolic-elliptic-elliptic) or $\tau_1 = 1$ and $\tau_2 = 0$ (parabolic-parabolic-elliptic) of (1.2) have also been considered. For the parabolic-elliptic-elliptic case without growth term, it was shown in [32, 34] that the solutions are globally bounded when $n \ge 1$ and the repulsion dominates over the attraction; while blow-up may occur if the attraction dominates the repulsion when $n = 2$. In [44], the parabolic-elliptic-elliptic case of (1.2) with logistic source was considered, and the global existence of solutions and asymptotic behavior of solutions were obtained under some additional conditions. The parabolic-parabolic-elliptic case was considered recently in [17]: again when the repulsion dominates the attraction, the global existence of uniformly-in-time bounded classical solutions with large initial data was proved, and if the attraction dominates. solution blow-up may occur. Such model without repulsive signalling chemicals were also studied by many people, see [3, 7, 8, 9, 21, 37, 45, 46, 47] for example.

Furthermore there have been many other work on other aspects of the attraction-repulsion chemotaxis model (1.2): traveling wave [31], steady states [23, 25], time-periodic solutions and pattern formation [25], global attractor and convergence to stationary solution [16].

In this paper, we consider the parabolic-elliptic-elliptic attraction-repulsion chemotaxis model with growth source (1.1). Our main global existence results are as follows:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \ge 1$) with smooth boundary $\partial\Omega$. Suppose that the parameters $\chi, \xi, \gamma, \delta, \eta, \rho > 0$, the functions $f(u)$, $\psi(u)$ and $\phi(u)$ satisfy (H_1), (H_2) and (H_3), and the parameters in (H_2) and (H_3) satisfy $a, b, p, q > 0$ and $r \ge 2$; the initial condition $u_0 \in W^{1,\infty}(\Omega)$, and $u_0(x) \ge 0$ for $x \in \Omega$. If one of the following sets of conditions holds:*

(i)

$$p \ge p_0 = \max\left\{1, \frac{2}{n}\right\}, \quad 0 < q \le p \le r - 1, \quad b > b_0, \tag{1.3}$$

where

$$b_0 = 3\chi\delta\frac{pn-2}{pn+2p-2}; \tag{1.4}$$

or

(ii)

$$0 < p < \frac{2}{n}, \quad 0 < q \le p \le r-1, \tag{1.5}$$

then the system (1.1) *possesses a unique global classical solution that is bounded in* $\Omega \times (0,\infty)$.

Under different assumptions on $f(u)$ and $\phi(u)$, we have another different global existence result:

Theorem 1.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *(*$n \ge 1$*) with smooth boundary* $\partial\Omega$. *Suppose that the parameters* $\chi, \xi, \gamma, \delta, \eta, \rho > 0$, *the functions* $f(u)$, $\psi(u)$ *and* $\phi(u)$ *satisfy* (H_1), (H_2) *and* (H_4), *and the parameters in* (H_2) *and* (H_4) *satisfy* $a, b, p, q > 0$ *and* $r \ge 2$; *the initial condition* $u_0 \in W^{1,\infty}(\Omega)$, *and* $u_0(x) \ge 0$ *for* $x \in \Omega$. *If one of the following sets of conditions holds:*

(iii)

$$q \ge q_0 = \max\left\{1, \frac{2}{n}\right\}, \quad 0 < p \le q \le r-1, \quad b > b_1, \tag{1.6}$$

where

$$b_1 = \frac{qn-2}{qn-2+2q}\left(\delta\chi - \xi\rho + \frac{2(q-p)\delta\chi}{qn-2+2p}\right); \tag{1.7}$$

or

(iv)

$$0 < q < \frac{2}{n}, \quad 0 < p \le q \le r-1, \tag{1.8}$$

Then the system (1.1) *possesses a unique global classical solution that is bounded in* $\Omega \times (0,\infty)$.

We can apply Theorems 1.1 and 1.2 to the following model with power chemotactic sensitivity functions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi\nabla\cdot(u^p\nabla v) + \xi\nabla\cdot(u^q\nabla w) + f(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \gamma v + \delta u, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - \eta w + \rho u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = \dfrac{\partial w(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.9}$$

where $f(u) \le a - b(1+u)^r$ with some $a > 0$, $b > 0$ and $r \ge 2$. Apparently Theorem 1.1 covers the case of $p \ge q$, while Theorem 1.2 deals with the case of $p \le q$. Combining Theorems 1.1 and 1.2, we have a quite complete picture for the question of global existence and boundedness of solutions to (1.9) in the following diagram and corollary:

Corollary 1.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *(*$n \ge 1$*) with smooth boundary* $\partial\Omega$. *Suppose that the parameters* $\chi, \xi, \gamma, \delta, \eta, \rho > 0$, $f \in C^1([0,\infty))$, $f(0) \ge 0$, *and* $f(u) \le a - b(1+u)^r$ *with some* $a > 0$, $b > 0$ *and* $r \ge 2$. *Then the system* (1.9) *possesses a unique global classical solution that is bounded in* $\Omega \times (0,\infty)$ *if (see Fig. 1)*

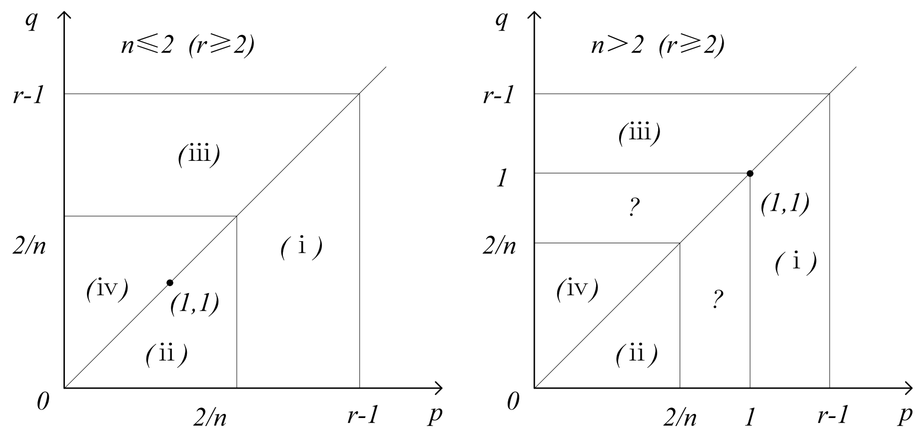

FIGURE 1. Regions in (p, q) plane where the global existence and boundedness of solutions to (1.9) are proved. The regions labelled by (i), (ii), (iii) and (iv) correspond to the ones defined in Theorems 1.1 and 1.2, and for the region labelled with ?, the result is not known. Left: $n \le 2$; Right: $n > 2$.

1. $n = 1, 2$, and $0 \le p, q \le 2/n$, or $2/n \le \max\{p, q\} \le r - 1$ and b large; or
2. $n \ge 3$, and $0 \le p, q \le 2/n$, or $1 \le \max\{p, q\} \le r - 1$ and b large.

Note that our results for system (1.9) are symmetric with respect to the two exponents p and q, and our results in Corollary 1.3 generalize earlier results with linear sensitivity functions in [44, Theorem 1.1] and [22, Theorem 1.1, 1.2] in which the case of $p = q = 1$ is considered. For the case of $n = 1$ or 2, our results cover all small p, q values ($\le r - 1$). On the other hand, for the case of $n \ge 3$, there is still a gap region (see Fig. 1 right panel) for which the global existence and boundedness of solutions to (1.9) is not known.

The organization of the remaining part of the paper is as follows. In Section 2, we recall some preliminaries and also obtain the local existence of the solution. In Section 3, global existence and boundedness of the solution under one sets of assumptions is obtained, while in Section 4, results under another sets of assumptions are obtained. We use $\|\cdot\|_{L^p(\Omega)}$ as the norm of $L^p(\Omega)$, $1 \le p \le \infty$; and $\|\cdot\|_{W^{m,p}(\Omega)}$ as the norm of $W^{m,p}(\Omega)$, $m = 1, 2$, $1 \le p \le \infty$.

2. **Local existence and preliminaries.**

First we state the local-in-time existence result of a classical solution of (1.1), which is similar to the ones in [34, 37, 45, 6, 35, 42].

Lemma 2.1. *Assume that the initial data satisfies $u_0 \ge 0$ and $u_0 \in W^{1,\infty}(\Omega)$, the function $\psi(u)$ and $\phi(u)$ are nonnegative which satisfy (H_2) and (H_3) (or (H_2) and (H_4)), and the function f satisfies (H_1). Then there exists a positive constant T_{max} (the maximal existence time) such that the system (1.1) has a unique non-negative classical solution $(u(x,t), v(x,t), w(x,t))$ which belongs to $C^0(\overline{\Omega}\times(0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max}))$. If $T_{max} < \infty$, then*

$$\lim_{t\to T_{max}} ||(u(\cdot,t)||_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Proof. Let $T \in (0,1)$ be a positive constant which is to be determined below. Define the Banach space

$$X := C^0(\overline{\Omega} \times [0,T]), \tag{2.2}$$

and we consider the closed bounded convex subset of X,

$$S := \{\overline{u} \in X : ||\overline{u}(\cdot,t)||_{L^\infty(\Omega)} \le R \text{ for all } t \in [0,T]\}, \tag{2.3}$$

where $R = ||u_0||_{L^\infty(\Omega)} + 1$. We define a mapping $\Theta : S \to S$ so that $\Theta(\overline{u}) = u$, where $\overline{u} \in X$ and u is the unique solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi\nabla\cdot(\psi(u)\nabla v) + \xi\nabla\cdot(\phi(u)\nabla w) + f(u), & x \in \Omega,\ t \in [0,T], \\ \dfrac{\partial u(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in [0,T], \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

with v being the solution of

$$\begin{cases} 0 = \Delta v - \gamma v + \delta\overline{u}, & x \in \Omega,\ t \in [0,T], \\ \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in [0,T], \end{cases} \tag{2.5}$$

and w being the solution of

$$\begin{cases} 0 = \Delta w - \eta w + \rho\overline{u}, & x \in \Omega,\ t \in [0,T], \\ \dfrac{\partial w(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in [0,T]. \end{cases} \tag{2.6}$$

Then we will use the Schauder fixed point theorem to show that Θ has a fixed point in S for T small enough. From the elliptic regularity theory in [11, Theorem 8.34], there exists a unique solution $v(x,t) \in C^{1+\iota,0}(\overline{\Omega} \times [0,T])$ to (2.5) for each $\iota \in (0,1)$. Similarly, there is a unique solution $w(x,t) \in C^{1+\iota,0}(\overline{\Omega} \times [0,T])$ to (2.6) . Then by elliptic L^p-estimates and the Sobolev embedding theorem, there exist positive constants c_1, c_2 such that

$$||\nabla v||_{L^\infty(\Omega\times(0,T))} \le c_1||\Delta v||_{L^\infty((0,T);W^{2,p}(\Omega))} \le c_2||\overline{u}||_{L^\infty((0,T);L^p(\Omega))} \tag{2.7}$$

where $p > n$. Similarly, we have

$$||\nabla w||_{L^\infty(\Omega\times(0,T))} \le c_3||\Delta w||_{L^\infty((0,T);W^{2,p}(\Omega))} \le c_4||\overline{u}||_{L^\infty((0,T);L^p(\Omega))} \tag{2.8}$$

where $c_3, c_4 > 0$. Thus, according the classical parabolic regularity theory in [19, Theorem V6.1], there exist $\iota \in (0,1)$ and $A > 0$ such that

$$||u||_{C^{\iota,\frac{\iota}{2}}(\overline{\Omega}\times[0,T])} \le A \tag{2.9}$$

for $u \in C^{\iota,\frac{\iota}{2}}(\overline{\Omega} \times [0,T])$, where A depends on $||\nabla v||_{L^\infty((0,T);C^\iota(\overline{\Omega}))}$ and $||\nabla w||_{L^\infty((0,T);C^\iota(\overline{\Omega}))}$. Hence, we have

$$\max_{0\le t\le T} ||u(\cdot,t)||_{L^\infty(\Omega)} \le ||u_0||_{L^\infty(\Omega)} + At^{\frac{\iota}{2}}. \tag{2.10}$$

Now, choosing $T < A^{-\frac{2}{\iota}}$ in (2.10), then we get

$$\max_{0\le t\le T} ||u(\cdot,t)||_{L^\infty(\Omega)} \le ||u_0||_{L^\infty(\Omega)} + 1. \tag{2.11}$$

So Θ maps S into itself for $0 \le t \le T$, which is shown to be a compact mapping from (2.9). Therefore, the Schauder fixed point theorem ensures the existence of a fixed point $u \in S$ of Θ. Using the regularity theory for elliptic equations we have $(v(\cdot,t), w(\cdot,t)) \in (C^{2+\iota}(\overline{\Omega}))^2$. Then we get $(v,w) \in (C^{2+\iota,\frac{\iota}{2}}(\overline{\Omega} \times [\tau,T]))^2$ for all

$\tau \in (0,T)$ by (2.9). The regularity theory for parabolic equations [19, Theorem V6.1] ensures that $u \in C^{2+\iota,1+\frac{\iota}{2}}(\overline{\Omega} \times [\tau,T])$. The solution may be extended to a maximal interval $[0,T_{max})$, and either if $T_{max}=\infty$ or $T_{max}<\infty$, where the latter case entails that (2.1) holds.

Since $f(0)\geq 0$, then the parabolic comparison principle ensures that u is nonnegative. Moreover applying the elliptic comparison principle to the second and third equations in (1.1) implies v and w are nonnegative. □

Next we recall some preliminary estimates which will be used in our proof. First we review some well-known estimates for the diffusion semigroup for the homogeneous Neumann boundary conditions (see [14]).

Lemma 2.2. *Assume that $m\in\{0,1\}$, $i\in[1,\infty]$ and $j\in(1,\infty)$, then there exists some positive constant c_5, for any $u\in D((-\Delta+1)^\theta)$ such that*

$$||u||_{W^{m,i}(\Omega)} \leq c_5||(-\Delta+1)^\theta u||_{L^j(\Omega)}, \tag{2.12}$$

where $\theta\in(0,1)$ satisfies

$$m-\frac{n}{i}<2\theta-\frac{n}{j}.$$

If in addition $j\geq i$, then there exist $c_6>0$ and $\gamma>0$ such that for any $u\in L^i(\Omega)$,

$$||(-\Delta+1)^\theta e^{t(\Delta-1)}u||_{L^j(\Omega)} \leq c_6 t^{-\theta-\frac{n}{2}(\frac{1}{i}-\frac{1}{j})}e^{-\gamma t}||u||_{L^i(\Omega)}, \tag{2.13}$$

where the associated diffusion semigroup $\{e^{t(\Delta-1)}\}_{t\geq 0}$ maps $L^i(\Omega)$ into $D((-\Delta+1)^\theta)$. Moreover, for given $i\in(1,\infty)$ and any $\varepsilon>0$, there exist $c_7>0$ and $\mu>0$ such that

$$||(-\Delta+1)^\theta e^{t\Delta}\nabla\cdot u||_{L^i(\Omega)} \leq c_7 t^{-\theta-\frac{1}{2}-\varepsilon}e^{-\mu t}||u||_{L^i(\Omega)} \tag{2.14}$$

holds for all $\mathbb{R}^n$-valued $u\in L^i(\Omega)$.

The following Gagliardo-Nirenberg inequality also plays a key role in our proof (see [28] for detail).

Lemma 2.3. *Suppose that $u\in W^{k,j}(\Omega)\cap L^i(\Omega)$, h,k are nonnegative integers satifying $h/k\leq\alpha\leq 1$, $1\leq i,j\leq\infty$, and $m>0$. Then there exists a constant $c_8>0$ such that*

$$||D^h u||_{L^m(\Omega)} \leq c_8||D^k u||^{\alpha}_{L^j(\Omega)}||u||^{1-\alpha}_{L^i(\Omega)} + c_8||u||_{L^i(\Omega)}, \tag{2.15}$$

where

$$\frac{1}{m}-\frac{h}{n}=\alpha\left(\frac{1}{j}-\frac{k}{n}\right)+(1-\alpha)\frac{1}{i}.$$

Finally, we give the following two elementary inequalities (see [37, 42] for detail).

Lemma 2.4. *Assume that $y,z\geq 0$ and $b>0$, then we have*

$$(y+z)^b \leq 2^b(y^b+z^b). \tag{2.16}$$

Lemma 2.5. *Let $x(t):[0,\infty)\to\mathbb{R}_+$ be a continuously differentiable function satisfying*

$$\begin{cases} x'(t)+Bx^k(t)\leq C, & t>0,\\ x(0)=x_0, \end{cases} \tag{2.17}$$

with $B>0$, $k>0$, $C\geq 0$ and $x_0\geq 0$. Then we have

$$x(t)\leq \max\left\{x_0, \left(\frac{C}{B}\right)^{\frac{1}{k}}\right\}, \quad \text{for } t\in(0,\infty). \tag{2.18}$$

3. **Global existence and boundedness of solutions under** (H_3)**.** In this section, we assume that $f(u)$, $\psi(u)$ and $\phi(u)$ satisfy (H_1), (H_2) and (H_3), and we study the global existence and boundedness of solutions. Before proving the main result, let us give some *a priori* estimates for $u(x,t)$, which are vital ingredients for our proofs. First we prove the following $L^1(\Omega)$ estimates.

Lemma 3.1. *Assume that* (H_1) *and* (H_3) *are satisfied, then there exists a positive constants* C_0 *such that*

$$\begin{aligned} &\int_\Omega u(x,t)dx \le C_0 \quad \text{for all } \ t\in(0,T_{max}), \\ &\int_\Omega v(x,t)dx \le \frac{\delta}{\gamma}C_0 \quad \text{for all } \ t\in(0,T_{max}), \\ &\int_\Omega w(x,t)dx \le \frac{\rho}{\eta}C_0 \quad \text{for all } \ t\in(0,T_{max}). \end{aligned} \tag{3.1}$$

Proof. Integrating the first equation of (1.1) and using (H_1), (H_3), we have

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega f(u(x,t))dx \le \int_\Omega (a-b(1+u(x,t))^r)\,dx \le \int_\Omega (a-bu^r(x,t))\,dx. \tag{3.2}$$

Then using Hölder inequality, we obtain that

$$\frac{d}{dt}\int_\Omega u(x,t)dx + b|\Omega|^{1-r}\left(\int_\Omega u(x,t)dx\right)^r \le a|\Omega|, \tag{3.3}$$

where $r\ge 2$. Thus from Lemma 2.5, we get

$$\int_\Omega u(x,t)dx \le \max\left\{\int_\Omega u_0(x,t)dx, \left(\frac{a}{b}\right)^{\frac{1}{r}}|\Omega|\right\} := C_0. \tag{3.4}$$

Then integrating the second equation of (1.1), and using (3.4), we obtain that

$$\int_\Omega v(x,t)dx = \frac{\delta}{\gamma}\int_\Omega u(x,t)dx \le \frac{\delta}{\gamma}C_0. \tag{3.5}$$

Similarly, we obtain that

$$\int_\Omega w(x,t)dx = \frac{\rho}{\eta}\int_\Omega u(x,t)dx \le \frac{\rho}{\eta}C_0. \tag{3.6}$$

□

Next we prove that $u(x,t)$ is bounded in $L^k(\Omega)$ for certain positive k. Inspired by the work in [36], we have the following lemma.

Lemma 3.2. *Assume that* $f(u)$, $\psi(u)$, *and* $\phi(u)$ *satisfy* (H_1), (H_2), (H_3) *and condition* (i) *in Theorem 1.1 are satisfied, then for all* $k\in(k_0,k_1)$, *where*

$$k_0 = \max\left\{1, \frac{n}{2}+1-p\right\}, \tag{3.7}$$

and

$$k_1 = \begin{cases} \dfrac{3\chi\delta-b+bp}{3\chi\delta-b} & \text{if } \ \chi>\dfrac{b}{3\delta}, \\ \infty & \text{if } \ 0<\chi\le\dfrac{b}{3\delta}. \end{cases} \tag{3.8}$$

there exists a positive constant $C_1>0$ *such that*

$$||u(\cdot,t)||_{L^k(\Omega)} \le C_1 \quad \text{for all } \ t\in(0,T_{max}). \tag{3.9}$$

Proof. If $\chi > b/(3\delta)$, from the assumptions on p and b in (i), we have

$$k_0 \le \frac{pn}{2} < \frac{3\chi\delta - b + bp}{3\chi\delta - b} := k_1, \tag{3.10}$$

where k_0 and k_1 are defined as (3.7) and (3.8). So the interval (k_0, k_1) is nonempty. In the following we assume that $k \in (k_0, k_1)$. Multiplying the first equation of (1.1) by $(1+u)^{k-1}$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega (1+u)^k \\
=&\int_\Omega (1+u)^{k-1}\Delta u - \int_\Omega (1+u)^{k-1}\chi\nabla\cdot(\psi(u)\nabla v) \\
&+ \int_\Omega (1+u)^{k-1}\xi\nabla\cdot(\phi(u)\nabla w) + \int_\Omega (1+u)^{k-1} f(u) \\
=&-(k-1)\int_\Omega (1+u)^{k-2}|\nabla u|^2 + \chi(k-1)\int_\Omega (1+u)^{k-2}\psi(u)\nabla u\cdot\nabla v \\
&+ \int_\Omega (1+u)^{k-1} f(u) - \xi(k-1)\int_\Omega (1+u)^{k-2}\phi(u)\nabla u\cdot\nabla w.
\end{aligned} \tag{3.11}$$

Since $k > 1$, then from (3.11), we obtain that

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega (1+u)^k \\
\le&\chi(k-1)\int_\Omega \nabla\Psi_1(u)\cdot\nabla v - \xi(k-1)\int_\Omega \nabla\Phi_1(u)\cdot\nabla w \\
&+ a\int_\Omega (1+u)^{k-1} - b\int_\Omega (1+u)^{k+r-1} \\
=&-\chi(k-1)\int_\Omega \Psi_1(u)\Delta v + \xi(k-1)\int_\Omega \Phi_1(u)\Delta w \\
&+ a\int_\Omega (1+u)^{k-1} - b\int_\Omega (1+u)^{k+r-1} \\
=&-\chi(k-1)\int_\Omega \Psi_1(u)(\gamma v - \delta u) + \xi(k-1)\int_\Omega \Phi_1(u)(\eta w - \rho u) \\
&+ a\int_\Omega (1+u)^{k-1} - b\int_\Omega (1+u)^{k+r-1},
\end{aligned} \tag{3.12}$$

where from (H_2) and (H_3),

$$\Psi_1(u) := \int_0^u (1+z)^{k-2}\psi(z)dz \le \int_0^u (1+z)^{k+p-2}dz = \frac{1}{k+p-1}(1+u)^{k+p-1}, \tag{3.13}$$

$$\begin{aligned}
\Phi_1(u) :=& \int_0^u (1+z)^{k-2}\phi(z)dz \le \int_0^u (1+z)^{k-2}(1+z)^q dz \\
\le& \int_0^u (1+z)^{k+q-2}dz = \frac{1}{k+q-1}(1+u)^{k+q-1}.
\end{aligned} \tag{3.14}$$

Then we have,

$$-\chi(k-1)\int_\Omega \Psi_1(u)(\gamma v - \delta u) \le \chi\delta(k-1)\int_\Omega \Psi_1(u)u \le \frac{\chi\delta(k-1)}{k+p-1}\int_\Omega (1+u)^{k+p}, \tag{3.15}$$

and

$$\xi(k-1)\int_\Omega \Phi_1(u)(\eta w-\rho u)\le \xi\eta(k-1)\int_\Omega \Phi_1(u)w\le \frac{\xi\eta(k-1)}{k+q-1}\int_\Omega (1+u)^{k+q-1}w. \tag{3.16}$$

Combining (3.11), (3.15) and (3.16), we obtain that

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\int_\Omega (1+u)^k \le \frac{\chi\delta(k-1)}{k+p-1}\int_\Omega (1+u)^{k+p}+\frac{\xi\eta(k-1)}{k+q-1}\int_\Omega (1+u)^{k+q-1}w\\ +a\int_\Omega (1+u)^{k-1}-b\int_\Omega (1+u)^{k+r-1}.\end{aligned} \tag{3.17}$$

By using the assumption $p\ge q$, we estimate the second term on the right hand side of (3.17) by Young's inequality,

$$\frac{\xi\eta(k-1)}{k+q-1}\int_\Omega (1+u)^{k+q-1}w\le \frac{\chi\delta(k-1)}{k+p-1}\int_\Omega (1+u)^{k+p}+d_1\int_\Omega w^{k+p}, \tag{3.18}$$

where

$$d_1=\frac{\xi\eta(k-1)}{k+q-1}\left(\frac{\xi\eta(k+p-1)^2}{\chi\delta(k+p)(k+q-1)}\right)^{k+p-1}\frac{1}{k+p}. \tag{3.19}$$

And since $r-1\ge p$, then we have

$$\int_\Omega (1+u)^{k+p}\le \frac{k+p}{k+r-1}\int_\Omega (1+u)^{k+r-1}+d_2\le \int_\Omega (1+u)^{k+r-1}+d_2, \tag{3.20}$$

where

$$d_2=\left(\frac{k+r-2}{k+p}\right)^{k+r-2}\frac{|\Omega|}{k+r-1}. \tag{3.21}$$

Applying the Agmon-Douglis-Nirenberg L^p estimates (see [1, 2] for detail) on linear elliptic equations with Neumann boundary condition, for any $l>1$ we have that there exists a positive constant d_3 such that

$$||w(\cdot,t)||_{W^{2,l}(\Omega)}\le d_3||u(\cdot,t)||_{L^l(\Omega)} \quad \text{for all} \quad t\in(0,T_{max}). \tag{3.22}$$

Then by using Lemma 2.3 and Lemma 3.1, the last term in (3.18) can be estimated as

$$\begin{aligned}&\int_\Omega w^{k+p}=||w||^{k+p}_{L^{k+p}(\Omega)}\\ &\le d_4||D^2w||^{(k+p)\lambda}_{L^{k+p-1}(\Omega)}||w||^{(k+p)(1-\lambda)}_{L^1(\Omega)}+d_4||w||^{k+p}_{L^1(\Omega)}\le d_5||u||^{(k+p)\lambda}_{L^{k+p-1}(\Omega)}+d_5,\end{aligned} \tag{3.23}$$

where

$$d_4>0,\ d_5>0,\ k+p>\frac{n}{2}+1 \text{ and } \lambda:=\frac{1-\frac{1}{k+p}}{1+\frac{2}{n}-\frac{1}{k+p-1}}\in(0,1). \tag{3.24}$$

Because of (3.24), we have

$$(k+p)\lambda<k+p-1. \tag{3.25}$$

So we obtain that

$$\int_\Omega w^{k+p}\le d_5||u||^{(k+p)\lambda}_{L^{k+p-1}(\Omega)}+d_5\le d_6(||u||^{k+p-1}_{L^{k+p-1}(\Omega)}+1)+d_6=d_6\int_\Omega u^{k+p-1}+2d_6. \tag{3.26}$$

By using Young's inequality, we have

$$\int_\Omega u^{k+p-1}\le \varepsilon_0\int_\Omega u^{k+p}+d_7, \tag{3.27}$$

where

$$d_7 = \left(\frac{k+p-1}{\varepsilon_0(k+p)}\right)^{k+p-1} \frac{|\Omega|}{k+p}, \quad \varepsilon_0 = \frac{\chi\delta(k-1)}{d_1 d_6(k+p-1)}. \tag{3.28}$$

Combining (3.26) and (3.27), we arrive at

$$\int_\Omega w^{k+p} \le \varepsilon_0 d_6 \int_\Omega u^{k+p} + d_8 \le \varepsilon_0 d_6 \int_\Omega (1+u)^{k+p} + d_8, \tag{3.29}$$

where $d_8 = d_6 d_7 + 2d_6$.

Substituting (3.18) , (3.20) and (3.29) into (3.17), we find that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega (1+u)^k \le -\left(b - \frac{3\chi\delta(k-1)}{k+p-1}\right)\int_\Omega (1+u)^{k+p} + a\int_\Omega (1+u)^{k-1} + bd_2 + d_1 d_8, \tag{3.30}$$

where from (3.8), we have

$$b - \frac{3\chi\delta(k-1)}{k+p-1} > 0, \tag{3.31}$$

Since $p \ge 0$, then from Young's inequality, for some $d_9 > 0$ we also have

$$a\int_\Omega (1+u)^{k-1} \le \frac{1}{2}\left(b - \frac{3\chi\delta(k-1)}{k+p-1}\right)\int_\Omega (1+u)^{k+p} + d_9. \tag{3.32}$$

Hence, we obtain that

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\frac{k}{2}\left(b - \frac{3\chi\delta(k-1)}{k+p-1}\right)\int_\Omega (1+u)^{k+p} + d_{10}, \tag{3.33}$$

where $d_{10} = k(bd_2 + d_1 d_8 + d_9)$.

By the Hölder inequality, we have

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\frac{k}{2}\left(b - \frac{3\chi\delta(k-1)}{k+p-1}\right)|\Omega|^{-\frac{p}{k}}\left(\int_\Omega (1+u)^k\right)^{\frac{k+p}{k}} + d_{10}. \tag{3.34}$$

Then according to Lemma 2.5, the boundedness of $||u(\cdot,t)||_{L^k(\Omega)}$ is obtained. □

From the proof of Lemma 3.2, we find that if $b > 3\chi\delta$, then for any $k > k_0$ the boundedness of $||u(\cdot,t)||_{L^k(\Omega)}$ is obtained. In the following we consider the boundedness of $||u(\cdot,t)||_{L^k(\Omega)}$ for large k in the case of $b \le 3\chi\delta$.

Lemma 3.3. *Assume that* (H_1), (H_2), (H_3) *and condition* (*i*) *in Theorem 1.1 are satisfied, then for all* $k > k_0$ *which is defined in* (3.7), *there exists a positive constant* $C_2 > 0$ *such that*

$$||u(\cdot,t)||_{L^k(\Omega)} \le C_2 \quad \text{for all} \ \ t \in (0, T_{max}). \tag{3.35}$$

Proof. If $b > 3\chi\delta$, then this is proved in Lemma 3.2. In the following, we assume that $0 < b \le 3\chi\delta$. For any $k > 0$, from (3.11) and (3.12), we have

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\int_\Omega (1+u)^k = &-\frac{4(k-1)}{k^2}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 - \chi(k-1)\int_\Omega \Psi_1(u)(\gamma v - \delta u)\\ &+ \xi(k-1)\int_\Omega \Phi_1(u)(\eta w - \rho u) + \int_\Omega (1+u)^{k-1} f(u),\end{aligned} \tag{3.36}$$

where $\Psi_1(u)$ and $\Phi_1(u)$ are defined as in (3.13) and (3.14).

The last term in (3.36) can be estimated as

$$\begin{aligned}&\int_\Omega (1+u)^{k-1} f(u) \le \int_\Omega (1+u)^{k-1}(a-b(1+u)^r) \le \int_\Omega (1+u)^{k-1}(a-bu^r)\\ \le &\int_\Omega (1+u)^{k-1}(a-b+rb-rbu) = \int_\Omega (1+u)^{k-1}(a-b+2rb) - \int_\Omega rb(1+u)^k.\end{aligned} \tag{3.37}$$

Then by Young's inequality, we have

$$\int_\Omega (1+u)^{k-1}(a-b+2rb) \le \int_\Omega rb(1+u)^k + \frac{e_1}{k}, \tag{3.38}$$

where e_1 is a positive constant.

Hence, from (3.15), (3.16), (3.18) , (3.29), (3.37) and (3.38), we obtain that for any $k>1$,

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\frac{4(k-1)}{k}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 + \frac{3\chi\delta k(k-1)}{k+p-1}\int_\Omega (1+u)^{k+p} + e_2. \tag{3.39}$$

where $e_2 = e_1 + kd_1d_8$.

Next, we consider the first two terms on the right side of (3.39). From the assumption on b in (i), we have

$$1 \le \frac{pn}{2} < \frac{3\chi\delta - b + bp}{3\chi\delta - b} := k_1. \tag{3.40}$$

We fix some $k^* \in (pn/2, k_1)$. Then from Lemma 3.2, there exists $C_1>0$ such that

$$||u(\cdot,t)||_{L^{k^*}(\Omega)} \le C_1, \quad \text{for all } t\in(0,T_{max}). \tag{3.41}$$

Now we choose $k>k^*$ and using Lemma 3.2 and Lemma 2.3, there exist $e_2>0$ and $e_3>0$ such that

$$\begin{aligned}&\int_\Omega (1+u)^{k+p} = ||(1+u)^{\frac{k}{2}}||_{L^{\frac{2(k+p)}{k}}(\Omega)}^{\frac{2(k+p)}{k}}\\ \le& e_2\left(||\nabla(1+u)^{\frac{k}{2}}||_{L^2(\Omega)}^{\alpha_1}||(1+u)^{\frac{k}{2}}||_{L^{\frac{2k^*}{k}}(\Omega)}^{1-\alpha_1} + ||(1+u)^{\frac{k}{2}}||_{L^{\frac{2k^*}{k}}(\Omega)}\right)^{\frac{2(k+p)}{k}}\\ \le& e_2\left(||\nabla(1+u)^{\frac{k}{2}}||_{L^2(\Omega)}^{\alpha_1}||1+u||_{L^{k^*}(\Omega)}^{\frac{k}{2}(1-\alpha_1)} + ||1+u||_{L^{k^*}(\Omega)}^{\frac{k}{2}}\right)^{\frac{2(k+p)}{k}}\\ \le& e_3\left(||\nabla(1+u)^{\frac{k}{2}}||_{L^2(\Omega)}^{\frac{2\alpha_1(k+p)}{k}} + 1\right),\end{aligned} \tag{3.42}$$

where

$$\begin{aligned}\alpha_1 =& \frac{\frac{kn}{2k^*} - \frac{kn}{2(k+p)}}{1-\frac{n}{2}+\frac{kn}{2k^*}} = \frac{(k+p-k^*)kn}{k^*(k+p)(2-n)+(k+p)kn}\\ =& \frac{(k+p-k^*)kn}{(k+p-k^*)kn + k^*(2k-pn+2p)}.\end{aligned} \tag{3.43}$$

Since $k>k^*>\frac{pn}{2}\ge k_0$, then $\alpha_1\in(0,1)$ and

$$\frac{2\alpha_1(k+p)}{k} = 2\frac{kn-k^*n+pn}{kn-k^*n+2k^*} < 2. \tag{3.44}$$

Hence, combining (3.42) with (3.43) and using Young's inequality, we have

$$\left(\frac{3\chi\delta k(k-1)}{k+p-1}+1\right)\int_\Omega (1+u)^{k+p} \le \frac{4(k-1)}{k}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 + e_4, \tag{3.45}$$

where e_4 is a positive constant.

Inserting (3.45) into (3.39), we find that there exists a constant $e_5 > 0$ such that

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\int_\Omega (1+u)^{k+p} + e_5. \tag{3.46}$$

Then from Hölder inequality, we derive that

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -|\Omega|^{-\frac{p}{k}}\left(\int_\Omega (1+u)^k\right)^{\frac{k+p}{k}} + e_5. \tag{3.47}$$

Therefore, the desired result is obtained from Lemma 2.5, that is,

$$\int_\Omega (1+u)^k \le \max\left\{\int_\Omega (1+u_0)^k, (e_5|\Omega|^{\frac{p}{k}})^{\frac{k}{k+p}}\right\}. \tag{3.48}$$

□

Finally we prove the $L^k(\Omega)$ boundedness of $u(x,t)$ under the condition (ii).

Lemma 3.4. *Assume that* (H_1), (H_2), (H_3) *and condition* (ii) *in Theorem 1.1 hold, then for all* $k > k_0$ *which is defined in* (3.7), *there exists a positive constant* $C_3 > 0$ *such that*

$$||u(\cdot,t)||_{L^k(\Omega)} \le C_3 \quad \textit{for all} \ \ t \in (0, T_{max}). \tag{3.49}$$

Proof. In this part, we consider the case of $0 < p < 2/n$. In this case, (3.39) still holds as $k > k_0 \ge 1$. By using Lemmas 2.3 and 3.1, we obtain that

$$\begin{aligned}
&\int_\Omega (1+u)^{k+p} = ||(1+u)^{\frac{k}{2}}||^{\frac{2(k+p)}{k}}_{L^{\frac{2(k+p)}{k}}(\Omega)} \\
\le& e_6\left(||\nabla(1+u)^{\frac{k}{2}}||^{\beta_1}_{L^2(\Omega)}||(1+u)^{\frac{k}{2}}||^{1-\beta_1}_{L^{\frac{2}{k}}(\Omega)} + ||(1+u)^{\frac{k}{2}}||_{L^{\frac{2}{k}}(\Omega)}\right)^{\frac{2(k+p)}{k}} \\
\le& e_6\left(||\nabla(1+u)^{\frac{k}{2}}||^{\beta_1}_{L^2(\Omega)}||1+u||^{\frac{k}{2}(1-\beta_1)}_{L^1(\Omega)} + ||1+u||^{\frac{k}{2}}_{L^1(\Omega)}\right)^{\frac{2(k+p)}{k}} \\
\le& e_6\left(||\nabla(1+u)^{\frac{k}{2}}||^{\beta_1}_{L^2(\Omega)}(|\Omega|+C_0)^{\frac{k}{2}(1-\beta_1)} + (|\Omega|+C_0)^{\frac{k}{2}}\right)^{\frac{2(k+p)}{k}} \\
\le& e_7\left(||\nabla(1+u)^{\frac{k}{2}}||^{\frac{2\beta_1(k+p)}{k}}_{L^2(\Omega)} + 1\right),
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned}
\beta_1 &= \frac{\frac{kn}{2} - \frac{kn}{2(k+p)}}{1-\frac{n}{2}+\frac{kn}{2}} = \frac{(k+p-1)kn}{(k+p)(2-n+kn)} = \frac{(k+p-1)kn}{(k+p-1)kn + (2k-pn+2p)} \\
&\le \frac{(k+p-1)kn}{(k+p-1)kn+(2-pn+2p)} < 1,
\end{aligned} \tag{3.51}$$

since $k > k_0$ and $2 - pn > 0$, then $\beta_1 \in (0,1)$. And we have

$$\frac{2\beta_1(k+p)}{k} = 2n\frac{k+p-1}{2+nk-n} < 2. \tag{3.52}$$

Thus, combining (3.50) with (3.52) and using Young's inequality, we have

$$\left(\frac{3\chi\delta k(k-1)}{k+p-1}+1\right)\int_\Omega (1+u)^{k+p} \le \frac{4(k-1)}{k}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 + e_8. \tag{3.53}$$

Then substituting (3.53) into (3.39), we arrive at

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\int_\Omega (1+u)^{k+p} + e_9. \tag{3.54}$$

Thus the L^k-bound of $u(x,t)$ is obtained by using Lemma 2.5 and we complete the proof. □

By using Lemmas 3.1-3.4, we now can obtain the L^∞-bound of $u(x,t)$.

Lemma 3.5. *Assume that all assumptions in Theorem 1.1 hold, then there exists a positive constant $C_4 > 0$ such that*

$$||u(\cdot,t)||_{L^\infty(\Omega)} \le C_4 \quad \textit{for all} \;\; t\in(0,T_{max}). \tag{3.55}$$

Proof. We use semigroup arguments similar to the ones in [39, 40, 14] to get the L^∞-bound of u. First, $u(x,t)$ can be expressed by

$$\begin{aligned} u(\cdot,t) = {}& e^{t(\Delta-1)}u_0 - \chi\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot(\psi(u(\cdot,t))\nabla v)ds \\ & + \xi\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot(\phi(u(\cdot,t))\nabla w)ds + \int_0^t e^{(t-s)(\Delta-1)}\varphi(u(\cdot,t))ds \\ = {}& U_1+U_2+U_3+U_4, \end{aligned} \tag{3.56}$$

where $\varphi(u) = f(u) + u$. Then we study the L^∞-bound of U_1, U_2, U_3 and U_4 respectively.

For U_1, we find that

$$U_1(\cdot,t) \le ||u_0||_{L^\infty(\Omega)} \quad \text{for all} \;\; t\in(0,T_{max}). \tag{3.57}$$

For U_2, let $j > n$, $i=\infty$ in Lemma 2.2, so there exists $\theta\in(\frac{n}{2j},\frac{1}{2})$; and in this case $\varepsilon\in(0,\frac{1}{2}-\theta)$ in Lemma 2.2. Then there exist positive constants κ_1,κ_2 and μ such that

$$\begin{aligned} ||U_2(\cdot,t)||_{L^\infty(\Omega)} &\le \kappa_1||(-\Delta+1)^\theta U_2(\cdot,t)||_{L^j(\Omega)} \\ &\le \chi\kappa_1\int_0^t ||e^{(t-s)(\Delta-1)}\nabla\cdot(\psi(u(\cdot,t))\nabla v(\cdot,t))||_{L^j(\Omega)}ds \\ &\le \chi\kappa_1\int_0^t e^{-(t-s)}||e^{(t-s)\Delta}\nabla\cdot(\psi(u(\cdot,t))\nabla v(\cdot,t))||_{L^j(\Omega)}ds \\ &\le \kappa_2\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}e^{-(\mu+1)(t-s)}||\psi(u(\cdot,t))\nabla v(\cdot,t)||_{L^j(\Omega)}ds \end{aligned} \tag{3.58}$$

for all $t\in(0,T_{max})$.

Then from Lemma 3.3 or Lemma 3.4, we have $||u(\cdot,t)||_{L^i(\Omega)} \le C_5$ for $i > k_0$ and using the elliptic regularity theory to the second equation in (1.1), we have

$$\sup_{0<t<T_{max}} ||v(\cdot,t)||_{W^{2,i}(\Omega)} \le \kappa_3 \;\; \text{for all } i > k_0. \tag{3.59}$$

Choosing $i > n$, from the Sobolev embedding theorem, we obtain that

$$\sup_{0<t<T_{max}} ||\nabla v(\cdot,t)||_{L^\infty(\Omega)} \le \kappa_4 \text{ for all } t \in (0, T_{max}). \tag{3.60}$$

Hence, there exists $\kappa_5 > 0$ such that

$$||\psi(u(\cdot,t))\nabla v(\cdot,t)||_{L^i(\Omega)} \le \kappa_5 \text{ for all } t \in (\tau, T_{max}). \tag{3.61}$$

Therefore, from (3.57) and (3.61), we obtain that, for all $t \in (\tau, T_{max})$

$$\begin{aligned} ||U_2(\cdot,t)||_{L^\infty(\Omega)} &\le \kappa_2\kappa_5 \int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon} e^{-(\mu+1)(t-s)} ds \\ \le &\kappa_2\kappa_5 \int_0^\infty \sigma^{-\theta-\frac{1}{2}-\varepsilon} e^{-(\mu+1)\sigma} d\sigma \le \kappa_6 \Gamma(\frac{1}{2}-\theta-\varepsilon), \end{aligned} \tag{3.62}$$

where $\Gamma(\frac{1}{2}-\theta-\varepsilon) > 0$ for $\frac{1}{2}-\theta-\varepsilon > 0$. Similarly, the L^∞-bound of U_3 can be obtained.

Finally, for U_4, by using (2.12), (2.14) and $\varphi(u) \le \hat{a} - \hat{b}(1+u)^r$ for all $u > 0$ with any $\hat{b} < b$ and some $\hat{a} > a$, we have

$$U_4(\cdot,t) \le \int_0^t e^{(t-s)(\Delta-1)} \hat{a} ds \le \kappa_6 \int_0^t e^{-\gamma(t-s)} ds \le \frac{\kappa_7}{\gamma} \text{ for all } t \in (0, T_{max}). \tag{3.63}$$

Therefore, by (3.57), (3.58) and (3.62), we obtain that u is bounded in $\Omega \times (0, T_{max})$. Along with (2.1), this proves that $T_{max} = \infty$ and hence, (u, v, w) is bounded in $\Omega \times (0, \infty)$. □

Next we prove Theorem 1.1.

Proof of Theorem 1.1. From Lemma 3.5 and the blowup criterion (2.1), we obtain that there exists a constant $C_4 > 0$ such that

$$||u(\cdot,t)||_{L^\infty(\Omega)} \le C_4 \text{ for all } t \in (0, \infty). \tag{3.64}$$

This completes the proof of Theorem 1.1. □

4. **Global existence and boundedness of solutions under (H_4).** In this section, we assume that $f(u)$, $\psi(u)$ and $\phi(u)$ satisfy (H_1), (H_2) and (H_4), and we consider the global existence and boundedness of solutions of (1.1). Similar to the proof in Section 3, we also need the $L^1(\Omega)$ bounds of u, v and w. The following lemma can be proved in the same way as that of Lemma 3.1, and we omit the proof.

Lemma 4.1. *Assume that (H_1) and (H_4) hold, then there exists a positive constant C_5 such that*

$$\begin{aligned} &\int_\Omega u(x,t)dx \le C_5 \quad \text{for all } t \in (0, T_{max}), \\ &\int_\Omega v(x,t)dx \le \frac{\delta}{\gamma} C_5 \quad \text{for all } t \in (0, T_{max}), \\ &\int_\Omega w(x,t)dx \le \frac{\rho}{\eta} C_5 \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{4.1}$$

Next we prove that $u(x,t)$ is $L^k(\Omega)$ bounded for certain positive k.

Lemma 4.2. *Assume that* (H_1), (H_2), (H_4) *and condition* (iii) *in Theorem 1.2 are satisfied. Define*

$$k_2 = \max\left\{1, \frac{n}{2}+1-q\right\}, \qquad k_3 = \frac{-F+\sqrt{F^2-4EG}}{2E}, \tag{4.2}$$

where

$$\begin{aligned} &E = b-\delta\chi+\xi\rho, \\ &F = b(p-1)+\xi\rho(p-2)+b(q-1)-\delta\chi(q-2), \\ &G = b(p-1)(q-1)+\delta\chi(q-1)-\xi\rho(p-1). \end{aligned} \tag{4.3}$$

Then for $k\in(\max\{k_2,k_3\},\infty)$ *if* $b-\delta\chi+\xi\rho\geq 0$($b-\delta\chi+\xi\rho=0$, *then* $k_3=-\frac{G}{F}$), *or for* $k\in(k_2,k_3)$ *if* $b-\delta\chi+\xi\rho<0$, *there exists a positive constant* $C_6>0$ *such that*

$$||u(\cdot,t)||_{L^k(\Omega)} \leq C_6 \quad \textit{for all} \quad t\in(0,T_{max}). \tag{4.4}$$

Proof. Multiplying the first equation of (1.1) by u^{k-1} and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{k}\frac{d}{dt}\int_\Omega u^k \\ =&\int_\Omega u^{k-1}\Delta u-\int_\Omega u^{k-1}\chi\nabla\cdot(\psi(u)\nabla v)+\int_\Omega u^{k-1}\xi\nabla\cdot(\phi(u)\nabla w)+\int_\Omega u^{k-1}f(u) \\ =&-(k-1)\int_\Omega u^{k-2}|\nabla u|^2+\chi(k-1)\int_\Omega u^{k-2}\psi(u)\nabla u\cdot\nabla v \\ &-\xi(k-1)\int_\Omega u^{k-2}\phi(u)\nabla u\cdot\nabla w+\int_\Omega u^{k-1}f(u) \\ \leq&\chi(k-1)\int_\Omega \nabla\Psi_2(u)\cdot\nabla v-\xi(k-1)\int_\Omega\nabla\Phi_2(u)\cdot\nabla w+a\int_\Omega u^{k-1}-b\int_\Omega u^{k+r-1} \\ =&-\chi(k-1)\int_\Omega\Psi_2(u)\Delta v+\xi(k-1)\int_\Omega\Phi_2(u)\Delta w+a\int_\Omega u^{k-1}-b\int_\Omega u^{k+r-1} \\ =&-\chi(k-1)\int_\Omega\Psi_2(u)(\gamma v-\delta u)+\xi(k-1)\int_\Omega\Phi_2(u)(\eta w-\rho u) \\ &+a\int_\Omega u^{k-1}-b\int_\Omega u^{k+r-1}, \end{aligned} \tag{4.5}$$

where

$$\Psi_2(u):=\int_0^u y^{k-2}\psi(y)dy\leq\int_0^u y^{k+p-2}dy=\frac{1}{k+p-1}u^{k+p-1}, \tag{4.6}$$

$$\Phi_2(u):=\int_0^u y^{k-2}\phi(y)dy=\int_0^u y^{k-2}y^q dy=\int_0^u y^{k+q-2}dy=\frac{1}{k+q-1}u^{k+q-1}. \tag{4.7}$$

Then we have

$$-\chi(k-1)\int_\Omega\Psi_2(u)(\gamma v-\delta u)\leq\chi\delta(k-1)\int_\Omega\Psi_2(u)u\leq\frac{\chi\delta(k-1)}{k+p-1}\int_\Omega u^{k+p}, \tag{4.8}$$

and

$$\xi(k-1)\int_\Omega\Phi_2(u)(\eta w-\rho u)=\frac{\xi\eta(k-1)}{k+q-1}\int_\Omega u^{k+q-1}w-\frac{\xi\rho(k-1)}{k+q-1}\int_\Omega u^{k+q}. \tag{4.9}$$

Combining (4.5), (4.6), (4.7), (4.8) and (4.9), we obtain that

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\int_\Omega u^k \leq \frac{\chi\delta(k-1)}{k+p-1}\int_\Omega u^{k+p} + \frac{\xi\eta(k-1)}{k+q-1}\int_\Omega u^{k+q-1}w \\ - \frac{\xi\rho(k-1)}{k+q-1}\int_\Omega u^{k+q} + a\int_\Omega u^{k-1} - b\int_\Omega u^{k+r-1}.\end{aligned} \tag{4.10}$$

From $q \geq p$ in (iii) and Young's inequality, there exists a constant $h_1 > 0$ such that

$$\int_\Omega u^{k+p} \leq \frac{k+p}{k+q}\int_\Omega u^{k+q} + h_1 \leq \int_\Omega u^{k+q} + h_1, \tag{4.11}$$

and also from $q \leq r-1$ in (iii) and Young's inequality, we have

$$\int_\Omega u^{k+q} \leq \frac{k+q}{k+r-1}\int_\Omega u^{k+r-1} + h_2 \leq \int_\Omega u^{k+r-1} + h_2, \tag{4.12}$$

where h_2 is a positive constant. Combining (4.10), (4.11) and (4.12), we obtain that

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\int_\Omega u^k &\leq \frac{\chi\delta(k-1)}{k+p-1}\int_\Omega u^{k+q} + \frac{\xi\eta(k-1)}{k+q-1}\int_\Omega u^{k+q-1}w \\ &\quad - \frac{\xi\rho(k-1)}{k+q-1}\int_\Omega u^{k+q} + a\int_\Omega u^{k-1} - b\int_\Omega u^{k+q} + \frac{\chi\delta(k-1)}{k+p-1}h_1 + bh_2 \\ &= \left(\frac{\chi\delta(k-1)}{k+p-1} - \frac{\xi\rho(k-1)}{k+q-1} - b\right)\int_\Omega u^{k+q} + \frac{\xi\eta(k-1)}{k+q-1}\int_\Omega u^{k+q-1}w \\ &\quad + a\int_\Omega u^{k-1} + \frac{\chi\delta(k-1)}{k+p-1}h_1 + bh_2.\end{aligned} \tag{4.13}$$

From our assumption for k, we have

$$\frac{\chi\delta(k-1)}{k+p-1} - \frac{\xi\rho(k-1)}{k+q-1} - b < 0. \tag{4.14}$$

Indeed for the case of $b-\delta\chi+\xi\rho \geq 0$, (4.14) holds when $k \in (\max\{k_2, k_3\}, \infty)$, and for the case of $b-\delta\chi+\xi\rho < 0$, (4.14) holds $k \in (k_2, k_3)$ as from the assumptions on q and b in (iii), we have $k_2 \leq \frac{qn}{2} < k_3$, hence the interval (k_2, k_3) is not empty.

We estimate the second term in the right side of (4.13) by Young's inequality,

$$\frac{\xi\eta(k-1)}{k+q-1}\int_\Omega u^{k+q-1}w \leq \frac{1}{4}\left(b - \frac{\chi\delta(k-1)}{k+p-1} + \frac{\xi\rho(k-1)}{k+q-1}\right)\int_\Omega u^{k+q} + h_3\int_\Omega w^{k+q}, \tag{4.15}$$

where h_3 is a positive constant. Similar to the proof of (3.26), there exists a constant $h_4 > 0$ such that

$$\int_\Omega w^{k+q} \leq h_4\int_\Omega u^{k+q-1} + 2h_4, \tag{4.16}$$

Then by Young's inequality, we arrive at

$$\int_\Omega w^{k+q} \leq \varepsilon_1 h_4\int_\Omega u^{k+q} + h_5. \tag{4.17}$$

where

$$\varepsilon_1 = \frac{b - \dfrac{\chi\delta(k-1)}{k+p-1} + \dfrac{\xi\rho(k-1)}{k+q-1}}{4h_3h_4} \tag{4.18}$$

and h_5 is a positive constant. Substituting (4.15) and (4.17) into (4.13), we find that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k \le \frac{1}{2}\left(\frac{\chi\delta(k-1)}{k+p-1}-\frac{\xi\rho(k-1)}{k+q-1}-b\right)\int_\Omega u^{k+q}+a\int_\Omega u^{k-1}+h_6. \quad (4.19)$$

where $h_6=h_1+bh_2+h_3h_5$. Since $q\ge 0$, then there exists a constant $h_7>0$ such that

$$a\int_\Omega u^{k-1}\le \frac{1}{4}\left(b-\frac{\chi\delta(k-1)}{k+p-1}+\frac{\xi\rho(k-1)}{k+q-1}\right)\int_\Omega u^{k+q}+h_7. \quad (4.20)$$

Hence from (4.19) and (4.20), we obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^k &\le -\frac{k}{4}\left(b-\frac{\chi\delta(k-1)}{k+p-1}+\frac{\xi\rho(k-1)}{k+q-1}\right)\int_\Omega u^{k+q}+h_8,\\ &\le -\frac{k}{4}\left(b-\frac{\chi\delta(k-1)}{k+p-1}+\frac{\xi\rho(k-1)}{k+q-1}\right)|\Omega|^{-\frac{q}{k}}\left(\int_\Omega u^k\right)^{\frac{k+q}{k}}+h_8.\end{aligned} \quad (4.21)$$

where $h_8=k(h_1+bh_2+h_3h_5+h_7)$. Then according to Lemma 2.5, the boundedness of $||u(\cdot,t)||_{L^k(\Omega)}$ is obtained. $\square$

In Lemma 4.2, if $F^2-4EG<0$, then (4.14) hold for any k, so we only consider the case that k_3 is a real number. And in this Lemma, we have shown that if $\xi\rho+b\ge\chi\delta$, then for any $k>k_2$, $||u(\cdot,t)||_{L^k(\Omega)}$ is bounded. In the following we prove the boundedness of $L^k(\Omega)$ in the case of $\xi\rho+b<\chi\delta$ under condition (iii).

Lemma 4.3. *Assume that (H_1), (H_2), (H_4) and condition (iii) in Theorem 1.2 are satisfied, then for all $k>\max\{k_2,k_3\}$, there exists a positive constant $C_7>0$ such that*

$$||u(\cdot,t)||_{L^k(\Omega)}\le C_7 \quad \textit{for all} \ \ t\in(0,T_{max}). \quad (4.22)$$

Proof. If $b-\delta\chi+\xi\rho\ge 0$, then this is proved in Lemma 4.2. In the following, we assume that $b-\delta\chi+\xi\rho<0$. Similar to (3.36), we have

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\int_\Omega(1+u)^k =&-\frac{4(k-1)}{k^2}\int_\Omega|\nabla(1+u)^{\frac{k}{2}}|^2-\chi(k-1)\int_\Omega\Psi_2(u)\Delta v\\ &+\xi(k-1)\int_\Omega\Phi_2(u)\Delta w+\int_\Omega(1+u)^{k-1}f(u),\end{aligned} \quad (4.23)$$

where $\Psi_2(u)$ and $\Phi_2(u)$ are defined in (4.6) and (4.7). The last term in (4.23) can be estimated as follows:

$$\begin{aligned}\int_\Omega(1+u)^{k-1}f(u)&\le\int_\Omega(1+u)^{k-1}(a-bu^r)\le\int_\Omega(1+u)^{k-1}(a-b+rb-rbu)\\ &=(a-b+2rb)\int_\Omega(1+u)^{k-1}-rb\int_\Omega(1+u)^k.\end{aligned} \quad (4.24)$$

From Young's inequality, there exists a constant $h_9>0$ such that

$$(a-b+2rb)\int_\Omega(1+u)^{k-1}\le rb\int_\Omega(1+u)^k+\frac{h_9}{k}. \quad (4.25)$$

Hence, from (4.8), (4.9), (4.11), (4.17), (4.23), (4.24) and (4.25), we obtain that

$$\frac{d}{dt}\int_\Omega(1+u)^k\le-\frac{4(k-1)}{k}\int_\Omega|\nabla(1+u)^{\frac{k}{2}}|^2+\frac{3\chi\delta k(k-1)}{k+p-1}\int_\Omega(1+u)^{k+q}+h_9. \quad (4.26)$$

Next, we consider the first two terms in the right side of (4.26). Since condition (iii) is satisfied, then $b > b_1$ where b_1 is defined as in (1.7), and recalling k_3 defined in (4.2), we have

$$\frac{qn}{2} < k_3. \tag{4.27}$$

We select $k' \in (qn/2, k_3)$, where $q \geq 2/n$ which implies $qn/2 \geq 1$. Therefore from Lemma 4.2, there exists $C_6 > 0$ such that

$$||u(\cdot,t)||_{L^{k'}(\Omega)} \leq C_6, \quad \text{for all } t \in (0, T_{max}). \tag{4.28}$$

Hence, choosing $k > k'$ and using Lemma 4.2 and Lemma 2.3, similar to (3.42), we obtain that there exists $h_{10} > 0$ such that

$$\int_\Omega (1+u)^{k+q} = ||(1+u)^{\frac{k}{2}}||_{L^{\frac{2(k+q)}{k}}(\Omega)}^{\frac{2(k+q)}{k}} \leq h_{10}\left(||\nabla(1+u)^{\frac{k}{2}}||_{L^2(\Omega)}^{\frac{2\alpha_2(k+q)}{k}} + 1\right), \tag{4.29}$$

where

$$\begin{aligned}\alpha_2 =& \frac{\dfrac{kn}{2k'} - \dfrac{kn}{2(k+q)}}{1 - \dfrac{n}{2} + \dfrac{kn}{2k'}} = \frac{(k+q)kn - k'kn}{k'(k+q)(2-n) + (k+q)kn} \\ =& \frac{(k+q)kn - k'kn}{(k+q)kn - k'kn + k'(2k - qn + 2q)}.\end{aligned} \tag{4.30}$$

Since $k > k' > \dfrac{qn}{2} \geq k_2$, then $\alpha_2 \in (0,1)$ and

$$\frac{2\alpha_2(k+q)}{k} = 2\frac{kn - k'n + qn}{kn - k'n + 2k'} < 2. \tag{4.31}$$

Hence, combining (4.29) with (4.31) and using Young's inequality, we find

$$\left(\frac{3\chi\delta k(k-1)}{k+q-1} + 1\right)\int_\Omega (1+u)^{k+q} \leq \frac{4(k-1)}{k}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 + h_{11}, \tag{4.32}$$

where h_{11} is a positive constant. Inserting (4.32) into (4.26), from Hölder inequality, we derive that

$$\frac{d}{dt}\int_\Omega (1+u)^k \leq -|\Omega|^{-\frac{q}{k}}\left(\int_\Omega (1+u)^k\right)^{\frac{k+q}{k}} + h_{12}. \tag{4.33}$$

where $h_{12} = h_9 + h_{11}$. Therefore, from Lemma 2.5, we have

$$\int_\Omega (1+u)^k \leq \max\left\{\int_\Omega (1+u_0)^k, (h_{12}|\Omega|^{\frac{q}{k}})^{\frac{k}{k+q}}\right\}. \tag{4.34}$$

□

The proof of the following lemma is similar to that of Lemma 3.4, and for the sake of completeness, we give the proof here.

Lemma 4.4. *Assume that (H_1), (H_2), (H_4) and condition (iv) in Theorem 1.2 hold, then for all $k > \max\{k_2, k_3\}$, there exists a positive constant $C_8 > 0$ such that*

$$||u(\cdot,t)||_{L^k(\Omega)} \leq C_8 \quad \textit{for all } \ t \in (0, T_{max}). \tag{4.35}$$

Proof. In this part, we will consider $0 < q < 2/n$. Similar to the proof of Lemma 3.3, we also obtain (4.26). By Lemmas 2.3 and 4.1, we obtain that

$$\int_\Omega (1+u)^{k+q} = ||(1+u)^{\frac{k}{2}}||^{\frac{2(k+q)}{k}}_{L^{\frac{2(k+q)}{k}}(\Omega)} \le h_{13}\left(||\nabla(1+u)^{\frac{k}{2}}||^{\frac{2\beta_2(k+q)}{k}}_{L^2(\Omega)}+1\right), \quad (4.36)$$

where

$$\begin{aligned}\beta_2 &= \frac{\frac{kn}{2}-\frac{kn}{2(k+q)}}{1-\frac{n}{2}+\frac{kn}{2}} = \frac{(k+q-1)kn}{(k+q)(2-n+kn)} = \frac{(k+q-1)kn}{(k+q-1)kn+(2k-qn+2q)} \\ &\le \frac{(k+q-1)kn}{(k+q-1)kn+(2-qn+2q)} < 1,\end{aligned} \quad (4.37)$$

since $k > k_2$ and $2 - qn > 0$, then $\beta_2 \in (0, 1)$. Also the following inequality holds:

$$\frac{2\beta_2(k+q)}{k} = 2n\frac{k+q-1}{2+nk-n} < 2. \quad (4.38)$$

Thus, combining (4.36) with (4.38) and using Young's inequality, we have

$$\left(\frac{3\chi\delta k(k-1)}{k+q-1}+1\right)\int_\Omega (1+u)^{k+q} \le \frac{4(k-1)}{k}\int_\Omega |\nabla(1+u)^{\frac{k}{2}}|^2 + h_{14}. \quad (4.39)$$

Then substituting (4.39) into (4.26), we arrive at

$$\frac{d}{dt}\int_\Omega (1+u)^k \le -\int_\Omega (1+u)^{k+q} + h_{15}. \quad (4.40)$$

So from Lemma 2.5, the L^k-bound of u is also obtained and we complete the proof. □

Finally we prove Theorem 1.2.

Proof. From Lemmas 4.3 and 4.4, we can show the L^∞-bound of $u(x,t)$ using the exactly same way as in Lemma 3.5. Now using the L^∞-bound and the blowup criterion (2.1), we obtain that there exists a constant $C_9 > 0$ such that

$$||u(\cdot,t)||_{L^\infty(\Omega)} \le C_9 \quad \text{for all } t \in (0,\infty). \quad (4.41)$$

Hence we complete the proof of Theorem 1.2. □

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