A global normal form for two-dimensional mode conversion

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A Global Normal Form for Two-Dimensional Mode Conversion

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ABSTRACT

Mode conversion is a phenomenon that is of interest as a method for heating in fusion reactors. A magnetosonic wave with dispersion relation $D_{\text{MS}}$ propagates toward the interior of the plasma, where it excites an ion-hybrid wave with dispersion relation $D_{\text{IH}}$ and thereby transfers energy to the plasma. We wish to study this process using ray-based methods. The $2 \times 2$ dispersion matrix $D$, which is, in general, a function of the phase space variables $(x,y,k_x,k_y)$, must be put into normal form, in which the diagonals of $D$, identified as the uncoupled dispersion relations, $D_{\text{MS}}$ and $D_{\text{IH}}$, Poisson-commute with the off-diagonals, identified as the coupling constants. Once in this form, we look at the points in the four-dimensional phase space that satisfy $D_{\text{MS}}(x,y,k_x,k_y) = 0 = D_{\text{IH}}(x,y,k_x,k_y)$, which, in general, will be a two-dimensional surface we call the conversion surface.

We implement our normal form algorithm on two models. First, we consider a slab model, in which $D$ depends only on $(x,k_x,k_y)$. Then we consider a two-dimensional model of the poloidal cross section of a tokamak reactor with a DT plasma with a density ratio of one-to-one. Using numerical methods, we put $D$ into normal form and identify the conversion surface. For both models, we find that there are regions in the four-dimensional phase space where the normal form transformation is well behaved and the conversion surface is what we expect. These are where the two dispersion surfaces $D_{\text{MS}}(x,y,k_x,k_y) = 0$ and $D_{\text{IH}}(x,y,k_x,k_y) = 0$ intersect transversely. However, there are also regions in the phase space where the normal form transformation is not well behaved. These coincide with tangential conversions, that is, where the two dispersion surfaces intersect tangentially. In this case, we must revisit the normal form theory and adapt it to this non-generic situation. Finally, we compute the transmission and conversion coefficients for such tangential conversions.
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In memory of my father, Jeffrey Johnston.
CHAPTER 1

Introduction

Waves are everywhere. Mode conversions are everywhere. For the purposes of this thesis, we take **mode conversion** to mean that the properties of a smoothly varying, background medium support the propagation of two or more *different* wave types within some region. Therefore, when a propagating wave of type (or mode) $A$ enters this region it may encounter a *resonance* where there is a possibility of it exciting and launching a wave of mode $B$. Thus, some of the energy in mode $A$ will be transferred to the wave of mode $B$. The physical world is replete with examples of this type of mode conversion. It occurs in the ocean where a water wave of one type excites a water wave of a different type [6]. In magnetohelioseismology, the phenomenon of mode conversion is used to explain how an acoustic wave traveling in the interior of the Sun is (partially) converted into a magnetic wave [2, 3]. They also occur in plasmas, (which, of course, the Sun is).

In particular, the existence of mode conversion in plasmas is exploited to heat magneti-
cally confined plasmas in nuclear fusion reactors. The basic idea is fairly straightforward. An electromagnetic wave, that is capable of propagating through the plasma, is launched from an antenna into the interior of the plasma. As it propagates through the plasma, it encounters a resonance,\textsuperscript{1} where, ideally, some of the energy is transferred to the plasma, i.e., it heats up! In order to understand where and how this energy is deposited, and to determine the optimal properties the incoming wave should have in order to heat the plasma as efficiently as possible, we need to understand this mode conversion process. Since plasmas are among the most complicated systems, this is not an easy problem to model and understand. Currently, a common approach to analyze the interaction of the wave modes in a magnetically confined plasma is to simply discretize the physical space and solve the (many) coupled differential equations. These are called full-wave simulations, and often take up to a week to run on the world's fastest supercomputers.

However, there are very good reasons, some of which we will talk about later in this thesis, to approach this problem using WKB methods and ray-tracing techniques. These techniques have been developed and employed over the last two decades by Kaufman, Tracy, et. al, to make the problem of mode conversion more tractable, and are wonderfully elaborated in the book \textit{Ray Tracing and Beyond} \textsuperscript{[15]}, on which this thesis heavily relies. A key point of these techniques is the use of phase space methods, which allow the problem of solving multi-component wave equations to be recast in such a way that it is reduced to solving coupled, first-order, ordinary differential equations, which, naturally, are easier to solve.\textsuperscript{2} Central to the application of these

\textsuperscript{1}We shall define more precisely what \textit{exactly} we mean by a resonance in chapter 2.

\textsuperscript{2}This is not meant to imply that the task is trivial; quite the opposite, in fact. However, with a great deal of work at the beginning, to recast the problem, we can use these tools to make the calculations much more
phase space methods, is the necessity of transforming our system into a particular form, the so-called normal form, of which we will have much more to say later.\(^3\) Once the system is in this ‘simplest’ form we can make use of the vast array of ray tracing techniques, some of which we will discuss later. It is precisely the development of this transformation, and its application to a model problem in plasma physics, with which this thesis is concerned.

First, in Chapter 2, we shall develop the physical plasma model which we will want to transform. This is going to be a cold plasma model, and we will guide the reader through the development of this model and the physical assumptions we are making, as well as our arguments for why this is still a physically valid model. Once we have our cold plasma model, we will introduce the different modes that can propagate in this plasma. We begin by examining a uniform, unmagnetized plasma; then a uniform, magnetized plasma; a nonuniform, unmagnetized plasma; and, finally, a nonuniform, magnetized plasma, for which we will derive the system that we will ultimately transform. Lastly, we present a motivation for our desire to transform the system.

Next, in Chapter 3, which is largely motivational, we introduce WKB methods and the mathematics underlying much of this work. We also introduce a variational principle that is key to understanding what class of transformations we should consider when transforming our system. We then introduce the concept of rays and apply them to the situation of resonances in a nonuniform magnetized plasma. Lastly, we summarize these results and begin to apply them to a general system of multi-component wave equations to motivate the necessity of straightforward.

\(^3\)The idea of this transformation is analogous to a coordinate transformation one performs on a system to recast the problem and make the physics more apparent.
transforming our system into the normal form.

In Chapter 4, we formally introduce the normal form concept and develop the algorithm for putting a self-adjoint system that exhibits two-dimensional mode conversion into normal form.

In Chapter 5, we apply the normal form transformation to the model developed in Chapter 2. After we apply the transformation, we show that it has, in spite of problems which need to be resolved, succeeded in uncoupling the two distinct modes; and that we are able to identify where the conversion occurs.

Lastly, in Chapter 6, we summarize the results of this research and discuss future work that needs to be done, as well as future extensions and possible applications.
CHAPTER 2

The Cold Plasma Model and $2 \times 2$ Dispersion Matrix

As mentioned in the introduction, we are interested in investigating the heating of plasma in a tokamak reactor via mode conversion. Before we can begin to explore this heating mechanism we need to develop a model for the plasma. First, we shall introduce the model, the so-called cold plasma model, and discuss the physical assumptions that we are making. Then, we shall proceed to linearize the model about some assumed background equilibrium. Since it is the nature of the background equilibrium that governs what type of behavior is dominant, we shall examine several cases.

First, we will consider a *uniform, unmagnetized* plasma, that is, a system in which the plasma properties are independent of position and the background magnetic field is zero. Next, a *uniform, magnetized* plasma. Third, a *nonuniform, unmagnetized* plasma, and; lastly,
a nonuniform, magnetized plasma. In each of these cases, we follow the usual theoretical development and write all variables in terms of the electric field, since it is the most readily measurable quantity. For the last case, a nonuniform, magnetized plasma, our derivation will lead us to the standard Stix model. We then begin to diverge from the standard theoretical progression by reducing the Stix model to a $2 \times 2$ case by projecting out the Langmuir wave. We will explain later why this reduction is necessary. Finally, in the last section of this chapter, we shall apply a transformation to our $2 \times 2$ system as a means to motivate our approach. But first, we need to develop the governing equations of our system.

2.1 The Cold Plasma Model

There are many different models that are used to analyze the behavior of plasma in a tokamak reactor. If we were interested, for example, in modeling the plasma inside a tokamak at equilibrium, in order to study the question of confinement, we might use a magnetohydrodynamic (MHD) model, in which, the entire plasma is thought of as a fluid that can carry a current. However, since we are interested in the question of heating the plasma, we shall use a cold plasma model. Because we will be using RF waves in a frequency range where ion motions are important to model, there can be significant differences in the dynamics of each species. Therefore, in this case, each particle species is treated as a cold, ideal fluid. By cold, we mean that we are assuming there is no pressure, and by ideal, that there are no collisions, i.e., no viscosity.$^{1}$ For a deuterium-tritium (DT) plasma, which is the plasma we will be modeling, the

$^{1}$At this point, one might ask, how can we make the assumption of a collisionless plasma, when, after all, we are modeling the plasma inside a fusion reactor, and for fusion to occur there must be collisions? The answer
particle species are, electrons \((s = e)\), protons \((s = H)\), deuterium nuclei \((s = D)\), and tritium nuclei \((s = T)\).

Since we are treating the plasma as a collection of cold, ideal fluids, for each particle species, \(s\), we have, \(n_s(x, t)\), the number density with units \(\text{cm}^{-3}\), and the velocity, \(v_s(x, t)\), with units \(\text{cm/s}\). In general, the number density and velocity, for each species, is a smooth function of the three spatial dimensions and time. Since each species is charged, we can write the total charge and current density as:

\[
\rho(x, t) = \sum_s \rho_s(x, t) = \sum_s q_s n_s(x, t) \tag{2.1.1}
\]

\[
J(x, t) = \sum_s J_s(x, t) = \sum_s q_s n_s(x, t) v_s(x, t). \tag{2.1.2}
\]

It is worthwhile to emphasize, that, although \(q_s\) is the charge for a single particle of species \(s\), we are not treating the plasma as a collection of particles, but rather as a collection of cold, ideal fluids. The fundamental fluid equations for a cold model are derived from the principles of number density and momentum conservation. These conservation laws hold for each species separately, with each cold fluid interacting with the other species only through the macroscopic electric and magnetic fields, (since there are no particle collisions). is, that the timescale for the behavior we are modeling, and the timescale on which collisions occur are very different. Hence, they can be separated and treated independently.
2.1.1 Number Density Conservation

Consider any fixed volume \( V_0 \), with surface \( S_0 \), in the space containing the plasma. The charge contained within this volume, due to species \( s \), is

\[
Q^V_0(t) = \int_{V_0} \rho_s(x,t) \, dV.
\]  

(2.1.3)

Therefore, the total derivative, with respect to time, of the charge contained in volume \( V_0 \) is

\[
\frac{dQ^V_0(t)}{dt} = \int_{V_0} \frac{\partial \rho_s}{\partial t} \, dV,
\]  

(2.1.4)

where we have suppressed the dependence of \( \rho_s \) on space and time for compactness. The charge, due to species \( s \), contained within the volume \( V_0 \), can only change if there is a flow of charge into or out of the volume, i.e., a current, through the boundary \( S_0 \):

\[
\frac{dQ^V_0(t)}{dt} = -\oint_{S_0} \mathbf{J}_s(x,t) \cdot d\mathbf{A}.
\]  

(2.1.5)

If current is flowing out of the volume, \( \mathbf{J} \cdot d\mathbf{A} > 0 \), the charge contained in \( V_0 \) is decreasing, and if current is flowing into the volume, \( \mathbf{J} \cdot d\mathbf{A} < 0 \), the charge is increasing. Thus, we choose the standard sign convention such that the oriented area \( d\mathbf{A} \) points outward from the enclosed volume and is normal to the surface. We can transform the surface integral into a volume
integral by invoking Gauss’ law to get:

\[
\frac{dQ_{V_0}(t)}{dt} = - \int_{S_0} J_s(x,t) \cdot dA = - \int_{V_0} \nabla \cdot J_s \, dV. \tag{2.1.6}
\]

Thus, combining Equations (2.1.4) and (2.1.6), we get,

\[
\int_{V_0} \left( \frac{\partial \rho_s}{\partial t} + \nabla \cdot J_s \right) \, dV = 0. \tag{2.1.7}
\]

Since equation (2.1.7) is true for any arbitrary volume \( V_0 \), the integrand must vanish, i.e.,

\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot J_s = 0. \tag{2.1.8}
\]

Using Equations (2.1.1) and (2.1.2) for each species \( s \), we can write this as,

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s v_s) = 0. \tag{2.1.9}
\]

Thus, Equation (2.1.9) is our first fundamental equation. Since Equation (2.1.9) is actually a statement of number density conservation, moving forward, we shall refer to this equation as the number density conservation equation.
2.1.2 Momentum Conservation

Consider, again, a fixed volume $V_0$, bounded by the surface $S_0$. The momentum of the fluid species $s$, contained within the volume $V_0$, is

$$P_s^{V_0}(t) = \int_{V_0} m_s n_s(x,t)v_s(x,t) \, dV, \quad (2.1.10)$$

where $m_s$ is the mass per particle of species $s$. The rate of change of momentum contained within this volume is

$$\frac{dP_s^{V_0}(t)}{dt} = \int_{V_0} \left( m_s v_s \frac{\partial n_s}{\partial t} + m_s n_s \frac{\partial v_s}{\partial t} \right) \, dV. \quad (2.1.11)$$

We can use Equation (2.1.9) to rewrite the first term as,

$$\frac{dP_s^{V_0}(t)}{dt} = \int_{V_0} \left[ -m_s v_s (\nabla \cdot n_s v_s) + m_s n_s \frac{\partial v_s}{\partial t} \right] \, dV. \quad (2.1.12)$$

We know that the rate of change of momentum of fluid species $s$, contained within volume $V_0$, is the surface integral of the momentum flux tensor for that species, $\Pi_s$, which flows through the bounding surface $S_0$, plus the external force acting on the fluid species, $F_s$:

$$\frac{dP_s^{V_0}(t)}{dt} = F_s - \int_{S_0} \Pi_s \cdot dA. \quad (2.1.13)$$
The flux of any quantity advected by a fluid with velocity \( v \) is given by the density of that quantity \( (\rho) \) times \( v \). Therefore, the flux of the \( j^{th} \) component of momentum density is:

\[
[\Pi_s]_j = m_s n_s v_s j v_s. \tag{2.1.14}
\]

Thus, the flux of the \( j^{th} \)-component of the momentum density through the surface \( S_0 \) is

\[
\left[ \int_{S_0} \Pi_s \cdot dA \right]_j = \int_{S_0} m_s n_s v_s j v_s \cdot dA. \tag{2.1.15}
\]

Transforming the surface integral to a volume integral, we get,

\[
\left[ \int_{S_0} \Pi_s \cdot dA \right]_j = \int_{V_0} \sum_k \partial_k (m_s n_s v_s j v_s k) \, dV. \tag{2.1.16}
\]

Using the product rule for differentiation, we have,

\[
\left[ \int_{S_0} \Pi_s \cdot dA \right]_j = \int_{V_0} \sum_k \left[ m_s v_s j \partial_k (n_s v_s k) + m_s n_s v_s k \partial_k v_s j \right] \, dV \tag{2.1.17}
\]

\[
= \int_{V_0} \left[ m_s v_s j (\nabla \cdot n_s v_s) + m_s n_s v_s \cdot \nabla v_s j \right] \, dV.
\]

Thus, the term for the surface integral of the momentum flux tensor becomes,

\[
\left[ \int_{S_0} \Pi_s \cdot dA \right]_j = \int_{V_0} \left[ m_s v_s (\nabla \cdot n_s v_s) + m_s n_s v_s \cdot \nabla v_s \right] \, dV. \tag{2.1.18}
\]
Since we are assuming that the fluids are cold and ideal, i.e., there is no pressure due to a finite temperature, (physically this means that the $\nabla p$ force term is small compared to the Lorentz forces due to the electric and magnetic fields), or collisions, (which would appear as finite viscosity), the only force acting on the fluids is the force from the electric and magnetic fields. Furthermore, by assumption, we are neglecting any gravitational forces acting on the plasma, which, for laboratory plasmas, are generally insignificant, but can be significant in astrophysical plasmas. Therefore, the net force acting on the fluid species $s$, in the volume $V_0$ is,

$$F_s = \int_{V_0} q_s n_s(x,t) \left[ E(x,t) + \frac{1}{c} v_s(x,t) \times B(x,t) \right] \, dV. \tag{2.1.19}$$

If we substitute Equations (2.1.11), (2.1.18), and (2.1.19) into Equation (2.1.13), and combine, we get,

$$\int_{V_0} \left[ m_sn_s \frac{\partial v_s}{\partial t} + m_sn_s v_s \cdot \nabla v_s - q_sn_s \left( E + \frac{1}{c} v_s \times B \right) \right] \, dV = 0. \tag{2.1.20}$$

Since this must hold for any volume $V_0$, the integrand must be zero, which leads to,

$$\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = q_s \frac{1}{m_s} \left( E + \frac{1}{c} v_s \times B \right), \tag{2.1.21}$$

the \textit{momentum conservation equation}.

\section*{2.1.3 The Maxwell Equations}

Recall, that we are treating each species as an ideal fluid, which means that we are ignoring collisions. Therefore, the only way in which the fluids interact is through the electric
and magnetic fields, for which they are also sources. Thus, in addition to the two families of conservation Equations, (2.1.21) and (2.1.9), we also have the Maxwell equations:

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 4\pi \rho \quad (2.1.22) \\
\nabla \cdot \mathbf{B} &= 0 \quad (2.1.23) \\
\n\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (2.1.24) \\
\n\nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad (2.1.25)
\end{align*}
\]

where \( \rho \) and \( \mathbf{J} \) are the total charge density and current density respectively, as given by Equations (2.1.1) and (2.1.2).

It is worthwhile to note, that, in some applications it is useful to split the current and charge density into internal and external pieces:

\[
\begin{align*}
\mathbf{J} &= \mathbf{J}_{\text{int}} + \mathbf{J}_{\text{ext}} \quad (2.1.26) \\
\rho &= \rho_{\text{int}} + \rho_{\text{ext}}. \quad (2.1.27)
\end{align*}
\]

The external quantities are used to model antennas and other sources, such as beams. Internal quantities are computed by self-consistently solving for the charged particle dynamics under the action of electromagnetic fields. In our work we assume the external terms are given. The computation of the internal terms requires further work, as outlined in the next few sections.

Let us take a moment to count variables and say a few words about the equations governing the dynamics of our cold plasma. First, for each fluid species, \( s \), in our plasma we have the
number density, \( n_s \), which is a scalar field, and the velocity, \( \mathbf{v}_s \), which is a three-component vector field. Thus, for each species, there are 4 dependent variables. Furthermore, we have the electric field, \( \mathbf{E} \), and the magnetic field, \( \mathbf{B} \), which are both three-component vectors. Therefore, in a plasma with \( n \) species, the total number of dependent variables is: \( 6 + 4n \).

Additionally, all variables are functions of four independent variables, three spatial dimensions and time. The equations governing the evolution of the dependent variables are, number density conservation for each species, Equation (2.1.9), (a scalar equation), momentum conservation for each species, Equation (2.1.21), (a three-component vector equation), and the Maxwell equations, Equations (2.1.22) to (2.1.25), two of which are scalar equations, Gauss’s law, Equation (2.1.22), and the no-name law, Equation (2.1.23), and two of which are three-component vector equations, Faraday’s law, Equation (2.1.24), and Ampere’s Law, Equation (2.1.25). Therefore, for a plasma with \( n \) species, we have \( 8 + 4n \) equations.

However, the two scalar Maxwell equations, Gauss’s law and \( \mathbf{E} \cdot \mathbf{B} = 0 \), are actually constraints on the initial conditions of \( \mathbf{E} \) and \( \mathbf{B} \). We can see this by taking the time derivative of both equations. For Gauss’s law, we get,

\[
\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}.
\]

(2.1.28)

We then use Ampere’s law, Equation (2.1.25), to substitute in for \( \frac{\partial \mathbf{E}}{\partial t} \):

\[
\nabla \cdot (\epsilon \nabla \times \mathbf{B} - 4\pi \mathbf{J}) = 4\pi \frac{\partial \rho}{\partial t}.
\]

(2.1.29)
We can write this as,
\[ c \nabla \cdot (\nabla \times \mathbf{B}) = 4\pi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right). \quad (2.1.30) \]

The left hand side of the above equation is zero, since the divergence of a curl is always zero, and the right hand side is zero from Equation (2.1.8). Now, for \( \nabla \cdot \mathbf{B} = 0 \), we have,
\[ \nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (2.1.31) \]

We then use Faraday’s law, Equation (2.1.24), to substitute in for \( \frac{\partial \mathbf{B}}{\partial t} \):
\[ -c \nabla \cdot (\nabla \times \mathbf{E}) = 0. \quad (2.1.32) \]

Therefore, as long as the electric and magnetic fields \textit{initially} satisfy the conditions that the divergence of the electric field is proportional to the \textit{initial} charge distribution and the divergence of the magnetic field is zero, they will satisfy them at all times. Thus, we actually have 6 + 4n dynamical equations \textit{and} 6 + 4n dynamical variables (and two constraints on the initial conditions). In principle, we could put this system of equations on a computer and numerically compute the solutions, given appropriate boundary and initial conditions. However, not only are the equations highly nonlinear, for a multiple species plasma the number of variables and equations is quite large, and it is highly non-trivial to solve the system numerically. Next, we shall proceed to linearize our model.
2.1.4 Linearization of the Maxwell and Fluid Equations

Recall, as we mentioned earlier, we want to eliminate all other variables in favor of the electric field, since it is the most straightforward to measure. Therefore, let us first eliminate $B$ in favor of $E$. We start by differentiating Ampere’s law, Equation (2.1.25), with respect to time:

$$\nabla \times \frac{\partial B}{\partial t} = \frac{4\pi}{c} \frac{\partial J}{\partial t} + \frac{1}{c} \frac{\partial^2 E}{\partial t^2}.$$  \hspace{1cm} (2.1.33)

Then, use Faraday’s law, Equation (2.1.24), to substitute for $\partial B/\partial t$:

$$\nabla^2 E - \nabla (\nabla \cdot E) - \frac{1}{c^2} \frac{\partial E}{\partial t} = \frac{4\pi}{c^2} \frac{\partial J}{\partial t}.$$  \hspace{1cm} (2.1.34)

Now, we will linearize the six equations under the following three assumptions. First, the background plasma is time-stationary, i.e., there is no zeroth order time dependence. Physically, this comes from the fact that the time scale of the behavior of the background plasma is much longer than the time scale of the wave physics we will be investigating. Hence, we treat the background plasma as time-stationary. Second, there is no unperturbed electric field, $E^0(x) = 0$, which implies that the background is charge neutral. Charge neutrality implies,

$$\sum_s q_s n_s^0(x) = 0.$$  \hspace{1cm} (2.1.35)

Third, we assume there is no background flow, i.e., $v_s^0(x) = 0 \Rightarrow J_s^0(x) = 0$. We should note that, although in real tokamak plasmas there is a non-zero background flow (and, thus,
non-zero background currents), which is utilized to generate a polodial magnetic field, we exclude it because, as we shall see, including a zeroth order velocity term leads to a much more complicated set of equations, and we want to understand the basics first, before adding additional difficulties.

Thus, let us write all the variables as zeroth order plus the perturbed quantities:

\begin{align}
E(x,t) &\approx E^1(x,t) & (2.1.36) \\
B(x,t) &\approx B^0(x) + B^1(x,t) & (2.1.37) \\
v_\text{s}(x,t) &\approx v^1_\text{s}(x,t) & (2.1.38) \\
n_\text{s}(x,t) &\approx n^0_\text{s}(x) + n^1_\text{s}(x,t). & (2.1.39)
\end{align}

Next, we substitute the above quantities into our six equations and collect them by order. At zeroth order, Gauss’s law, Equation (2.1.22), becomes

\begin{equation}
\nabla \cdot E^0(x) = 4\pi \sum_s q_s n^0_\text{s}(x),
\end{equation}

which is trivially satisfied, since $E^0 = 0$ by assumption, and $\sum_s q_s n^0_\text{s} = 0$ by charge neutrality. The divergence of $B$, Equation (2.1.23), at zeroth order, is simply

\begin{equation}
\nabla \cdot B^0(x) = 0,
\end{equation}

which implies that the background magnetic field must be divergence-free. The zeroth order
part of Faraday’s law, Equation (2.1.24), becomes

\[ \nabla \times \mathbf{E}^0(x) = -\frac{1}{c} \frac{\partial \mathbf{B}^0(x)}{\partial t}, \quad (2.1.42) \]

which is satisfied, since \( \mathbf{E}^0 = 0 \) and \( \mathbf{B}^0 \) does not depend on time. Finally, for Ampere’s law, Equation (2.1.25), at zeroth order, we have,

\[ \nabla \times \mathbf{B}^0(x) = \frac{4\pi}{c} \mathbf{J}^0(x), \quad (2.1.43) \]

which leads to the constraint that

\[ \nabla \times \mathbf{B}^0(x) = 0, \quad (2.1.44) \]

since \( \mathbf{J}^0 = 0 \). Thus, \( \mathbf{B}^0 \) should be curl-free.

Let us note that, for three out of the four cases of background equilibrium we will consider, namely, a uniform, unmagnetized plasma; a uniform, magnetized plasma; and a nonuniform, unmagnetized plasma, the curl-free condition of \( \mathbf{B}^0 \) is trivially satisfied. In the unmagnetized plasmas \( \mathbf{B}^0 = 0 \), and in the uniform, magnetized plasma \( \mathbf{B}^0 \) is constant. However, in the case of a nonuniform, magnetized plasma, the example which we shall use in the application of our normal form theory, we do not force \( \mathbf{B}^0 \) to be curl-free. This is done purely as a simplification, so as not to add additional layers of complexity. However, despite the fact that our background magnetic field violates Ampere’s law, our model does predict the existence of certain wave types which have been observed in tokamaks in approximately the correct spatial position and with about the right dispersion characteristics.
Continuing, the fluid equation derived from charge conservation, Equation (2.1.9), at lowest order is,

\[
\frac{\partial n^0_s(x)}{\partial t} + \nabla \cdot [n^0_s(x)v^0_s(x)] = 0,
\]

(2.1.45)

which is true since \(v^0_s = 0\) and \(n^0_s\) does not depend on time. Finally, at zeroth order, the fluid equation derived from momentum conservation, Equation (2.1.21), becomes:

\[
\frac{\partial v^0_s(x)}{\partial t} + v^0_s(x) \cdot \nabla v^0_s(x) = \frac{q_s}{m_s} \left[ E^0(x) + \frac{1}{c} v^0_s(x) \times B^0(x) \right],
\]

(2.1.46)

which is trivially satisfied, since \(v^0_s = 0\) and \(E^0 = 0\). Thus, the zeroth order equations leave us with two conditions on \(B^0(x)\), that it be curl- and divergence-free.

Now, let us examine the first order equations, where we will use only Equation (2.1.34), a combination of Faraday's and Ampere's law, and the fluid equation derived from momentum conservation, Equation (2.1.21). We do not need the other equations since, recall, the first two Maxwell Equations, (2.1.22) and (2.1.23), are merely constraints on the initial conditions, and the charge conservation fluid equation, (2.1.9), is encompassed in Gauss's law and Ampere's Law. Hence, at first order, Equation (2.1.34) becomes,

\[
\nabla^2 E^1(x,t) - \nabla \left[ \nabla \cdot E^1(x,t) \right] - \frac{1}{c^2} \frac{\partial^2 E^1(x,t)}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial J^1(x,t)}{\partial t},
\]

(2.1.47)

where

\[
J^1(x,t) = \sum_s q_s n^0_s(x) v^1_s(x,t).
\]

(2.1.48)
The momentum conservation fluid equation, at first order, is

\[ \frac{\partial \mathbf{v}_s^1(x,t)}{\partial t} + \frac{q_s}{m_s c} \mathbf{B}^0(x) \times \mathbf{v}_s^1(x,t) = \frac{q_s}{m_s} \mathbf{E}^1(x,t). \]  \hspace{1cm} (2.1.49)

Notice that there is no advective term in Equation (2.1.49) because the first order terms, which involve products of \( \mathbf{v}_s^0 \) and \( \mathbf{v}_s^1 \), are zero since \( \mathbf{v}_s^0 \) is zero, and the term \( \mathbf{v}_s^1 \cdot \nabla \mathbf{v}_s^1 \) is of second order. This is precisely why we assumed that there is no background flow, for, if we had not, then Equation (2.1.49) would have the term \( \mathbf{v}_s^0 \cdot \nabla \mathbf{v}_s^1 + \mathbf{v}_s^1 \cdot \nabla \mathbf{v}_s^0 \). Then, in a moment, when we want to invert Equation (2.1.49) in order to express \( \mathbf{v}_s^1 \) in terms of \( \mathbf{E}^1 \), we would no longer have a straightforward, algebraic equation to deal with. It is this additional complication that we wish to avoid by choosing \( \mathbf{v}_s^0 \) to be zero.

Going forward, let us write

\[ \mathbf{B}^0(x) \equiv \mathbf{B}^0(x) \mathbf{\hat{b}}^0(x), \]  \hspace{1cm} (2.1.50)

and define the gyrofrequency of species \( s \), \( \Omega_s(x) \), as:

\[ \Omega_s(x) \equiv \frac{q_s \mathbf{B}^0(x)}{m_s c}. \]  \hspace{1cm} (2.1.51)

In the case of a plasma in a uniform magnetic field, the gyrofrequency is a global constant. However, in a nonuniform plasma, the gyrofrequency depends upon position and is a local quantity. Thus, using Equations (2.1.50) and (2.1.51), Equation (2.1.49) can be written as

\[ \left[ \frac{\partial}{\partial t} + \Omega_s(x) \mathbf{\hat{b}}^0(x) \times \right] \mathbf{v}_s^1(x,t) = \frac{q_s}{m_s} \mathbf{E}^1(x,t). \]  \hspace{1cm} (2.1.52)
Now, if we assume that the time dependence of all time-varying quantities is harmonic and with the same frequency, we may Fourier analyze in time and write all quantities in the form:

\[ E(x,t) \equiv \tilde{E}_\omega(x)e^{-i\omega t}. \]  

(2.1.53)

Thus, restricting attention to one frequency, \( \omega \), and after dropping tildes and subscripts, we have two coupled, partial differential equations, which we can solve for the perturbed velocities and electric fields:

\[
\nabla^2 E^1(x) - \nabla \left[ \nabla \cdot E^1(x) \right] + \frac{\omega^2}{c^2} E^1(x) = -\frac{4\pi i\omega}{c^2} J^1(x) \\
\left[ -i\omega + \Omega_s(x) i \right] \tilde{v}_s^1(x) = \frac{q_s}{m_s} E^1(x).
\]  

(2.1.54) (2.1.55)

Next, we want to derive the *dispersion matrix*, \( D \), which is going to be an object of central importance throughout this thesis.

### 2.2 The Dispersion Matrix

We shall consider the case where \( \mathbf{B}^0(x) \) lies in the positive \( z \)-direction, with a magnitude that depends only on \( x \) and \( y \),

\[ \mathbf{B}^0(x) = B^0(x,y) \hat{\mathbf{z}}. \]  

(2.2.1)
Thus, $\mathbf{b}^0(x) = \mathbf{\dot{z}}$. Therefore, we can write Equation (2.1.55) as,

$$\left[-i\omega + \Omega_s(x,y)\mathbf{\dot{z}} \times \right] \mathbf{v}_x^1(x) = \begin{pmatrix} -i\omega & -\Omega_s(x,y) & 0 \\ \Omega_s(x,y) & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \mathbf{v}_{s,x}^1(x) \\ \mathbf{v}_{s,y}^1(x) \\ \mathbf{v}_{s,z}^1(x) \end{pmatrix}. \quad (2.2.2)$$

The determinant of the matrix in Equation (2.2.2) is $i\omega[\omega^2 - \Omega_s^2(x,y)]$, which is zero if $\omega = \Omega_s(x,y)$ (a gyroresonance). Therefore, as long as we are not in the vicinity of a gyroresonance, so $\omega \neq \Omega_s(x,y)$, we can invert the matrix in Equation (2.2.2) to get:

$$\begin{pmatrix} \mathbf{v}_{s,x}^1(x) \\ \mathbf{v}_{s,y}^1(x) \\ \mathbf{v}_{s,z}^1(x) \end{pmatrix} = \frac{q_s}{m_s} \begin{pmatrix} -i\omega & \Omega_s(x,y) & 0 \\ -\Omega_s(x,y) & i\omega & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_x^1(x) \\ E_y^1(x) \\ E_z^1(x) \end{pmatrix}. \quad (2.2.3)$$

Substituting the above equation into Equation (2.1.48), we get an expression for the perturbed current $\mathbf{J}^1$,

$$\begin{pmatrix} J_{s,x}^1(x) \\ J_{s,y}^1(x) \\ J_{s,z}^1(x) \end{pmatrix} = \sum_s q_s^2 n_s^0(x) \frac{1}{m_s} \begin{pmatrix} \Omega_s(x,y) & -\Omega_s(x,y) & 0 \\ -\Omega_s(x,y) & \Omega_s(x,y) & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} E_x^1(x) \\ E_y^1(x) \\ E_z^1(x) \end{pmatrix}. \quad (2.2.4)$$

Now, define the species plasma frequency,

$$\omega_s(x) \equiv \left(\frac{4\pi n_s^0(x) q_s^2}{m_s}\right)^{\frac{1}{2}}, \quad (2.2.5)$$
and the plasma frequency $\omega_p$,

$$\omega_p^2(x) \equiv \sum_s \omega_s^2(x) = 4\pi \sum_s n_s^0(x) q_s^2/m_s. \quad (2.2.6)$$

Because of the assumption of charge neutrality and the fact that the mass of the electrons is much smaller than the mass of the ions, the sum over species, in Equation (2.2.6), will be dominated by the electron term. Thus, to a good approximation, the plasma frequency will equal the electron plasma frequency:

$$\omega_p^2(x) \approx \frac{4\pi n_e^0(x) q_e^2}{m_e} = \omega_e^2(x). \quad (2.2.7)$$

Then, equation (2.2.4) becomes,

$$\mathbf{J}^1(x) = \frac{1}{4\pi} \sum_s \omega_s^2(x) \mathbf{\Omega}(x,y) \cdot \mathbf{E}^1(x), \quad (2.2.8)$$

where we define

$$\mathbf{\Omega}(x,y) \equiv \begin{pmatrix}
-i\omega & \frac{\Omega_s(x,y)}{\Omega_s^2(x,y)-\omega^2} & 0 \\
\frac{-\Omega_s(x,y)}{\Omega_s^2(x,y)-\omega^2} & \frac{\Omega_s(x,y)}{\Omega_s^2(x,y)-\omega^2} & 0 \\
0 & 0 & i/\omega
\end{pmatrix}. \quad (2.2.9)$$

Thus, Equation (2.1.54) is:

$$\nabla^2 \mathbf{E}^1(x) - \nabla[\nabla \cdot \mathbf{E}^1(x)] + \frac{\omega^2}{c^2} \mathbf{E}^1(x) + \frac{i\omega}{c^2} \sum_s \omega_s^2(x) \mathbf{\Omega}(x,y) \cdot \mathbf{E}^1(x) = 0. \quad (2.2.10)$$
If we multiply by $c^2$ and convert back to the curl notation for the spatial derivatives $\nabla \times \nabla \times \mathbf{E}^1$, we have

$$\hat{D} \cdot \mathbf{E}^1(\mathbf{x}) = \left[ -c^2 \nabla \times \nabla \times + \omega^2 + i \omega \sum_s \omega_s^2(\mathbf{x}) \Omega(\mathbf{x}, y) \right] \mathbf{E}^1(\mathbf{x}) = 0, \quad (2.2.11)$$

where we have introduced the wave operator $\hat{D}$, which is:

$$\hat{D} = -c^2 \nabla \times \nabla \times + \omega^2 + i \omega \sum_s \omega_s^2(\mathbf{x}) \begin{pmatrix} -\frac{i\omega}{\Omega_s^2 - \omega^2} & \frac{\Omega_s}{\Omega_s^2 - \omega^2} & 0 \\ -\frac{\Omega_s}{\Omega_s^2 - \omega^2} & \frac{-i\omega}{\Omega_s^2 - \omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}. \quad (2.2.12)$$

Note, that although we have suppressed the explicit argument of $\Omega_s$ in Equation (2.2.12), it still depends on $x$ and $y$. Let us look at the last term in the expression for $\hat{D}$ and bring the summation inside the matrix:

$$i \omega \sum_s \omega_s^2(\mathbf{x}) \Omega = \begin{pmatrix} \sum_s \frac{\omega_s^2 \omega_s^2(\mathbf{x})}{\Omega_s^2 - \omega^2} & i \sum_s \frac{\omega \omega_s^2(\mathbf{x}) \Omega_s}{\Omega_s^2 - \omega^2} & 0 \\ -i \sum_s \frac{\omega \omega_s^2(\mathbf{x}) \Omega_s}{\Omega_s^2 - \omega^2} & \sum_s \frac{\omega^2 \omega_s^2(\mathbf{x})}{\Omega_s^2 - \omega^2} & 0 \\ 0 & 0 & -\sum_s \omega_s^2(\mathbf{x}) \end{pmatrix}. \quad (2.2.13)$$

If we factor out an $\omega^2$ and write the second and third terms of $\hat{D}$ in matrix form, we get,

$$\omega^2 + i \omega \sum_s \omega_s^2 \Omega = \omega^2 \begin{pmatrix} 1 - \sum_s \frac{\omega_s^2}{\omega^2 - \Omega_s^2} & -i \sum_s \frac{\omega_s^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} & 0 \\ i \sum_s \frac{\omega_s^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} & 1 - \sum_s \frac{\omega_s^2}{\omega^2 - \Omega_s^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_s^2}{\omega^2} \end{pmatrix}, \quad (2.2.14)$$
where we have used Equation (2.2.6), the definition of the plasma frequency.

Now, let us define the three Stix functions, $S(x)$, $H(x)$ (Stix’s $D(x)$), and $P(x)$:

\[
S(x) \equiv 1 - \sum_s \frac{\omega_s^2(x)}{\omega^2 - \Omega_s^2(x,y)} \tag{2.2.15}
\]

\[
H(x) \equiv \sum_s \frac{\omega_s^2(x)\Omega_s(x,y)}{\omega(\omega^2 - \Omega_s^2(x,y))} \tag{2.2.16}
\]

\[
P(x) \equiv 1 - \frac{\omega_p^2(x)}{\omega^2}. \tag{2.2.17}
\]

Using these functions, Equation (2.2.11) becomes,

\[
\hat{D} \cdot E^1(x) = -c^2 \nabla \times \nabla \times + \omega^2 \begin{pmatrix}
S & -iH & 0 \\
iH & S & 0 \\
0 & 0 & P
\end{pmatrix} \begin{pmatrix}
E^1_x \\
E^1_y \\
E^1_z
\end{pmatrix} = 0, \tag{2.2.18}
\]

where we have dropped the explicit arguments of the functions. Since the wave operator $\hat{D}$ has no terms with mixed products of $x$ and $-i\nabla$, the Weyl mapping allows us to replace $-i\nabla$ with $k$, to get

\[
\hat{D} \xrightarrow{\Sigma} D(x,k;\omega) = \begin{pmatrix}
\omega^2 S - c^2(k_x^2 + k_y^2) & -i\omega^2 H + c^2 k_x k_y & c^2 k_x k_z \\
i\omega^2 H + c^2 k_x k_y & \omega^2 S - c^2(k_x^2 + k_z^2) & c^2 k_y k_z \\
c^2 k_x k_z & c^2 k_y k_z & \omega^2 P - c^2(k_x^2 + k_y^2)
\end{pmatrix}, \tag{2.2.19}
\]

where $\Sigma$ denotes the invertible Weyl symbol mapping. (More will be said about the Weyl symbol in Appendix A). The Weyl symbol $D$ of the operator $\hat{D}$ is called the dispersion matrix.
It is a matrix-valued function on ray-phase space, which is the six-dimensional space \((x, k)\).

Note, \(D\) is a self-adjoint matrix for each choice of real \(x, k,\) and \(\omega\).

Next, we shall explore the properties of the dispersion matrix assuming, first, a uniform plasma, and, second, a nonuniform plasma. In each instance, we shall examine the cases when the background plasma is magnetized, and when it is unmagnetized. Developing each of these cases shall allow us to introduce several concepts that will be of central importance.

### 2.3 Uniform Plasma

First, let us consider the case of a uniform plasma. A uniform plasma is a plasma where the zeroth order quantities, \(n_s^0\) and \(B^0\), do not depend upon position, i.e., they are constants. Therefore, the plasma frequency of each species, \(\omega_s\), and the plasma frequency \(\omega_p\) (defined by Equation (2.2.6)) are also constant, that is, they are global quantities of the plasma. We begin our examination with the simplest case: a uniform, unmagnetized plasma.

#### 2.3.1 Uniform, Unmagnetized Plasma

For our purposes, an unmagnetized plasma is one where the zeroth order magnetic field is zero, \(B^0 = 0\). (This is also the high-frequency limit, where the wave frequency \(\omega \gg \Omega_s\). Thus, we see from Equation (2.1.55), that the term including \(\omega\) will dominate.) Hence, the gyrofrequency of each species, \(\Omega_s\), is also zero by virtue of (2.1.51). Using our expressions for
the Stix functions $S, H,$ and $P$, Equations (2.2.15), (2.2.16), and (2.2.17), respectively, we get

$$S = 1 - \frac{\omega_p^2}{\omega^2},$$ \hspace{1cm} (2.3.1) \\
$$H = 0,$$ \hspace{1cm} (2.3.2) \\
$$P = 1 - \frac{\omega_p^2}{\omega^2},$$ \hspace{1cm} (2.3.3)

where we used the definition of the plasma frequency, Equation (2.2.6), to simplify $S$.

In the case of a uniform unmagnetized plasma, the Weyl symbol mapping reduces to the usual Fourier transform and the dispersion matrix is

$$D(\mathbf{k}, \omega) = \begin{pmatrix}
\omega^2 - \omega_p^2 - c^2(k_x^2 + k_z^2) & c^2k_xk_y & c^2k_xk_z \\
c^2k_xk_y & \omega^2 - \omega_p^2 - c^2(k_x^2 + k_y^2) & c^2k_yk_z \\
c^2k_xk_z & c^2k_yk_z & \omega^2 - \omega_p^2 - c^2(k_x^2 + k_y^2)
\end{pmatrix}. \hspace{1cm} (2.3.4)

Without the background magnetic field, the plasma is isotropic and we can, without loss of generality, choose a coordinate system in which the $z$ axis is in the direction of the vector $\mathbf{k}$, following the approach of Ref. [4]. In this case, we have, $\mathbf{k} = (0, 0, k)$ and $\mathbf{E} = (\tilde{E}_x, \tilde{E}_y, \tilde{E}_z)$. In this coordinate system, the Fourier transform of Equation (2.2.18) becomes

$$\begin{pmatrix}
\omega^2 - \omega_p^2 - c^2k^2 & 0 & 0 \\
0 & \omega^2 - \omega_p^2 - c^2k^2 & 0 \\
0 & 0 & \omega^2 - \omega_p^2
\end{pmatrix}
\begin{pmatrix}
\tilde{E}_x \\
\tilde{E}_y \\
\tilde{E}_z
\end{pmatrix} = 0, \hspace{1cm} (2.3.5)
where we have dropped the superscript on $E$ for simplicity.

Let us denote the eigenvalues $D_\alpha$ and the associated eigenvectors $\tilde{E}_\alpha$, with $\alpha = (1, 2, 3)$.

Thus, we have,

\[
D_{1,2} = c^2 k^2 - \omega^2 + \omega_p^2 \quad \tilde{E}_{1,2} = (\tilde{E}_x, \tilde{E}_y, 0) \quad (2.3.6)
\]

\[
D_3 = \omega^2 - \omega_p^2 \quad \tilde{E}_3 = (0, 0, \tilde{E}_z). \quad (2.3.7)
\]

Note that there are two modes appearing here, one that is perpendicular to the vector $k$, $(k \cdot \tilde{E}_{1,2}=0)$, which is called the transverse mode, and one that is parallel to $k$, $(k \times \tilde{E}_3 = 0)$, called the longitudinal mode. We will examine each mode and see what is happening in the plasma in each case. In order for Equation (2.3.5) to be valid for nontrivial $E$, the eigenvalues must be zero. Setting $D_{1,2}$ or $D_3$ from Equations (2.3.6) and (2.3.7) equal to zero gives us the following dispersion relations:

\[
c^2 k^2 = \omega^2 - \omega_p^2, \quad (2.3.8)
\]

\[
\omega^2 = \omega_p^2. \quad (2.3.9)
\]

A plot of these dispersion relations, called dispersion curves, is shown in Figure 2.1.

**The Transverse Mode**

First, let us examine the transverse mode, which has the electric field eigenvector $\tilde{E}_{1,2} = (\tilde{E}_x, \tilde{E}_y, 0)$. As we can see, this eigenvector has two independent components, $\tilde{E}_x$ and $\tilde{E}_y$. The
corresponding magnetic field is given by the relation $\mathbf{B} = (\mathbf{k}/\omega) \times \mathbf{E}_{1,2}$, which is simply the the Fourier transform of Faraday’s Law. Since the electric field is such that $\mathbf{k} \cdot \mathbf{E}_{1,2} = 0$, the Fourier transformation of Gauss’s law, $4\pi \mathbf{\rho} = i\mathbf{k} \cdot \mathbf{E}$, tells us that, although this mode does generate an electrical current, there is no charge density oscillation associated with the transverse mode.

The current generated by the transverse mode is, modifying Equation (2.2.8) for the uniform, unmagnetized plasma case,

$$\mathbf{J}_{1,2} = \frac{i}{4\pi} \frac{\omega_p^2}{\omega} \mathbf{E}_{1,2}, \quad (2.3.10)$$

where the frequency $\omega$ satisfies Equation (2.3.8). The fact that Fourier transform of the current is purely imaginary implies that the current and the electric field are out of phase by $\pi/2$. Furthermore, notice, that as the frequency increases the current decreases. This occurs because at higher frequencies the particles have less time to respond to the variations of the
electric field, resulting in smaller particle velocities and currents.

The dispersion relation for the transverse mode is plotted in Figure 2.1 ($D_{1,2} = 0$, the purple curve). We observe that for large $\omega$ and $k$ the dispersion curve asymptotes to the lines $\omega = \pm ck$, which are the dispersion curves for an electromagnetic wave in free space. Thus, at frequencies far above the plasma frequency, the transverse mode reduces to the free space electromagnetic mode. Note, that there is no real solution for $D_{1,2} = 0$ at frequencies below the plasma frequency $\omega_p$. We can examine the nature of this cutoff by solving the dispersion relation (2.3.8) for the wave number

$$k = \pm \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}.$$  \hspace{1cm} (2.3.11)

If $|\omega| > \omega_p$, the wave number is real and there is a propagating wave. However, if $|\omega| < \omega_p$, the wave number is purely imaginary, which corresponds to spatial fluctuations that decay exponentially. Such a non-propagating wave is called an evanescent wave.

The Longitudinal Mode

Next, let us examine the characteristics of the longitudinal mode. Recall that the electric field eigenvector for the longitudinal mode is $\hat{E}_3 = (0, 0, \tilde{E}_z)$, which is in the direction of the wave vector $\mathbf{k}$ such that $\mathbf{k} \times \hat{E}_3 = 0$. This implies that the corresponding magnetic field is zero, (since the magnetic field is given by the Fourier transform of Faraday’s law, $\hat{B} = (\mathbf{k}/\omega) \times \hat{E}_3$, and $\mathbf{k} \times \hat{E}_3 = 0$). However, unlike the transverse mode, the perturbed charge density is not zero since $\mathbf{k} \cdot \hat{E}_3 \neq 0$, and the Fourier transform of Gauss’s law tells us $4\pi \hat{\rho} = \mathbf{k} \cdot \hat{E}_3$. Thus,
the longitudinal mode is associated with a non-zero charge density. That is, if the ions or
electrons in the plasma are displaced from their equilibrium positions they will be subjected to
a linear restoring force and they will oscillate at the plasma frequency $\omega_p$, which, since we are
assuming the plasma to be charge neutral, is approximately the electron plasma frequency $\omega_e$.
Furthermore, since $k \times \mathbf{E}_3 = 0 \ (\nabla \times \mathbf{E}_3 = 0)$, we can write the electric field for the longitudinal
mode as the gradient of a potential, $\mathbf{E}_3 = -\nabla \Phi$. The absence of a magnetic field, and the fact
that $\mathbf{E}$ can be derived as the gradient of a scalar potential, is why these types of oscillations
are called electrostatic. Next, let us see what happens when we add a background magnetic
field.

2.3.2 Uniform, Magnetized Plasma

For the case of a uniform, magnetized plasma, we are still assuming that the zeroth order
quantities, $n^0_s$ and $\mathbf{B}^0$, do not depend on position; however, the background magnetic field is
non-zero, $\mathbf{B}^0 \neq 0$. Therefore, the gyrofrequency, $\Omega_s$, of each ion species $s$ is non-zero; hence,
the Stix functions $S$ and $H$ are no longer trivial.

Recall from Section 2.2 that we are choosing the $z$-axis to point in the direction of the
uniform magnetic field, i.e.,

$$\mathbf{B}^0 = B_0 \hat{z}. \quad (2.3.12)$$

Additionally, we may, without loss of generality, rotate our coordinate system so that the
wavevector $\mathbf{k}$ lies in the $x,z$ plane ($k_y = 0$) at an angle $\theta$ relative to the direction of the
magnetic field (the \( z \)-axis). Therefore, using the definition of the index of refraction,

\[
n = \frac{c}{\omega},
\]

(2.3.13)

and expressing the components of \( n \) in polar form,

\[
n = (n \sin \theta, 0, n \cos \theta),
\]

(2.3.14)

the dispersion matrix for a uniform, magnetized plasma is

\[
D(n, \omega) = \begin{pmatrix}
S - n^2 \cos^2 \theta & -iH & n^2 \cos \theta \sin \theta \\
iH & S - n^2 & 0 \\
n^2 \cos \theta \sin \theta & 0 & P - n^2 \sin^2 \theta
\end{pmatrix}.
\]

(2.3.15)

As in the case of the uniform, unmagnetized plasma, we need to solve the matrix equation

\[
\begin{pmatrix}
S - n^2 \cos^2 \theta & -iH & n^2 \cos \theta \sin \theta \\
iH & S - n^2 & 0 \\
n^2 \cos \theta \sin \theta & 0 & P - n^2 \sin^2 \theta
\end{pmatrix}
\begin{pmatrix}
\tilde{E}_x \\
\tilde{E}_y \\
\tilde{E}_z
\end{pmatrix}
= 0,
\]

(2.3.16)

which will only have nontrivial solutions when the determinant of the dispersion matrix is equal to zero. That is, when one or more of the eigenvalues of the dispersion matrix are zero, at which point the associated eigenvectors will determine the electric field. We will not examine the most general case of arbitrary \( \theta \), but, rather, we will investigate two limiting cases. First, we
will analyze the properties of the dispersion matrix when $\theta = 0$, i.e., a wave that is propagating \textit{parallel} to the external magnetic field. Then, we will examine the case when $\theta = \pi/2$, which corresponds to a wave that is propagating \textit{perpendicular} to the external magnetic field.

**Propagation Parallel to the Magnetic Field, $\theta = 0$**

For the case when $\theta = 0$, Equation (2.3.16) reduces to

$$
\begin{pmatrix}
  S - n^2 & -iH & 0 \\
  iH & S - n^2 & 0 \\
  0 & 0 & P
\end{pmatrix}
\begin{pmatrix}
  \tilde{E}_x \\
  \tilde{E}_y \\
  \tilde{E}_z
\end{pmatrix} = 0.
$$

(2.3.17)

The eigenvalues and associated eigenvectors are

$$
D_1 = n^2 - R \quad \tilde{E}_1 = (E_0, iE_0, 0)
$$

(2.3.18)

$$
D_2 = n^2 - L \quad \tilde{E}_2 = (E_0, -iE_0, 0)
$$

(2.3.19)

$$
D_3 = P \quad \tilde{E}_3 = (0, 0, E_0)
$$

(2.3.20)

where

$$
R \equiv S + H
$$

(2.3.21)

$$
L \equiv S - H.
$$

(2.3.22)
Setting each of the eigenvalues equal to zero gives us three dispersion relations:

\[ n^2 = R, \quad (2.3.23) \]
\[ n^2 = L, \quad (2.3.24) \]
\[ P = 0. \quad (2.3.25) \]

First, let us note that the \( D_3 = 0 \) mode, whose electric field oscillates in the direction of propagation, is a longitudinal wave and is, in fact, the same longitudinal electrostatic mode from the uniform, unmagnetized plasma case. Since the plasma particles are oscillating along the magnetic field lines, there is no magnetic \( \mathbf{v} \times \mathbf{B} \) force and the introduction of a background magnetic field has no effect on this mode. As in the case of the uniform, magnetized plasma, we see from Equation (2.2.17), that these oscillations occur at the plasma frequency, \( \omega = \omega_p \).

For the \( D_1 = 0 \) and the \( D_2 = 0 \) modes, the electric field eigenvectors are perpendicular to the wavevector \( \mathbf{k} \), which for the \( \theta = 0 \) case is in the direction of the background magnetic field along the \( z \)-axis. Therefore, these modes correspond to transverse electromagnetic waves with \( \tilde{\mathbf{B}} = \mathbf{k} \times \tilde{\mathbf{E}} / \omega \). As in the case of the transverse wave in the uniform, unmagnetized plasma, there is no charge density fluctuation associated with these waves since \( \mathbf{k} \cdot \tilde{\mathbf{E}} = 0 \). Additionally, it is fairly straightforward to see that the \( D_1 = 0 \) mode with electric field eigenvector \( \tilde{\mathbf{E}} = (E_0, iE_0, 0) \) corresponds to a right circularly polarized wave and the \( D_2 = 0 \) mode with electric field eigenvector \( \tilde{\mathbf{E}} = (E_0, -iE_0, 0) \) corresponds to a left circularly polarized wave.

The dispersion relation for the \( D_1 = 0 \) and \( D_2 = 0 \) modes are \( n^2 = R \) and \( n^2 = L \), respectively, where \( R \) and \( L \) are defined in Equations (2.3.21) and (2.3.22). Expressing the functions
and $L$ in terms of the plasma properties, we arrive at

$$R = 1 - \sum_s \frac{\omega_s^2}{\omega(\omega + \Omega_s)}$$  \hspace{1cm} (2.3.26)

$$L = 1 - \sum_s \frac{\omega_s^2}{\omega(\omega - \Omega_s)}.$$  \hspace{1cm} (2.3.27)

For our purposes, we are interested in a plasma which consists of three types of particle species; positively charged deuterium nuclei ($s = D$), positively charged tritium nuclei ($s = T$), and negatively charged electrons ($s = e$). The plasma frequencies and gyrofrequencies (often called the cyclotron frequencies) of the particle species are, using Equations (2.2.5) and (2.1.51),

$$\omega_e = \left( \frac{4\pi n_e^0 q_e^2}{m_e} \right)^{1/2}, \quad \Omega_e = q_e B^0 \frac{e}{m_e c},$$  \hspace{1cm} (2.3.28)

$$\omega_D = \left( \frac{4\pi n_D^0 q_D^2}{m_D} \right)^{1/2}, \quad \Omega_D = q_D B^0 \frac{e}{m_D c},$$  \hspace{1cm} (2.3.29)

$$\omega_T = \left( \frac{4\pi n_T^0 q_T^2}{m_T} \right)^{1/2}, \quad \Omega_T = q_T B^0 \frac{e}{m_T c}.$$  \hspace{1cm} (2.3.30)

Now, let us look at the frequency dependence of the index of refraction for the $R$ and $L$ modes for the case of a 50/50 DT plasma. A plot of $n^2 = R$ as a function of frequency is shown in Figure 2.2. The branch of the dispersion curve in the region $0 \leq \omega < |\Omega_e|$ is called the Whistler mode. The branch of the dispersion curve in the region $\omega > |\Omega_e|$ is called the right-hand polarized Free space mode, since the index of refraction approaches one, the free space value (the gray, dashed line in Figure 2.2), as the frequency approaches infinity. Since the electron gyrofrequency is negative, ($\Omega_e < 0$ because $q_e = -e$), the function $R$ goes to
FIG. 2.2: Plot of the dispersion relation $n^2 = R$ for the $D_1 = 0$ mode of a uniform, magnetized plasma composed of 50% deuterium and 50% tritium. The red, vertical, dashed line indicates the location of the electron gyroresonance, which occurs at $\omega = \Omega_e$. The horizontal, gray, dashed line denotes the index of refraction for a free space electromagnetic wave, $n^2 = 1$.

infinity when $\omega = |\Omega_e|$. A condition when the index of refraction goes to infinity is called a resonance, in this case, it is the electron cyclotron resonance. Since the electrons rotate about the static magnetic field in the same right-handed sense as the rotation of the wave, when the wave frequency $\omega$ is near the electron cyclotron resonance the wave interacts strongly with the electrons in the plasma. Note that there are no corresponding resonances at the ion gyrofrequencies, $\Omega_D$ and $\Omega_T$, since the ions rotate about the static magnetic field in a left-handed direction. The frequency at which the index of refraction goes to zero is called a cutoff and is denoted on Figure 2.2 as $\omega_{R=0}$. In the range of frequencies $\Omega_e < \omega < \omega_{R=0}$, $n^2$ is less than zero, which corresponds to an imaginary index of refraction; hence, an imaginary
wavenumber $k$. As in the case of the transverse mode in a uniform, unmagnetized plasma when $|\omega| < \omega_p$, this corresponds to an evanescent (non-propagating) wave. When the frequency is zero, the index of refraction becomes

$$n^2_A = 1 + \sum_s \frac{\omega_s^2}{\Omega_s^2}.$$  \hfill (2.3.31)

This is called the Alfvén index of refraction and waves that obey this dispersion relation are called Alfvén waves.

A plot of $n^2 = L$ as a function of frequency is shown in Figure 2.3. Notice that in this

![FIG. 2.3: A plot of the dispersion relation $n^2 = L$ for the $D_2 = 0$ mode of a uniform, magnetized plasma composed of 50\% deuterium and 50\% tritium. The red, vertical, dashed lines indicate the location of the ion gyroresonances, which occur at $\omega = \Omega_T$ and $\omega = \Omega_D$. The horizontal, gray, dashed line denotes the index of refraction for a free space electromagnetic wave, $n^2 = 1$.](image)
case there are two resonances, one at each of the ion gyrofrequencies, $\Omega_D$ and $\Omega_T$, and no resonance at the electron gyrofrequency, $\Omega_e$. Since the positively charged ions rotate about the magnetic field in the same left-handed sense as the wave, it will interact strongly (i.e. resonate) with the particles when the wave frequency is near either of the ion gyrofrequencies. There is no resonance at the electron gyrofrequency since the electrons rotate about the magnetic field in the opposite sense. Furthermore, we see that for the left-handed mode there are two cutoffs and, therefore, two evanescent regions; and as the frequency approaches zero the index of refraction once again approaches the Alfvén index of refraction. The two branches of the dispersion curve in the regions $0 \leq \omega < \Omega_T$ and $\Omega_T < \omega < \Omega_D$ are called the ion cyclotron modes and the branch in the region $\omega > \Omega_D$ is the left-hand polarized Free space mode, since the index of refraction approaches one as the frequency gets very large. (The gray, dashed line in Figure 2.3).

**Propagation Perpendicular to the Magnetic Field, $\theta = \pi/2$**

Now, let us turn our attention to the case of a wave propagating perpendicular to the magnetic field. For the case when $\theta = \pi/2$, i.e. the wavevector $\mathbf{k}$ is in the $x$-direction, Equation (2.3.16) becomes

\[
\begin{pmatrix}
S & -iH & 0 \\
 iH & S-n^2 & 0 \\
0 & 0 & P-n^2 \\
\end{pmatrix}
\begin{pmatrix}
\tilde{E}_x \\
\tilde{E}_y \\
\tilde{E}_z \\
\end{pmatrix} = 0.
\]  

(2.3.32)
Examining the nullspace of the dispersion matrix, we find the following two dispersion relations and null eigenvectors

\[ n^2 = \frac{RL}{S} \quad \vec{E}_{1,2} = \left( \frac{iH}{S}E_0, E_0, 0 \right) \]  
\[ n^2 = P \quad \vec{E}_3 = (0, 0, E_0). \]  

(2.3.33)  
(2.3.34)

Note that the null eigenvector, \( \vec{E}_{1,2} \), is degenerate.

First, notice that the \( n^2 = P \) mode, where \( P \) is defined by Equation (2.2.17), is identical to the transverse electromagnetic wave encountered in the case of a uniform, unmagnetized plasma. This transverse electromagnetic wave is not affected by the presence of a background magnetic field because the particle motion is parallel to the direction of the static background magnetic field; hence, as in the case of the longitudinal mode when \( \theta = 0 \), there is no \( \mathbf{v} \times \mathbf{B} \) force. Furthermore, in this case the electric field is \textit{linearly} polarized.

The \( n^2 = RL/S \) mode is more complex since the electric field has both longitudinal (parallel to \( \mathbf{k} \)) and transverse (perpendicular to \( \mathbf{k} \)) components. Hence, this mode has both electrostatic and electromagnetic characteristics. The frequency dependence of the index of refraction for this mode is determined by the functions \( R, L, \) and \( S \). We have already examined and plotted the functions \( R \) and \( L \) in Figures 2.2 and 2.3. A plot of \( S \) as a function of frequency is shown in Figure 2.4, once again for the case of a 50/50 DT plasma. The function \( S \), defined in Equation (2.2.15), consists of alternating poles and zeros, with a pole located at each gyrofrequency. For the case we are modeling, a cold plasma with two ion species (deuterium and tritium), the function \( S \) has three poles, one at each of the ion gyrofrequencies and one at the electron
as can be seen in Figure 2.4. The index of refraction, \( n^2 = RL/S \), has zeros at the zeros of \( R \) and \( L \) and resonances at the zeros of \( S \). A plot of \( n^2 = RL/S \) is shown in Figure 2.5. The resonance located between the two ion gyrofrequencies \( \Omega_T \) and \( \Omega_D \) is called the ion hybrid resonance, \( \omega_{IH} \), the resonance between the larger ion gyrofrequency \( \Omega_D \) and the electron gyrofrequency \( \Omega_e \) is called the lower hybrid resonance, \( \omega_{LH} \), and the resonance above the electron gyrofrequency is called the upper hybrid gyrofrequency, \( \omega_{UH} \). These resonances are called hybrid resonances because they involve a combination (hybrid) of motions of two different types of particle species. If we examine the electric field eigenvector for this mode, \( \tilde{E}_{1,2} = [(iH/S)E_0, E_0, 0] \), we see that at frequencies near the hybrid resonances (when \( S = 0 \)) the longitudinal component \( \tilde{E}_x = (iH/S)E_0 \) becomes arbitrarily large compared to the transverse component \( \tilde{E}_y = E_0 \). Therefore, in this limit, the wave is mostly electrostatic in nature, similar to
FIG. 2.5: A plot of the dispersion relation $n^2 = RL/S$ for the case of a uniform, unmagnetized plasma composed of 50% deuterium and 50% tritium. The red, vertical, dashed lines indicated the locations of the hybrid resonances, and the horizontal, gray, dashed line once again denotes the index of refraction for a free space electromagnetic wave, $n^2 = 1$.

the electron plasma oscillations of the longitudinal mode in the case of a uniform, unmagnetized plasma; however, in this instance, magnetic forces need to be considered. Finally, note that, as in the case of wave propagation parallel to the magnetic field ($\theta = 0$), when the wave frequency $\omega$ approaches zero, the index of refraction approaches the Alfvén index of refraction $n_A^2$ defined in Equation (2.3.31). Additionally, as the frequency gets very large, much higher than the electron gyrofrequency, the index of refraction once again approaches unity (the gray, dashed line in Figure 2.5).

Before we proceed to the analysis of a nonuniform plasma, let us return briefly to a discussion of the hybrid resonances, in particular, the ion hybrid resonance located between the
two ion gyrofrequencies. Waves with a frequency just below the ion hybrid resonance, which are propagating perpendicular to the background magnetic field, will interact strongly with the ions in the plasma and can transfer energy, resulting in an increase in the kinetic energy and thermal motions of the ions. This is one method of heating a plasma confined in a tokamak reactor, and the method that we will be investigating. In order to investigate how the incident wave interacts with the ions in the plasma it is necessary to find the value of $\omega_{\text{IH}}$, which will be the smallest, positive root of $S$. This can always be done numerically; however, for the situation we are modeling, we can express this frequency (within about a 4% error) as

$$\omega_{\text{IH}} \approx \left( \frac{\Omega_D^2 + \Omega_T^2}{2} \right)^{1/2}. \quad (2.3.35)$$

Notice that the ion hybrid frequency $\omega_{\text{IH}}$ depends on the magnitude of the background magnetic field $B_0$, through the ion gyrofrequencies as seen from Equation (2.1.51). As we shall see shortly, in the case of a nonuniform plasma, the ion hybrid frequency will not only depend on the magnitude of the magnetic field, but also on the spatial properties of the plasma.

### 2.4 Nonuniform Plasma

Before we begin our analysis of a nonuniform plasma, let us take a moment to examine Equation (2.2.18). First, note that we are still assuming that all time-varying quantities are harmonic and with the same frequency $\omega$. In order to find solutions to Equation (2.2.18) we are going to make use of the eikonal approximation, in which we assume that the electric field
$\mathbf{E}(x)$ is of the form

$$\mathbf{E}(x) = A(x)e^{i\theta(x)}\hat{e}(x),$$

(2.4.1)

where $A(x)$ is the amplitude, $\theta(x)$ is the phase, and $\hat{e}(x)$ is the unit polarization vector. Furthermore, if we assume that the phase, $\theta(x)$, varies much more rapidly than the amplitude and the polarization, we can go from the operator $\hat{D}(x, \nabla; \omega)$ in Equation (2.2.18) to the symbol $D(x, k; \omega)$ of Equation (2.2.19), with the essential awareness that

$$k(x) = -i\nabla\theta(x).$$

(2.4.2)

We shall discuss the eikonal approximation in further detail in the next chapter, and the Weyl symbol is discussed in Appendix A, but we want to keep in mind the explicit relationship expressed in Equation (2.4.2) as we proceed to discuss nonuniform plasmas. Finally, let us note that for an electric field of the form in Equation (2.4.1), Equation (2.2.18) will have non-trivial solutions if and only if the polarization vector $\hat{e}(x)$ is a null eigenvector of the operator $\hat{D}$. Setting the associated eigenvalue to zero will give us a dispersion relation involving $x, k$, and $\omega$.

### 2.4.1 Nonuniform, Unmagnetized Plasma

The analysis of a nonuniform, unmagnetized plasma is similar to the analysis for a uniform, unmagnetized plasma. The significant difference between a uniform, unmagnetized plasma and nonuniform, unmagnetized plasma is that the species plasma frequency $\omega_s$, and, hence, the
plasma frequency $\omega_p$, are no longer \textit{global} constants, but depend upon the density distribution of the plasma. Thus, the Stix functions $S, H,$ and $P$ will be given by Equations (2.3.1), (2.3.2), and (2.3.3), respectively. First, it is straightforward to see that we will again have \textit{two} distinct modes. The \textit{transverse} mode, corresponding to the electric field eigenvector which is perpendicular to the wave vector, $\mathbf{k}(x) \cdot \tilde{\mathbf{E}}_{1,2} = 0$, and the \textit{longitudinal} mode, corresponding to the electric field eigenvector which is parallel to the wave vector, $\mathbf{k}(x) \times \tilde{\mathbf{E}}_3 = 0$. (The important difference here, is that the wave vector $\mathbf{k}$ is a function of $x$!)

Thus, we can once again, without loss of generality, choose a coordinate system in which the $z$-axis is in the direction of the vector $\mathbf{k}$. Then our wave Equation, (2.2.18), becomes

$$
\begin{pmatrix}
D_1 & 0 & 0 \\
0 & D_2 & 0 \\
0 & 0 & D_3
\end{pmatrix}
\begin{pmatrix}
E_x(x) \\
E_y(x) \\
E_z(x)
\end{pmatrix} = 0.
$$

(2.4.3)

The eigenvalues, and their associated eigenvectors, are, using the same notation as the previous section,

$$D_{1,2} = c^2 k^2(x) - \omega^2 + \omega_p^2(x) \quad \mathbf{e}_{1,2} = [E_x(x), E_y(x), 0]$$

(2.4.4)

$$D_3 = \omega^2 - \omega_p^2(x) \quad \mathbf{e}_3 = [0, 0, E_z(x)].$$

(2.4.5)
Furthermore, the two dispersion relations, obtained by setting $D_{1,2} = 0$ and $D_3 = 0$, are now

\[ c^2 k^2 = \omega^2 - \omega_p^2(x), \quad \text{(2.4.6)} \]
\[ \omega^2 = \omega_p^2(x). \quad \text{(2.4.7)} \]

Recall, that since we are assuming that our plasma has no net charge, that the plasma frequency is approximately equal to the electron plasma frequency, $\omega_p(x) \approx \omega_e(x)$. We can see from Equation (2.2.5), that the only term in the electron plasma frequency that can depend upon position is the zeroth order number density $n_0^e(x)$. Hence, if we wish to plot the dispersion relations we need to specify $n_0^e(x)$. We are going to assume that the plasma density only varies in the $x$ and $y$ directions. Specifically,

\[ n_s^0(x) = n_s^0(x, y) = n_s^0 \exp \left[ - (x^2 + y^2)^2 \right]. \quad \text{(2.4.8)} \]

This type of density profile is shown in Figure 2.6. Thus, using this number density profile, we can plot the dispersion relations given by Equations (2.4.6) and (2.4.7). Note that if we use polar coordinates such that $r^2 = x^2 + y^2$ then the dispersion functions are functions of three variables $(r, k, \omega)$. Hence, the plots of the dispersion relations will be two-dimensional dispersion surfaces imbedded in a three-dimensional space. Plots of these surfaces are shown in Figures 2.8 through 2.10. The number density profile is also shown in polar coordinates in Figure 2.7.
FIG. 2.6: A plot of the number density profile, $n^0(x,y) = n_0^y \exp \left[ - (x^2 + y^2)^2 \right]$, we are assuming for our plasma. Note that the density is symmetric about the origin.

FIG. 2.7: A plot of the number density profile in polar coordinates, $r^2 = x^2 + y^2$. 
FIG. 2.8: Dispersion surfaces for a nonuniform, unmagnetized plasma: (a) Plots of the dispersion surfaces $D_{1,2}(r,k,\omega) = c^2k^2(x) - \omega^2 + \omega_p^2(x) = 0$ and $D_3(r,k,\omega) = \omega^2 - \omega_p^2(x) = 0$. (b) Plots of the dispersion surfaces $D_{1,2}(r,k,\omega) = 0$ and $D_3(r,k,\omega) = 0$ with the surface $\omega = ck$, which is the free space electromagnetic dispersion surface. The discoloration of the surface at large $r$ is simply an effect of the image rendering, since at large $r$, $D_{1,2} = 0 \approx \omega = ck$. 
FIG. 2.9: Plots of the dispersion surfaces $D_{1,2}(r,k,\omega) = c^2k^2(x) - \omega^2 + \omega_p^2(x) = 0$ and $D_3(r,k,\omega) = \omega^2 - \omega_p^2(x) = 0$. Note the spatial dependence of both the transverse dispersion surface and the longitudinal dispersion surface, and the fact that the transverse mode is still bounded below by the longitudinal mode.
FIG. 2.10: Plots of the dispersion surfaces $D_{1,2}(r,k,\omega) = c^2 k^2(x) - \omega^2 + \omega_p^2(x) = 0$ and $D_3(r,k,\omega) = \omega^2 - \omega_p^2(x) = 0$ at three different values of $r$. The surface $\omega = \pm ck$ (the vacuum dispersion surfaces with positive frequencies) is also shown for reference. (a) The dispersion surfaces at the center of the plasma, $r = 0$. (b) The dispersion surfaces at an intermediate value of $r$, between the center and the edge of the plasma. (c) The dispersion surfaces near the edge of the plasma.

We see from the figures that the two modes, the transverse mode $(k \cdot e_{1,2} = 0)$, which is the orange dispersion surface, and the longitudinal mode $(k \times e_3 = 0)$, which is the blue dispersion
surface, have the same properties as in the unmagnetized, uniform plasma case. Namely, the transverse dispersion surface is bounded below by the longitudinal dispersion surface, which indicates that the transverse mode cannot propagate if \( \omega < \omega_p(r) \), Figure 2.8a. Furthermore, we see again that the transverse mode asymptotes to the free space electromagnetic mode as the frequency \( \omega \) gets very large. This can best be seen in Figure 2.8b, where the orange, transverse dispersion surface is getting very close to the \( \omega = \pm ck \) plane, which, recall, is the free-space electromagnetic dispersion surface (for \( \omega > 0 \)). Additionally, the longitudinal mode is once again independent of the wave number \( k \), and only exists for the frequency \( \omega = \omega_p(r) \). The rest of the analysis from the previous section still holds for both the transverse and longitudinal modes.

The most significant difference, of course, is the addition of the spatial dependence. If we look at Figure 2.9, we see that the plasma frequency, which is the blue surface, is maximum at the center of the plasma and asymptotes to zero as one moves away from the center. This is because, as can be seen from Figure 2.7, that the plasma density approaches zero at large values of \( r \), and the plasma frequency \( \omega_p^2 \propto n^0(r) \). Therefore, if we look at the dispersion surfaces at \( r = 0 \), Figure 2.10a, we see they are the same as in the unmagnetized, uniform case. Then, as the plasma density drops off and the plasma frequency decreases, the transverse mode can exist at lower and lower frequencies, as can be seen in Figure 2.10b. Finally, as the plasma density and the plasma frequency approach zero, the longitudinal mode disappears and the transverse mode goes over to the free space electromagnetic mode, shown in Figure 2.10c.
2.4.2 Nonuniform, Magnetized Plasma

Now, let us proceed to discuss the most general case; a nonuniform magnetized plasma. Recall, from Equation (2.2.1) that we are taking the direction of the background magnetic field to be in the $z$ direction and the magnitude of the magnetic field depends only on the $x$ and $y$ position. In fact, in the model we are considering, the magnetic field strength depends only on the $x$ coordinate and is of the form:

$$B(x) = B_0 \frac{1}{1+x/10} \hat{z},$$

where $B_0$ is the strength of the magnetic field on the magnetic axis ($x,y = 0,0$). This field strength profile is sometimes used to model the poloidal plane of a tokamak in simplified models, see Refs. [5, 13, 14]. A plot of the magnetic field strength over the range of interest is shown in Figure 2.11. We should note, that this expression for the magnetic field $B$ is not curl-free.

Recall, that as a result of our linearization about the background plasma, one of the constraints was that the background magnetic field, $B^0$, be curl-free, Equation (2.1.44). However, as we noted earlier, insisting that $B^0(x)$ be curl-free would introduce additional complexities into the physical model that will not be dealt with in this thesis, though it can be incorporated in future work. For example, in Juan, et. al., where the first ray-tracing algorithm (RAYCON) was reported that can treat mode conversion in a tokamak cavity, a full Soloviev toroidal equilibrium was used, which includes toroidal currents and poloidal magnetic fields [5]. The topic
FIG. 2.11: A plot of $B(x) = \frac{B_0}{1+\frac{x}{10}}$ over the range of interest in $x$.

of the current thesis, the use of normal form methods to identify mode conversion regions and then compute the local uncoupled ray Hamiltonians and polarizations, could then be inserted into RAYCON as a numerical module.

Since the background magnetic field is no longer uniform, the species gyrofrequencies, which depend on $B(x)$ through Equation (2.1.51), are no longer global constants, but vary with position. Hence, the various resonances that we encountered in the previous sections no longer occur at specific frequencies, rather, they will also depend upon position, similar to the behavior of the plasma frequency we observed in the previous section. A plot of the deuterium and tritium gyrofrequencies (along with the hydrogen gyrofrequency for reference) is shown in Figure 2.12. The quantity $\Omega_0$ that appears in Figure 2.12 is defined to be the proton
2.12: A plot of the ion gyrofrequencies $\Omega_H(x)$, $\Omega_D(x)$, and $\Omega_T(x)$.

Gyrofrequency on the magnetic axis (i.e. at $(x=0, y=0)$):

$$\Omega_0 = \frac{eB_0}{m_0c^2},$$

(2.4.10)

where $m_0$ is the mass of the proton and $c$ is the speed of light.

Recall, from Section 2.3.2, where we discussed waves propagating perpendicular to the magnetic field in a uniform magnetized plasma, that we mentioned that we are interested in the propagation of waves with frequencies near the ion hybrid resonance. Furthermore, recall that the hybrid resonances occur when the Stix function $S = 0$, where $S$ is defined in Equation (2.2.15). In the case of a nonuniform magnetized plasma, the function $S$ depends on the wave frequency $\omega$ and the $x$ and $y$ position. Thus, the condition $S(x, y; \omega) = 0$ is one condition on
three variables, hence generically defines a series of two-dimensional surfaces in the three-space \((x, y, \omega)\). Since we are ultimately interested in the heating of a magnetically confined plasma through ion hybrid interactions, we choose our wave frequency \(\omega\) to be just below the ion hybrid resonance at the center of the plasma \((x = 0, y = 0)\):

\[
\omega = \omega_{rf} = 0.95 \frac{\sqrt{\Omega_D^2(0,0) + \Omega_T^2(0,0)}}{\sqrt{2}} \approx \frac{2}{5} \Omega_0. \tag{2.4.11}
\]

A plot of the ion hybrid resonance surface, along with the surface \(\omega = \omega_{rf}\), is shown in Figure 2.13. We do not include the upper or lower hybrid resonances because we are only interested in the frequencies near the ion hybrid resonance.
FIG. 2.13: A plot of the ion hybrid resonance, which is a root of the function $S(x, y; \omega) = 0$. We have also included a plot of the function $\omega = \omega_H$. 
The intersection of the two surfaces, \( S = 0 \) and \( \omega = \omega_{tf} \), indicate the points where \( S(x, y; \omega = \omega_{tf}) = 0 \). Since this is an intersection of two, two-dimensional surfaces, these points will, generally, lie on a one-dimensional curve in \((x, y)\) space. The intersection of the two surfaces in Figure 2.13 is emphasized in Figure 2.14a. We also show the projection of the intersection onto the \( xy \)-plane in Figure 2.14b.
FIG. 2.14: The intersection of the surfaces $S(x, y, \omega) = 0$ and $\omega = \omega_f$. (a) The intersection (highlighted in green) in the space $(x, y, \omega)$. (b) The intersection projected onto the $xy$-plane. Also shown, is the location in physical space where the plasma density drops to 1% of its maximum value at the center, denoted $n_0$.

Let us take a moment to say that we are not interested in anything that happens very near
or at the edge of the plasma because we are not attempting to include any boundary effects into our model. For our purposes, we shall define the plasma boundary to be the points at which the plasma density is 1% of its maximum value.\(^2\) (The golden line shown in Figure 2.14b.) Thus, we shall ignore the vertical line that occurs in Figure 2.14b, since it falls outside the plasma. What happens if we plot the surface \(S(x,y; \omega = \omega_{\text{rf}}) = 0\) in ray phase space \((x,y,k_x,k_y)\)? (We will often use \(\mathbf{n}\) and \(\mathbf{k}\) interchangeably since they are proportional to each other.) Since \(S\) does not depend on \(k\), we expect the surface \(S(x,y; \omega = \omega_{\text{rf}}) = 0\) to be vertical when plotted in ray phase space. The plot of \(S(x,y; \omega = \omega_{\text{rf}}) = 0\) in the space \((x,y,n)\), where \(n\) is the magnitude of the vector \(\mathbf{n}\) projected onto the \(xy\)-plane, is shown in Figure 2.15. Recall, in our discussion of waves in a \textit{uniform} magnetized plasma, that we defined a resonance to occur at a location where the index of refraction went to infinity. In that sense, we can see from Figure 2.15, that this is indeed a resonance since the surface can be extended arbitrarily far in the \(+n\) or \(-n\) directions. However, for our purposes, it is more fundamental to realize that the ion hybrid resonance occurs when \(S = 0\) for any choice of \(\mathbf{n}\) or \(\mathbf{k}\).

Now, let us see what the dispersion matrix is, Equation (2.2.19), for a nonuniform, magnetized plasma (using the index of refraction notation defined in Equation (2.3.13))

\[
D(x, \mathbf{n}; \omega) = \begin{pmatrix}
S - (n_y^2 + n_z^2) & -iH + n_x n_y & n_x n_z \\
-iH + n_x n_y & S - (n_x^2 + n_z^2) & n_y n_z \\
n_x n_z & n_y n_z & P - (n_x^2 + n_y^2)
\end{pmatrix},
\]

\(2.4.12\)

\(^2\)Our choice of defining the plasma edge to be where the density is 1% of the maximum value is completely arbitrary and not motivated by any physical arguments. We note that there are plasma pedestals for which the density of the edge is greater than 1%; however, since we are not attempting to model the edge of the plasma \textit{at all}, we merely acknowledge this and continue with our choice for the edge.
where $S$, $H$, and $P$ are each functions of $(x, y; \omega)$. Let us note, that since no properties of the plasma depend on the $z$ coordinate, $k_z$ (hence, $n_z$) is simply a fixed parameter. That is, we can express the eikonal ansatz, Equation (2.4.1), as

$$E(x) = A(x, y) e^{i\theta(x, y)} e^{-ik_z z} e^{i\zeta(x, y)}.$$  \hspace{1cm} (2.4.13)

Because we are interested in behavior near the ion hybrid resonance we shall set $\omega = \omega_{rf}$, $n_z = 0$ (recall, this is when we encounter the ion hybrid resonance), and $n_x = n \cos \theta$ and $n_y = n \sin \theta$. 

FIG. 2.15: A plot of the ion hybrid resonance in ray phase space when $\omega = \omega_{rf}$.
Then, Equation (2.4.12) becomes

\[
D(x, y, n, \theta; \omega_r) = \begin{pmatrix}
S - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta & 0 \\
0 & 0 & 0 \\
iH + n^2 \cos \theta \sin \theta & S - n^2 \cos^2 \theta & 0
\end{pmatrix},
\] (2.4.14)

where we have dropped the explicit arguments of the functions \(S, H,\) and \(P.\)

From Section 2.3.2, we already know that the two modes which propagate perpendicular to the magnetic field, which is the case if \(n_z = 0,\) have the dispersion relations

\[
D_{1,2} = n^2 S - RL = 0, \tag{2.4.15}
\]

\[
D_3 = n^2 - P = 0. \tag{2.4.16}
\]

Let us look at plots of the dispersion relations in ray phase space, \((x, y, k).\)
FIG. 2.16: Plots of the dispersion surfaces (in ray phase space) for a nonuniform, magnetized plasma when \( k_z = 0 \). \( k_1 \) is the vacuum value, such that \( k_1 = \pm \omega / c \). (a) \( D_{1,2} = n^2 S - RL = 0 \) and the plasma boundary. We also include the ion hybrid surface, \( S = 0 \). (b) The surface \( D_3 = n^2 - P = 0 \) and the plasma boundary.
As you can see in Figure 2.16a, at large $k$ the dispersion surface $D_{1,2} = 0$ approaches the $S = 0$ surface, as expected. Additionally, note that beyond the boundary of the plasma, where the density is effectively zero, the magnitude of the wave vector approaches the $k_1 = \pm \omega/c$ plane, the free space value. Looking at Figure 2.16b, which shows the dispersion surface for the $n^2 = P$ mode, the transverse electromagnetic wave previously encountered in our discussion of a nonuniform unmagnetized plasma, we see that, at the selected frequency, $\omega = \omega_f$, this mode only exists outside the plasma and quickly goes over to the free space, linearly polarized, electromagnetic mode.

Let us return to the plot in Figure 2.16a, and, in particular, consider the ion-hybrid surface ($S = 0$), and the cutoff for this mode. Recall, that a cutoff occurs whenever the index of refraction goes to zero. In the case of a fixed frequency, that is equivalent to the condition of the wave vector going to zero. Setting the index of refraction equal to zero in Equation (2.4.15), we see that the cutoff occurs when $RL = 0$. Since $R$ and $L$ are both functions of $x$ and $y$, the condition $RL = 0$, will, generally, produce a one-dimensional curve in the two-dimensional $xy$-space. The cutoff is plotted in Figure 2.17, along with the $S = 0$ curve and the plasma boundary. We notice that the cutoff, the $RL = 0$ curve, extends beyond the boundary of the plasma, which means that this mode is not confined to the plasma. Additionally, we can see that the left part of the ion hybrid resonance, the $S = 0$ curve, is very close to the plasma edge. Recall, that we make no claims about the veracity of our model at or near the plasma boundary. Therefore, we do not want the ion hybrid resonance to be this close to the boundary. Furthermore, we require that the mode be entirely confined to the interior of the
FIG. 2.17: Plots of the ion hybrid resonance curve, the cutoff for the \( D_{1,2} = 0 \) mode, and the plasma boundary. Note that the cutoff extends beyond the plasma boundary and that the ion hybrid resonance curve is very close to the boundary.

plasma. We shall see that this can be achieved if we consider a small, but finite value of \( k_z \).

In the case of a non-zero, but fixed, value of \( k_z \), \((n_z \neq 0)\), Equation (2.4.12) becomes

\[
D(x, y, n, \theta; \omega_{\text{ref}}, n_z) = \begin{pmatrix}
S - n_z^2 - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta & (n \cos \theta)n_z \\
iH + n^2 \cos \theta \sin \theta & S - n_z^2 - n^2 \cos^2 \theta & (n \sin \theta)n_z \\
(n \cos \theta)n_z & (n \sin \theta)n_z & P - n^2
\end{pmatrix}. \tag{2.4.17}
\]

The determinant of the above matrix is

\[
\det D = H^2(n^2 - P) + [n^2 - (S - n_z^2)][n^2S - P(S - n_z^2)], \tag{2.4.18}
\]
where $n^2 = n_x^2 + n_y^2$. We note that this determinant is not easily factorable into a product of the eigenvalues of the dispersion matrix in Equation (2.4.17). However, since the determinant is the product of the eigenvalues, and we are interested in knowing where the eigenvalues go to zero, we can plot the zero surface of the determinant in $(x,y,k)$-space, which will show us where one or more of the eigenvalues goes to zero. Hence, the determinant contains all of the information we need, and can be used as the dispersion function. Three different views of the surface $\det D = D(x,y,k;n_z) = 0$ (for a particular choice of $n_z$) are shown in Figure 2.18.
FIG. 2.18: Plots of the dispersion surface \( \det D = D(x,y,k;z) = 0 \), along with the plasma boundary (in green). (a) Note that one branch of the dispersion surface is outside of the plasma. (b) Looking down the \( k \)-axis in the \( -k \) direction. Notice the white area on the right between the circular part of the dispersion surface and the cylindrical part. This is an avoided crossing. (c) The dispersion surface with the \( x > 0, y < 0 \) quadrant removed to allow us to view the interior of the structure. Notice that the hemispherical ellipsoid is hollow and only connects to the other part of the surface at one point. Furthermore, we should note that this is in fact one surface, which connects at larger values of \( k \).

Now, we want to show that, throughout the region of interest in ray phase space (away from the plasma boundary and for not exceedingly large values of \( k(n) \)) we can ‘factor’ the dispersion surface into two parts, one which involves only the electrons and the other which involves only the ions. Therefore, let us take a closer look at the determinant of the matrix \( D \),
Equation (2.4.18), and the Stix functions $S$, $H$, and $P$. First, it is fairly straightforward to see that the terms in the definition of the function $S$, Equation (2.2.15), are proportional to the mass of the plasma particle species, $m_s$. Furthermore, the terms in $H$, Equation (2.2.16), are proportional to the mass of the particle species squared. Thus, since the mass of the electrons is much smaller than the mass of the deuterium and tritium ions, we can essentially ignore the contribution of the electrons to the functions $S$ and $H$. Alternatively, we see that the terms in the function $P$, Equation (2.2.17), are proportional to $1/m_s$. Hence, as mentioned earlier, we can ignore the ion contribution to $P$ and include only the electron term. Plots of the three Stix functions are shown in Figure 2.19. As can be seen from Figures 2.19a to 2.19c the magnitude of the function $P$ is much greater than the magnitude of $S$ or $H$ near the center of the plasma, (which is the region of interest). Using this fact, let us reexamine the determinant of the dispersion matrix $D$.

First, defining the following quantity,

$$\tilde{S}(x,y) = S - n_z^2,$$

(2.4.19)

Equation (2.4.18) becomes

$$D(x, y, n) = H^2(n^2 - P) + (n^2 - \tilde{S})(n^2 S - P\tilde{S})$$

$$= H^2(n^2 - P) + n^4 S - n^2 P\tilde{S} - n^2 \tilde{S}S + P\tilde{S}^2.$$  

(2.4.20)

We can change $S \rightarrow \tilde{S}$ in the fourth term of the second line of (2.4.20) without significantly
FIG. 2.19: Plots of the Stix functions $S$, $H$, and $P$ for the case of 50/50 DT, cold, nonuniform, magnetized plasma. (a) The function $S$ ranges from about -5 to 5. (b) The function $H$ ranges from approximately -2 to 13. (c) The function $P$ ranges from $-4 \times 10^4$ to 1.

changing the function $D$ or, more importantly, the dispersion surface defined by $D = 0$. Arranging Equation (2.4.20) by powers of $n$, we get

$$D(x, y, n) = n^4 S + n^2 (H^2 - P \tilde{S} - \tilde{S}^2) + P \tilde{S}^2 - pH^2$$

$$= n^4 S - n^2 (\tilde{S}^2 - H^2 + P \tilde{S}) + P(\tilde{S}^2 - H^2)$$

$$= n^4 S - n^2 (\tilde{R} \tilde{L} + P \tilde{S}) + P \tilde{R} \tilde{L},$$

where we have used the fact that $\tilde{S}^2 - H^2 = \tilde{R} \tilde{L}$, with $\tilde{R} = R - n_c^2$ and $\tilde{L} = L - n_c^2$. Recall that, near the center of the plasma, the function $P$ dominates over the functions $S(\tilde{S})$ and $H$, (and,
hence, also over the functions $\tilde{R}$ and $\tilde{L}$). Therefore, to a good approximation, $\tilde{R}\tilde{L} + P\tilde{S} \approx P\tilde{S}$ and

$$D(x,y,n) = n^4 S - n^2 P\tilde{S} + P\tilde{R}\tilde{L}. \quad (2.4.22)$$

Since we are interested in the case when $D(x,y,n) = 0$, let us set Equation (2.4.22) equal to zero and solve for $n^2$:

$$n^2 = \frac{P\tilde{S} \pm \sqrt{P^2\tilde{S}^2 - 4PS\tilde{R}\tilde{L}}}{2S}. \quad (2.4.23)$$

If we factor out a $P^2$ from under the radical, we get

$$n^2 = \frac{P\tilde{S} \pm P\sqrt{\tilde{S}^2 - 4S\tilde{R}\tilde{L}/P}}{2S}. \quad (2.4.24)$$

Once again, since the magnitude of $P$ is much greater than the other Stix functions, we see that $4S\tilde{R}\tilde{L}/P \ll 1$. Letting $x = 4S\tilde{R}\tilde{L}/P$ and Taylor expanding Equation (2.4.24) about $x = 0$ to first order gives

$$n^2 = \frac{P\tilde{S} \pm P(\tilde{S} - 2S\tilde{R}\tilde{L}/P\tilde{S})}{2S}. \quad (2.4.25)$$

For the ‘+’ root we have

$$n^2 = \frac{2P\tilde{S} - 2S\tilde{R}\tilde{L}/\tilde{S}}{2S}$$

$$n^2 S = P\tilde{S} - \frac{S\tilde{R}\tilde{L}}{\tilde{S}} \quad (2.4.26)$$

$$n^2 \tilde{S}\tilde{S} = P\tilde{S}^2 - S\tilde{R}\tilde{L}.$$

---

At this point, one might ask, what happens when $\tilde{S} = 0$? Then, in that case, $P\tilde{S} = 0$ and it is no longer true that $P\tilde{S} \gg \tilde{R}\tilde{L}$. However, we are not simply interested in the function $D(x,y,n)$, but rather, in the surface defined by the condition $D(x,y,n) = 0$. It turns out that this surface is not significantly altered by the approximation $\tilde{R}\tilde{L} + P\tilde{S} \approx P\tilde{S}$. 
Using the fact that the magnitude of $P$ is much greater than the other Stix functions, we can drop the $S\tilde{R}\tilde{L}$ term

$$n^2S\tilde{S} = P\tilde{S}^2$$

$$n^2S = P\tilde{S}.$$  \hspace{1cm} (2.4.27)

Recall, $\tilde{S} = S - n_z^2$, so

$$n^2S = P\tilde{S} = P(S - n_z^2) = PS - n_z^2P.$$  \hspace{1cm} (2.4.28)

Since the magnitude of the function $P$ is much greater than the magnitude of $S$, (near the center of the plasma), the influence of $S$ will not be important unless $n$ is very large.\(^4\) Therefore, if we ‘ignore’ $S$, that is, restrict our attention to large but not infinite values of $n$, we can write Equation (2.4.28) as

$$n^2 = (1 - n_z^2)P \approx -P,$$  \hspace{1cm} (2.4.29)

where we have used that fact that $n_z > 1$. Thus, we have

$$n^2 + P = 0.$$  \hspace{1cm} (2.4.30)

A plot of the surface defined by Equation (2.4.30) (along with the plasma boundary) is shown in Figure 2.20. Notice that this is the same surface which appears in Figure 2.18, also outside the plasma boundary. Since this surface is entirely determined by the Stix function $P$, which is, in turn, almost entirely determined by the electron contribution, we conclude that this ‘mode’

\(^4\)In fact, as can be seen from Equation (2.4.27), the surface defined by the condition $n^2S - P\tilde{S} = 0$, must asymptote to the surface defined by $S(x,y) = 0$ as $n$ approaches infinity. However, as we mentioned earlier and as shown in Figure 2.17, the $S = 0$ surface lies very close to the edge of the plasma and we do not purport to describe anything that happens close to the plasma boundary.
FIG. 2.20: A plot of the surface defined by the condition \( n^2 + P = 0 \) in \( x, y, k \)-space. Note that, over this range of \( k \) the surface is entirely outside the plasma boundary.

will involve only the electrons. Although, recall, we will not draw any conclusions from this since our model is not valid near the plasma boundary. Now, let us look at the negative root of Equation (2.4.25).

Taking the ‘−’ root of (2.4.25) gives us

\[
\begin{align*}
n^2 &= \frac{P\bar{S} - P\bar{S} + 2S\bar{R}L/\bar{S}}{2s} \\
    &= \frac{\bar{R}L}{\bar{S}}
\end{align*}
\]

or

\[
\begin{align*}
n^2\bar{S} - \bar{R}L &= 0.
\end{align*}
\]

(2.4.31)  

(2.4.32)
Note, that this is of the *same* form as Equation (2.4.15), with the substitutions $S \rightarrow \tilde{S}$, $R \rightarrow \tilde{R}$, and $L \rightarrow \tilde{L}$. The plot of the surface defined by Equation (2.4.32) is shown in Figure 2.21. As can be seen in the figure, the surface $n^2 \tilde{S} - \tilde{R}\tilde{L} = 0$ is the same surface that we first encountered in Figure 2.18. Additionally, since the functions $\tilde{S}$, $\tilde{R}$, and $\tilde{L}$ are determined almost entirely by the properties of the ions (for this particular frequency), the surface $n^2 \tilde{S} - \tilde{R}\tilde{L} = 0$ involves only the plasma ions and *not* the electrons. Hence, we have succeeded in showing that we can ‘factor’ our initial surface $\det D = 0$ into two surfaces, one involving only the electrons and the other involving only the ions.

Finally, let us plot both of the surfaces, $n^2 + P = 0$ and $n^2 \tilde{S} - \tilde{R}\tilde{L} = 0$, together and shown.
next to the ‘original’ surface $\det D = 0$. 
\( (n^2 \mathcal{S} - \mathcal{R}\mathcal{L})(n^2 + P) = 0 \) and \( \det D = 0 \). Over this region of phase space they are the same surface. (a) The surface defined by the condition \( (n^2 \mathcal{S} - \mathcal{R}\mathcal{L})(n^2 + P) = 0 \), along with the plasma boundary. (b) The surface defined by the condition \( \det D = 0 \) and the plasma boundary.

**FIG. 2.22:** Comparison of the surfaces \( (n^2 \mathcal{S} - \mathcal{R}\mathcal{L})(n^2 + P) = 0 \) and \( \det D = 0 \).
As you can see from Figure 2.22, over this region of phase-space, the zero surfaces look very similar, and it can be shown that they are in fact almost identical. But how does this relate back to our original problem of identifying mode conversion regions in the plasma?

Well, let us take another look at the matrix from Equation (2.4.17). If we use the definition of \( \tilde{S} \) given by (2.4.19), the dispersion matrix becomes

\[
D(x, y, n, \theta; n_z) = 
\begin{pmatrix}
\tilde{S} - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta & (n \cos \theta)n_z \\
iH + n^2 \cos \theta \sin \theta & \tilde{S} - n^2 \cos^2 \theta & (n \sin \theta)n_z \\
(n \cos \theta)n_z & (n \sin \theta)n_z & P - n^2
\end{pmatrix}.
\] (2.4.33)

Since we have shown that it is possible to factor the dispersion surface \( \text{det}D = 0 \) into two parts, this implies that we can also separate the null space, the space spanned by the null eigenvectors of the \( 3 \times 3 \) dispersion matrix \( D \), into two spaces. Notice what happens if we act with the projection operator

\[
\hat{P} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (2.4.34)
on \( D \):

\[
D' = \hat{P}D\hat{P} = 
\begin{pmatrix}
\tilde{S} - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta & 0 \\
iH + n^2 \cos \theta \sin \theta & \tilde{S} - n^2 \cos^2 \theta & 0 \\
0 & 0 & 0
\end{pmatrix}.
\] (2.4.35)
Note that the matrix $D'$ is in block diagonal form, with the lower-right block equal to zero and the upper-left block equal to the $2 \times 2$ matrix

$$
\tilde{D} = \begin{pmatrix}
\tilde{S} - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta \\
iH + n^2 \cos \theta \sin \theta & \tilde{S} - n^2 \cos^2 \theta
\end{pmatrix}.
$$

Taking the determinant of $\tilde{D}$ and setting it equal to zero, we get

$$
\det(\tilde{D}) = n^2 \tilde{S} - (\tilde{S}^2 - H^2) = n^2 \tilde{S} - \tilde{R}\tilde{L} = 0,
$$

which is identical to Equation (2.4.32). Hence, we have effectively reduced the dimensionality of the problem by preforming a Galerkin projection onto the space spanned by the null eigenvectors of the dispersion matrix $\tilde{D}$. Thus, we find that the dispersion matrix $\tilde{D}(x,y,n_x,n_y)$ is a matrix-valued function on the four-dimensional ray phase space $(x,y,k_x,k_y)$, and the dispersion function $\tilde{D}(x,y,n_x,n_y) = \det(\tilde{D}) = 0$ defines, in general, a three-dimensional surface embedded in the four-dimensional ray phase space.

Let us take one last look at the dispersion surface $\tilde{D} = n^2 \tilde{S} - \tilde{R}\tilde{L} = 0$ shown in Figures 2.23a and 2.23b in $(x,y,k)$-space, (where $k = \sqrt{k_x^2 + k_y^2}$); and when $n = 0$ in Figure 2.23c. As we can see from Figures 2.23a and 2.23b, we have, (again), the dispersion surface inside the plasma that we first encountered when we plotted the zero surface of the determinant of the $3 \times 3$ dispersion matrix. We should note, as can be seen in the cutaway plot in Figure 2.18c, that the enclosed, hemispherical shape seen on the right side of Figure 2.23b is hollow when
plotted in \((x, y, k)\)-space.\(^5\) Finally, if we compare Figure 2.23b with Figure 2.17, we see that by choosing \(n_z\) to be nonzero, the cutoff for this mode is now \emph{inside} the plasma, as desired, and the new ion-hybrid surface, \(\tilde{S} = 0\), is further from the edge of the plasma. Before moving on to a discussion of the role played by the dispersion surfaces in solving our wave equation, let us look a bit closer at the \(2 \times 2\) dispersion matrix, \(\tilde{D}\), defined in Equation (2.4.36).

\(^5\)Since the dispersion function \(D(x, y, n_x, n_y) = 0\) is a function of \(x, y, n_x,\) and \(n_y\) we could (more naturally) plot the dispersion surface in \((x, y, k_x; k_y)\) space, that is, treat \(n_y\) as a parameter and plot \(D(x, y, n_x; n_y) = 0\) for each value of \(n_y\). In that case, there are regions of the surface that are indeed \emph{three}-dimensional, i.e., they are solid. In fact, in \((x, y, k_x; k_y)\), both the hemispherical bulge and the hemispherical enclosed shape are solid; however, the elongated cylinder remains hollow.
FIG. 2.23: Plots of the dispersion surface $\tilde{D}(x,y,n_x,n_y) = 0$ from the $2 \times 2$ dispersion matrix $\tilde{D}$, defined in Equation (2.4.36), for a nonuniform magnetized plasma. (a) The dispersion surface in $(x,y,k)$-space, along with the plasma boundary. (b) A top down view of the dispersion surface in $(x,y,k)$-space to see the avoided crossing. (c) The dispersion surface when $n = 0$ in $(x,y)$-space. We see that the cutoff, the location where $\tilde{R}\tilde{L} = 0$, is entirely within the plasma, and that the ion-hybrid resonance, defined by the condition $\tilde{S} = 0$, is not as close to the plasma boundary as in Figure 2.17.
2.5 The $2 \times 2$ Dispersion Matrix

Now that we have reduced the dimension of our problem, let us recall what it is we are trying to solve. We want to solve the multicomponent wave equation

$$\hat{D}(x, -i\nabla; \omega, k_z) \cdot \mathbf{E}(x) = 0,$$  
(2.5.1)

where $\hat{D}$ is a $2 \times 2$ operator valued matrix that depends on the parameters $\omega$ and $k_z$, and $\mathbf{E}(x)$, the electric field, is

$$\mathbf{E}(x) = \begin{pmatrix} E_x(x,y) \\ E_y(x,y) \end{pmatrix}.$$  
(2.5.2)

Utilizing WKB methods, Equation (2.5.1) reduces to

$$D(x, y, k_x, k_y; \omega, k_z) \cdot \mathbf{E}(x) = 0,$$  
(2.5.3)

with $D$ being the $2 \times 2$ symbol of the operator $\hat{D}$, such that

$$D(x, y, k_x, k_y; \omega, k_z) = \begin{pmatrix} \omega^2 - c^2(k_y^2 + k_z^2) & -i\omega^2 H + c^2 k_x k_y \\ i\omega^2 H + c^2 k_x k_y & \omega^2 - c^2(k_x^2 + k_z^2) \end{pmatrix},$$  
(2.5.4)

6Remember, we are assuming that neither the plasma nor the background magnetic field depend on either time or the $z$-spatial coordinate. Hence, we have already Fourier transformed in both time and the $z$-direction, to get the parameters $\omega$ and $k_z$, respectively.
where \( S \) and \( H \) both depend upon \( x \) and \( y \), and we express \( \mathbf{E}(x) \) as

\[
\mathbf{E}(x) = E(x)\hat{\mathbf{e}}(x),
\]

that is, as a slowly varying amplitude times a slowly varying polarization.

Using the definition of the index of refraction vector, (2.3.13), and the function \( \tilde{S} \), (2.4.19), we can express \( D \) as

\[
D(x,y,n_x,n_y) = \begin{pmatrix}
\tilde{S} - n_y^2 & -iH + n_x n_y \\
-iH + n_x n_y & \tilde{S} - n_x^2 \\
\end{pmatrix},
\]

(2.5.6)

where the \( \omega \) and \( k_z \) dependencies are in the definitions of \( n \) and \( \tilde{S} \), respectively. Once again, recall, a condition for a solution of Equation (2.5.3) to exist is that the determinant of the matrix \( D \) must equal zero. Taking the determinant of the \( 2 \times 2 \) matrix \( D \), as written in (2.5.6), and setting it equal to zero gives us

\[
D(x,y,n_x,n_y) = \det D = (n_x^2 + n_y^2) \tilde{S} - (\tilde{S}^2 - H^2) = 0
\]

(2.5.7)

\[
D(x,y,n_x,n_y) = (n_x^2 + n_y^2) \tilde{S} - \tilde{R}\tilde{L} = 0,
\]

(2.5.8)

with \( \tilde{R} \) and \( \tilde{L} \) as defined in Equation (2.4.21). Since the dispersion function \( D \) is a function on the four-dimensional ray phase space \((x,y,k_x,k_y)\), the condition \( D(x,y,n_x,n_y) = 0 \) will, in general, define a three-dimensional surface embedded in the four-dimensional ray phase space. If we set \( n_y = 0 \), we can plot the surface \( D(x,y,n_x) = 0 \) in \((x,y,k_x)\)-space, Figure 2.24. First,
notice that this is the same surface we originally encountered in Figure 2.18. Additionally, we should point out, that the two pieces of the dispersion surface, the squashed cylinder that bulges out around $k_x = 0$, and the inner hemispherical portion, intersect at only two points in the four-dimensional ray phase space. These intersection points occur at the location $(x, y, 0, 0)$, such that $\tilde{R}(x, y) = \tilde{L}(x, y) = \tilde{S}(x, y) = 0$. For our present configuration, there are two such points, as can be seen in Figure 2.23c.

![FIG. 2.24: Plots of the dispersion surface $D(x, y, n_x) = n_x^2 \tilde{S} - \tilde{R}L = 0$ when $n_y = 0$. (a) The surface plotted in $(x, y, k_x)$-space. (b) The dispersion surface with the octant $(x > 0, y < 0, k_x > 0)$ removed to show the interior structure.](image)

Now, let us plot the dispersion surface $D(x, y, n_x) = 0$, along with the surfaces $\tilde{S}(x, y) = 0$, $n_x^2 = \tilde{R}(x, y)$, and $n_x^2 = \tilde{L}(x, y)$, all with $n_y = 0$, in $(x, y, k_x)$-space.
FIG. 2.25: Plots of the surfaces $n^2 S - R L = 0$ (orange), $S = 0$ (blue), $n^2_x = R$ (green), and $n^2_x = L$ (red). Notice how the surface $D = 0$ appears to be a ‘combination’ of the other three surfaces.

First, note that the dispersion surface $D = n^2_x S - R L = 0$, (the orange surface in Figure 2.25) is a conglomeration\(^7\) of the surfaces $S = 0$, (the blue surface in Figure 2.25), $n^2_x - R = 0$, (the green surface in Figure 2.25), and $n^2_x - L = 0$, (the red surface in Figure 2.25). That is, the dispersion surface $D = 0$, (where, recall, $D$ is the determinant of the $2 \times 2$ dispersion matrix

\(^7\) We are using conglomeration in the colloquial, non-technical sense, to mean a loose collection of different things.
from (2.5.6), asymptotes to the surfaces defined by the relations $\tilde{S} = 0$, $n_x^2 = \tilde{R}$, and $n_x^2 = \tilde{L}$, in different regions of $x,y,k_x$-space. For instance, as can be seen in Figure 2.25, in the limit where the magnitude of $k_x$ is large, the orange dispersion surface merges with the blue surface defined by $\tilde{S} = 0$. Additionally, near $k_x = 0$, we see in Figure 2.25 that the orange dispersion surface goes over to both the green surface defined by $n_x^2 - \tilde{R} = 0$ and the red surface defined by $n_x^2 - \tilde{L} = 0$, depending on where one is looking in the $x,y$-plane. Figure 2.26 shows a slice of the surfaces in Figure 2.25 at $y = 0$.

---

8In fact, as we can see in Figure 2.25, the orange $D = 0$ surface goes over to the green $n_x^2 = \tilde{R}$ surface and the red $n_x^2 = \tilde{L}$ surface in two regions of $x,y,k_x$-space. See also Figure 2.26
As you may have already realized, the surfaces $\tilde{S} = 0$, $n_x^2 = \tilde{R}$, and $n_x^2 = \tilde{L}$ are not arbitrary. They are, in fact, the dispersion surfaces of different wave modes that we have previously encountered. The surface $\tilde{S} = 0$ corresponds to the ion-hybrid resonance first encountered in Section 2.3.2 and seen again in 2.4.2. The $n_x^2 = \tilde{R}$ surface is the dispersion surface corresponding to the $n^2 = R$ mode introduced in Section 2.3.2, Equation (2.3.23). The wave mode corresponding to $n^2 = \tilde{R}$ is the low-temperature, low-frequency limit of the fast magnetosonic
mode. Lastly, the surface defined by \( n_r^2 = \tilde{L} \) corresponds to the \( n^2 = L \) mode also introduced in Section 2.3.2, Equation (2.3.24), and is the low-temperature, low-frequency limit of the transverse or shear Alfvén mode.

We can explain the shape of the dispersion surface \( D = 0 \), shown in Figures 2.24, 2.25, and 2.26, as being due to avoided crossings, which are the result of coupling between the \( n^2 = \tilde{R}(\tilde{L}) \) mode and the ion-hybrid resonance \( \bar{S} = 0 \). To illustrate this idea, let us consider a simple example.

Let \( D \) be a 2 \( \times \) 2 dispersion matrix, (i.e., \( D \) is the Weyl symbol of an operator), such that

\[
D(x, k; \eta) = \begin{pmatrix} x + k & \eta \\ \eta^* & x - k \end{pmatrix},
\]

where \( \eta \) is, in general, a complex constant, which is called the coupling constant. For this example, we are working in a two-dimensional ray phase space, \((x, k)\). First, let us compute the determinant of the matrix \( D \):

\[
D(x, k; \eta) \equiv \det D = x^2 - k^2 - |\eta|^2.
\]

Since the matrix \( D \) appears in an equation such as (2.5.3), we want to know where in our two-dimensional ray phase space the determinant of \( D \) is zero, which is a condition on the existence of solutions to equations of the form of (2.5.3). Therefore, let us plot the dispersion surface defined by the condition \( D(x, k; \eta) = x^2 - k^2 - |\eta|^2 = 0 \), for a nonzero value of \( \eta \).

\footnote{Since we are in a two-dimensional ray space, the 'surface' defined by the condition \( D(x, k; \eta) = 0 \) will, in}
dispersion curve $D = 0$ is shown in Figure 2.27, along with the *uncoupled* dispersion curves.

**FIG. 2.27:** A plot of the dispersion curve $x^2 - k^2 - |\eta|^2 = 0$, (for a particular value of $\eta$). Also shown are the uncoupled dispersion curves $x + k = 0$ and $x - k = 0$. The conversion point, i.e., the point where the uncoupled dispersion curves intersect, is labeled, as well as the avoided crossings.

The first thing we notice about the dispersion curve $D = 0$ is that there are two distinct branches that do not intersect. However, if we look at the dispersion matrix $D$ in (2.5.9) and set the coupling constant, $\eta$, to zero, we immediately find that the eigenvalues are $x + k$ and $x - k$. Then, plotting the curves $x + k = 0$ and $x - k = 0$ gives us the green and orange dashed fact, be one-dimensional, i.e., a curve.
lines seen in Figure 2.27. These curves are called the *uncoupled* dispersion curves, since they arise when the coupling constant is zero, i.e. \( \eta = 0 \). Their intersection is called a *conversion point*, since a wave of one type, (say \( x + k \)), traveling towards the conversion point, can excite a wave of the other mode, (\( x - k \) in this case), when it passes through the conversion point.

Therefore, we conclude that the avoided crossing behavior, (i.e., the fact that the two branches do not intersect, but rather approach one another and then turn away, as seen in Figure 2.27), is due to the nonzero coupling between the \( x + k \) mode and the \( x - k \) mode. Additionally, it is straightforward to see, that as the coupling constant decreases the two branches of the dispersion surface \( D = 0 \) move closer together and the avoided crossing becomes smaller, until finally, when \( \eta = 0 \), the branches of the dispersion surface become the uncoupled dispersion surfaces.

Applying the above analysis to our dispersion surface, \( n^2 \tilde{S} - \tilde{R} \tilde{L} = 0 \), as seen in Figure 2.25 and, in particular, Figure 2.26, we draw the following conclusions. First, as labeled in Figure 2.28, there is an avoided crossing due to the coupling between the \( n^2 = \tilde{R} \) mode, (which, recall, is the fast magnetosonic mode), and the ion-hybrid resonance, \( \tilde{S} = 0 \). Additionally, there is a small avoided crossing due to the coupling between the \( n^2 = \tilde{L} \) mode (the Alfvén mode) and the ion-hybrid resonance. Since this avoided crossing is much smaller than the avoided crossing between the fast magnetosonic mode and the ion-hybrid resonance, we conclude that the coupling strength between the Alfvén mode and the ion-hybrid resonance is weaker than the coupling between the fast magnetosonic wave and the ion-hybrid resonance. Lastly, since the dispersion surface \( n^2 \tilde{S} - \tilde{R} \tilde{L} = 0 \) (the solid blue curve in Figure 2.28) passes directly through
the intersection of the $n_x^2 = \tilde{R}$ dispersion and the $n_x^2 = \tilde{L}$ dispersion curve (the green and red dashed lines, respectively), there must be no coupling between the $n^2 = \tilde{R}$ mode and the $n^2 = \tilde{L}$ mode, as expected.

However, the question that remains, and the main focus of this thesis, is how do we identify the uncoupled dispersion relations and the conversion points? That is, can we cast the 2 $\times$ 2 dispersion matrix from Equation (2.5.8), into a form that is reminiscent of (2.5.9), where
the diagonals are the uncoupled dispersion functions and the off-diagonals are the coupling constants. On a hunch, let us transform the $2 \times 2$ dispersion matrix $D$ from (2.5.8) via a congruence transformation, such that,

$$D' = Q^\dagger DQ,$$

(2.5.11)

where $Q$ is the matrix

$$Q = \begin{pmatrix} -\frac{1}{2^{1/4}} & -i2^{1/4} \\ -i & 0 \\ -\frac{1}{2^{1/4}} \end{pmatrix}.$$ 

(2.5.12)

Note, that since the determinant of the congruence transformation $Q$ is 1, we have

$$\det D' = \det D.$$ 

(2.5.13)

Throughout this thesis, we shall restrict our attention to congruence transformations with unit determinant. However, as we shall discuss in further detail in later chapters, the eigenvalues of $D$ are not invariant under congruence transformation. Now, the transformed matrix $D'$ is

$$D' = \begin{pmatrix} \frac{1}{\sqrt{2}}(2\hat{R} - n_x^2 - n_y^2) & i(\hat{R} - n_x^2) + n_x n_y \\ -i(\hat{R} - n_x^2) + n_x n_y & \sqrt{2}(\hat{S} - n_y^2) \end{pmatrix},$$

(2.5.14)

10In general, the coupling terms will not be global constants. However, as we mention in a moment and explain in the following chapter, we want the off-diagonals to be constant following a ray.

11The reason we are considering the class of congruence transformations rather than another class, such as similarity transformations, will be explained in the next chapter.
where we have used the fact that $\tilde{R} = \tilde{S} + H$. If we set $n_y = 0$ we get

$$D' = \begin{pmatrix} \frac{1}{\sqrt{2}}(2\tilde{R} - n_x^2) & i\tilde{R} \\ -i\tilde{R} & \sqrt{2}\tilde{S} \end{pmatrix}. \quad (2.5.15)$$

Setting the diagonals of $D'$ equal to zero gives us

$$D_{11} = n_x^2 - 2\tilde{R} = 0 \quad (2.5.16)$$

$$D_{22} = \tilde{S} = 0. \quad (2.5.17)$$

The following three figures show the plots of $D_{11} = 0$ and $D_{22} = 0$, along with the surface $\det D' = \det D = 0$, (where the equality follows from the fact that the determinant is invariant under our congruence transformation).
FIG. 2.29: Plots of the zero surfaces of the diagonals of the transformed matrix $D'$. (a) As you can see from this plot of $D_{11} = 0$ and $D_{22} = 0$, the two ‘uncoupled’ dispersion surfaces intersect. (b) Plots of $D_{11} = 0$, $D_{22} = 0$, and the original dispersion surface $D = 0$. Note that the avoided crossing is gone, as if the coupling were set to zero. (c) The $y = 0$ slice of Figure 2.29b. Again, note that the avoided crossing is gone and the surfaces $D_{11} = 0$ and $D_{22} = 0$ intersect.

As can be seen from the plots in Figure 2.29, the transformation $Q$ has succeeded in producing surfaces that look like the uncoupled dispersion surfaces we are after, from which we should be able to identify the mode conversion regions. However, this is only true for the case when $n_y = 0$. If we truly want the diagonals to be the uncoupled dispersion relations
and the off-diagonals to be the coupling ‘constants’, then we need the Poisson brackets of the diagonals with the off-diagonals to vanish, i.e., we require \( \{D_{11}, D_{12}\} = 0 \) and \( \{D_{22}, D_{12}\} = 0 \).

As can be seen from Equation (5.1.47), the transformed matrix \( D' \) does not satisfy these requirements. We need to follow a systematic approach to develop a transformation that will produce a transformed matrix, \( D' \), that does meet our requirements. We shall introduce this technique, known as the normal form transformation, in Chapter 4. However, before we get into that, let us talk, briefly, about ray tracing techniques. We have alluded to this topic, and it is time to introduce it formally.
In this chapter we are going to formalize the concept of a *dispersion function*, which is distinct from a *dispersion relation*. We shall see that both of these objects can be used to define the *dispersion surface*. Namely, the dispersion surface is defined *implicitly* as the roots of the dispersion function, or *explicitly* through the dispersion relation. The former is a *global* characterization of the surface, the latter is usually only local.

We will also introduce the concept of *eikonality* and restrict our attention to narrow-banded solutions (narrow frequency range) to wave equations, such as wave packets or modulated wave trains. We can then derive an evolution equation for the envelope, which leads to the notion of the *group velocity*. Additionally, we will see how we can use ray tracing techniques to reconstruct the wave field using the dispersion function.
3.1 Wave Equations

First, we will consider the example of a scalar wave equation as a means to introduce the concepts of the dispersion function and the dispersion relation. Suppose we have a linear wave equation in one spatial dimension:

\[ D(-i\partial_x, i\partial_t)\psi(x,t) = \sum_{m,n=0}^{N} d_{mn}(-i\partial_x)^m(i\partial_t)^n \psi(x,t) = 0. \]  

(3.1.1)

For our purposes, we shall assume \( N \) is finite. The expansion coefficients \( d_{mn} \) are constant and assumed to be real. This ensures that the operator \( D(-i\partial_x, i\partial_t) \) is self-adjoint with respect to the usual inner product

\[ \langle \phi | \psi \rangle \equiv \int dx \, \phi^*(x)\psi(x). \]  

(3.1.2)

3.1.1 Electromagnetic Waves in a Uniform, Unmagnetized Plasma

To illustrate these concepts, let us return to the wave equation for electromagnetic waves in a uniform, unmagnetized plasma (encountered in the previous chapter):

\[ D(-i\partial_x, i\partial_t) = \partial_t^2 - c^2 \partial_x^2 + \omega_p^2, \]  

(3.1.3)

with

\[ \omega_p^2 = 4\pi \sum_{s} \frac{n_s q_s^2}{m_s} \]  

(3.1.4)
being the plasma frequency, and \( s \) runs over all the species in the plasma. In this instance the coefficients \( a_{mn} \) are,

\[
d_{00} = \omega_p^2, \quad (3.1.5)
\]
\[
d_{20} = c_s^2, \quad (3.1.6)
\]
\[
d_{02} = -1, \quad (3.1.7)
\]

with all others are zero.

We are only interested in freely propagating solutions that are coherent, i.e., that have a well-defined decomposition into a slowly varying amplitude and a phase at each point \((x,t)\), so that we can write:

\[
\psi(x,t) = A(x,t)e^{i\vartheta(x,t)}, \quad (3.1.8)
\]

where the \( A(x,t) \) and \( \vartheta(x,t) \) are real valued functions called the amplitude (or envelope) and the phase, respectively. Thus, we can define a local wave number,

\[
k(x,t) \equiv \frac{\partial \vartheta}{\partial x}, \quad (3.1.9)
\]

and frequency,

\[
\omega(x,t) \equiv -\frac{\partial \vartheta}{\partial t}. \quad (3.1.10)
\]

For the case of the uniform, unmagnetized plasma, the operator \( \hat{D} \) is independent of \( x \) and \( t \). Hence, we can Fourier analyze in both \( x \) and \( t \) and look for special solutions in the form
of planes waves:

$$\psi(x,t) = A(x,t) \exp(i[kx - \omega t]).$$  \hfill (3.1.11)

Inserting the plane wave ansatz, Equation (3.1.11), into the wave Equation (3.1.1), we arrive at,

$$D(k, \omega) = \sum_{m,n=0}^{N} d_{mn} k^m \omega^n = 0. \hfill (3.1.12)$$

We identify \(D(k, \omega)\) as the dispersion function.

For real \(k\) and \(\omega\), the condition \(D(k, \omega) = 0\) implicitly defines a one-dimensional curve in the \((k, \omega)\) plane. This is called a dispersion curve, or in higher dimensions, a dispersion surface.

For most points on the curve we can locally find an explicit representation either in the form,

$$\omega = \Omega(k), \text{ such that } D[k, \Omega(k)] = 0,$$ \hfill (3.1.13)

or as,

$$k = \kappa(\omega), \text{ such that } D[\kappa(\omega), \omega] = 0.$$ \hfill (3.1.14)

These are the dispersion relations.

Returning to the example of electromagnetic waves in a uniform, unmagnetized plasma and inserting the plane wave ansatz into Equation (3.1.3), we find that the dispersion function is:

$$D(k, \omega) = -\omega^2 + c^2 k^2 + \omega_p^2. \hfill (3.1.15)$$

(This is the same function as Equation (2.3.6).) The condition \(D(k, \omega) = 0\) defines the disper-
sion curves implicitly in the \((k, \omega)\) plane, as seen in Figure 3.1. We can also find the dispersion curves, but we must keep in mind that they are \textit{local} relations. For \(\omega > 0\), we have,

\[
\omega = \Omega_+(k) = \sqrt{c^2k^2 + \omega_p^2}. \tag{3.1.16}
\]

For \(\omega < 0\),

\[
\omega = \Omega_-(k) = -\sqrt{c^2k^2 + \omega_p^2}. \tag{3.1.17}
\]

Instead of writing \(\omega\) as a function of \(k\), what happens if we solve for \(k\) as a function of \(\omega\)? We get,

\[
k = k_\pm(\omega) = \pm \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}. \tag{3.1.18}
\]
However, at \( \omega = \pm \omega_p \), the derivative of \( \kappa \) with respect to \( \omega \) goes to infinity. We see this explicitly if we attempt to project the dispersion curve onto the \( \omega \) axis. The projection is changing character from double-valued, (when \( \omega > \omega_p \)), to single-valued, (when \( \omega = \omega_p \)), to not having a real solution, (when \( \omega < \omega_p \)). This is an example of the fact that the dispersion relation, (3.1.18), is a local result.

### 3.1.2 The Eikonal Approximation

Let us now consider the free propagation of a wave packet. This means that \( k \) and \( \omega \) are roots of the dispersion function \( D(k, \omega) \). We assume the wave function \( \psi(x,t) \) has a carrier oscillation of the form,

\[
\exp[i(k_0x - \omega_0t)], \text{ with } D(k_0, \omega_0) = 0, \tag{3.1.19}
\]

but with a varying amplitude. The envelope \( A(x,t) \) is assumed to vary on length and time scales that are long compared with the carrier wavelength and period. Thus,

\[
k_0 \gg \frac{1}{A} \frac{\partial A}{\partial x} \tag{3.1.20}
\]

\[
\omega_0 \gg \frac{1}{A} \frac{\partial A}{\partial t}. \tag{3.1.21}
\]

Going forward, let us introduce a formal small parameter \( \varepsilon \) to keep track of the orders in the asymptotic series. We will place an \( \varepsilon \) in front of each derivative, so that when we take the limit \( \varepsilon \to 0 \) it will emphasize the most rapidly varying terms, i.e., ones with large derivatives.
Since the phase is assumed to vary rapidly with respect to the amplitude, we write,

$$\psi(x,t) = A(x,t) \exp \left[ \frac{i}{\varepsilon} (k_0 x - \omega_0 t) \right].$$  \hspace{1cm} (3.1.22)

Now, note what happens when we take the partial derivative of $\psi(x,t)$ with respect to $x$:

$$-i\varepsilon \partial_x \psi(x,t) = -i\varepsilon \partial_x A(x,t) \exp \left[ \frac{i}{\varepsilon} (k_0 x - \omega_0 t) \right]$$

$$= \exp \left[ \frac{i}{\varepsilon} (k_0 x - \omega_0 t) \right] (k_0 - i\varepsilon \partial_x) A(x,t),$$  \hspace{1cm} (3.1.23)

and with respect to $t$:

$$i\varepsilon \partial_t \psi(x,t) = i\varepsilon \partial_t A(x,t) \exp \left[ \frac{i}{\varepsilon} (k_0 x - \omega_0 t) \right]$$

$$= \exp \left[ \frac{i}{\varepsilon} (k_0 x - \omega_0 t) \right] (\omega_0 + i\varepsilon \partial_t) A(x,t).$$  \hspace{1cm} (3.1.24)

Thus, Equations (3.1.23) and (3.1.24) show us, that when the derivatives act on the full wave solution (including the carrier), Equation (3.1.22), we can move the carrier to the left of the derivatives if we make the following replacements:

$$-i\varepsilon \partial_x \rightarrow k_0 - i\varepsilon \partial_x$$  \hspace{1cm} (3.1.25)

$$i\varepsilon \partial_t \rightarrow \omega_0 + i\varepsilon \partial_t.$$  \hspace{1cm} (3.1.26)
Hence, we can write the wave Equation (3.1.1) in the form,

$$D(-i\varepsilon \partial_x, i\varepsilon \partial_t) \psi(x,t) = 0$$  \hspace{1cm} (3.1.27)

$$D(k_0 - i\varepsilon \partial_x, \omega_0 + i\varepsilon \partial_t) A(x,t) = 0,$$  \hspace{1cm} (3.1.28)

which is an equation for the envelope $A(x,t)$.

Now, we want to expand $D(k_0 - i\varepsilon \partial_x, \omega_0 + i\varepsilon \partial_t)$ as a power series in $\varepsilon$. Let us use the dispersion function $D(k, \omega)$ and say that we want to expand $D(k_0 + \varepsilon \kappa, \omega_0 + \varepsilon \Omega)$ where $\kappa$ and $\Omega$ are just real variables. Then, we have, to first order in $\varepsilon$,

$$D(k_0 + \varepsilon \kappa, \omega_0 + \varepsilon \Omega) \approx D(k_0, \omega_0) + \varepsilon \kappa \frac{\partial D(k_0, \omega_0)}{\partial k} + \varepsilon \Omega \frac{\partial D(k_0, \omega_0)}{\partial \omega} + O(\varepsilon^2).$$  \hspace{1cm} (3.1.29)

If we reintroduce the operators using:

$$\kappa \rightarrow -i\partial_x,$$  \hspace{1cm} (3.1.30)

$$\Omega \rightarrow i\partial_t,$$  \hspace{1cm} (3.1.31)

we get,

$$D(k_0 - i\varepsilon \partial_x, \omega_0 + i\varepsilon \partial_t) A(x,t) \approx [D(k_0, \omega_0) + i\varepsilon(D\omega \partial_t - D_k \partial_x)] A(x,t)$$

$$= 0,$$  \hspace{1cm} (3.1.32)
where,

\[ D_k = \left. \frac{\partial D}{\partial k} \right|_{(k_0, \omega_0)} \]  

(3.1.33)

\[ D_\omega = \left. \frac{\partial D}{\partial \omega} \right|_{(k_0, \omega_0)} \]  

(3.1.34)

Since \( D(k_0, \omega_0) = 0 \) by construction, we have, to order \( \varepsilon \):

\[ (D_\omega \partial_t - D_k \partial_x) A(x, t) \approx 0. \]  

(3.1.35)

Assuming \( D_\omega \neq 0 \), we can write Equation (3.1.35) as an advection equation:

\[ (\partial_t + v_g \partial_x) A(x, t) \approx 0, \]  

(3.1.36)

where the advection velocity is the group velocity:

\[ v_g \equiv -\frac{\partial D/\partial k}{\partial D/\partial \omega}, \]  

(3.1.37)

with all derivatives evaluated at \((k_0, \omega_0)\).

Let us verify that this expression is indeed equal to the usual expression for the group velocity:

\[ v_g = \frac{d\Omega}{dk}, \]  

(3.1.38)

where \( \Omega(k) \) is the dispersion relation. If we expand the dispersion function \( D(k, \omega) \) about the
Since \( D(k_0, \omega_0) = 0 \), we have, to linear order,

\[
(k - k_0)D_k + (\omega - \omega_0)D_\omega = 0.
\] (3.1.40)

Thus, if \( D_\omega \neq 0 \):

\[
\Omega(k) = \omega \approx -\frac{D_k}{D_\omega}(k - k_0) + \omega_0,
\] (3.1.41)

and

\[
\frac{d\Omega}{dk} = -\frac{D_k}{D_\omega}.
\] (3.1.42)

### 3.1.3 Electromagnetic Waves in a One-Dimensional Nonuniform Unmagnetized Plasma

Now we want to carry out this asymptotic expansion on the wave equation for the transverse electric field in a nonuniform unmagnetized plasma. We shall use the wave equation for the electric field we derived using the cold plasma model:

\[
c^2 \nabla^2 \mathbf{E} - c^2 \nabla (\nabla \cdot \mathbf{E}) - \frac{\partial^2 \mathbf{E}}{\partial t^2} = \omega_p^2 \mathbf{E}.
\] (3.1.43)
Recall, that

$$\omega_p^2 = 4\pi \sum_s \frac{n_s g_s^2}{m_s}, \quad (3.1.44)$$

which depends on \(x\) in a non-uniform plasma through the number density \(n_s(x)\). We shall assume appropriate initial and boundary conditions for a unique solution to exist. Furthermore, suppose the background plasma is a smooth function of \(x\) and in the \(\hat{z}\) direction:

$$\mathbf{E}(x,t) = E(x,t)\hat{z}. \quad (3.1.45)$$

It is worthwhile to note that, for this choice of \(\mathbf{E}\), \(\nabla \cdot \mathbf{E} = 0\). This means that the second term on the left hand side of Equation (3.1.43) drops out. Inserting our small parameter \(\varepsilon\), we get,

$$\left[ \varepsilon^2 c^2 \frac{\partial^2}{\partial x^2} - \varepsilon^2 \frac{\partial^2}{\partial t^2} - \omega_p^2(x) \right] E(x,t) = 0. \quad (3.1.46)$$

Once again, we insert our eikonal ansatz:

$$E(x,t) = A(x,t) \exp \left\{ \frac{i}{\varepsilon} \theta(x) - \omega t \right\}. \quad (3.1.47)$$
Now, let us take the first and second partial derivatives with respect to $x$ and $t$:

\[
\varepsilon \frac{\partial E}{\partial x} = \left( \frac{i\theta_x + \varepsilon \frac{A_x}{A}}{A_x} \right) E(x, t) \quad (3.1.48)
\]

\[
\varepsilon \frac{\partial E}{\partial t} = \left( -i\omega + \varepsilon \frac{A_t}{A} \right) E(x, t) \quad (3.1.49)
\]

\[
\varepsilon^2 \frac{\partial^2 E}{\partial x^2} = \varepsilon \frac{\partial}{\partial x} \left[ \left( i\theta_x + \varepsilon \frac{A_x}{A} \right) E \right] = \varepsilon \left( i\theta_x + \varepsilon \frac{A_x}{A} \right)_x E + \left( i\theta_x + \varepsilon \frac{A_x}{A} \right)^2 E \quad (3.1.50)
\]

\[
\varepsilon^2 \frac{\partial^2 E}{\partial t^2} = \varepsilon \frac{\partial}{\partial x} \left[ \left( -i\omega + \varepsilon \frac{A_t}{A} \right) E \right] = \varepsilon \left( -i\omega + \varepsilon \frac{A_t}{A} \right)_t E + \left( -i\omega + \varepsilon \frac{A_t}{A} \right)^2 E, \quad (3.1.51)
\]

where $f_x = \partial f / \partial x$ for some function $f(x,t)$. Let us put these expressions into Equation (3.1.46) and collect powers of $\varepsilon$. At order $\varepsilon^0$ we have,

\[
[-c^2 \theta_x^2 + \omega^2 - \omega_p^2(x)] E(x, t) = 0. \quad (3.1.52)
\]

Thus, for $E(x,t) \neq 0$, Equation (3.1.52) implies:

\[
\omega^2 - c^2 \theta_x^2 - \omega_p^2(x) = 0. \quad (3.1.53)
\]

This is a nonlinear, ordinary differential equation for the phase function $\theta(x)$. Notice, if we define the local wavenumber as

\[
k(x,t) \equiv \theta_x, \quad (3.1.54)
\]
we get an expression that looks like a local version of the dispersion function:

\[ D(x, k = \theta(x), \omega) = \omega^2 - c^2 k^2 - \omega^2_p(x) = 0. \]  

(3.1.55)

Next, at order \( \epsilon \), we get:

\[
(\omega A^2) + (c^2 k A^2)_x = \frac{1}{2} \left( \frac{\partial D}{\partial \omega} A^2 \right)_t + \frac{1}{2} \left( -\frac{\partial D}{\partial k} A^2 \right)_x = 0,
\]

(3.1.56)

where we have used Equations (3.1.54) and (3.1.55). Thus, we have an equation for the amplitude \( A(x,t) \).

It is important to note, that \( D \) is treated as a function of the independent variables \((x,k,\omega)\), and the partial derivatives of \( D \) are evaluated at \((x,k = \theta(x), \omega)\).

### 3.1.4 A Conservation Law

We define the wave-action flux density as:

\[ \mathcal{J}(x,t) \equiv A^2 D_{\omega}. \]  

(3.1.57)

Given this definition, Equation (3.1.56) can be put into the form of a conservation law:

\[
\frac{\partial}{\partial t} \mathcal{J} + \frac{\partial}{\partial x} (\mathcal{J} v_g) = 0,
\]

(3.1.58)
where \( v_g \) is the local group velocity:

\[
v_g(x,t) \equiv -\frac{\partial D/\partial k}{\partial D/\partial \omega}
\]  

(3.1.59)

Away from caustics, (where the group velocity goes to zero), we can integrate \( J(x,t) \) over a spatial region to get a function only of \( t \):

\[
J(t) = \int_a^b dx \ J(x,t).
\]  

(3.1.60)

We take the time derivative, and not being particularly careful about the order of integration and differentiation, we get,

\[
\frac{d}{dt} \tilde{J} = \int_a^b dx \ \frac{\partial J}{\partial t}
= -\int_a^b dx \ \frac{\partial}{\partial x}(J v_g)
= (J v_g)_a^b.
\]  

(3.1.61)

Thus, the total action contained in the segment can only change if it is advected through the boundaries at the local group velocity.

It is worthwhile to mention that we needed to have computed the phase function \( \theta(x) \), which allows us to compute the local group velocity \( v_g \), before we can solve for \( J \). Given \( J \), or once it is computed, we can use \( J = A^2 D_\omega \) to compute the amplitude \( A(x,t) \). We note in passing that this algorithm for constructing the amplitude \( E(x,t) \) will be badly behaved where \( D_\omega = 0 \). However, this usually only occurs at isolated points or regions in the plasma, so they
are special, and require a local treatment.

Therefore, away from those problem regions, let us go back to Equation (3.1.53), and integrate to get $\theta(x)$:

$$\theta(x) = \theta(x_0) \pm \frac{1}{c} \int_{x_0}^{x} dx' \left[ \omega^2 - \omega_p^2(x') \right]^{1/2}, \quad (3.1.62)$$

with the plus/minus sign fixed using the boundary conditions, where we assume we know the value of $k(x) = d\theta/dx$. Also, notice that $\theta(x)$ is only well defined in regions of $x$ where

$$\omega > \omega_p(x), \quad (3.1.63)$$

since $\theta(x)$ is assumed to be real. We can now use this expression for $\theta(x)$ to compute the group velocity. First, we compute the local wavenumber:

$$k(x) = \theta_x = \pm \frac{1}{c} \left[ \omega^2 - \omega_p^2(x) \right]^{1/2}. \quad (3.1.64)$$

Next, recall that the dispersion function is:

$$D(x, k; \omega) = \omega^2 - c^2 k^2 - \omega_p^2(x). \quad (3.1.65)$$

Thus, we can find the group velocity:

$$v_g = -\frac{\partial D / \partial k}{\partial D / \partial \omega} = \frac{c^2 k}{\omega} = \pm \frac{c}{\omega} \left[ \omega^2 - \omega_p^2(x) \right]^{1/2}. \quad (3.1.66)$$
Now, if we assume that the background plasma is stationary, that is, does not depend on time, we can seek a time-stationary solution of the wave action conservation law, Equation (3.1.58):

$$J(x) = \frac{\omega J_0}{c \left[ \omega^2 - \omega_p^2(x) \right]^{1/2}}, \quad (3.1.67)$$

for $x$, such that, $\omega \geq \omega_p(x)$. Note that, $J(x)$ blows up at $\omega = \omega_p(x)$, i.e., at the caustic where the group velocity, (3.1.66), goes to zero.

### 3.1.5 Dispersion Functions

The solution of the equation for the phase, Equation (3.1.62), in the time independent example is deceptively simple. We need to develop general methods that will allow us to work with wave equations that are not so amenable. Therefore, let us stay with our current example but return to the dispersion function

$$D(x,k;\omega) = \omega^2 - c^2 k^2 - \omega_p^2(x) \quad (3.1.68)$$

and introduce ideas that will be useful in multiple dimensions. Note that, for the rest of this chapter, we will treat $\omega$ as a parameter.

First, consider the character of the dispersion function $D(x,k)$, where we have left out the dependence on $\omega$ and assume it is given. $D$ is a smooth function of $x$ and $k$, which we treat as a two-dimensional ray phase space, $(x,k)$. Since $D$ is smooth, it has well-defined level sets
that foliate the two-dimensional ray phase space, and we can take the gradient:

\[ \nabla D = (D_x, D_k). \]  

(3.1.69)

As we know from previous discussion, we are most interested in the level set

\[ D(x, k) = 0. \]  

(3.1.70)

However, bear in mind, that \( D(x, k) = 0 \) implicitly defines one particular one-dimensional surface in a two-dimensional space. As we have seen before, this is the dispersion surface. Near points on the dispersion surface where \( D_x \neq 0 \), we can find \( k(x) \), such that:

\[ D[x, k(x)] = 0. \]  

(3.1.71)

Similarly for \( x(k) \) where \( D_k \neq 0 \).

Let us now introduce a parametrization of the dispersion surface by writing \( x \) and \( k \) as smooth functions of some parameter, \( \sigma \), \([x(\sigma), k(\sigma)]\), with,

\[ D(\sigma) = D[x(\sigma), k(\sigma)] = 0. \]  

(3.1.72)

Since we are on the surface \( D = 0 \), we require, \( dD/d\sigma = 0 \). This implies:

\[ \frac{\partial D}{\partial x} \frac{dx}{d\sigma} + \frac{\partial D}{\partial k} \frac{dk}{d\sigma} = 0. \]  

(3.1.73)
Equation (3.1.73) fixes a relation between the way $x$ and $k$ vary with $\sigma$. Since we are free to specify the parametrization of the dispersion surface, let us choose:

$$\frac{dx}{d\sigma} = -\frac{\partial D}{\partial k} = +2c^2k. \tag{3.1.74}$$

Then Equation (3.1.73) requires,

$$\frac{dk}{d\sigma} = \frac{\partial D}{\partial x} = -2\omega_p^2 \omega'_p(x). \tag{3.1.75}$$

The above pair of ordinary differential equations are Hamilton’s equations, where $x$ and $k$ form a canonical pair and the dispersion function, $D(x, k)$ is the ray hamiltonian. Let us mention, although we will not derive the result here, that we can relate the ray parameter $\sigma$ to the physical time $t$ by integrating:

$$\frac{dt}{d\sigma} = \frac{\partial D}{\partial \omega} = 2\omega. \tag{3.1.76}$$

### 3.2 Variational Formulation of Wave Equations

Before proceeding on to the discussion of using Hamilton’s equations to generate rays, let us talk about how we can derive our wave equations from an action principle. The importance of the variational formulation is two-fold; first, it shows us directly why we should use congruence transformations when we want to transform the dispersion matrix, and, second, the conservation of the wave action density falls out as a result of a symmetry in the action.

Let us continue considering the example of electromagnetic waves in a nonuniform, un-
magnetized plasma. As we did in Chapter 2, we assume that the plasma properties only vary in the $x$ and $y$ directions. Additionally, for the moment, let us assume that the electric field points only in the $z$-direction. Thus, $E(x,y,t) = E(x,t)\hat{z}$. In this case, our wave equation is

$$\hat{D}(x,-i\nabla,i\partial_t)E(x,t) = \left[ \frac{\partial^2}{\partial t^2} - c^2\nabla^2 + \omega_p^2(x) \right] E(x,t) = 0.$$  \hfill (3.2.1)

Let us define the action $\mathcal{A}$ as

$$\mathcal{A}[E,E^*] \equiv \langle E|\hat{D}|E^* \rangle = \int d^2x dt\ E^*(x,t)\hat{D}(x,-i\nabla,i\partial_t)E(x,t).$$  \hfill (3.2.2)

When can recover the wave Equation (3.2.1) from the condition

$$\frac{\delta \mathcal{A}}{\delta E^*} = 0,$$  \hfill (3.2.3)

where $\frac{\delta \mathcal{A}}{\delta E^*}$ is the variational derivative of $\mathcal{A}[E,E^*]$ with respect to the function $E^*$.

It is important to note that the action $\mathcal{A}[E,E^*]$ is a bilinear form. That is, it takes in two vectors, $|E\rangle$ and $\langle E|$, (or functions $E(x,t)$ and $E^*(x,t)$), and produces a scalar. In fact, if, and only if, $E(x,t)$ is a solution of the wave Equation (3.2.1), will $\mathcal{A}[E,E^*] = 0$. Now, suppose we transform the vector $|E\rangle \rightarrow |E'\rangle$ via a general, invertable transformation $\hat{Q}$, such that,

$$|E'\rangle = \hat{Q}|E\rangle,$$  \hfill (3.2.4)

$$\langle E'| = \langle E|\hat{Q}^\dagger,$$  \hfill (3.2.5)
where the last line follows from the definition of the adjoint. As a result of this transformation, the action \( \mathcal{A} \) becomes

\[
\mathcal{A}[E', E^{**}] = \langle E'| \hat{D}' E' \rangle \\
= \langle E| \hat{Q}^\dagger \hat{D} \hat{Q}|E \rangle \\
= \langle E| \hat{D}'|E \rangle \quad (3.2.6) \\
= \int d^2x dt \ E^*(x,t) \hat{D}'(x, -i\nabla, i\partial_t) E(x,t) \\
= \mathcal{A}'[E, E^*].
\]

Since the wave operator \( \hat{D} \) transforms as \( \hat{D} \rightarrow \hat{D}' = \hat{Q}^\dagger \hat{D} \hat{Q} \), which is the definition of a congruence transformation, we see that the bilinear form \( \mathcal{A} \) transforms under the action of a congruence transformation! Therefore, when considering how to transform the dispersion matrix \( D \), it is appropriate that we use congruence transformations.

We shall discuss the properties of congruence transformations in more detail in the next chapter; but, for now, let us note that unlike similarity transformations, which leave the entire eigen-spectrum of an operator unchanged, (i.e., the eigenvalues are invariant under the action of a similarity transformation), congruence transformations do not change the signature of the operator. That is, the number of negative, zero, and positive eigenvalues is invariant under the action of a congruence transformation. This means that the null space of the the wave operator \( \hat{D} \), which is the space spanned by the eigenvectors of \( \hat{D} \) which have eigenvalue 0, is invariant under congruency transformations. Since the null space is precisely what we are interested in, (after all we are trying to solve equations of the form \( \hat{D} \cdot E = 0 \)), it is logical to
use congruence transformations rather than similarity transformations because they comprise a ‘larger’ group of transformations, i.e., the congruence transformations offer us more flexibility.

Now that we have shown that the use of congruence transformations is forced upon us when we take the wave equation to be derived from an action principle. Let us show that the results of the previous sections, namely that there exists a conserved quantity, can be derived much more elegantly through the use of the action.

### 3.2.1 A Reduced Action Principle

Recall, that we are only interested in solutions to our wave equation that are eikonal in form, that is,

\[
E(x, t) = A(x, t)e^{i\theta(x,t)}. \tag{3.2.7}
\]

Let’s insert the eikonal ansatz into our expression for the action, Equation (3.2.2). Keeping only the leading order terms, we get

\[
\mathcal{A}[A, \theta] = \int d^2x dt \left[ -\left(\partial_x \theta \right)^2 + c^2 \nabla \theta \cdot \nabla \theta + \omega^2_p(x) \right] A^2(x, t), \tag{3.2.8}
\]

where \(\mathcal{A}\) is the reduced action. The fields to be varied are the amplitude, \(A\), and the phase, \(\theta\), both of which are real-valued functions.
First, let us vary the amplitude:

\[ \frac{d}{d\varepsilon} \mathcal{S}[A + \varepsilon a, \theta] \bigg|_{\varepsilon=0} = \int d^2x dt \frac{\delta \mathcal{S}}{\delta a}(x,t) \]

\[ = 2 \int d^2x dt \left[ - (\partial_t \theta)^2 + c^2 \nabla \theta \cdot \nabla \theta + \omega_p^2(x) \right] A(x,t) a(x,t). \quad (3.2.9) \]

Hence, Equation (3.2.9) implies

\[ \frac{\delta \mathcal{S}}{\delta A} = 2 \left[ - (\partial_t \theta)^2 + c^2 \nabla \theta \cdot \nabla \theta + \omega_p^2(x) \right] A(x,t). \quad (3.2.10) \]

Since we want the variation of the reduced action with respect to \( a \) to be zero, i.e., \( \frac{\delta \mathcal{S}}{\delta A} = 0 \), we have

\[ - (\partial_t \theta)^2 + c^2 \nabla \theta \cdot \nabla \theta + \omega_p^2(x) = 0, \quad (3.2.11) \]

which is a partial differential equation for the phase \( \theta(x,t) \).

Now, varying the phase, we get

\[ \frac{d}{d\varepsilon} \mathcal{S}[A, \theta + \varepsilon \phi] \bigg|_{\varepsilon=0} = 2 \int d^2x dt \left[ - \partial_t \theta \partial_t \phi + c^2 \nabla \theta \cdot \nabla \phi \right] A^2(x,t) \]

\[ = 2 \int d^2x dt \left[ \partial_t \left( \partial_t \phi \right) - c^2 \nabla \cdot \left( \nabla \phi \right) \right] \phi(x,t), \quad (3.2.12) \]

where, to get from the first line to the second line, we integrate by parts and assume that the variation of \( \phi \) on the boundary is zero. Thus,

\[ \frac{\delta \mathcal{S}}{\delta \theta} = 2 \left[ \partial_t \left( \partial_t \phi \right) - c^2 \nabla \cdot \left( \nabla \phi \right) \right]. \quad (3.2.13) \]
Again, since we are interested in the case when $\delta S / \delta \theta = 0$, we have

$$\partial_t (\partial_t A^2) - c^2 \nabla \cdot (\nabla A^2) = 0,$$  \hspace{1cm} (3.2.14)

which is a nonlinear, partial differential equation for the amplitude $A(x,t)$.

Notice that the reduced action, Equation (3.2.8), has a symmetry. Namely, $\mathcal{A}$ is invariant under a shift of the phase: $\theta(x,t) \rightarrow \theta(x,t) + \alpha$, where $\alpha$ is a real constant. Thus, as a consequence of Noether’s theorem, we expect there to be an associated conserved quantity. That conserved quantity is the wave action density, as defined in Equation (3.1.57). If we identify the local frequency as

$$\omega \equiv -\partial_t \theta,$$  \hspace{1cm} (3.2.15)

and the local wave-vector as

$$k \equiv \nabla \theta,$$  \hspace{1cm} (3.2.16)

then equation (3.2.11) becomes

$$-\omega^2 + c^2 k^2 + \omega_p^2(x) = 0,$$  \hspace{1cm} (3.2.17)

which is identical to Equation (3.1.55), which is the dispersion function $D(x,k,\omega)$. Additionally, Equation (3.2.14) becomes

$$\partial_t (\omega A^2) - c^2 \nabla \cdot (k A^2) = 0,$$  \hspace{1cm} (3.2.18)

which is equivalent to Equation (3.1.56).
Now, since

\[ D(x, k, \omega) = \omega^2 - c^2 k^2 - \omega_p^2(x), \quad (3.2.19) \]

\[ \mathcal{J}(x,t) = A^2 \partial_\omega D = \omega A^2, \quad (3.2.20) \]

and

\[ v_g = -\nabla_k D \partial_\omega D = c^2 \frac{k}{\omega}, \quad (3.2.21) \]

Equation (3.2.18) can be expressed as

\[
\frac{\partial \mathcal{J}}{\partial t} + \nabla \cdot (\mathcal{J} v_g) = \mathcal{J}_t + v_g \cdot \nabla \mathcal{J} + \mathcal{J} \nabla \cdot v_g = 0,
\quad (3.2.22)
\]

which is the multi-dimensional analogue of Equation (3.1.58). This action conservation law can be recast as an evolution equation for \( \mathcal{J} \), following a ray, by using the fact that

\[
\partial_t + v_g \cdot \nabla
\quad (3.2.23)
\]

is the total time derivative following a ray in position space:

\[
\frac{d \ln \mathcal{J}}{dt} = (\partial_t + v_g \cdot \nabla) \ln \mathcal{J}
\]

\[
= \frac{1}{\mathcal{J}} (\mathcal{J}_t + v_g \cdot \nabla \mathcal{J})
\]

\[
= -\frac{1}{\mathcal{J}} (\mathcal{J} \nabla \cdot v_g)
\]

\[
= -\nabla \cdot v_g,
\quad (3.2.24)
\]
where we have used Equation (3.2.22). Thus, following a ray, the wave action flux density can only change if the rays converge or diverge.

### 3.3 Rays

Now, let us choose a specific form for the plasma frequency, \( \omega_p(x) \), and apply the above methods in one spatial dimension, and then examine the extension to two spatial dimensions. First, recall that the plasma frequency is given by

\[
\omega_p^2(x) = 4\pi \sum_s n_s(x) q_s^2 / m_s.
\] (3.3.1)

Since we assume that the plasma is charge-neutral, i.e., \( \sum_s q_s n_s(x) = 0 \), then \( \omega_p^2(x) \) can be approximated by the electron-plasma frequency:

\[
\omega_p^2(x) \approx \omega_e^2(x) = \frac{4\pi n_e(x) q_e^2}{m_e}.
\] (3.3.2)

We shall assume the same number density profile as we did in the previous chapter, namely,

\[
n_e(x) = n_e e^{-x^4},
\] (3.3.3)

which results in a plasma frequency of the form

\[
\omega_p(x) = n_e^{1/2} e^{-x^4/2}.
\] (3.3.4)
Plots of the number density and the plasma frequency are shown in Figures 3.2a and 3.2b, respectively. Plots of the rays, for several values of the carrier frequency $\omega$, are shown in Figure 3.3.

The dispersion function for the nonuniform, unmagnetized plasma is

$$D(x, k; \omega) = \omega^2 - c^2 k^2 - n_e e^{-x^4}, \quad (3.3.5)$$

where we are continuing to approximate the plasma frequency by the electron plasma frequency. Now, from $dx/d\sigma = 2c^2 k$, we see that when $k < 0$, $x$ should be decreasing, i.e. the ray is moving to the left, and when $k > 0$, $x$ is increasing, (moving to the right). Furthermore, since $dk/d\sigma = -2\omega_p \omega_p' = 2x^3 \exp(-x^4)$, and $x$ is always positive, $k$ must always be increasing along a ray. Note that the rays with large $\omega$ are only slightly perturbed by the presence of the plasma, but, otherwise, propagate like free space electromagnetic rays. This follows from the fact that at large $\omega$, the dispersion function of the transverse mode approaches the dispersion function of
FIG. 3.3: Rays in a nonuniform, unmagnetized plasma, for different values of ω: (a) Rays in (x,k) space for various values of ω. The rays for k < 0 travel from right to left and the rays for k > 0 travel from left to right. Each of the rays satisfies the condition that D(x,k;ω) = 0. Note that the rays with ω < ωp(0), encounter a turning point, hence they are reflected. Rays with ω > ωp(0) are not reflected. (b) Rays in (x,k,ω) space. Note that the rays live on the dispersion surface D = 0. Also, note that because the background is time-stationary, the wave frequency is constant, which is present in the fact that the rays live on curves of constant omega in the dispersion surface. Additionally, notice that the rays do not extend into the region where ωp > ω.

a free-space electromagnetic wave, as can be seen in Figure 2.8b, where the dispersion surface of the transverse mode asymptotes to the free-space mode at large ω. Additionally, for some values of the carrier frequency, we see that the rays travel into the plasma and are then reflected back and exit the plasma. The point at which the rays turn around is when ω = ωp(x).

We shall consider the extension of these concepts to a multidimensional plasma in a bit; but, first, in an effort to highlight the usefulness of eikonal methods and rays, let us examine the behavior of the plasma near the location of gyro- and ion-hybrid resonances.
3.4 Resonances in a Nonuniform Magnetized Plasma

Gyroresonances are of great importance in plasma physics. For example, see Refs. [15] and [17] for more detailed discussions of gyroresonances. Although we will not be focusing on gyroresonances, but rather on ion-hybrid resonances, gyroresonances provide an easily visualized, conceptual example to talk about the actual motion of the particles in the plasma. First, consider gyroresonances in the cold plasma model. For a nonuniform plasma, the resonance condition is a local statement:

\[ D_s(x; \omega) \equiv \omega - \Omega_s(x) = 0, \quad (3.4.1) \]

where

\[ \Omega_s(x) = \frac{q_s B(x)}{m_s c}. \quad (3.4.2) \]

This condition defines a three-dimensional surface imbedded in the four-dimensional ray phase space \((x, k) = (x, y, k_x, k_y)\). It is significant to note that this surface is 'vertical' in the sense that it does not depend on \(k\). See Figure 3.4.

Let us use the dispersion function \(D_s\) as the ray Hamiltonian to generate rays,

\[ \frac{dx}{d\sigma} = -\nabla_k D_s = 0, \quad (3.4.3) \]
where, recall, $\sigma$ is the ray parameter, which, in this case, is equal to the physical time $t$ since

$$\frac{dt}{d\sigma} = \frac{\partial D}{\partial \omega} = 1. \tag{3.4.4}$$

Looking at the group velocity as defined in Equation (3.1.37), we see that these waves have zero group velocity. Additionally,

$$\frac{dk}{d\sigma} = \nabla D = -\nabla \Omega_s(x), \tag{3.4.5}$$

which is proportional to $\nabla B$, since the gyrofrequency $\Omega_s$ depends on position only through the
magnetic field. Thus, we see that the rays evolve only in the \( k \)-direction.

Now, let us use the relation

\[
\omega \rightarrow i \partial_t \tag{3.4.6}
\]

to derive an evolution equation for a scalar field using the dispersion function (3.4.1). Based on the derivation of (3.4.1) from the cold plasma model in chapter 2, let us introduce a scalar field \( \psi(x,t) \), which is related to the left-circular part of the velocity field of gyrating particles of ion species \( s \). Therefore, we have

\[
[i \partial_t - \Omega(x)] \psi(x,t) = 0. \tag{3.4.7}
\]

The general solution to this equation is, of course,

\[
\psi(x,t) = e^{i \Omega(x)t} \psi_0(x), \tag{3.4.8}
\]

where \( \psi_0(x) = \psi(x,0) \). Assuming that the initial wave function is eikonal in form, such that,

\[
\psi_0(x) = A(x)e^{ik_0 \cdot x}, \tag{3.4.9}
\]

where \( A(x) \) is a smooth, slowly varying envelope, we can write

\[
\psi(x,t) = A(x)e^{i\theta(x,t)}. \tag{3.4.10}
\]
This implies

$$\theta(x,t) = k_0 \cdot x - \Omega(x)t,$$

(3.4.11)

and, since \( k = \nabla \theta \) by definition,

$$k(x,t) = \nabla \theta = k_0 - t \nabla \Omega(x).$$

(3.4.12)

Taking the time derivative of Equation (3.4.12), we get,

$$\frac{dk}{dt} = -\nabla \Omega(x),$$

(3.4.13)

which is identical to Equation (3.4.5), since, as we noted above, \( t \equiv \sigma \).

Let us take a moment and discuss what is going on: First, at \( t = 0 \), the ions of species \( s \) have some initial velocity distribution given by \( \mathcal{V}_0 \). As the particles gyrate with \textit{local} frequency \( \Omega_s(x) \) the phase pattern evolves in time. Because the particles are \textit{not} streaming, (recall, we are assuming that zeroth order velocities are zero), the envelope of the disturbance, \( A(x) \) does not change with time. This is reflected in the fact that the group velocity is zero! The only quantity that evolves is the \textit{local} wavevector, \( k = \nabla \theta(x,t) \). This information is contained in the ray equations, and the picture of ray evolution in phase space shows rays evolving straight 'up' the 'vertical' surface defined by \( D_s(x) = 0 \).

Now, we want to consider the \textit{physical} motion of the plasma particles themselves. Suppose, at \( t = 0 \) the particles in the region defined by the envelope function \( A(x) \) have an initial velocity \( v_0(x) \) in the \( x,y \)-plane, such that they are gyrating, and \( v_\perp^2 = v_x^2 + v_y^2 \) is constant. (Recall, we
are taking the direction of the magnetic field to be the $z$-direction, and we are only interested in the motion of the particles in the plane perpendicular to the magnetic field, i.e., in the $x,y$-plane.) Note, that if the magnetic field were constant, the particles would continue to gyrate at their original velocities, i.e, the velocity field $v(x)$ would not evolve in time, since

$$\frac{dk}{dt} = -\nabla \Omega(x) = 0,$$  \hspace{1cm} \text{for a constant magnetic field.} \hspace{1cm} (3.4.14)$$

Let us assume that the magnetic field points in the positive $z$-direction, and only varies in the $x$-direction, such that

$$B(x) = B(x)\hat{z}. \hspace{1cm} (3.4.15)$$

Additionally, let $\nabla B(x)$ point in the positive $x$-direction. Since the instantaneous radius of curvature is

$$r_c = \frac{mv_r}{|q|B}, \hspace{1cm} (3.4.16)$$

we see that the radius of curvature increases in regions where the magnetic field is weaker and decreases in regions where the magnetic field is stronger. For simplicity, we assume that the variation of the magnetic field over a single particle orbit is small, so that we may separate the cyclotron motion from the drift motion.

First, let us consider the drift motion. Since we are separating the gyration and the drift, we can treat the drift motion as the drift of the guiding center for each particle. The drift
velocity due to the gradient of the magnetic field is\(^1\)

\[
v_G = \frac{w_\perp}{qB} \left( \frac{\hat{B} \times \nabla B}{B} \right),
\]

(3.4.17)

where

\[
w_\perp = \frac{1}{2} mv_\perp^2,
\]

(3.4.18)

which is constant since magnetic fields do no work. For our system, \(v_G\) will be in the positive \(y\)-direction since \(B\) is in the positive \(z\)-direction and \(\nabla B\) is in the positive \(x\)-direction.

Now, we return to the discussion of the circular motion of the particles. Since we are assuming that the variation of \(B(x)\) over a single orbit is small, each particle’s gyration will be constant about its guiding center. However, since \(B(x)\) is not constant and varies from one guiding center to another, there will be an evolution in the velocity field \(v(x,t)\), (where we are now evaluating \(x\) at the guiding centers). Thus, since the particles are gyrating at different speeds, the phase function

\[
\theta(x,t) = k_0 \cdot x - \Omega(x)t
\]

(3.4.19)

will evolve.

Let us examine this evolution more closely. Suppose at \(t = 0\), all the particles are in phase, which implies

\[
k_0 = k(x,t = 0) = \nabla \theta(x,t = 0) = 0,
\]

(3.4.20)

\(^1\)We are following the notation of Ref. [4], where \(v_G\) is the gradient drift velocity and \(v_C\) is the curvature drift velocity.
since the gradient of $\theta(x, t = 0)$ is zero. Now, since $k_0 = 0$, the phase function, Equation (3.4.19), evolves as

$$\theta(x, t) = -\Omega(x)t.$$  \hfill (3.4.21)

Thus, at a later time, $t = t_1$, the particles in the stronger magnetic field region will be further ahead. This leads to

$$k(x, t) = \nabla \theta(x, t),$$  \hfill (3.4.22)

which is non-zero and positive. It is straightforward to see that at later times there will be a greater discrepancy between the phases, which leads to a larger value of $|k|$. Hence, as $t \to \infty$, $|k| \to \infty$, which follows directly from the fact that

$$k(x, t) = -t \nabla \Omega(x).$$  \hfill (3.4.23)

As $|k|$ becomes very large, there are other effects such as phase mixing and Landau damping that need to be taken into account.

Remember, also that the guiding centers are moving in the positive $y$-direction, (for positively charged particles; negatively charged particles will be moving in the $-y$-direction), with velocity $|v_G|$. Therefore, for this configuration, when the particles reach the boundary of the envelope, $A(x)$, their velocities must go to zero. Hence, this is a non-physical model. However, we are only interested in the case where the evolution of $k(x, t)$ happens much faster than the drift velocity. In this case, we can ignore the effect of the drift velocity all together.

The above discussion was concerned with gyroresonances in a cold plasma. However, we
are ultimately interested in hybrid resonances in a cold plasma. In our current model, the hybrid resonances are associated with roots of the dispersion function, in the uniform case,

\[ D(k, \omega) = \omega^2 R L - \frac{1}{2} c^2 (k_x^2 + k_y^2) S, \]  

(3.4.24)

where \( R(\omega) \), \( L(\omega) \), and \( S(\omega) \) are all functions of the carrier frequency \( \omega \). Recall, from the previous chapter, that this dispersion function is the determinant of the dispersion matrix \( D(k, \omega) \). Also remember, that the hybrid resonances occur at frequencies satisfying the condition, \( S(\omega) = 0 \). Whereas, the gyroresonances occur at frequencies satisfying \( S(\omega) = \pm \infty \). (See Figure 2.4 on page 40.) Note, as we saw in the previous chapter, if \( R(\omega)L(\omega) \neq 0 \) when \( S(\omega) = 0 \), then \( (k_x^2 + k_y^2) \to \infty \), as seen in Figure 2.5 on page 41.

If we consider the case of a nonuniform, magnetized plasma, the condition \( S(x, \omega) = 0 \) is now local, i.e., it depends on position within the plasma. In this case, the dispersion function is

\[ D(x, k; \omega) = \omega^2 R(x; \omega) L(x; \omega) - \frac{1}{2} c^2 (k_x^2 + k_y^2) S(x; \omega). \]  

(3.4.25)

It is tempting to use the above dispersion function as the ray Hamiltonian; however, our position is that this is not what you want to do because in the vicinity of the resonance, there are two modes present which are nearly degenerate. This is an example of mode conversion. What we want to do is find a polarization basis of the electric field which isolates the two interacting
modes and casts the dispersion matrix into the form:

\[
D(x, k, \omega) = \begin{pmatrix} D_{11}(x, k, \omega) & \eta \\ \eta^* & S(x, \omega) \end{pmatrix}
\]  

(3.4.26)

where, in general, \( \eta(x, k, \omega) \) is a function of \( x, k, \) and \( \omega \). In this form, the wave equation emphasizes that there is a coupling, (namely \( \eta \)), between the hybrid wave, with dispersion function

\[
D_H \equiv S(x, \omega),
\]  

(3.4.27)

and a collective wave with dispersion function \( D_{11}(x, k, \omega) \).

Let us examine the nature of the hybrid wave satisfying the condition

\[
D_H = S(x, \omega) = 0.
\]  

(3.4.28)

First, note, as in the case of a gyroresonance, the dispersion function for the ion hybrid mode is independent of \( k \). Thus, the dispersion surface, in the four-dimensional ray phase space, \((x, y, k_x, k_y)\), should be 'vertical', as seen in Figure 3.5. Hamilton's equations for the rays that live on this dispersion surface are

\[
\frac{dx}{d\sigma} = -\nabla_k D_H = 0,
\]  

(3.4.29)
FIG. 3.5: A plot of the dispersion surface, $D_H = 0$, in $(x, y, k_x)$ space. Note that the surface is indeed vertical in the $k$-direction.

which implies that ion-hybrid waves have zero group velocity, and

$$\frac{dk}{d\sigma} = \nabla_x D_H = \nabla_x S(x, \omega).$$  \hspace{1cm} (3.4.30)

Thus, the rays evolve only in the $k$-direction.

The physical interpretation of these results, in terms of the motion of the plasma and the fields, is very similar to the gyroresonance case. The main difference is that the hybrid motion involves more than one ion species and is not a simple gyromotion. However, the evolution in $k$-space is still due to the nonuniformity of the background magnetic field, which causes the hybrid resonance frequency to vary from point to point, causing the phase relationship among
neighboring oscillations to change with time.

Now, suppose we change the polarization basis of the electric field as follows:

\[
\mathbf{\hat{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathbf{\hat{e}}'_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
\]

(3.4.31)

\[
\mathbf{\hat{e}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathbf{\hat{e}}'_2 = \mathbf{\hat{e}}_2,
\]

(3.4.32)

where \(\alpha_1\) and \(\alpha_2\) are complex numbers. Then, as we saw in the section on deriving wave equations from a variational principle, the dispersion matrix \(D\) changes via a congruence transformation:

\[
D' = Q^\dagger D Q,
\]

(3.4.33)

where

\[
Q = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.
\]

(3.4.34)

Under this choice of \(Q\), we see that \(D_{11} \rightarrow D'_{11}\) and the coupling \(\eta \rightarrow \eta'\), but \(S(x, \omega)\) has been left isolated. Thus, we see that we can transform the dispersion matrix \(D\) by a congruence transformation to get a new dispersion matrix \(D'\). The question is, which choice of polarization, i.e., which congruence transformation, is correct?

Mathematically, all representations of the dispersion matrix are equal. However, since we are interested in applying WKB methods to calculate rays in the vicinity of ion-hybrid resonances, certain choices will be more appropriate for asymptotic matching. Thus, we choose
the dispersion matrix in which the diagonals are the *uncoupled* dispersion functions and the
off-diagonal terms represent the coupling between the two modes. Specifically, we want the
diagonals to *Poisson-commute* with the off-diagonal terms; such that, when we take the diago-
inals to be the ray Hamiltonians, the coupling will be *constant* following the rays. We call this
particular choice the **normal form**.

Before we can begin to discuss how to put the dispersion matrix $D$ into normal form we
need to know how to go from the matrix $\hat{D}(x,-i\nabla;\omega)$, whose entries are *operators*, to the
matrix $D(x,k;\omega)$, whose elements are *functions* of the four-dimensional phase space $(x,k)$. In
our current, very simple model, we can simply make the replacement $-i\nabla \rightarrow k$. However, if
we want to be able to apply these methods to more physical and complex models of plasmas,
as we hope to do, we will not, in general, be able to make the simple identification above.
We need a systematic and rigorous method to go from operators, which do *not* commute, to
functions. The tool we need is the Weyl symbol calculus, which is discussed in Appendix A.

### 3.5 General Wave Equations

Now, let us combine the various concepts that we introduced in this chapter; namely,
dispersion functions, action principles, the Weyl symbol (discussed in Appendix A), and the
eikonal approximation and apply them to the case of a very general wave equation. This will
accomplish two things: first, we will be able to derive Hamilton's equations for the evolution
of rays in a general, multidimensional case; second, it will help to illuminate the power of the
normal form transformation, the discussion of which will encompass the remaining chapters.
3.5.1 The Eikonal Approximation in Configuration Space

Consider a general form for a wave equation that encompasses partial differential equations or integro-differential wave equations. Additionally, we consider an arbitrary, but finite, number of components, such that the wave equation can be expressed in the form

\[ \int d^n x' D(x, x'; \omega) \cdot \Psi(x') = 0 \]

\[ \int d^n x' D_{mn}(x, x'; \omega) \psi_n(x') = 0. \]  

We are assuming that the background is independent of time and Fourier analyze in time to restrict our attention to a single frequency. Hence, we are treating the frequency \( \omega \) as a parameter. Furthermore, we are specifically interested in the case where the wave operator is self-adjoint, i.e.,

\[ D(x, x') = D^\dagger(x', x) \]

\[ D_{mn}(x, x') = D^*_{nm}(x', x). \] 

Then, using the methods of the Weyl symbol mapping, we can write (3.5.1) as a pseudo-differential equation

\[ D(x, -i\nabla) \cdot \Psi(x) = 0 \]

\[ D_{mn}(x, -i\nabla) \psi_n(x) = 0. \] 

Since we are assuming that the wave operator is Hermitian, we can introduce an action
principle:

\[ \mathcal{A}[\Psi, \Psi^\dagger] = \int d^n x \, \Psi^\dagger(x) \cdot D(x, -i \nabla) \cdot \Psi(x) \]  
\[ \mathcal{A}[\psi, \psi^*] = \int d^n x \, \psi^*_m(x) D_{mn}(x, -i \nabla) \psi_n(x). \]  

(3.5.4)

The wave Equation (3.5.3) is recovered through the stationarity condition

\[ \frac{\delta \mathcal{A}}{\delta \Psi^\dagger} = 0 \]  
\[ \frac{\delta \mathcal{A}}{\delta \psi^*_m} = 0. \]  

(3.5.5)

Now, as we did in the one-dimensional case, we introduce a reduced action principle by assuming that the field \( \Psi(x) \) is eikonal, i.e., that we can write

\[ \Psi(x) = \bar{\Psi}(x) e^{i \theta(x)} \]  
\[ \psi_m(x) = \bar{\psi}_m(x) e^{i \theta(x)}, \]  

(3.5.6)

where the phase \( \theta(x) \) varies over a length scale that is small compared to the background, and the amplitude \( \bar{\Psi}(x) \) \( (\bar{\psi}_m(x)) \) varies over the length scale of the background. Since the phase \( \theta(x) \) varies more rapidly than the amplitude \( \bar{\psi}_m(x) \), we can take the derivatives to act only on the phase, and express the reduced action as

\[ \mathcal{A}[\bar{\Psi}, \theta] = \int d^n x \, \bar{\Psi}^\dagger(x) \cdot D(x, k = \nabla \theta) \cdot \bar{\Psi}(x) \]  
\[ \mathcal{A}[\bar{\psi}, \theta] = \int d^n x \, \bar{\psi}^*_m(x) D_{mn}(x, k = \nabla \theta) \bar{\psi}_n(x). \]  

(3.5.7)
The pseudo-differential form of the operator that appears in Equations (3.5.3) and (3.5.4), can be thought of as having been arrived at by replacing \( k \) wherever it appears in the symbol \( D(x, k) \) with \(-i \nabla\), provided we fully symmetrize all terms involving products of \( \hat{x} \) and \( \hat{k} \). Since we are using the eikonal ansatz, and all derivatives but those acting on the phase are dropped, we simply replace all \( k \)'s in the symbol with \(-q\).

The variation with respect to \( \Psi^\dagger (\psi_m^\ast) \) gives the condition

\[
\frac{\delta \mathcal{S}[\Psi, \theta]}{\delta \Psi(x)} = D(x, k = \nabla \theta) \cdot \Psi(x) = 0
\]

\[
\frac{\delta \mathcal{S}[\psi, \theta]}{\delta \psi_n(x)} = D_{mn}(x, k = \nabla \theta) \psi_n(x) = 0.
\]

Hence, the phase function \( \theta(x) \), (if it exists), must have the property that, at each point \( x \), if we insert \( k = \nabla \theta \) into the \( N \times N \) symbol matrix \( D(x, k) \), it must have a zero eigenvalue. Let us suppose that the phase \( \theta(x) \) is known. Thus, the multicomponent amplitude \( \Psi(x) \) must be proportional to the associated null eigenvector of \( D(x, k = \nabla \theta) \). We are also assuming that there is only one null eigenvalue at each point \( x \), i.e., the null eigenvalues are non-degenerate. In general, this is true. However, there may be isolated points in the ray phase space where the null eigenvalues are degenerate, such as at the location of a mode conversion. These can be dealt with following the asymptotic matching techniques developed by Tracy, et. al., Refs. [12–14]. Away from such degeneracies, let us write

\[
\Psi(x) = A(x) \hat{e}(x),
\]
where $A(x)$ is real and $\hat{e}(x)$, (which is the polarization), is the unit null eigenvector. The condition $D \cdot \hat{e} = 0$ fixes $\hat{e}$ up to a phase, and $\hat{e}$ is assumed to vary on the length scale of the background.

Now, let us consider the variation of the reduced action (3.5.7) with respect to $\theta$. After performing the Fréchet derivative, integrating by parts, and applying the stationarity condition $\delta A / \delta \theta = 0$, we have

\begin{equation}
\nabla \cdot \left[ \tilde{\Psi}^\dagger(x) \cdot \nabla_k D(x, k) \cdot \tilde{\Psi}(x) \right]_{k=\nabla \theta} = 0
\end{equation}

where $j = 1, 2, \ldots, N$ and $m, n = 1, 2, \ldots, N$. Using (3.5.9) we can write

\begin{equation}
\tilde{\Psi}^\dagger(x) \cdot D(x, k) \cdot \tilde{\Psi}(x) = A^2(x) \left[ \hat{e}^\dagger(x) \cdot D(x, k) \cdot \hat{e}(x) \right].
\end{equation}

Notice, that the quantity $\hat{e}^\dagger(x) \cdot D(x, k) \cdot \hat{e}(x)$ appears to be a projection onto the null-space of $D$, thereby isolating the null eigenvalue. However, the matrix valued function, $D(x, k)$, lives in ray phase space, whereas the polarization vector, $\hat{e}(x)$, lives only in $x$-space. Therefore, let us bring our discussion into the full ray phase space.

### 3.5.2 The Eikonal Approximation in Phase Space

Consider the $N \times N$ self-adjoint matrix, $D(x, k)$, which is the dispersion matrix. Since the dispersion matrix is an $N \times N$ Hermitian matrix, it has $N$ real eigenvalues at each point $(x, k)$
in the ray phase space. We will label the eigenvalues

\[ D_\alpha(x, k), \quad (3.5.12) \]

where \( \alpha = 1, 2, \ldots, N \). We are continuing to assume that the eigenvalues are not degenerate.

Recall, as we mentioned in the previous section, this assumption will not be true at all points in phase space, (such as at the location of a mode conversion), but it will be true at most points.

The eigenvector associated with the eigenvalue \( D_\alpha(x, k) \) is denoted

\[ \hat{e}_\alpha(x, k). \quad (3.5.13) \]

Because the eigenvalues are assumed to be non-degenerate, the Hermiticity of \( D \) tells us that the associated eigenvectors are orthogonal. Furthermore, we can always normalize the eigenvectors so that they are of unit length. Thus, the eigenvectors form an orthonormal basis:

\[ \hat{e}_\alpha^\dagger(x, k) \cdot \hat{e}_\beta(x, k) = \delta_{\alpha\beta}. \quad (3.5.14) \]

Now, at each point in phase space, we can decompose the self-adjoint dispersion matrix \( D \) in terms of its eigenvalues and eigenvectors:

\[ D(x, k) = \sum_{\beta=1}^{N} D_\beta(x, k) \hat{e}_\beta(x, k) \hat{e}_\beta^\dagger(x, k). \quad (3.5.15) \]
Using the orthonormality condition (3.5.14), we have

$$\hat{e}_\alpha^\dagger(x,k) \cdot D(x,k) \cdot \hat{e}_\alpha(x,k) = D_\alpha(x,k).$$

(3.5.16)

If we take the derivative of (3.5.16) with respect to $k_j$, we get

$$\frac{\partial D_\alpha(x,k)}{\partial k_j} = \frac{\partial}{\partial k_j} \left[ \hat{e}_\alpha^\dagger(x,k) \cdot D(x,k) \cdot \hat{e}_\alpha(x,k) \right]$$

$$= \frac{\partial \hat{e}_\alpha^\dagger}{\partial k_j} \cdot D \cdot \hat{e}_\alpha + \hat{e}_\alpha^\dagger \cdot \frac{\partial D}{\partial k_j} \cdot \hat{e}_\alpha + \hat{e}_\alpha^\dagger \cdot D \cdot \frac{\partial \hat{e}_\alpha}{\partial k_j}.$$  

(3.5.17)

If we are at a point in phase space where $D_\alpha(x,k) = 0$, i.e., $D_\alpha$ is a null eigenvalue at that point, then

$$\frac{\partial D_\alpha(x,k)}{\partial k_j} = \hat{e}_\alpha^\dagger \cdot D \cdot \hat{e}_\alpha, \quad \text{iff} \quad D_\alpha(x,k) = 0.$$  

(3.5.18)

This is an identity on phase space; and, if we are at a point in phase space where $(x,k = \nabla \theta)$ and $D_\alpha(x,k = \nabla \theta) = 0$, we can use (3.5.18) to write (3.5.10) as

$$\frac{\partial}{\partial x_j} \left\{ A^2(x,k = \nabla \theta) \left[ \hat{e}_\alpha^\dagger(x,k) \cdot \frac{\partial D(x,k)}{\partial k_j} \cdot \hat{e}(x,k) \right]_{k = \nabla \theta} \right\} =$$

$$\nabla_x \cdot \left[ A^2(x,k) \nabla_k D_\alpha(x,k) \right]_{k = \nabla \theta}.$$  

(3.5.19)

Additionally, as in Section 3.1.4, if we identify the quantities

$$\mathcal{J} \equiv A^2 \frac{\partial D_\alpha}{\partial \omega},$$  

(3.5.20)
which is the wave action density, and

\[ v_g \equiv -\frac{\nabla_k D_\alpha}{\partial D_\alpha / \partial \omega}, \]  

(3.5.21)

which is the group velocity, we get a multidimensional analog of the conservation law (3.1.58):

\[ \nabla_x \cdot \left( J v_g \right) = 0, \]  

(3.5.22)

where \( J \) and \( v_g \) are both evaluate at \((x, k = \nabla \theta)\) prior to being acted on by \( \nabla_x \). There are several things to note about Equation (3.5.22). First, the time derivative is absent since we have assumed a fixed carrier frequency \( \omega \). Second, the null eigenvalue \( D_\alpha \) is the dispersion function; hence, it will be the ray Hamiltonian.

Let us now consider the task of computing the eikonal phase \( \theta(x) \). First, note, if \( \theta(x) \) is a smooth function, then

\[ \frac{\partial^2 \theta}{\partial x^\mu \partial x^v} = \frac{\partial^2 \theta}{\partial x^v \partial x^\mu}, \]  

(3.5.23)

where \( \mu \) and \( \nu \) run over the number of spatial dimensions of the system. Since \( k_\mu = \partial \theta / \partial x^\mu \), we have

\[ \frac{\partial k_\nu}{\partial x^\mu} = \frac{\partial k_\mu}{\partial x^\nu}. \]  

(3.5.24)

Thus, following any smooth curve \( x(\sigma) \),

\[ \frac{dk_\nu}{d\sigma} = \frac{\partial k_\nu}{\partial x^\mu} \frac{dx^\mu}{d\sigma} = \frac{\partial k_\mu}{\partial x^\nu} \frac{dx^\mu}{d\sigma}. \]  

(3.5.25)
Since \( D_\alpha(x, k) \) is the ray Hamiltonian, we can choose

\[
\frac{dx^\mu}{d\sigma_\alpha} = -\frac{\partial D_\alpha}{\partial k_\mu}, \quad \frac{dk}{d\sigma_\alpha} = -\nabla_k D_\alpha, \tag{3.5.26}
\]

where we have included the \( \alpha \) subscript on the parameter \( \sigma \) to emphasize that the parameterization is associated with the eigenvalue \( D_\alpha \). Combining the results of (3.5.25) and (3.5.26) we get

\[
\frac{dk_\nu}{d\sigma_\alpha} = \frac{\partial k_\mu}{\partial x^\nu} \frac{dx^\mu}{d\sigma_\alpha} = -\frac{\partial k_\mu}{\partial x^\nu} \frac{\partial D_\alpha}{\partial k_\mu}. \tag{3.5.27}
\]

Recall that we require

\[
D_\alpha(x, k(x)) = 0. \tag{3.5.28}
\]

This implies that,

\[
\frac{dD_\alpha}{dx^\nu} = 0 \Rightarrow \frac{\partial D_\alpha}{\partial x^\nu} + \frac{\partial D_\alpha}{\partial k_\mu} \frac{\partial k_\mu}{\partial x^\nu} = 0, \tag{3.5.29}
\]

which, when combined with (3.5.27), gives

\[
\frac{dk_\nu}{d\sigma_\alpha} = \frac{\partial D_\alpha}{\partial x^\nu} \Rightarrow \frac{dk}{d\sigma_\alpha} = \nabla_x D_\alpha. \tag{3.5.30}
\]

Equations (3.5.26) and (3.5.30) are the multi-dimensional analogs of (3.1.74) and (3.1.75), with the eigenvalue \( D_\alpha \) as the ray Hamiltonian and \((x, k)\) as conjugate variables. Note, the phase space is filled with trajectories that are solutions to Hamilton’s equations where \( D_\alpha(x, k) \) is the Hamiltonian. However, only the trajectories on the surface \( D_\alpha(x, k) = 0 \) are called rays.

Now, suppose we fix an initial condition \((x_0 = x(0), k_0 = k(0))\), such that \( D_\alpha(x_0, k_0) = 0 \).
Thus, since $k = \nabla \theta$, we have that $k_0 \equiv \nabla \theta(x_0)$. Then, using Hamilton’s Equations, (3.5.26) and (3.5.30), we can integrate (typically numerically) to generate the rays that start at the point $(x_0, k_0)$ in phase space. Once we have the solution $(x(\sigma_\alpha), k(\sigma_\alpha))$, we can find $\theta(\sigma_\alpha)$:

$$
\theta(\sigma_\alpha) = \int_0^{\sigma_\alpha} d\sigma'_\alpha \frac{d\theta(x(\sigma'_\alpha))}{d\sigma'_\alpha} = \theta(x_0) + \int_0^{\sigma_\alpha} d\sigma'_\alpha \nabla_x \theta(x(\sigma'_\alpha)) \cdot \frac{dx(\sigma'_\alpha)}{d\sigma'_\alpha} = \theta(x_0) - \int_0^{\sigma_\alpha} d\sigma'_\alpha \ k(\sigma'_\alpha) \cdot \nabla_k D_\alpha(x(\sigma'_\alpha), k(\sigma'_\alpha)).
$$

(3.5.31)

The ray $(x(\sigma_\alpha), k(\sigma_\alpha))$ lives in ray phase space, but we can project it down to $x$-space by ‘forgetting’ the wavevector $k$. For each $\sigma_\alpha$ we have $x(\sigma_\alpha)$, and, therefore, we can assign, (assuming the projection is well-behaved), $\theta(\sigma_\alpha) = \theta(x(\sigma_\alpha))$. In this way, we can use (3.5.31) and a family of rays to construct the phase $\theta(x)$.

The ray orbit parameter, $\sigma_\alpha$, is related to the physical time, $t$, by another pair of Hamilton equations, reflecting the fact that they are also a conjugate pair$^2$:

$$
\frac{dt}{d\sigma_\alpha} = \frac{\partial D_\alpha}{\partial \omega}.
$$

(3.5.32)

---

$^2$We only quote the result here and do not derive it. For a complete discussion and derivation see Ref. [15].
Recall, we require $D_\alpha = 0$, from which it follows that

$$\frac{dD_\alpha}{d\sigma_\alpha} = 0$$

$$= \frac{\partial D_\alpha}{\partial t} \frac{dt}{d\sigma_\alpha} + \frac{\partial D_\alpha}{\partial \omega} \frac{d\omega}{d\sigma_\alpha}$$

$$= \frac{\partial D_\alpha}{\partial t} \frac{\partial D_\alpha}{\partial \omega} + \frac{\partial D_\alpha}{\partial \omega} \frac{d\omega}{d\sigma_\alpha}$$

$$\Rightarrow \frac{d\omega}{d\sigma_\alpha} = -\frac{\partial D_\alpha}{\partial t}.$$  (3.5.33)

Since we are assuming that $D_\alpha$ does not explicitly depend on time, we have

$$\frac{d\omega}{d\sigma_\alpha} = -\frac{\partial D_\alpha}{\partial t} = 0,$$  (3.5.34)

which is consistent with our restriction to a specific frequency $\omega$ for all rays. However, we can treat $\omega$ as a variable, rather than a fixed parameter. Then we have,

$$D_\alpha = D_\alpha(x, k, \omega),$$  (3.5.35)

which is a function of $2n + 1$ variables. Now, the condition $D_\alpha = 0$ defines the surface

$$\omega = \Omega_\alpha(x, k),$$  (3.5.36)

which is a local dispersion relation.

Now, let us see what happens to the ray evolution equations if we change the parameter-
ization from $\sigma_\alpha$ to $t$. First, we multiply Equation (3.5.26) by $d\sigma_\alpha/dt$:

$$\left( \frac{dx}{d\sigma_\alpha} = -\nabla_k D_\alpha \right) \frac{d\sigma_\alpha}{dt} \Rightarrow \frac{dx}{dt} = -\nabla_k D_\alpha = v_\text{g}, \quad (3.5.37)$$

where we have used (3.5.32) and the definition of the group velocity. If we multiply (3.5.30) by $d\sigma_\alpha/dt$ and, again, make use of (3.5.32), we have

$$\left( \frac{dk}{d\sigma_\alpha} = \nabla_x D_\alpha \right) \frac{d\sigma_\alpha}{dt} \Rightarrow \frac{dk}{dt} = \frac{\nabla_x D_\alpha}{\partial D_\alpha/\partial \omega}. \quad (3.5.38)$$

Additionally, the condition that $D_\alpha(x, k, \omega = \Omega_\alpha(x, k)) = 0$ implies

$$\nabla_k D_\alpha + \frac{\partial D_\alpha}{\partial \omega} \nabla_k \Omega_\alpha = 0, \quad (3.5.39)$$

and

$$\nabla_x D_\alpha + \frac{\partial D_\alpha}{\partial \omega} \nabla_x \Omega_\alpha. \quad (3.5.40)$$

Using (3.5.39) and (3.5.40) we can write (3.5.37) and (3.5.38) as

$$\frac{dx}{dt} = \nabla_k \Omega_\alpha, \quad (3.5.41)$$

and

$$\frac{dk}{dt} = -\nabla_x \Omega_\alpha, \quad (3.5.42)$$

which are the more familiar forms of Hamilton’s equations.
3.5.3 Ray Equations for a Nonuniform, Unmagnetized Plasma

As a final example, let us consider the case of a nonuniform, unmagnetized plasma in three-dimensions. Recall, the wave equation for the perturbed electric field, \( E^1(x) \), is

\[
\nabla^2 E^1(x) - \nabla \left[ \nabla \cdot E^1(x) \right] + \frac{\omega^2}{c^2} E^1(x) - \frac{\omega_p^2(x)}{c^2} E^1(x) = 0, \tag{3.5.43}
\]

where the plasma frequency \( \omega_p(x) \) can usually be approximated by the electron plasma frequency \( \omega_e(x) \). Assuming that \( E^1(x) \) is of eikonal form, we can easily compute the symbol of the wave operator and write it in matrix form:

\[
D(x, k; \omega) = \begin{pmatrix}
D_0(x, k; \omega) + k_x^2 & k_xk_y & k_xk_z \\
k_xk_y & D_0(x, k; \omega) + k_y^2 & k_yk_z \\
k_xk_y & k_yk_z & D_0(x, k; \omega) + k_z^2
\end{pmatrix}, \tag{3.5.44}
\]

where

\[
D_0(x, k; \omega) = \frac{\omega^2}{c^2} - \frac{\omega_p^2(x)}{c^2} - k^2, \tag{3.5.45}
\]

and \( k^2 = k_x^2 + k_y^2 + k_z^2 \).

In order to use Hamilton’s equations we need to find the eigenvalues of the dispersion matrix \( D(x, k; \omega) \), which are

\[
D_{1,2}(x, k; \omega) = D_0(x, k; \omega) = \omega^2 - \omega_p^2(x) - c^2 k^2 \tag{3.5.46}
\]

\[
D_3(x, k; \omega) = \omega^2 - \omega_p^2(x). \tag{3.5.47}
\]
Note, the eigenvalue \( D_{1,2} \) corresponds to the eigenvector perpendicular to the wavevector \( \mathbf{k} \), and \( D_3 \) corresponds to the eigenvector parallel to \( \mathbf{k} \). Now, we can use the eigenvectors to write out Hamilton's equations for the evolution of rays. For \( D_{1,2} = \omega^2 - \omega_p^2(x) - c^2 k^2 \) we have

\[
\frac{dx}{d\sigma_{1,2}} = -\frac{\partial D_{1,2}}{\partial k_{\mu}} = 2k_{\mu} \quad \Rightarrow \quad \frac{dx}{d\sigma_{1,2}} = 2k \tag{3.5.48}
\]

\[
\frac{dk_{\mu}}{d\sigma_{1,2}} = \frac{\partial D_{1,2}}{\partial x_{\mu}} = -2\omega_p(x) \frac{\partial \omega_p}{\partial x_{\mu}} \quad \Rightarrow \quad \frac{dk}{d\sigma_{1,2}} = -2\omega_p(x)\nabla \omega_p(x), \tag{3.5.49}
\]

and \( D_3(x, k; \omega) = \omega^2 - \omega_p^2(x) \) we get

\[
\frac{dx}{d\sigma_3} = -\frac{\partial D_3}{\partial k_{\mu}} = 0 \quad \Rightarrow \quad \frac{dx}{d\sigma_3} = 0 \tag{3.5.50}
\]

\[
\frac{dk_{\mu}}{d\sigma_3} = \frac{\partial D_3}{\partial x_{\mu}} = -2\omega_p(x) \frac{\partial \omega_p}{\partial x_{\mu}} \quad \Rightarrow \quad \frac{dk}{d\sigma_3} = -2\omega_p(x)\nabla \omega_p(x). \tag{3.5.51}
\]

Furthermore, for both \( D_{1,2} \) and \( D_3 \),

\[
\frac{dt}{d\sigma_{1,2,3}} = \frac{\partial D_{1,2,3}}{\partial \omega} = 2\omega. \tag{3.5.52}
\]

Thus, the ray equations, become

\[
\frac{dx}{dt} = \frac{2k}{2\omega} = \frac{k}{\omega}, \tag{3.5.53}
\]

\[
\frac{dk}{dt} = -\frac{\omega_p(x)\nabla \omega_p(x)}{\omega}, \tag{3.5.54}
\]
when $D_\alpha = D_{1,2}$, and

\[
\frac{dx}{dt} = 0
\]
\[
\frac{dk}{dt} = -\frac{\omega_p(x) \nabla \omega_p(x)}{\omega},
\]

when $D_\alpha = D_3$.

Given a specific form for the plasma frequency $\omega_p(x)$, one can solve (typically numerically) Hamilton’s equations for the rays, subject to the appropriate boundary conditions. Once a family of ray trajectories are known, the eikonal phase can be computed throughout that region of space. Additionally, it is possible to reconstruct the amplitude of the wave following the rays. For the complete details see Ref. [15].

### 3.6 Ray Tracing, Mode Conversion, and the Normal Form

The purpose of the previous sections was to introduce some of the underlying structure that will serve as our foundation as we move on to the discussion of mode conversion and the normal form. Additionally, it was necessary to explain the relationship between rays and the dispersion surfaces, and there usefulness in solving wave equations. Furthermore, throughout our previous discussions in this chapter, we have constrained ourselves to the cases where the eigenvalues, (which serve as the ray Hamiltonians), were not degenerate. However, if we recall the discussion of the physical application underpinning all of this work, it is precisely a conversion between two wave modes that is of interest to us. That is, there are regions in ray
phase space where the eigenvalues of the dispersion matrix are degenerate. The consequence for ray tracing is that one ray traveling into a mode conversion region can split into two rays, a transmitted ray and a converted ray. Therefore, it is necessary to say a few words about ray tracing and mode conversion.

First, it should be noted that the eikonal assumption is not valid in a mode conversion region, due to the fact that the polarization vectors (i.e., the eigenvectors \( \hat{e}_\alpha \)) can change rapidly in the mode conversion region, which violates our assumption that the polarization basis is slowly-varying. However, this does not mean we have to abandon ray tracing techniques. Indeed, Tracy et. al. have shown extensively, (see Refs. [12–14]), that ray tracing techniques are extremely useful provided that one accepts the fact that the eikonal assumption breaks down in the mode conversion region. In particular, one can asymptotically match the incoming and outgoing waves (of each mode) through the mode conversion region since the eikonal ansatz is valid away from the conversion area. Thus, the question becomes, what is the appropriate function to use as the Hamiltonian for computing the rays?

We could use the determinant of the dispersion matrix \( D(x,k;\omega) \), since the determinant is just the product of the eigenvalues. Thus, for a \( 3 \times 3 \) dispersion matrix,

\[
\det D = D_1 D_2 D_3,
\]

where \( D_\alpha \) are the eigenvalues of \( D \). The advantage of using the determinant of the dispersion matrix as the ray Hamiltonian is that it has the same properties as the eigenvalues, namely, \( \det D = 0 \) when \( D_1 = 0 \) or \( D_2 = 0 \) or \( D_3 = 0 \), but we do not have to find the eigenvalues.
themselves, which saves computational time. Additionally, it is fairly straightforward to show that using the determinant as the ray Hamiltonian, rather than the eigenvalues, results in the same rays but with a change in the ray parameter.³

Although it is possible to use the determinant of the dispersion matrix as the ray Hamiltonian, this leads to the avoided crossings as seen in Figure 2.28 in the previous chapter. This results in an incoming ray connecting smoothly with a converted ray, but not with its transmitted ray. In order to ‘see’ how and where the rays convert we need dispersion functions (i.e., ray Hamiltonians) that represent the uncoupled modes. This is where the normal form transformation comes in, as demonstrated in Figure 2.29b in Chapter 2. Notice how away from the mode conversion region the uncoupled dispersion surfaces asymptote to the zero surface of the determinant of the original matrix. It is this normal form transformation that is the subject of the next chapter and the remainder of this thesis.

³For details, consult Ref. [15].
CHAPTER 4

The Normal Form

Now that we have (briefly) discussed ray tracing as a method for solving multicomponent, vector wave equations, and have seen the importance of the dispersion functions, (they are the ray Hamiltonians), we can return to where we left off at the end of Chapter 2, namely how to put a $2 \times 2$ dispersion matrix into normal form. First, let us briefly talk about the concept of the normal form in a broader mathematical sense.

4.1 The Normal Form Concept

The main idea of the normal form is to cast a system, which, in the most general sense is just a set of mathematical objects, into its simplest form.\textsuperscript{1} What one means by ‘simplest form’ is, of course, a vague notion and can vary widely from system to system and situation to situation. As an example, consider the set of dispersion functions in $x$ and $k$ up to second

\textsuperscript{1}This section is based largely on Appendix F in Ref. [15].
order. An example of a member of this set would be the polynomial

\[ D(x,k) = c_0 + c_1 x + c_2 k + c_3 x^2 + c_4 xk + c_5 k^2. \]  \hspace{1cm} (4.1.1)

If we focus on the parameters of \( D(x,k) \) and write \( D = D(c) \), treating \((1, x, k, x^2, xk, k^2)\) as basis vectors, then the set of polynomials in \( x \) and \( k \) up to second order is isomorphic to \( \mathbb{R}^6 \), the six-dimensional real space. Furthermore, since \( D(x,k) \) depends linearly on the parameters, this is a linear vector space.

Now, let us introduce a group of transformations, (such as the group of coordinate transformations), that act on this vector space. If we define \( G = \{ g \} \) as the group of these transformations, then, for a particular transformation, \( g \in G \), we can think of this as a transformation acting upon the six-dimensional vector \( c \), which are the parameters for a certain \( D(x,k) \). In this case, the action of \( g \) on \( c \) is

\[ c \xrightarrow{g} c'. \]  \hspace{1cm} (4.1.2)
FIG. 4.1: A conceptual diagram showing the vector space $c$. The action of a particular transformation $g$ on an arbitrary, but fixed, point $c_0$ takes it to the point $c_0' : c_0 \xrightarrow{g} c_0'$.

Suppose $G$ is the group of linear canonical transformations of the type $a \xrightarrow{g(a)} a^{-1}x$ and $k \xrightarrow{g(a)} ak$, where $a \in \mathbb{R}^1$ and is non-zero. In this case, the vector $c$ is transformed as

$$c \xrightarrow{g(a)} c'(a) = \left( c_0, \frac{c_1}{a}, ac_2, \frac{c_3}{a^2}, c_4, a^2 c_5 \right). \quad (4.1.3)$$

Now, let us consider the action of the entire group of transformations $G$ on an arbitrary, but fixed point $c_0$ in the six-dimensional vector space, which is the parameter space of quadratic dispersion functions. The full set of points

$$c'_0(a) = \left( c_{00}, \frac{c_{01}}{a}, ac_{02}, \frac{c_{03}}{a^2}, c_{04}, a^2 c_{05} \right), \quad (4.1.4)$$

\footnote{We can think of this as a \textit{passive} transformation on the basis vectors.}
where $a$ ranges over the whole real line with zero removed, is called the orbit of the point $c_0$ under the group of canonical dilations. As can be seen in Figure 4.2, the orbit traces out a one-dimensional curve.

**FIG. 4.2:** A conceptual diagram showing the action of the group $G$ on an arbitrary, but fixed point $c_0$ of the parameter space. The action of $G$ on $c_0$ generates a set of points called the orbit of $c_0$. Every point on the curve is an image of $c_0$ under some $g \in G$.

The next step is to use the orbits to introduce an equivalence relation on the set of points in our six-dimensional vector space. Namely, we shall say that any two points $c_0$ and $c'_0(a)$ that lie on the same orbit are equivalent, since there is some element of the group $G$ that transforms one point into the other. Furthermore, points that do not lie on the same orbit are not equivalent. For example, if the transformations are associated with a change in representation, then the transformations do not alter the underlying physics. Hence, the introduction of this equivalence relation shifts the emphasis from the space of points to the space of orbits. As we shall see later in the chapter, there are certain combinations of the
parameters which do not change along an orbit, but vary between orbits; and it is with these *invariants* that we can label the orbits.

Now that we have declared all of the points on an orbit equivalent, we can choose ‘the simplest’ one with which to work. This is, to be sure, a vague notion; however, we are often guided by the underlying physics of the problem as to which is the simplest. For instance, we may want to eliminate particular variables or display the invariants in a certain way. Rather than continuing with an abstract example, let us move on to discussing the normal form for $2 \times 2$ dispersion matrices.

### 4.2 The Normal Form of the Reduced $2 \times 2$ Dispersion Matrix

Rather than starting with our specific $2 \times 2$ dispersion matrix from Chapter 2, let us begin our discussion of the normal form procedure with a generic $2 \times 2$ dispersion matrix $D$, in a four-dimensional space $(x, y, k_x, k_y)$. Namely,

$$D(z) = \begin{pmatrix} D_{11}(z) & D_{12}(z) \\ D_{12}^*(z) & D_{22}(z) \end{pmatrix}, \quad (4.2.1)$$

where $z = (x, y, k_x, k_y)$. We assume that $D$ is an Hermitian matrix, $(D = D^\dagger)$, and that entries of the matrix are smooth functions of the four-dimensional phase space.
4.2.1 The Spinor Decomposition of $D$

We now want to show that the space of $2 \times 2$ Hermitian matrices is isomorphic to the real vector space $\mathbb{R}^4$. First, we utilize the Pauli spin matrices (with the addition of the $2 \times 2$ identity matrix) as a basis for the space of $2 \times 2$ Hermitian matrices. Second, we turn the vector space into an inner product space by defining the inner product of two $2 \times 2$ Hermitian matrices as

$$ (A, B) = \frac{1}{2} \text{Tr}[A^\dagger B]. \tag{4.2.2} $$

Using the above inner product, we decompose our dispersion matrix $D$ onto the basis of Pauli spin matrices, and define the following real valued functions of our four-dimensional phase space:

$$ B_0(z) = \frac{1}{2} \text{Tr}[D \cdot \sigma^0] = \frac{1}{2} (D_{11} + D_{22}) \tag{4.2.3} $$

$$ B_1(z) = \frac{1}{2} \text{Tr}[D \cdot \sigma^1] = \frac{1}{2} (D_{12} + D_{21}) = \text{Re}(D_{12}) \tag{4.2.4} $$

$$ B_2(z) = \frac{1}{2} \text{Tr}[D \cdot \sigma^2] = \frac{i}{2} (D_{21} - D_{12}) = \text{Im}(D_{12}) \tag{4.2.5} $$

$$ B_3(z) = \frac{1}{2} \text{Tr}[D \cdot \sigma^3] = \frac{1}{2} (D_{11} - D_{22}), \tag{4.2.6} $$
where

\[
\begin{align*}
\sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\
\sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

and we have used the fact that \(D\) is Hermitian. The functions \(B_i(z)\) are the entries of the four-dimensional real vector \(B(z)\), such that

\[
B(z) = \begin{pmatrix} B_0(z) \\ B_1(z) \\ B_2(z) \\ B_3(z) \end{pmatrix}. \tag{4.2.7}
\]

Thus, the vector \(B(z)\) is an element of the vector space \(\mathbb{R}^4\), with the usual Euclidean inner product. Additionally, we can go from the vector \(B\) to the (Hermitian) matrix \(D\) by writing

\[
D(z) = B_\mu \sigma^\mu = B_0 \sigma_0 + B_1 \sigma_1 + B_2 \sigma_2 + B_3 \sigma_3 = \begin{pmatrix} B_0(z) + B_3(z) & B_1(z) + iB_2(z) \\ B_1(z) - iB_2(z) & B_0(z) - B_3(z) \end{pmatrix}. \tag{4.2.8}
\]
This is called the ‘B-representation’ of the dispersion matrix. Note, that every $2 \times 2$ Hermitian matrix $D$ is a point in the real, four-dimensional space $(B_0, B_1, B_2, B_3)$. Thus, we have shown that the space of $2 \times 2$ Hermitian matrices with the inner product, $(A, B) = \frac{1}{2} \text{Tr}[A^\dagger B]$, is indeed isomorphic to the vector space $\mathbb{R}^4$.

Recall, we are interested in the dispersion relation defined by the condition

$$
\text{det} D = 0. \quad (4.2.9)
$$

This relation defines the dispersion manifold, which is, generically, a three-dimensional surface imbedded in the larger four-dimensional phase space. Using Equations (4.2.3)-(4.2.6), we can express the dispersion relation in terms of $B_0, B_1, B_2,$ and $B_3$,

$$
\text{det} D = D_{11}D_{22} - D_{12}D_{21} = B_0^2 - B_1^2 - B_2^2 - B_3^2. \quad (4.2.10)
$$

Notice that, in terms of the $B'$s, $\text{det} D = B_\mu B^\mu = B_\mu \eta^{\mu\nu}B_\nu$, where $\eta^{\mu\nu} = \eta_{\mu\nu}$ is the Minkowski tensor

$$
\eta^{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \quad (4.2.11)
$$

Thus, the condition that $\text{det} D = 0$, is equivalent to the condition that $B_\mu B^\mu = 0$, i.e., that the inner product of the 4-vector $B_\mu$ with itself vanish, where $B_\mu$ is a 4-vector in the usual
four-dimensional spacetime with metric $\eta^{\mu\nu}$.

Now, as you may recall, our ultimate goal is to find a polarization basis for the dispersion matrix $D$ in which the diagonal entries of the transformed dispersion matrix $D'$ are the uncoupled dispersion relations. Then, using the techniques outlined in the previous chapter, the diagonals will serve as the ray Hamiltonians for the uncoupled rays. Additionally, as mentioned in Chapters 2 and 3, we want the off-diagonal terms to be constant along a ray generated by the uncoupled ray Hamiltonian. Thus, we seek a representation where the diagonals Poisson commute with the off-diagonals. That is,\footnote{Since $D'$ is Hermitian, and $D_{21} = D_{12}^\dagger$, $\{D'_{11}, D'_{12}\} = 0,$ is implied.}

$$\{D'_{11}, D'_{12}\} = \{D'_{22}, D'_{12}\} = 0.$$ \hspace{1cm} (4.2.12)

The dispersion matrix expressed in this polarization basis is said to be in the normal form. From our discussion in the previous chapter, Section 3.2, the class of congruence transformations is the logical class to use in order to put $D$ into normal form. Under congruence transformation

$$D \rightarrow D'(\mathbf{z}) = Q^\dagger(\mathbf{z})D(\mathbf{z})Q(\mathbf{z}).$$ \hspace{1cm} (4.2.13)

This implies that

$$D' = Q^\dagger B_\mu \sigma^\mu Q = B_\mu Q^\dagger \sigma^\mu Q.$$ \hspace{1cm} (4.2.14)

Since $D'$ is Hermitian, $Q^\dagger \sigma^\mu Q$ must also be Hermitian; thus, it has a spinor composition. We
define the expansion coefficients of $Q^{\dagger} \sigma^\mu Q$ such that

$$Q^{\dagger} \sigma^\mu Q \equiv \Lambda^\mu_v \sigma^v. \quad (4.2.15)$$

It is important to note that the mapping from $Q$ to $\Lambda$ is not one-to-one; it is independent of the overall phase of $Q$. Hence, from equation (4.2.15), together with Equation (4.2.14) and the fact that $D' = B'_v \sigma^v$, it follows that

$$B'_v(z) = \Lambda^\mu_v(z) B_\mu(z). \quad (4.2.16)$$

Thus, we can interpret $\Lambda$ as a linear transformation acting on the vector $B$.

Now, let us consider what happens to the determinant of $D$ under the action of a congruence transformation. First, following from Equation (4.2.13), we know that

$$\det D' = |\det Q|^2 \det D. \quad (4.2.17)$$

In the $B$-representation, this becomes

$$B'_\mu B'^\mu = B'_\mu \eta^{\mu\nu} B'_\nu = |\det Q|^2 B_\alpha B^\alpha = |\det Q|^2 B_\alpha \eta^{\alpha\beta} B_\beta. \quad (4.2.18)$$

Hence, if we restrict our congruence transformations to have $\det Q = 1$, we have

$$B'_\mu \eta^{\mu\nu} B'_\nu = B_\alpha \eta^{\alpha\beta} B_\beta. \quad (4.2.19)$$
Additionally, if $\det Q = 1$, the determinant of $D$ is left invariant under this class of congruence transformations. Connecting this back to our discussion of the normal form concept in Section 4.1, and in particular to our discussion surrounding Figure 4.2 on Page 150, the determinant plays the role of the invariant that labels the orbits\(^4\) and the group of congruence transformations, (with $\det Q = 1$), are the operators that take one point on an orbit and carry it to a different point on the same orbit. Returning to Equation (4.2.19) and using (4.2.16) we get,

$$
(B_\alpha^n \Lambda^\alpha_\mu) \eta^{\mu \nu} (B_\beta^n \Lambda^\beta_\nu) = B_\alpha \eta^{\alpha \beta} B_\beta.
$$

This implies that

$$
\Lambda^\alpha_\mu \eta^{\mu \nu} \Lambda^\beta_\nu = \eta^{\alpha \beta},
$$

or, expressed in matrix notation,

$$
\Lambda^T \eta \Lambda = \eta.
$$

Thus, the transformation $\Lambda$ leaves the Minkowski tensor invariant and is, therefore, a Lorentz transformation! This means we can use the theory of the Lorentz group to develop the normal form theory for $2 \times 2$ dispersion matrices. The step first is to define the, so called, $\Omega$-tensor!\(^4\)

\(^4\)The determinant of the dispersion matrix is one particular invariant. There are other invariants, such as the so-called invariant coupling constant, with which we could also label the orbits. See Ref. [15], Appendix F.
4.2.2 The $\Omega$-Tensor

The $\Omega$-tensor is a $4 \times 4$ anti-symmetric (second rank) tensor, whose entries are defined by the relation

$$\Omega^\mu_\nu(z) \equiv \{B^\mu(z), B^\nu(z)\},$$

(4.2.23)

where $\{f(z), g(z)\}$ is the Poisson bracket for a four-dimensional phase space. Written out explicitly it is,

$$\{f(z), g(z)\} = (\nabla_z f)^T J (\nabla_z g),$$

(4.2.24)

where $J$ is the $4 \times 4$ symplectic matrix. Note that the $\Omega$-tensor is invariant under a canonical transformation of the four-dimensional phase space, $z \to z'(z)$, since its entries, the pairwise Poisson brackets of the four $B$'s, are symplectic invariants.

We want to consider how the $\Omega$-tensor transforms when we transform the dispersion matrix, $D$, via a congruence transformation. First, let us write the tensor as the matrix

$$\Omega = \{B, B^T\},$$

(4.2.25)

where

$$B = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} \quad \text{and} \quad B^T = \begin{pmatrix} B^0 & B^1 & B^2 & B^3 \end{pmatrix}.$$
Under a constant congruence transformation, the vectors $\mathbf{B}$ and $\mathbf{B}^T$, transform as

\begin{align*}
\mathbf{B}' &= \Lambda \mathbf{B} \quad (4.2.27) \\
\mathbf{B'}^T &= \mathbf{B}^T \Lambda^T, \quad (4.2.28)
\end{align*}

where the entries of the Lorentz matrix $\Lambda$ are constant. Therefore,

\begin{align*}
\Omega' &= \{ \mathbf{B}',\mathbf{B'}^T \} = \Lambda \Omega \Lambda^T. \quad (4.2.29)
\end{align*}

Note, that since $\Omega_{\psi}^\mu$ is a bilinear form it is natural that it transforms via congruence, as Equation (4.2.29) implies.

Now consider the tensor $\eta \Omega$. Since the elements of the Lorentz group satisfy the identity

\begin{align*}
\Lambda^T \eta \Lambda &= \eta, \quad (4.2.30)
\end{align*}

it follows that

\begin{align*}
\eta \Lambda &= (\Lambda^T)^{-1} \eta. \quad (4.2.31)
\end{align*}

Hence,

\begin{align*}
\eta \Omega' &= \eta \Lambda \Omega \Lambda^T = (\Lambda^T)^{-1} (\eta \Omega) \Lambda^T, \quad (4.2.32)
\end{align*}

that is, the tensor $\eta \Omega$ transforms by a similarity transformation, (hence, $\eta \Omega$ is an operator). As such, the coefficients of its characteristic polynomial are both symplectic and congruence
invariants:

\[ \mathcal{P}_\Omega(\lambda) = \det(\eta \Omega - \lambda) = \lambda^4 - \frac{1}{2} \text{tr}[(\eta \Omega)^2] \lambda^2 + \det(\eta \Omega) \]  

(4.2.33)

Let us write the \( \Omega \) tensor in the following way

\[ \Omega_{\alpha\beta} = \{B_\alpha, B_\beta\} = \begin{pmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 \\
-\gamma_1 & 0 & -\omega_3 & \omega_2 \\
-\gamma_2 & \omega_3 & 0 & -\omega_1 \\
-\gamma_3 & -\omega_2 & \omega_1 & 0
\end{pmatrix}, \]  

(4.2.34)

where

\[ \gamma_1 = \{B_0, B_1\} \]  

(4.2.35)

\[ \gamma_2 = \{B_0, B_2\} \]  

(4.2.36)

\[ \gamma_3 = \{B_0, B_3\} \]  

(4.2.37)

\[ \omega_1 = \{B_3, B_2\} \]  

(4.2.38)

\[ \omega_2 = \{B_1, B_3\} \]  

(4.2.39)

\[ \omega_3 = \{B_2, B_1\}. \]  

(4.2.40)
Additionally, we can organize the $\gamma_i$'s and the $\omega_j$'s into vectors that live in $\mathbb{R}^3$:

$$
\gamma = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{pmatrix} \quad (4.2.41)
$$

$$
\omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} . \quad (4.2.42)
$$

Written in this form, it is fairly straightforward to show that the invariant coefficients of the characteristic polynomial, Equation (4.2.33), are

$$
I_1 = -\frac{1}{2} \text{tr}[(\eta \Omega)^2] = \omega^2 - \gamma^2 \quad (4.2.43)
$$

$$
I_2 = \det(\eta \Omega) = -(\omega \cdot \gamma)^2, \quad (4.2.44)
$$

where $\omega^2 = \omega \cdot \omega$ and $\gamma^2 = \gamma \cdot \gamma$, as usual. (N.B., these invariants are extremely significant, and we will return to them later.

Recall that, our goal is to transform the dispersion matrix $D$, defined in Equation (4.2.1), into the normal form matrix $D'$, where the diagonals are the uncoupled dispersion relations and Poisson commute with the off-diagonals. As can be seen from Equation (4.2.8), this is equivalent to saying $\{B'_0, B'_1\} = \{B'_0, B'_2\} = \{B'_3, B'_1\} = \{B'_3, B'_2\} = 0$. This implies that we want to transform the $\Omega$-tensor from $\Omega$ to the tensor $\Omega'$ where $\gamma'_1 = \gamma'_2 = \omega'_1 = \omega'_2 = 0$. 

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4.2.3 The Normal Form Transformation

Before using the normal form transformation on our particular $2 \times 2$ dispersion matrix, Equation (2.5.6), we want to develop a general procedure for putting any $2 \times 2$ dispersion matrix into normal form. Following the results of the previous section, we know that the task of putting the reduced $2 \times 2$ dispersion matrix into normal form; that is, into the form where the diagonals are the uncoupled ray Hamiltonians and the off-diagonals are the coupling constants, can be recast, in terms of the $\Omega$-tensor, as defined in Equation (4.2.34), into the task of finding the appropriate combination of Lorentz transformations, (i.e., the combination of rotations and boosts), that transform the vectors $\gamma$ and $\omega$, Equations (4.2.41) and (4.2.42), into the vectors

$$
\gamma' = \begin{pmatrix}
0 \\
0 \\
\gamma'_3 \\
\end{pmatrix}
$$

(4.2.45)

$$
\omega' = \begin{pmatrix}
0 \\
0 \\
\omega'_3 \\
\end{pmatrix}
$$

(4.2.46)

Recall that the entries of the $\Omega$-tensor are functions of the four-dimensional ray phase space $(x, y, k_x, k_y)$. Hence, $\gamma(z)$ and $\omega(z)$ are also functions of phase space. Now, in general, i.e., at most points in our four-dimensional ray phase space, it will be true that $\gamma(z) \cdot \omega(z) \neq 0$. In fact, the condition $\gamma \cdot \omega = 0$ defines a three-dimensional manifold imbedded in the larger four-
dimensional phase space. Note, since $\omega \cdot \gamma = \sqrt{-I_2}$, the three-dimensional surface $\gamma \cdot \omega = 0$ is equivalent to the three-dimensional surface defined by the condition $I_2 = 0$. Hence, the surface $\gamma \cdot \omega = 0$ is invariant under any Lorentz or linear canonical transformation. It is important to identify this surface, as the following normal form procedure breaks down at points where $I_1$ and $I_2 = 0$. However, since $I_1 \neq 0$ and $I_2 \neq 0$ at nearly all the points in our ray-phase space, we may continue with the development of our normal form transformation.

The vectors $\omega$ and $\gamma$ define a plane in $\mathbb{R}^3$, and satisfy, in general, $\omega \cdot \gamma \neq 0$. An example of two such vectors is shown in Figures 4.3a and 4.3b.

![FIG. 4.3: An example of the vectors $\omega$ and $\gamma$ in $\mathbb{R}^3$, shown from two different viewpoints.](image)

First, we perform a rotation, denoted $\Lambda_1$, which puts $\omega'$ along the $z$-axis, i.e., it zeros out the $x$- and $y$-components of $\omega'$. The resulting vectors, $\omega'$ and $\gamma'$ are shown in Figure 4.4.
FIG. 4.4: The vectors $\omega'$ and $\gamma'$ after application of the Lorentz transformation $\Lambda_1$. Note, $\omega'$ lies entirely in the $z$-direction, and $\gamma'$ still has a non-zero $z$-component.

After the action of $\Lambda_1$, we have

$$\gamma \rightarrow \gamma' = \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \\ \gamma'_3 \end{pmatrix}$$

(4.2.47)

$$\omega \rightarrow \omega' = \begin{pmatrix} 0 \\ 0 \\ \omega'_3 \end{pmatrix},$$

(4.2.48)
and

\[ \Omega' = \Lambda_1 \Omega_1^T = \begin{pmatrix} 0 & \gamma_1' & \gamma_2' & \gamma_3' \\ -\gamma_1' & 0 & -\omega_3' & 0 \\ -\gamma_2' & \omega_3' & 0 & 0 \\ -\gamma_3' & 0 & 0 & 0 \end{pmatrix}. \] (4.2.49)

Second, we perform an additional rotation, \( \Lambda_2 \), this time about the \( z \)-axis, so that \( \gamma' \) lies in the \( x-z \) plane. Note, since this rotation is about the \( z \)-axis, it leaves \( \omega' \) invariant. We now have 5

\[ \gamma' \rightarrow \gamma'' = \begin{pmatrix} \gamma''_1 \\ 0 \\ \gamma''_3 \end{pmatrix} \] (4.2.50)

\[ \omega' \rightarrow \omega'' = \begin{pmatrix} 0 \\ 0 \\ \omega''_3 \end{pmatrix} \] (4.2.51)

\[ \Omega'' = \Lambda_2 \Omega' \Lambda_2^T = \begin{pmatrix} 0 & \gamma_1'' & 0 & \gamma_3'' \\ -\gamma_1'' & 0 & -\omega''_3 & 0 \\ 0 & \omega''_3 & 0 & 0 \\ -\gamma_3'' & 0 & 0 & 0 \end{pmatrix}. \] (4.2.52)

The vectors \( \omega'' \) and \( \gamma'' \) are shown in Figure 4.5.

---

5There is a slight difference between our \( \gamma'' \) and \( \omega'' \), as defined above, and the \( \gamma'' = (0, 0, \gamma''_3) \) and \( \omega'' = (\omega''_1, 0, \omega''_3) \) derived following Littlejohn’s procedure in Ref. [8]. As we shall see in the next chapter, when we construct the \( \Omega \)-tensor for our \( 2 \times 2 \) dispersion matrix, (2.5.6), \( \omega_3 = 0 \). Thus, it is more straightforward to construct a rotation that sets \( \omega''_1 \) and \( \gamma''_2 \) equal to zero, rather than \( \omega''_3 \) and \( \gamma''_3 \).
Third, we perform a boost in the $y$-direction, denoted $\Lambda_3$, which leaves $\omega^{(3)}$ and $\gamma^{(3)6}$ still

\footnote{The notation $\gamma^{(3)}$ is to be read as: “The vector gamma triple-prime.”}
lying in the $x$-$z$ plane, such that they are collinear:

$$
\gamma'' \rightarrow \gamma^{(3)} = \begin{pmatrix}
\gamma_1^{(3)} \\
0 \\
\gamma_3^{(3)}
\end{pmatrix},
$$  \hspace{1cm} (4.2.53)

$$
\omega'' \rightarrow \omega^{(3)} = \begin{pmatrix}
\omega_1^{(3)} \\
0 \\
\omega_3^{(3)}
\end{pmatrix},
$$  \hspace{1cm} (4.2.54)

$$
\Omega^{(3)} = \Lambda_3 \Omega'' \Lambda_3^T = \begin{pmatrix}
0 & \gamma_1^{(3)} & 0 & \gamma_3^{(3)} \\
-\gamma_1^{(3)} & 0 & -\omega_3^{(3)} & 0 \\
0 & \omega_3^{(3)} & 0 & -\omega_1^{(3)} \\
-\gamma_3^{(3)} & 0 & \omega_1^{(3)} & 0
\end{pmatrix},
$$  \hspace{1cm} (4.2.55)

and

$$
\gamma^{(3)} \times \omega^{(3)} = 0.
$$  \hspace{1cm} (4.2.56)

The resultant vectors, $\omega^{(3)}$ and $\gamma^{(3)}$, are shown in Figure 4.6.
FIG. 4.6: $\omega^{(3)}$ and $\gamma^{(3)}$ after the Lorentz boost $\Lambda_3$. Both vectors are still in the $x$-$z$ plane, but they are now collinear.

Now, in terms of the parameter $\beta$, (which, in special relativity, would be the relative velocity) a boost in the $y$-direction is

$$A = \begin{pmatrix} (1 - \beta^2)^{-1/2} & 0 & -\beta(1 - \beta^2)^{-1/2} & 0 \\ 0 & 1 & 0 & 0 \\ -\beta(1 - \beta^2)^{-1/2} & 0 & (1 - \beta^2)^{-1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.2.57)$$
Applying this boost to our tensor $\Omega''$ we get

$$\gamma^{(3)} = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix} \gamma'_1 - \beta \omega'_3 \\ 0 \\ \gamma'_3 \\ 0 \\ \gamma'_3 - \beta \gamma'_1 \end{pmatrix},$$

(4.2.58)

and

$$\omega^{(3)} = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix} \beta \gamma'_3 \\ 0 \\ \omega'_3 - \beta \gamma'_1 \end{pmatrix}.$$

(4.2.59)

We want to choose $\beta$ so that $\gamma^{(3)} \times \omega^{(3)} = 0$. As we can see from the form of $\gamma^{(3)}$ and $\omega^{(3)}$ in Equations (4.2.58) and (4.2.59), only the $y$-component of the cross product will be non-zero. Hence,

$$\left( \gamma^{(3)} \times \omega^{(3)} \right)_y = \frac{(1 + \beta^2)\gamma'_1 \omega'_3 - \beta (\gamma'_1 \gamma'_2 + \gamma'_2 \omega'_3 + \omega'_1 \gamma'_2)}{1 - \beta^2}.$$

(4.2.60)

Setting $\left( \gamma^{(3)} \times \omega^{(3)} \right)_y = 0$ and solving for $\beta$, we get

$$\beta = \frac{\gamma'^2 + \gamma'^2 + \omega'^2}{2 \gamma' \omega'} \pm \sqrt{\frac{(\gamma'^2 + \gamma'^2 + \omega'^2)^2 - 4 \gamma'^2 \omega'^2}{2 \gamma' \omega'^2}},$$

(4.2.61)

which we can write as

$$\beta = \frac{\gamma'^2 + \omega'^2 \pm \sqrt{(\omega'^2 - \gamma'^2)^2 + 4 (\omega' \cdot \gamma')^2}}{2 \gamma' \omega'},$$

(4.2.62)

where $\gamma'^2 = \gamma'_1^2 + \gamma'_3^2$ and $\omega'^2 = \omega'_3^2$. Using the invariants, as defined in Equations (4.2.43)
and (4.2.44), we can write \( \beta \) as

\[
\beta = \frac{\gamma'^2 + \omega''_3^2 \pm \sqrt{I_1'^2 - 4I_2'}}{2\gamma'\omega''_3}.
\] (4.2.63)

Note, we do not have to worry about the denominator of \( \beta \) going to zero; since that would only happen when either \( \omega''_3 = 0 \), which implies that \( I_2 = 0 \), (which, in turn, contradicts our assumption that \( I_2 \neq 0 \)), or \( \gamma''_3 = 0 \), in which case the system would already be in normal form after the first two Lorentz transformation \( \Lambda_1 \) and \( \Lambda_2 \); and we would not need to do a boost.

Finally, the last thing to mention about \( \beta \) is that, based on Equation (4.2.60), we know that \( \beta_+ \beta_- = 1 \). Therefore, one of the roots will always be less than 1 and greater than -1, which ensures that entries of the Lorentz boost are real. Hence, after choosing the root that lies between -1 and 1, and inserting it into Equation (4.2.57), we have constructed the Lorentz transformation, \( \Lambda_3 \),\(^7\) and the \( \Omega \)-tensor is in the form of Equation (4.2.55).

Lastly, we perform a rotation about the \( y \)-axis, denoted \( \Lambda_4 \), such that both \( \omega^{(4)} \) and \( \gamma^{(4)} \)

\(^7\)We reserve the notion \( \Lambda_3 \) to mean, explicitly, the Lorentz boost \( \Lambda \), after inserting the appropriate root of \( \beta \).
lie along the $z$-axis:

$$\gamma^{(3)} \rightarrow \gamma^{(4)} = \begin{pmatrix} 0 \\ 0 \\ \gamma_3^{(4)} \end{pmatrix}$$  \quad (4.2.64)$$

$$\omega^{(3)} \rightarrow \omega^{(4)} = \begin{pmatrix} 0 \\ 0 \\ \omega_3^{(4)} \end{pmatrix}$$  \quad (4.2.65)$$

$$\Omega^{(4)} = \Lambda_3 \Omega^{(3)} \Lambda_3^T = \begin{pmatrix} 0 & 0 & 0 & \gamma_3^{(4)} \\ 0 & 0 & -\omega_3^{(4)} & 0 \\ 0 & \omega_3^{(4)} & 0 & 0 \\ -\gamma_3^{(4)} & 0 & 0 & 0 \end{pmatrix}.$$  \quad (4.2.66)$$

The final state of the vectors $\omega^{(4)}$ and $\gamma^{(4)}$ is displayed in Figure 4.7.
FIG. 4.7: $\omega^{(3)}$ and $\gamma^{(3)}$ after the Lorentz boost $\Lambda_3$. Both vectors are still in the $x$-$z$ plane, but they are now collinear.

Thus, our normal form transformation, $\Lambda$, is the composite of the transformations $\Lambda_1, \Lambda_2, \Lambda_3$, and $\Lambda_4$:

$$\Lambda \equiv \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1. \quad (4.2.67)$$

We can then act with the transformation $\Lambda$ on the vector $\mathbf{B}$ to get the vector $\mathbf{B}'$, as in Equation (5.1.46), and construct the $2 \times 2$ dispersion matrix $D'$, using (4.2.8). By construction, the matrix $D'$ is in normal form! The last thing we may need to do, depending on our goals, is to construct the congruence transformation, $Q$, that corresponds to the Lorentz transformation, $\Lambda$. (This is fairly straightforward to do, as shown in Ref. [15], Exercise F.3.1). We need the congruence transformation to find the local polarization basis for the local field solution back
in the $x$-representation.

Now that we have introduced the normal form concept, and have discussed the procedure for constructing the normal form transformation; we shall apply it to our specific model of a $2 \times 2$ dispersion matrix, as given in Equation (2.5.6). Once in normal form, we will be able to identify the mode conversion surface, which is the surface in ray-phase space where both the diagonals are zero. The conversion surface is one of the main things we are after, since it tells us where (in the plasma) one type of wave mode can convert to another, and where the energy (of the initial magnetosonic wave) will be deposited within the plasma.
CHAPTER 5

Application of the Normal Form Transformation to the $2 \times 2$ Dispersion Matrix of a 50/50 Cold, Deuterium-Tritium Plasma

Now that we have introduced the procedure for constructing the normal form transformation, let us take a moment to step back and remind ourselves of our goals. Ultimately, we want to calculate, using ray tracing techniques, how the energy contained in an incident wave launched from an antenna, is deposited within a magnetically confined plasma; with the purpose of heating the plasma to thermonuclear conditions. First, after we have derived a $2 \times 2$ dispersion matrix from our favorite plasma model using the Weyl Symbol calculus dis-
cussed in Appendix A, we need to put the matrix into *normal form*, following the procedures outlined in the previous chapter. Recall that, the normal form is defined to be the form where the diagonal entries, (which are to be the *uncoupled* dispersion functions), Poisson-commute with the off-diagonals, (which represent the coupling ‘constants’). Thus, when the diagonals are used as the uncoupled ray Hamiltonians to generate the rays, the off-diagonal terms *will* be constant following those rays. To illustrate this, we shall construct and apply the normal form transformation of the $2 \times 2$ reduced dispersion matrix for a cold, 50/50 deuterium-tritium plasma, that we derived in Chapter 2, Equation (2.5.6).

### 5.1 The Normal Form of the Reduced $2 \times 2$ Dispersion Matrix

Before we begin to construct the normal form transformation for our $2 \times 2$ dispersion matrix, let us recall the properties of this matrix.

#### 5.1.1 Review: $2 \times 2$ Dispersion Matrix for Cold, 50/50-DT Plasma

To refresh our memories, the dispersion matrix that we are going to be working with, (i.e., the dispersion matrix we want to put into normal form), is

$$D(x,y,n_x,n_y) = \begin{pmatrix} \tilde{S} - n_y^2 & -iH + n_x n_y \\ iH + n_x n_y & \tilde{S} - n_x^2 \end{pmatrix}, \quad (5.1.1)$$
where $\tilde{S} = S - n_z^2$, and $\mathbf{n} = (nx, ny, nz) = c/\omega(k_x, ky, k_z) = c \mathbf{k}/\omega$ is the index of refraction vector. Recall, from Chapter 2, Section 2.2, that the Stix functions $S(x, y)$ and $H(x, y)$, defined in Equations (2.2.15) and (2.2.16), respectively, depend on the parameters of the plasma, (as well as on the wave frequency $\omega$). Since we are assuming that the plasma properties vary only in the $x$- and $y$-directions, the functions $S(x, y)$ and $H(x, y)$ are functions of $x$ and $y$ only. Additionally, the wave frequency $\omega$ is a parameter that we set independently, which, for our purposes, we choose to be 95% of the ion-hybrid frequency at the center of the plasma, Equation (2.4.11). Furthermore, recall from our discussion in Chapter 2, Section 2.4.2, that we choose $k_z$ to be small, but nonzero so that all of our modes are confined to the interior of the plasma and not near the plasma edge. Lastly, recall that one of the things we are interested in is the zero locus of the determinant of the matrix D. That is, when

$$\det \mathbf{D} = (n_x^2 + n_y^2)\tilde{S} - (\tilde{S}^2 - H^2) = (n_x^2 + n_y^2)\tilde{S} - \tilde{R}\tilde{L} = 0,$$

where, recall, $\tilde{R} = \tilde{S} + H$ and $\tilde{L} = \tilde{S} - H$. If we set $ny = 0$, we can plot this surface in $(x, y, k_x)$-space, as shown in Figure 2.24, and reproduced here for convenience, Figure 5.1.
FIG. 5.1: Plots of the dispersion surface $D(x,y,n_x) = n_x^2 S - \tilde{R}\tilde{L} = 0$ when $n_y = 0$. (a) The surface plotted in $(x,y,k_x)$-space. (b) The dispersion surface with the octant $(x > 0, y < 0, k_x > 0)$ removed to show the interior structure.

Note, as mentioned in the previous chapter, the determinant of $D$ and, hence, this surface, is invariant under congruence transformations with unit determinant, which the congruence transformations we are considering have. Now, let us begin the construction of the normal form transformation.

5.1.2 Constructing the $\Omega$-Tensor

We begin the construction of the normal form transformation, following the procedure laid out in the previous chapter, by constructing the ‘B’-representation of the dispersion matrix $D$. 
To that end, let us first define the entries of the dispersion matrix:

\[ D_{11}(x,y,n_x,n_y) = \tilde{S}(x,y) - n_y^2 \]  
(5.1.3)

\[ D_{12}(x,y,n_x,n_y) = n_x n_y - iH(x,y) \]  
(5.1.4)

\[ D_{21}(x,y,n_x,n_y) = D_{12}^* = n_x n_y + iH(x,y) \]  
(5.1.5)

\[ D_{22}(x,y,n_x,n_y) = \tilde{S}(x,y) - n_x^2. \]  
(5.1.6)

We construct the entries of the vector \( B = (B_0, B_1, B_2, B_3) \) using Equations (4.2.3)-(4.2.6),

\[ B_0(x,y,n_x,n_y) = \tilde{S}(x,y) - \frac{1}{2} (n_x^2 + n_y^2) \]  
(5.1.7)

\[ B_1(x,y,n_x,n_y) = n_x n_y \]  
(5.1.8)

\[ B_2(x,y,n_x,n_y) = -H(x,y) \]  
(5.1.9)

\[ B_3(x,y,n_x,n_y) = \frac{1}{2} (n_x^2 - n_y^2). \]  
(5.1.10)

Recall, that in the previous chapter we showed that \( \text{det} D = B_\mu B^\mu = B_0^2 - B_1^2 - B_2^2 - B_3^2 \), and, indeed,

\[ B_\mu B^\mu = \left[ \tilde{S} - \frac{1}{2} (n_x^2 + n_y^2) \right]^2 - (n_x n_y)^2 - (-H)^2 - \left[ \frac{1}{2} (n_x^2 - n_y^2) \right]^2 \]

\[ = \tilde{S}^2 - (n_x^2 + n_y^2) \tilde{S} + \frac{1}{4} (n_x^2 + n_y^2)^2 - n_x^2 n_y^2 - H^2 - \frac{1}{4} (n_x^2 - n_y^2)^2 \]

\[ = \tilde{S}^2 - (n_x^2 + n_y^2) \tilde{S} - H^2 = \text{det} D. \]  
(5.1.11)
Now, we shall proceed with the construction of the $\Omega$-tensor, where, recall, the elements of the tensor are defined by $\Omega^\mu_v = \{B^\mu, B_v\}$. Utilizing Equations (4.2.35)-(4.2.40), the entries of the $\Omega$-tensor are

$$g_1 = \{B_0, B_1\} = n_x \tilde{S}_y(x,y) + n_y \tilde{S}_x(x,y) \quad (5.1.12)$$
$$g_2 = \{B_0, B_2\} = -n_x H_x(x,y) - n_y H_y(x,y) \quad (5.1.13)$$
$$g_3 = \{B_0, B_3\} = n_x \tilde{S}_x(x,y) - n_y \tilde{S}_y(x,y) \quad (5.1.14)$$
$$w_1 = \{B_3, B_2\} = n_x H_y(x,y) - n_y H_x(x,y) \quad (5.1.15)$$
$$w_2 = \{B_1, B_3\} = 0 \quad (5.1.16)$$
$$w_3 = \{B_2, B_1\} = -n_x H_y(x,y) - n_y H_x(x,y), \quad (5.1.17)$$

where $\tilde{S}_x(x,y)$ denotes the partial derivate of $\tilde{S}(x,y)$ with respect to $x$, etc. Thus, the $\Omega$-tensor is as given in Equation (4.2.34), with the $\gamma$'s and $\omega$'s as defined by Equations (5.1.12)-(5.1.17)

Since the vector $B$ depends on the functions $\tilde{S}(x,y)$ and $H(x,y)$, and the $\gamma$ and $\omega$ vectors depend on the partial derivatives of $\tilde{S}$ and $H$ with respect to $x$ and $y$, let us plot both the functions and their derivatives. Figure 5.2 depicts the level sets of the function $\tilde{S}(x,y)$ along with the level sets of its partial derivatives, $\tilde{S}_x(x,y)$ and $\tilde{S}_y(x,y)$.
The Stix function $H(x,y)$, (for our case of a cold, nonuniform, magnetized, 50/50 deuterium-tritium plasma), along with its partial derivatives $H_{,y}(x,y)$ and $H_{,y}(x,y)$, is shown in Figure 5.3.
FIG. 5.3: Contour plots of the functions (a) $H(x,y)$, (b) $H_y(x,y)$, and (c) $H_y(x,y)$. In all plots, regions of blue shading indicate the function is negative, with the darker the blue the more negative the function; regions of beige shading indicate that the function is positive, with the lighter the beige the more positive the function. Also, for reference, the locations where the function is zero is indicated with a thick, red line.

Before we continue our discussion of the $\Omega$-tensor and the construction of the normal form transformation, we want to do a coordinate transformation from $(k_x, k_y)$ to $(k, \theta)$, where
\( k_x = k \cos \theta \) and \( k_y = k \sin \theta \), so that \( k = \sqrt{k_x^2 + k_y^2} \) is the magnitude of the wave vector (in the \( k_xk_y \)-plane) and \( \theta = \tan^{-1}(k_y/k_x) \) is the angle the vector makes with the \(+k_x\)-axis. In this case, \( n_x \to n \cos \theta \) and \( n_y \to n \sin \theta \). In this system, the dispersion matrix, which we will denote as \( \tilde{D} \), is

\[
\tilde{D}(x, y, n, \theta) = \begin{pmatrix}
\tilde{S} - n^2 \sin^2 \theta & -iH + n^2 \cos \theta \sin \theta \\
iH + n^2 \cos \theta \sin \theta & \tilde{S} - n^2 \cos^2 \theta
\end{pmatrix},
\]

and the determinant is

\[
\det \tilde{D} = n^2 \tilde{S} - \tilde{R} \tilde{L},
\]

the \( \tilde{B}_\mu \)'s are,

\[
\begin{align*}
\tilde{B}_0(x, y, n, \theta) &= \tilde{S}(x, y) - \frac{1}{2} n^2 & (5.1.20) \\
\tilde{B}_1(x, y, n, \theta) &= \frac{1}{2} n^2 \sin 2\theta & (5.1.21) \\
\tilde{B}_2(x, y, n, \theta) &= -H(x, y) & (5.1.22) \\
\tilde{B}_3(x, y, n, \theta) &= \frac{1}{2} n^2 \cos 2\theta, & (5.1.23)
\end{align*}
\]
and the entries of the $\Omega$-tensor are,

$$\gamma_1 = \{\vec{B}_0, \vec{B}_1\} = n[\vec{S}_y \cos \theta + \vec{S}_x \sin \theta]$$

(5.1.24)

$$\gamma_2 = \{\vec{B}_0, \vec{B}_2\} = -n[H_x \cos \theta + H_y \sin \theta]$$

(5.1.25)

$$\gamma_3 = \{\vec{B}_0, \vec{B}_3\} = n[\vec{S}_x \cos \theta - \vec{S}_y \sin \theta]$$

(5.1.26)

$$\omega_1 = \{\vec{B}_3, \vec{B}_2\} = n[H_x \cos \theta - H_y \sin \theta]$$

(5.1.27)

$$\omega_2 = \{\vec{B}_1, \vec{B}_3\} = 0$$

(5.1.28)

$$\omega_3 = \{\vec{B}_2, \vec{B}_1\} = -n[H_y \cos \theta + H_x \sin \theta].$$

(5.1.29)

For completeness, the zero locus of the determinant, in these coordinates, is shown in Figure 5.4. Notice how the entries of the $\Omega$-tensor are all directly proportional to the magnitude of the index of refraction vector, $n$. As we shall see, this means that our normal form transformation will be independent of the magnitude of $n$; this will greatly reduce the complexity of the transformation. Hence, for our model, the $\Omega$-tensor is\(^1\):

$$\Omega = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & 0 & -\omega_3 & 0 \\ -\gamma_2 & \omega_3 & 0 & -\omega_1 \\ -\gamma_3 & 0 & \omega_1 & 0 \end{pmatrix}.$$ 

(5.1.30)

\(^1\)For simplicity, we are dropping the tilde notation, since from now we will mostly be working in $(x,y,n,\theta)$-space.

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FIG. 5.4: Plots of the dispersion surface \( \tilde{D}(x,y,n,\theta) = n^2 \tilde{S} - \tilde{R}L = 0 \) for any \( \theta \). (a) The surface plotted in \( (x,y,k) \)-space. (b) The dispersion surface with the octant \( (x > 0, y < 0, k > 0) \) removed to show the interior structure.

The construction of the normal form transformation are the invariants \( I_1 \) and \( I_2 \), first introduced in Equations (4.2.43) and (4.2.44). In the space \( (x,y,k,\theta) \), the invariants are

\[
I_1(x,y,n,\theta) = \tilde{\omega}^2 - \tilde{\gamma}^2 = n^2 \left[ (H_y \cos \theta - H_x \sin \theta)^2 - (\tilde{S}_x^2 + \tilde{S}_y^2) \right] \quad (5.1.31)
\]

\[
I_2(x,y,n,\theta) = -(\tilde{\omega} \cdot \tilde{\gamma})^2 = -n^4 (\tilde{S}_y H_x - \tilde{S}_x H_y)^2. \quad (5.1.32)
\]

If we allow ourselves to be a little loose with notation, we can express the invariants as

\[
I_1(x,y,n,\theta) = n^2 \left[ (\hat{n} \times \nabla H)^2 - (\nabla \tilde{S})^2 \right] \quad (5.1.33)
\]

\[
I_2(x,y,n,\theta) = -n^4 (\nabla \tilde{S} \times \nabla H)^2, \quad (5.1.34)
\]
where $\nabla$ is the usual gradient operator in three spatial dimensions, and $\hat{n} = \cos \theta \, \hat{x} + \sin \theta \, \hat{y}$.

First, note that when $n = 0$, (that is, when $k_x$ and $k_y = 0$), that both $I_1$ and $I_2$ are zero; and, hence, the entire $\Omega$-tensor is zero. This suggests that we may not be able to put the dispersion matrix into normal form, or at the least, that we cannot follow the procedure outlined in Chapter 4 for constructing the normal form transformation.\(^2\) Moreover, this gives us another motivation for constructing the normal form transformation in the $(x, y, k, \theta)$ representation. Since, in $(x, y, k, \theta)$ space, the transformation is independent of the magnitude $n$, we will avoid potential issues when $n = 0$. We also want to know at what other 'points' $I_2$ is zero, where $I_1 = 0$, and where both $I_1$ and $I_2$ are zero. We want to know where $I_2 = 0$, as this implies that the vectors $\gamma$ and $\omega$ are orthogonal (and, hence, cannot be made parallel). Recall that one of our steps in the construction of the normal form transformation is a Lorentz boost that is meant to make $\gamma$ and $\omega$ parallel; however, when $I_2 = 0$, we will not be able to perform this step of the construction. Furthermore, $I_2 = 0$, (and, assuming $I_1 \neq 0$), implies that either $\omega_3$ or $\gamma_3$, of the transformed $\Omega$-tensor are zero. Which one is zero is arbitrary, in the sense that we can preform a transformation to make either $\omega_3$ or $\gamma_3$ zero, (when $I_2 = 0$). However, if want to be consistent with our generic normal form transformation, it turns out that $\omega_3$ is the entry that goes to zero. We will return to this matter a little later, as we will actually end up focusing on the normal form of the dispersion matrix expanded about the point where $I_2 = 0$.\(^3\)

Now that we have established the motivation for finding out where $I_2 = 0$, let us actually see where in our space $(x, y, k, \theta)$ this occurs. We already know about the case when $n = 0$, as

\(^2\)This is not very troubling, since the space $(x, y, k_x = 0, k_y = 0)$ is a two-dimensional subspace of the larger four-dimensional space $(x, y, k_x, k_y)$ and, hence, has Lebesgue measure zero.

\(^3\)In fact, when $I_2 = 0$, we have the so called Bramms-Duistermaat normal form, Ref. [1].
discussed above; and we do not have to be particularly concerned with this occurrence if we are working in \((x, y, k, \theta)\) space, (for the reasons explained previously). Upon examination of Equations (5.1.32) and (5.1.34), we see that \(I_2 = 0\) if the cross product of the gradient of \(\tilde{S}\) with the gradient of \(H\) vanishes, that is, where \(\nabla \tilde{S}\) is parallel to \(\nabla H\). Indeed, if we plot \(I_2(x,y)\) and locate the points where \(I_2 = 0\), we find that \(I_2 = 0\) when \(y = 0\), as shown in Figure 5.5.

Lastly, as we shall find later in the chapter, we will want to know where \(I_1 = 0\) and where both

---

**FIG. 5.5:** A contour plot of \(I_2(x,y)\). As before, regions of blue shading indicate the function is negative, with the darker the blue the more negative the function; regions of beige shading indicate that the function is positive, with the lighter the beige the more positive the function. The thick, red line indicates where \(I_2(x,y) = 0\); which occurs where \(y = 0\), exactly as expected.
$I_1$ and $I_2$ are zero. Unlike Equation (5.1.34), there is no simple geometric interpretation for when $I_1 = 0$, Equation (5.1.33). However, we are not particularly interested in understanding the shape of the zero contour of $I_1$, but merely identifying it, as it will show up again. The surfaces $I_1(x, y, \theta) = 0$ and $I_1(x, y, \theta = 0) = 0$ are shown in Figure 5.6.

![Plots of the zero loci of the invariant $I_1$.](image)

**FIG. 5.6:** Plots of the zero loci of the invariant $I_1$. (a) The surface $I_1(x, y, \theta) = 0$ plotted in $(x, y, \theta)$-space; along with the plane $\theta = 0$. Notice that the geometry is fairly complicated. Furthermore, note how the pincer part of the surface touches the $\theta = 0$ surface tangentially; this happens at $y = 0$. (b) The surface $I_1 = 0$ when $\theta = 0$, plotted in the space $(x, y, k_x)$. The invariant $I_1$ is also zero along the red line at $y = 0$. This is a result of the tangential intersection shown in (a).

### 5.1.3 Construction & Application of the Normal Form Transformation

Now that we have defined the $\Omega$-tensor, we can construct the normal form transformation following the algorithm set forth in Chapter 4, Section 4.2.3. First, we will focus on applying the normal form in regions where neither $I_1$ nor $I_2$ are zero, and then identify and discuss some
of the issues that appear in the vicinity of where \( I_1 \) or \( I_2 \) go to zero.

As laid out in Section 4.2.3, the first sequence of the normal form transformation is to perform a rotation, denoted \( \Lambda_1 \) which rotates the vector \( \omega \) along the \( z \)-axis. As we can see from Equations (5.1.27)-(5.1.29), \( \omega \) lies in the \( xz \)-plane, that is, the \( y \)-component of \( \omega \), \( \omega_2 \), is already zero. Hence, \( \Lambda_1 \) is a simple rotation about the \( y \)-axis, such that \( \omega_1 \to 0 \). Explicitly,

\[
\Lambda_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \omega_3/\omega & 0 & -\omega_1/\omega \\
0 & 0 & 1 & 0 \\
0 & \omega_1/\omega & 0 & \omega_3/\omega \\
\end{pmatrix}, \quad \text{(5.1.35)}
\]

where \( \omega = |\omega| = \sqrt{\omega_1^2 + \omega_3^2} \). After the application of the transformation \( \Lambda_1 \), the \( \Omega \)-tensor is in the form of (4.2.49), shown here again for ease of reference,

\[
\Omega' = \Lambda_1 \Omega \Lambda_1^T = \begin{pmatrix}
0 & \gamma_1' & \gamma_2' & \gamma_3' \\
-\gamma_1' & 0 & -\omega_3' & 0 \\
-\gamma_2' & \omega_3' & 0 & 0 \\
-\gamma_3' & 0 & 0 & 0 \\
\end{pmatrix}. \quad \text{(5.1.36)}
\]

Continuing with the normal form algorithm, we next perform a rotation, \( \Lambda_2 \) about the \( z \)-axis,
such that the (doubly) transformed vector $\gamma''$ lies in the $xz$-plane, i.e., $\gamma''_2 = 0$. Hence,

$$\Lambda_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\gamma'}{\sqrt{\gamma^2 + \gamma'_2}} & \frac{\gamma''}{\sqrt{\gamma^2 + \gamma'_2}} & 0 \\
0 & -\frac{\gamma'}{\sqrt{\gamma^2 + \gamma'_2}} & \frac{\gamma'}{\sqrt{\gamma^2 + \gamma'_2}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (5.1.37)

Thus, the $\Omega$-tensor becomes

$$\Omega'' = \Lambda_2 \Omega' \Lambda_2^T = \begin{pmatrix}
0 & \gamma'_1 & 0 & \gamma''_3 \\
-\gamma''_3 & 0 & -\omega''_3 & 0 \\
0 & \omega''_3 & 0 & 0 \\
-\gamma''_3 & 0 & 0 & 0
\end{pmatrix}. \hspace{1cm} (5.1.38)$$

The third part of the transformation, $\Lambda_3$, is the boost in the $y$-direction given by the matrix

$$\Lambda_3 = \begin{pmatrix}
(1 - \beta^2)^{-1/2} & 0 & -\beta (1 - \beta^2)^{-1/2} & 0 \\
0 & 1 & 0 & 0 \\
-\beta (1 - \beta^2)^{-1/2} & 0 & (1 - \beta^2)^{-1/2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \hspace{1cm} (5.1.39)$$

where the boost parameter, $\beta$ is

$$\beta = \frac{\gamma'^2 + \omega'^2 - \sqrt{l_1^2 - 4l_2}}{2\gamma'' \omega''_3}. \hspace{1cm} (5.1.40)$$
Notice, in Equation (5.1.40) we have explicitly chosen the negative root of $\beta$. Recall, from our discussion in Section 4.2.3, we must chose the root of $\beta$ such that $\beta < 1$, to ensure that the entries of $\Lambda_3$ are real. It turns out, that for almost all points in phase space, except for those where $I_2 = 0$, (which occurs at $y = 0$ for our model), it is the negative root of $\beta$ that is less than one. After application of the boost $\Lambda_3$, the $\Omega$-tensor is

$$
\Omega^{(3)} = \Lambda_3 \Omega'' \Lambda_3^T
$$

Finally, we perform a rotation about the $y$-axis, $\Lambda_4$, such that $\omega^{(4)}$ and $\gamma^{(4)}$ lie solely along the $z$-axis:

$$
\Lambda_4 = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \gamma_3^{(3)}/\gamma^{(3)} & 0 & -\gamma_1^{(3)}/\gamma^{(3)} \\
0 & 0 & 1 & 0 \\
0 & \gamma_1^{(3)}/\gamma^{(3)} & 0 & \gamma_3^{(3)}/\gamma^{(3)}
\end{pmatrix},
$$

where $\gamma^{(3)} = |\gamma^{(3)}| = \sqrt{\left[\gamma_1^{(3)}\right]^2 + \left[\gamma_3^{(3)}\right]^2}$. After application of the transformation $\Lambda_4$, the $\Omega$-tensor is in the form of Equation (4.2.66), where only the $z$-component of $\omega$ and $\gamma$ are non-zero. Thus, the complete normal form transformation, $\Lambda$, is

$$
\Lambda \equiv \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1.
$$
Before we apply the normal form transformation, Equation (5.1.43), it is imperative to be clear that this transformation will put the dispersion matrix $D$ into normal form at each point in the ray-phase space, **to leading order**. Recall, when we first introduced the $\Omega$-tensor in Section 4.2.2 and asked how it transformed under a congruence transformation, (which acts on the vector $\mathbf{B}$), we assumed that this transformation is constant. (That is why we were able to bring it outside the Poisson bracket in Equation (4.2.29).) Thus, before we apply the transformation, we must **evaluate** it at each point in the ray-phase space. We will denote a particular point in the phase space as $z_0 = (x_0, y_0, k_0, q_0)$. Thus, the normal form transformation, $\Lambda^{\mu}(z_0) \equiv \Lambda(z_0)$ is constant.\(^4\) We then apply the transformation point-wise to get

$$B'_\nu(z; z_0) = \Lambda^{\mu}_\nu(z_0) B_\mu(z).$$ \hspace{1cm} (5.1.44)

Now, in general, it will not be true that $\{B'_0, B'_1\} = \{B'_0, B'_2\} = \{B'_3, B'_1\} = \{B'_3, B'_2\} = 0$, which, as you may recall, was the central criteria for defining the normal form. Rather, if we expand each of the $B'_\mu$’s about the point $z_0$, such that

$$\bar{B}'_\mu(z; z_0) = B'_\mu(z_0) + (z - z_0) \cdot \nabla_z B'_\mu(z_0),$$ \hspace{1cm} (5.1.45)

and then take the Poisson bracket, we will find that $\{\bar{B}'_0, \bar{B}'_1\} = \{\bar{B}'_0, \bar{B}'_2\} = \{\bar{B}'_3, \bar{B}'_1\} = \{\bar{B}'_3, \bar{B}'_2\} = 0$.\(^5\)

Let us now apply the normal form transformation $\Lambda(z_0)$ to the vector $\mathbf{B}$, whose entries are

\(^4\)Recall, in $(x, y, k, \theta)$ coordinates the normal form transformation is independent of the magnitude $k$.

\(^5\)See Ref. [10] for a discussion of higher order corrections to the normal form.
given by Equations (5.1.20)-(5.1.23), to get $B'$, (this follows from Equation (4.2.16) on Page 156):

$$B'(z; z_0) = \Lambda(z_0)B(z). \quad (5.1.46)$$

Then, using the entries of $B'$, and following from Equation (4.2.8), we construct the transformed $2 \times 2$ dispersion matrix, such that,

$$D_0'(z) = \begin{pmatrix} B'_0(z) + B'_3(z) & B'_1(z) + iB'_2(z) \\ B'_1(z) - iB'_2(z) & B'_0(z) - B'_3(z) \end{pmatrix}. \quad (5.1.47)$$

Recall, from our discussion at the end of Chapter 2, that one of the goals for putting the dispersion matrix into normal form is to use the diagonals of the normal form dispersion matrix as the uncoupled dispersion surfaces of each mode. To that end, we want the avoided crossings that we first encountered at the end of Chapter 2, and that we see again in Figure 5.4, to be gone, and for the two dispersion surfaces to intersect. To see if we have achieved that goal, at least in regions where we expect the global normal form algorithm to be well behaved, (away from where the invariant $I_1 = 0$), let us plot the zero locus of the diagonals evaluated at $z = z_0$

$$D_{11}'(z_0) = B'_0(z_0) + B'_3(z_0) \quad (5.1.48)$$

$$D_{22}'(z_0) = B'_0(z_0) - B'_3(z_0), \quad (5.1.49)$$

when $\theta_0 = 0$, i.e., $k_{y0} = 0$. As we can see from Figure 5.7, the blue surface, which are the points

---

6We have dropped the explicit dependence on $z_0$, with the understanding that we have constructed the matrix $D'$ point-wise, as described above.
FIG. 5.7: Plots of the zero loci of the diagonals $D'_{11}(z_0)$ (the blue surface) and $D'_{22}(z_0)$ (the green surface) for $\theta = 0$. (a) The two dispersion surfaces plotted in a region of phase space where the normal form transformation is well-behaved. Notice that the dispersion surfaces do indeed intersect. (b) The uncoupled dispersion surfaces shown with the surface $\det D = 0$ (orange surface), in the same region as in Figure (a). Notice that the avoided crossings are indeed gone!

where $D'_{11}(z_0) = 0$, intersects the green surface, which are the points where $D'_{22}(z_0) = 0$; and, as we see from Figure 5.7b, which also shows $\det D = 0$ (orange surface), these intersections occur exactly where the avoided crossings used to be! Furthermore, we see that uncoupled dispersion surfaces are approaching the zero locus of the determinant as we go away from the conversion region. Additionally, now that we have identified these conversion points, (the points where the blue and green surfaces intersect), we can determine what the coupling constant is at those points. This is a crucial success since we need to know the magnitude of the coupling at the conversion points in order to calculate the transmitted rays!

However, all is not copacetic. For if we plot $D'_{11}(z_0) = 0$ and $D'_{22}(z_0) = 0$ over the full
region of physical interest, still at $\theta_0 = 0$, we discover that there are still issues that need to be dealt with. As we can see in Figure 5.8a, where the two uncoupled dispersion surfaces are plotted

FIG. 5.8: (a) Plots of the uncoupled dispersion surfaces $D'_{11}(z_0) = 0$ (blue surface) and $D'_{22}(z_0) = 0$ (green surface) in the full region of physical interest. Clearly, there are regions where the normal form transformation is not working as expected. (b) The uncoupled dispersion surfaces shown with the orange surface $\det D = 0$. (c) The uncoupled dispersion surfaces shown with the surface (and line) where $I_1(x, y, k, \theta = 0) = 0$.

over the full region of physical interest, there are some unexpected results. First, there appears to be a line, at large magnitudes of $k_x$, where the two surfaces intersect. Additionally, both the
blue dispersion surface \((D'_{11}(z_0) = 0)\) and the green dispersion surface \((D'_{22}(z_0) = 0)\) appear to have features of both the ion-hybrid dispersion surface and the magnetosonic dispersion surface, as can be seen in Figure 5.8b, where we plot both surfaces along with \(\text{det} D = 0\). If we look at Figure 5.8c, which shows the dispersion surfaces along with the surface where \(I_1 = 0\), it appears that this ‘mixing’ might be the result of a breakdown of the transformation at or near where \(I_1 = 0\). Indeed, \(I_1 = 0\) along the red line shown in Figure 5.8c, which is at \(y = 0\). This means that \(I_2\) is also zero at along this line, and, as mentioned previously, we expect the normal form algorithm to break down at those points. Hence, we suspect, that the non-physical appearance of the ‘pinching’, where the two surfaces come together along the red line, is an artifact of the plotting program. Additionally, it is interesting to note, that the portion of the blue dispersion surface \((D'_{11}(z_0) = 0)\), which is disjointed from the rest, is to the left of the red surface \((I_1 = 0)\). This might suggest that the normal form algorithm is having difficulty as it ‘passes through’ the \(I_1 = 0\) region. This requires further investigation and is an example of how rich this problem is.

In fact, we can begin to address this issue immediately. First, perform the normal form transformation about the base point \(z_0 = (x_0, y_0 = 0, k_0, \theta_0 = 0)\); then plot the points where the diagonals are zero in \((x, k_x)\)-space, (i.e., for \(y = 0\) and \(\theta = 0\)).
FIG. 5.9: (a) The $y = 0$ slice of the plot in Figure 5.8b, along with red, dashed lines that indicate where $I_1 = 0$. (b) The uncoupled dispersion curves after putting the dispersion matrix into normal form about the base point $(x_0, 0, k_0, 0)$.

The plot of the zero curves of the uncoupled diagonal terms, (where the normal form has been carried out about the point $(x_0, y_0 = 0, k_0, \theta_0 = 0)$), along with the $y = 0$ slice of the
surface $\det D = 0$, is shown in Figure 5.9b. For comparison, in Figure 5.9a, we include the $y = 0$ slice of the plot from Figure 5.8b. There are several small, but significant differences between the two plots, so let us go through them one at a time. First, notice how in Figure 5.9a the two uncoupled dispersion curves, (the blue and green curves), have properties indicative of both the ion-hybrid mode and the magnetosonic mode, as we mentioned earlier. In particular, the dispersion curve associated with the diagonal element $D'_{11}$, the blue curve, appears to have the left part of the ion-hybrid resonance, where the blue curve is completely vertical; but it also appears to incorporate portions of the magnetosonic dispersion curve, to the left, there is the blue closed curve, (which is disjointed from the rest of the curves by the negative root of $I_1 = 0$), and to the right, there is a portion that also has the shape of the magnetosonic dispersion curve. The left part of the dispersion curve associated with the diagonal element $D'_{22}$, the green curve, incorporates a large part of the magnetosonic dispersion curve; but, the right most green curve appears to be the right part of the ion-hybrid resonance. Additionally, as mentioned earlier, both the green and the blue dispersion curves appear to encounter some type of singularity about the positive root of $I_1 = 0$. Now, if we look at Figure 5.9b, we see that the blue dispersion curve, corresponding to $D'_{11} = 0$, is just the ion-hybrid resonance, $\tilde{S} = 0$! Furthermore, the green dispersion curve, $D'_{22} = 0$, exhibits only properties of the magnetosonic mode. Part of the green dispersion curve, the left-most closed curve, is still disjointed from the rest of the surface; but at least it is part of the same dispersion curve, i.e., it is green instead of blue! Lastly, we still have the singularity about the positive root of $I_1 = 0$, but, again, only one of the curves is encountering it. Clearly, this is an improvement, and suggests that in the full
phase space, when the normal form algorithm is being evaluated in the neighborhood of where
\( I_2 = 0 \), there is a flipping or mixing up of terms. This, together with the apparent problems
arising near points where \( I_1 = 0 \), are still areas in need of further investigation.

5.1.4 The Identification of the Conversion Surface

Putting the \( 2 \times 2 \) dispersion matrix \( D \) into the normal form dispersion matrix \( D' \) is not only
necessary in order to identify the uncoupled dispersion functions (the diagonals), which, recall,
will serve as the uncoupled ray Hamiltonians; we also need the dispersion matrix in normal form
to identify the conversion surface. The conversion surface is the set of points in ray-phase space
were both diagonals \( D'_{11} \) and \( D'_{22} \) are zero. Since we are working in a four-dimensional ray-phase
space, \((x, y, k_x, k_y)\), each of the uncoupled dispersion surfaces \( D'_{11}(z) = 0 \) and \( D'_{22}(z) = 0 \) are,
in general, three-dimensional. Hence, the points at which they intersect will, in general, define
a two-dimensional surface. The identification of this surface is paramount, perhaps even more
so than the complete construction of the three-dimensional dispersion surfaces, for, at least,
two reasons. First, we usually want to know where the conversion is happening physically. For
example, if we are trying to heat a plasma in a tokamak via mode conversion, we definitely
want to know where the conversion is occurring. If the conversion is happening too far from the
center of the plasma, where the density is typically lower, very little energy will be transferred to
the plasma. Second, in order to calculate the transmission and conversion coefficients using the
ray tracing methods developed by Tracy, et. al., and discussed extensively in Refs. [5, 7, 11–15],
we must identify the conversion points. For example, in order to calculate the magnitude of the
coupling, $|\eta|$, which we need to know if we wish to calculate the transmission and conversion coefficients, we need to know where the mode conversion points are, since\(^7\)

$$|\eta| = \frac{|\text{det} D(z_*)|^{1/2}}{\sqrt{2} \left| \sqrt{I_1^2 + 4I_2 - I_1} \right|^{1/2}},$$  \hspace{1cm} (5.1.50)

where the determinant of the dispersion matrix is evaluated at a point, $z_*$, on the conversion surface. As long as the conversion surface is ‘away’ from the regions of phase space where the normal form transformation breaks down, which, in general, it will be; we should be able to use the normal form dispersion matrix $D'$ to find the conversion surface.

We now want to develop a quick method to identify the mode conversion surface. As can be seen from the form of the dispersion matrix $D$ in Equation (5.1.18), the entries of the dispersion matrix are quadratic in $n$. Additionally, as we can see from Equations (5.1.20)-(5.1.23), the entries of the vector $B$ are all of the form

$$B_{\mu}(x, y, n, \theta) = a(x, y, \theta)n^2 + c(x, y, \theta),$$  \hspace{1cm} (5.1.51)

where $a(x, y, \theta)$ and $c(x, y, \theta)$ are some functions of $x$, $y$, and $\theta$. Since the entries of the normal form transformation, $\Lambda$, do not depend on $n$, and the elements of the transformed vector $B'$ are linear combinations of the entries of $\Lambda$ and the entries of the vector $B$, these elements must also have the form of Equation (5.1.51). Furthermore, as the entries of the normal form matrix $D'$ are combinations of the elements of $B'$, they must also have the form of Equation (5.1.51).

\(^7\)For the derivation of this expression for the magnitude of the coupling see Ref. [15].
Thus, we can express the diagonal elements as

\[ D'_{11}(x,y,n,\theta) = a_1(x,y,\theta)n^2 + c_1(x,y,\theta) \]  \hspace{1cm} (5.1.52)

\[ D'_{22}(x,y,n,\theta) = a_2(x,y,\theta)n^2 + c_2(x,y,\theta). \]  \hspace{1cm} (5.1.53)

Let a point on the conversion surface be donated as \( z_* = (x_*,y_*,n_*,\theta_*) \). By definition,

\[ D'_{11}(z_*) = a_1(x_*,y_*,\theta_*)n_*^2 + c_1(x_*,y_*,\theta_*) = 0 \]  \hspace{1cm} (5.1.54)

\[ D'_{22}(z_*) = a_2(x_*,y_*,\theta_*)n_*^2 + c_2(x_*,y_*,\theta_*) = 0, \]  \hspace{1cm} (5.1.55)

since both surfaces must be zero at a conversion point. Solving Equation (5.1.54) for \( n_*^2 \) gives

\[ n_*^2 = -\frac{c_1}{a_1}. \]  \hspace{1cm} (5.1.56)

Similarly, solving (5.1.55) for \( n_*^2 \), we get

\[ n_*^2 = -\frac{c_2}{a_2}. \]  \hspace{1cm} (5.1.57)

But, by definition Equations (5.1.56) and (5.1.57) must be equal. Thus,

\[ -\frac{c_1}{a_1} = -\frac{c_2}{a_2} \Rightarrow c_1a_2 = c_2a_1. \]  \hspace{1cm} (5.1.58)
Let us define the function $C(x, y, \theta)$, such that

$$
C(x, y, \theta) = c_1(x, y, \theta)a_2(x, y, \theta) - c_2(x, y, \theta)a_1(x, y, \theta),
$$

(5.1.59)

where

$$
c_1 = D_{11}'(x, y, 0, \theta), \quad a_1 = \frac{1}{2} \frac{\partial^2 D_{11}'}{\partial n^2},
$$

(5.1.60)

$$
c_2 = D_{22}'(x, y, 0, \theta), \quad a_2 = \frac{1}{2} \frac{\partial^2 D_{22}'}{\partial n^2}.
$$

(5.1.61)

In general, the surface $C = c_1a_2 - c_2a_1 = 0$, will include the conversion surface; but, it will also include points that are not on the conversion surface. Since we require $n_*$ to be real (and) finite, we can see from Equations (5.1.56) and (5.1.57), that in order for a point $z_* = (x_*, y_*, \theta_*)$ to be on the conversion surface it must satisfy all of the following conditions:

$$
c_1(z_*) < 0 \text{ AND } a_1(z_*) > 0 \text{ OR } c_1(z_*) > 0 \text{ AND } a_1(z_*) < 0
$$

(5.1.62)

$$
c_2(z_*) < 0 \text{ AND } a_2(z_*) > 0 \text{ OR } c_2(z_*) > 0 \text{ AND } a_2(z_*) < 0
$$

(5.1.63)

$$
a_1(z_*) \neq 0 \text{ AND } a_2(z_*) \neq 0
$$

(5.1.64)

$$
c_1(z_*) \neq 0 \text{ OR } c_2(z_*) \neq 0.
$$

(5.1.65)

Hence, a point $z_*$ is on the conversion surface if

$$
C(z_*) = c_1(x_*, y_*, \theta_*)a_2(x_*, y_*, \theta_*) - c_2(x_*, y_*, \theta_*)a_1(x_*, y_*, \theta_*) = 0,
$$

(5.1.66)
and it satisfies the conditions (5.1.62)-(5.1.65).

Now, we apply this algorithm to our model to find the conversion surface, which is shown, along with the surface $\tilde{S} = 0$, in Figure 5.10. First, let us mention, that in this particular case,

both $a_1$ and $a_2$ are less than zero everywhere in the plotted region of phase space. Therefore, the only conditions that we need to ensure are that $c_1 > 0$ and that $c_2 > 0$. (We do not worry about the last condition, (5.1.65), since it should only be violated at isolated points.) Examining the plots in Figure 5.10, we see that most of the conversion surface (orange) is contained within the $\tilde{S} = 0$ surface (purple). Indeed, a good portion of the conversion surface lies along the inner surface of $\tilde{S} = 0$, near the center of the plasma, which is what we intended!

FIG. 5.10: Plots of the conversion surface $C(x, y, \theta) = 0$ (orange) along with the surface $\tilde{S} = 0$ (purple). Of particular interest are the wing-like portions of the surface that extend beyond the $\tilde{S} = 0$ surface.
The curvature of the surface is interesting, but we do not yet understand this very well and need to investigate it further. Additionally, there is a defect on the inner surface at \( y = 0 \), which, of course, is not unexpected, since both \( I_1 \) and \( I_2 \) are zero there. However, the most interesting and unexpected feature of the conversion surface are the sloping ‘wings’ which extend beyond the \( \tilde{S} = 0 \) surface. These parts of the surface are not near points where \( I_1 = 0 \); and, hence, do not appear to be artifacts, but real pieces of the conversion surface. In fact, it is likely, although not proven, that these pieces of the conversion surface are where the two uncoupled dispersion surfaces intersect tangentially.

To further this example, let us plot the conversion surface for the case when \( y = 0 \), corresponding to the dispersion curves shown in Figure 5.9b. In this case, since the uncoupled dispersion surfaces are two-dimensional, the conversion surface should be one-dimensional, i.e., a curve. The conversion curve is plotted in Figure 5.11. First, note that the two vertical parts of the conversion curve (solid blue lines) are located at exactly the same points as the \( \tilde{S}(x,0) = 0 \) curve (purple, dashed lines). Second, we have the unexpected appearance of these horizontal lines at \( \theta = \pm \pi/2 \). It seems that these lines denote the location of tangential mode conversions; that is, points where the uncoupled dispersion relations do not intersect, but touch. However, due to recent investigation, it is not clear that the entirety of the horizontal lines are true parts of the conversion surface. It may be, that the lines are an artifact of the normal form transformation that is not entirely understood.\(^8\) Lastly, we see the \( I_1 = 0 \) surface rearing its ugly head once again. The blue dots that lie on the thin, green curve, (which is

\(^8\)We suspect that this might be the case, since the application of a Lorentz transformation that leaves the dispersion matrix in normal form, but appears to ‘smooth’ out the uncoupled dispersion surfaces, results in the disappearance of these lines.
FIG. 5.11: A plot of the conversion curve (blue, solid lines) when $y = 0$. We also plot the curves $\tilde{S}(x,0) = 0$, which are the dashed, purple lines coincident with the vertical parts of the conversion curve, and the curves $I_1(x,0,\theta) = 0$, which are the thin, green lines. The horizontal lines of the conversion surface identify the location of a tangential mode conversion, at $\theta = \pm \pi/2$. However, recent investigation suggests that the lines themselves may not be parts of the conversion surface.

where $I_1 = 0$, were produced by the algorithm to find the conversion surface. However, at those points, both $c_1$ and $c_2$ are zero, which implies these points are not part of the conversion surface. Furthermore, $a_1$ and $a_2$ are both nonzero throughout this region of phase space.
CHAPTER 6

Conclusion and Future Work

6.1 Conclusion

Starting with a particular $2 \times 2$ dispersion matrix, $D$, in the form of Equation (5.1.18), we apply the normal form transformation algorithm, introduced in Chapter 4, point-wise. This results in a transformed matrix $D'$, which is in normal form to leading order. This means that diagonal entries of $D'$ Poisson-commute with the off-diagonal entries, (to leading order). If we plot the points at which the diagonals are zero we find, at least in regions of the phase space where the normal form transformation is well behaved, that these do appear to be the uncoupled dispersion surfaces and the avoided crossings have disappeared, as seen in Figure 5.7b. Hence, although the algorithm breaks down in certain regions, and needs to be better understood and modified; it has achieved a great success. Namely, the removal of the avoided crossings, which, in turn, allows us to identify the mode conversion points that are essential to
the application of our ray tracing methods.

6.2 Future Work

Before this normal form algorithm can be applied to more physically realistic plasmas, or to any other two-dimensional mode conversion problem, we have to understand how to connect the normal form transformation across the boundaries in phase space where it is not well defined. Presently, it appears, that as the normal form transformation is applied across one of these boundaries, (which, recall, are defined by $I_1 = 0$ and $I_2 = 0$), there is a mixing of terms. Namely, on one side of the boundary, a particular dispersion surface appears to correspond to one mode, while on the other side of the boundary, the same dispersion function appears to correspond to the other mode. We suspect that this is due to the introduction of some kind of sign flip, and that it can be corrected by applying an additional transformation that leaves the system in normal form, but rearranges some of the terms. In fact, it will undoubtedly be useful to better understand the subspace of transformations that leave the system in normal form.

Once we know how to smoothly connect the algorithm across these problem regions, we would like to apply it to a more physically relevant model of a plasma in a tokamak reactor. For instance, we would like to include the algorithm as a module in the RAYCON code discussed in Refs. [15] and [5], to put the $2 \times 2$ system into normal form.

Lastly, it would be worthwhile to apply these techniques to other physical systems that exhibit two-dimensional mode conversion, such as the conversions that occur in the Sun, which we mentioned in the introduction. It would be interesting to see if this technique brings to
light any new features of that conversion.
APPENDIX A

An Introduction to the Weyl Symbol Calculus

The Weyl symbol calculus was first introduced by Hermann Weyl in his book, *The Theory of Groups and Quantum Mechanics*, in 1931, Ref. [16]. The Weyl symbol allows us to go from operators to functions on phase space. For our purposes, the phase space is the four-dimensional ray phase space, \((x,k) = (x,y,k_x,k_y)\). Hence, they symbol calculus is a means to go from a multicomponent wave operator, the operator-valued matrix \(\hat{D}\), to the matrix-valued function on ray phase space, \(D\). Furthermore, the Weyl symbol mapping provides a unique way to define the dispersion matrix. Our motivation for using the Weyl symbol is that it allows us to go from operators to symbols (and back again), in a way that treats \(x\) and \(k\) on equal footings, which compliments the appearance of Hamilton’s equations for ray evolution. To begin, let us discuss the Weyl symbol in one dimension.
A.1 The Weyl Symbol in One Dimension

We shall consider the simplest case first, that of one dimension. The extension to higher dimensions is straightforward. Let $\hat{a}$ be a general operator in some Hilbert space. In the $x$-representation, the matrix elements of $\hat{a}$ are given by

$$A(x,x') = \langle x|\hat{a}|x' \rangle.$$  \hfill (A.1.1)

The Weyl symbol is a better, (at least for our purposes), way of representing the operator, but now as a function of the ray phase space $(x,k)$. That is, the operator $\hat{a}$ is associated with a function $a(x,k)$ through the symbol mapping $\Sigma$:

$$\hat{a} \xrightarrow{\Sigma} a(x,k).$$  \hfill (A.1.2)

The bi-directional arrow emphasizes the fact that the symbol mapping goes both ways, i.e., allows us to go from the operator to the symbol, and from the symbol to the operator. In general, the symbol is a complex function of the real variables $x$ and $k$. Furthermore, as we shall see, self-adjoint operators produce self-adjoint symbols. Thus, for scalar, self-adjoint operators, the symbol is real, and for operator-valued, Hermitian matrices, the symbol is a self-adjoint matrix.

Now, let us define the Weyl mapping to go from an operator to a symbol, $\hat{a} \xrightarrow{\Sigma} a(x,k)$,
and look at a few examples:

\[ a(x,k) \equiv \int ds \ e^{-iks} \langle x + \frac{s}{2} | \hat{a} | x - \frac{s}{2} \rangle. \]  \hspace{1cm} (A.1.3)

First, consider the case where \( \hat{a} = \hat{x} \), i.e, the position operator. Then, from Equation (A.1.3), we have

\[
\begin{align*}
a(x,k) &= \int ds \ e^{-iks} \langle x + \frac{s}{2} | \hat{x} | x - \frac{s}{2} \rangle \\
&= \int ds \ e^{-iks} \left( x - \frac{s}{2} \right) \delta(s) \\
&= x.
\end{align*}
\]  \hspace{1cm} (A.1.4)

Second, consider the case when \( \hat{a} = \hat{k} \), where \( \hat{k} = -i\partial_x \), (in the \( x \)-representation),

\[
\begin{align*}
a(x,k) &= \int ds \ e^{-iks} \langle x + \frac{s}{2} | \hat{k} | x - \frac{s}{2} \rangle \\
&= \int ds \ e^{-iks} \left( x + \frac{s}{2} \right) \left( -i\partial_x \right) | x - \frac{s}{2} \rangle \\
&= -i \int ds \ e^{-iks} \delta'(s) \\
&= k,
\end{align*}
\]  \hspace{1cm} (A.1.5)

where we have integrated by parts. Now, to illustrate the true power of the Weyl symbol, we
shall consider the cases when \( \hat{a} = \hat{x} \hat{k} \) and \( \hat{a} = \hat{k} \hat{x} \). First, \( \hat{a} = \hat{x} \hat{k} \):

\[
a(x, k) = \int ds \ e^{-iks} \left(x + \frac{s}{2}\right) \hat{x} \hat{k} \left(x - \frac{s}{2}\right) \\
= -i \int ds \ e^{-iks} \left(x + \frac{s}{2}\right) \delta'(s) \\
= xk - \frac{i}{2} \int ds \ e^{-iks} s \delta'(s) \\
= xk + \frac{1}{2} \int ds \ \partial_k \left(e^{-iks}\right) \delta'(s) \\
= xk + \frac{1}{2} \partial_k (ik) \\
= xk + \frac{i}{2} .
\]

(A.1.6)

Note, that we have used the result of Equation (A.1.5) twice in our derivation. Now, for \( \hat{a} = \hat{k} \hat{x} \):

\[
a(x, k) = \int ds \ e^{-iks} \left(x + \frac{s}{2}\right) \hat{k} \hat{x} \left(x - \frac{s}{2}\right) \\
= -i \int ds \ e^{-iks} \left(x - \frac{s}{2}\right) \delta'(s) \\
= xk + \frac{i}{2} \int ds \ e^{-iks} s \delta'(s) \\
= xk - \frac{1}{2} \int ds \ \partial_k \left(e^{-iks}\right) \delta'(s) \\
= xk - \frac{1}{2} \partial_k (ik) \\
= xk - \frac{i}{2} .
\]

(A.1.7)

Notice how these results are completely consistent with the fact that the operators \( \hat{x} \) and \( \hat{k} \) do
not commute! Furthermore, we can combine Equations (A.1.6) and (A.1.7) to get

$$\frac{1}{2} (\hat{x}\hat{k} + \hat{k}\hat{x}) \rightarrow xk.$$ (A.1.8)

Let us return to the definition of the symbol, Equation (A.1.3). We can think of $A(x, x') = \langle x|\hat{a}|x'\rangle$ as an infinite-dimensional matrix. The entries $A(x, x) = \langle x|\hat{a}|x\rangle$ lie on the diagonal. Then the values $A(x+s/2, x-s/2) = \langle x+s/2|\hat{a}|x-s/2\rangle$, for a fixed $x$, lie on the anti-diagonal passing through the point labeled by $x$. Therefore, the Weyl symbol is the one-dimensional Fourier transform taken along this anti-diagonal, and indexed by $x$.

Now, we want to define the Weyl mapping that goes from a symbol to an operator, $\hat{a} \leftrightarrow a(x, k)$. First, we take the two-dimensional Fourier transform of the symbol $a(x, k)$ in both $x$ and $k$:

$$\tilde{a}(\sigma, \tau) = \frac{1}{2\pi} \int dxdk \ e^{-i(x-\sigma)k} a(x, k).$$ (A.1.9)

The $\tilde{a}(\sigma, \tau)$ are used as expansion coefficients over the set of phase space shifts generated by $\hat{x}$ and $\hat{k}$. That is,

$$\hat{a} = \frac{1}{2\pi} \int d\sigma d\tau \ e^{i(\sigma x - \tau k)} \tilde{a}(\sigma, \tau).$$ (A.1.10)

The Weyl symbol calculus is based on the fact that operators can be represented as a super-position of phase space shifts.\footnote{See Ref. [9] for a proof of this, which is related to the representation theory of the Heisenberg-Weyl group.} While not all operators can be decomposed onto the space of phase space shifts, the operators we are interested in can. We can use a truncated form of the
Campbell-Baker-Hausdorff theorem to write

\[ e^{i(\sigma \hat{x} - \tau \hat{k})} = e^{i\sigma \hat{x}} e^{-i\tau \hat{k}} = e^{-i\tau \hat{k}} e^{i\sigma \hat{x}} = e^{i\sigma \hat{x}} e^{-i\tau \hat{k}/2} = e^{-i\tau \hat{k}} e^{i\sigma \hat{x}} e^{i\tau/2}, \]  

(A.1.11)

where we have used

\[ [\hat{x}, \hat{k}] = \hat{x}\hat{k} - \hat{k}\hat{x} = i. \]  

(A.1.12)

Let us look at some examples to see how this works. First, consider the case when \( a(x,k) = x \). Then, Equation (A.1.9) gives us

\[
\tilde{a}(s,t) = \frac{1}{2\pi} \int dx dk \ e^{-i(\sigma x - \tau k)x}
= 2\pi i \delta'(\sigma) \delta(\tau).
\]  

(A.1.13)

Inserting this into Equation (A.1.10), we find

\[
\hat{a} = i \int d\sigma d\tau \ e^{i(\sigma \hat{x} - \tau \hat{k})} \delta'(\sigma) \delta(\tau)
= \hat{x}.
\]  

(A.1.14)

If we consider the case when \( a(x,k) = k \), we get

\[
\tilde{a}(\sigma, \tau) = \frac{1}{2\pi} \int dx dk \ e^{-i(\sigma x - \tau k)k}
= -2\pi i \delta(\sigma) \delta'(\tau)
\]  

(A.1.15)

\[
\hat{a} = -i \int d\sigma d\tau \ e^{i(\sigma \hat{x} - \tau \hat{k})} \delta(\sigma) \delta'(\tau)
= \hat{k}.
\]
Thus, we have shown, that under the Weyl symbol mapping

\[ \hat{x} \leftrightarrow x \]  \hspace{1cm} (A.1.16)

\[ \hat{k} \leftrightarrow k. \]  \hspace{1cm} (A.1.17)

Lastly, consider the case where \( a(x,k) = xk \). Then,

\[
\tilde{a}(\sigma, \tau) = \frac{1}{2\pi} \int dx dk \ e^{-i(\sigma x - \tau k)} xk \\
= 2\pi \delta'(\sigma) \delta'(\tau) \\
= \int d\sigma d\tau \ e^{i(\sigma \hat{x} - \tau \hat{k})} \delta'(\sigma) \delta'(\tau) \\
= - \int d\sigma d\tau \left( i\hat{x} - i\frac{\tau}{2} \right) e^{i\sigma \hat{x} - i\tau \hat{k}} e^{-i\sigma \tau/2} \delta'(\sigma) \delta'(\tau) \\
= - \int d\tau \left( i\hat{x} - i\frac{\tau}{2} \right) e^{-i\tau \hat{k}} \delta'(\tau) \\
= \int d\tau \left( \hat{x} \hat{k} - i \frac{\tau}{2} - \frac{1}{2} \hat{k} \hat{x} \right) e^{-i\tau \hat{k}} \delta'(\tau) \\
= \hat{x} \hat{k} - i \frac{1}{2} = \hat{x} \hat{k} - \frac{1}{2}(\hat{x} \hat{k} - \hat{k} \hat{x}) = \frac{1}{2}(\hat{x} \hat{k} + \hat{k} \hat{x}).
\]

Note that this agrees with our earlier result of Equation (A.1.6), and we have

\[
\frac{1}{2} (\hat{x} \hat{k} + \hat{k} \hat{x}) \leftrightarrow xk. \]  \hspace{1cm} (A.1.19)

Additionally, a key feature of the Weyl symbol mapping is that it preserves relationships between operators. For example, let \( \hat{a} = \hat{x} \) and \( \hat{b} = \hat{k} \). We know that \( [\hat{a}, \hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a} = i \), and
that $\hat{a}\hat{b} \rightarrow xk + i/2$ and $\hat{b}\hat{a} \rightarrow xk - i/2$. Hence, we have

$$\hat{a}\hat{b} - \hat{b}\hat{a} = i \rightarrow xk + \frac{i}{2} - \left(xk - \frac{i}{2}\right) = i.$$  \hspace{1cm} (A.1.20)

Furthermore, from Equation (A.1.18) and the commutation of $\hat{x}$ and $\hat{k}$, we can write

$$xk \leftarrow \hat{x}\hat{k} - \frac{i}{2} \hspace{1cm} \text{(A.1.21)}$$

$$xk \leftarrow \hat{k}\hat{x} + \frac{i}{2}. \hspace{1cm} \text{(A.1.22)}$$

Thus,

$$xk - xk = 0 \leftrightarrow \hat{x}\hat{k} - \frac{i}{2} - \left(\hat{k}\hat{x} + \frac{i}{2}\right) = i - i = 0. \hspace{1cm} \text{(A.1.23)}$$

### A.2 The Weyl Symbol in Multiple Dimensions

Let us now turn to the discussion of the Weyl symbol mapping in multiple dimensions, since we are interested in solving a multicomponent wave equation in multiple dimensions. Let $\hat{a}$ be an operator that acts on function of $n$ spatial dimensions, $x = (x_1, x_2, \ldots, x_n)$. To go from the operator to the symbol we have:

$$a(x, k) \equiv \int d^n s \ e^{-ik \cdot s} \left\langle x + \frac{1}{2}s \left| \hat{a} \left( x - \frac{1}{2}s \right) \right. \right\rangle. \hspace{1cm} \text{(A.2.1)}$$
To go from the symbol to the operator, we first calculate the $n$-dimensional Fourier transform of the symbol:

$$\tilde{a}(\sigma, \tau) = \frac{1}{(2\pi)^n} \int d^n x d^n k \ e^{-i(\sigma \cdot x - \tau \cdot k)} a(x, k),$$  \hspace{1cm} (A.2.2)

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$. Now, in the phase space associated with an $n$-dimensional configuration space, phase space shifts are generated by the operators $\hat{x}_m$ and $\hat{k}_m$, which obey the following commutation relations:

$$[\hat{x}_m, \hat{x}_n] = 0$$  \hspace{1cm} (A.2.3)

$$[\hat{k}_m, \hat{k}_n] = 0$$  \hspace{1cm} (A.2.4)

$$[\hat{x}_m, \hat{k}_n] = i\delta_{mn},$$  \hspace{1cm} (A.2.5)

for $m, n = 1, 2, \ldots, n$. If we define

$$\sigma \cdot \hat{x} - \tau \cdot \hat{k} \equiv \sigma_m \hat{x}_m - \tau_m \hat{k}_m,$$  \hspace{1cm} (A.2.6)

(where we are using the Einstein summation convention), we can write

$$\hat{a} = \frac{1}{(2\pi)^n} \int d^n x d^n \tau \ e^{i(\sigma \cdot \hat{x} - \tau \cdot \hat{k})} \tilde{a}(\sigma, \tau).$$  \hspace{1cm} (A.2.7)

Once again, using a corollary of the Campbell-Baker-Hausdorff theorem, and the fact that

$$[i\sigma \cdot \hat{x}, -i\tau \cdot \hat{k}] = \sigma_m \tau_n [\hat{x}_m, \hat{k}_n] = i\sigma_m \tau_n \delta_{mn} = i\sigma \cdot \tau,$$  \hspace{1cm} (A.2.8)
we can write
\[ e^{i(\sigma \cdot \hat{x} - \tau \cdot \hat{k})} = e^{i\sigma \cdot \hat{x}} e^{-i\tau \cdot \hat{k}} e^{-i(\sigma \cdot \tau)/2} \] (A.2.9)

Let us look at an example that nicely illustrates the power of the Weyl symbol mapping.

Consider a wave equation that is of the form
\[ \int d^n x' \ D(x - x') \psi(x') = 0. \] (A.2.10)

Such a wave equation can be derived from the abstract equation
\[ \hat{D} |\psi\rangle = 0. \] (A.2.11)

If we insert a complete set of states and come from the left with \( \langle x | \), we have
\[
0 = \int d^n x' \ \langle x | \hat{D} | x' \rangle \langle x' | \psi \rangle \\
= \int d^n x' \ D(x, x') \psi(x') \\
= \int d^n x' \ D(x - x') \psi(x'),
\] (A.2.12)

where \( D(x, x') \) is the \( x \)-representation of the abstract operator \( \hat{D} \), and \( D(x, x') = D(x - x') \) explicitly. We want to show that, in this case, the Weyl symbol of the operator \( \hat{D} \) is a function of \( k \) only, and that we want write the wave equation in the pseudodifferential form
\[ D(-i\nabla) \psi(x) = 0. \] (A.2.13)
First, we compute the symbol of $\hat{\mathcal{O}}$:

\[
D(x, k) = \int d^n s \ e^{-i k s} \left( x + \frac{1}{2} s \right) \left| x - \frac{1}{2} s \rightangle \langle \left. x + \frac{1}{2} s \right| \hat{\mathcal{O}} \cdot \left( x - \frac{1}{2} s \right) \\
= \int d^n s \ e^{-i k s} D(s).
\]  
(A.2.14)

Hence, we have

\[
\hat{\mathcal{O}} \xrightarrow{\Sigma} D(k) = \int d^n s \ e^{-i k s} D(s),
\]  
(A.2.15)

that is, the symbol $D$ of the operator $\hat{\mathcal{O}}$ is a function of $k$ only. Now, let us go from the symbol to the operator. First, we compute the $n$-dimensional Fourier transform of the symbol using Equation (A.2.2):

\[
\tilde{D}(\sigma, \tau) = \frac{1}{(2\pi)^n} \int d^n x d^n k \ e^{-i(\sigma \cdot x - \tau \cdot k)} D(k) \\
= \frac{1}{(2\pi)^n} \int d^n x d^n k \ e^{-i(\sigma \cdot x - \tau \cdot k)} \int d^n s \ e^{-i k s} D(s) \\
= \frac{1}{(2\pi)^n} \int d^n s d^n x d^n k \ e^{-i \sigma \cdot x} e^{-i k (s - \tau)} D(s) \\
= (2\pi)^n \delta(\sigma) \delta(s - \tau) D(s) \\
= (2\pi)^n \delta(\sigma) D(\tau).
\]  
(A.2.16)

Using Equation (A.2.7) to calculate the symbol $\hat{\mathcal{O}}$, we get

\[
\hat{\mathcal{O}} = \int d^n \sigma d^n \tau \ e^{i(\sigma \cdot \hat{x} - \tau \cdot \hat{k})} \delta(\sigma) D(\tau) \\
= \int d^n \tau \ e^{-i \tau \cdot \hat{k}} D(\tau) \\
= \int d^n \tau \ e^{-i \tau \cdot \hat{k}} D(\tau) \\
= D(\hat{k}).
\]  
(A.2.17)
Therefore, we can express Equation (A.2.10) as, (in the $x$-representation),

$$D(-i\nabla)\psi(x) = 0.$$  \hspace{1cm} (A.2.18)

Finally, we want to examine the symbols of multicomponent wave operators, which is what we are interested in after all! Consider the following abstract system:

$$\hat{\mathcal{D}}_{nm}\ket{\psi_m} = 0,$$  \hspace{1cm} (A.2.19)

with $n, m = 1, 2, \ldots, N$, and summation over repeated indices is implied. If we insert a complete set of position states and come from the left with the bra $\bra{x}$, we can write Equation (A.2.19) as the integral equation

$$\int d^n x' \bra{x} \hat{\mathcal{D}}_{nm} \ket{x'} \bra{x'} \psi_m \rangle = \int d^n x' \ D_{nm}(x, x') \psi_m(x') = 0.$$  \hspace{1cm} (A.2.20)

The collection of operators $\hat{\mathcal{D}}_{nm}$ forms an operator-valued, $N \times N$ matrix. Hence, each operator entry has an associated Weyl symbol as defined in the scalar operator case, with the appropriate spatial dimensions. The result is a matrix of symbols, such that,

$$\hat{\mathcal{D}}_{nm} \leftrightarrow D_{nm}(x, \mathbf{k}).$$  \hspace{1cm} (A.2.21)

We identify the matrix of symbols, $D(x, \mathbf{k})$, as the dispersion matrix. Because the symbol mapping gives a decomposition of the operators into phase space shifts, in the $x$-representation,
we can write:

\[ D_{nm}(x, -i\nabla) \overset{\sim}{\leftrightarrow} D_{nm}(x, k), \]  

(A.2.22)

without any ambiguity. *Note bene,* this implies much more than simply replacing \( k \) with \(-i\nabla\) in the symbol. The Weyl symbol calculus *automatically* symmetrizes the operator on the left! (As seen in Equations (A.1.18) and (A.1.19).)
APPENDIX B

Tangential Mode Conversions

In this last section we will briefly discuss our investigation of tangential mode conversions, such as those indicated by the appearance of the blue, horizontal lines in Figure 5.11. We start with a $2 \times 2$ dispersion matrix of the form in Equation (5.1.1), where $\tilde{S}(x,y)$ and $H(x,y)$ are both functions of the spatial coordinates $(x,y)$. To facilitate this investigation, and construct a model with an explicit tangential intersection, we make some simplifying assumptions. First, as can be seen from the level sets of the functions $\tilde{S}(x,y)$ and $H(x,y)$, as shown in Figures 5.2a and 5.3a, respectively, in the vicinity of $y = 0$ the functions do not vary significantly in the $y$-direction. Hence, our first simplification is to ignore the $y$-dependence of $\tilde{S}$ and $H$. Next, we Taylor expand the function $\tilde{S}$, (to first order in $x$), about the ion-hybrid resonance, the left-most vertical line in Figure 5.11, which we shall denote as $x_\lambda$. Shifting the origin to this point, and
going back to $k_x$ and $k_y$ notation, our dispersion matrix becomes

$$D(x, k_x, k_y) = \begin{pmatrix} x - k_y^2 & -i\eta + k_x k_y \\ i\eta + k_x k_y & x - k_x^2 \end{pmatrix}, \quad (B.0.1)$$

where $\eta \equiv H(x, 0)$, and is, indeed, constant. Furthermore, when $k_y = 0$, the matrix $D$ of Equation (B.0.1), is, in fact, in normal form. Therefore, let us plot the zero loci of the diagonals, $D_{11} = 0$ and $D_{22} = 0$, when $k_y = 0$. As can be seen from Figure B.1, this system

![Figure B.1](image)

**FIG. B.1:** A plot of the uncoupled dispersion relations for the matrix $D$ of Equation (B.0.1). The blue line, $D_{11} = x = 0$, corresponds to the ion-hybrid dispersion curve. The orange line, $D_{22} = x - k_x^2 = 0$, corresponds to the magnetosonic dispersion curve. Note, there is in fact a tangential intersection at $(x = 0, k_x = 0)$. 

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does exhibit a tangential intersection, and the conversion point is at \((x = 0, k_x = 0)\), as desired.

Lastly, using the Weyl symbol mapping, which, recall, is invertible, we can go from the symbol \(D\) to the operator \(\hat{D}\) in the \(k_x\)-representation:

\[
\hat{D}(\hat{x}, k_x; k_y, \eta) = \begin{pmatrix}
\hat{x} - k_y^2 & -i\eta + k_x k_y \\
i\eta + k_x k_y & \hat{x} - k_x^2
\end{pmatrix},
\]

where \(\hat{x} = i \frac{d}{dk_x}\), and \(k_y\) and \(\eta\) are parameters. We then want to solve the equation

\[
\hat{D}\Psi(k_x) = \begin{pmatrix}
\hat{x} - k_y^2 & -i\eta + k_x k_y \\
i\eta + k_x k_y & \hat{x} - k_x^2
\end{pmatrix} \begin{pmatrix}
\psi_1(k_x) \\
\psi_2(k_x)
\end{pmatrix} = 0.
\]

Once we solve this system numerically, we can proceed to match it to the asymptotic solutions in order to determine the WKB connection coefficients. When we do this, for various values of \(\eta\) and \(k_y\), we find that the conversion and transmission coefficients behave similarly to those corresponding to a more generic transverse intersection. Thus, even where there might be these tangential conversions, they should not introduce any new difficulties into our analysis.
BIBLIOGRAPHY


