5-2009

The Set of All Countable Ordinals: An Inquiry into Its Construction, Properties, and a Proof Concerning Hereditary Subcompactness

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A thesis submitted in partial fulfillment of the requirement for
the degree of Bachelor of Science with Honors in Mathematics
from the College of William and Mary in Virginia,

by

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Accepted for ______________________________

(Honors, High Honors, or Highest Honors)

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Williamsburg, Virginia

May 1, 2009

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Chapter 1: The Construction of the Set of All Countable Ordinals

1 Introduction

The purpose of this chapter is to construct the set of all countable ordinals, designated Ω. The set has the property that it is uncountable, well-ordered, and the subset of Ω consisting of any element of Ω along with its predecessors is at most countable. The set of all countable ordinals can be constructed immediately from any uncountable set and the Axiom of Choice. This chapter uses more familiar elements to construct Ω, and weaker axioms than the Axiom of Choice, namely the Countable Axiom of Choice and the Axiom of Dependent Choice.

2 Ordinals, Limit Ordinals, and Their Properties

Traditionally, ordinals can be thought of as indexing numbers. The purpose of ordinal numbers constrasts with the purpose of cardinal numbers. Cardinals help mathematicians gain a sense of measurement in terms of size and magnitude. Ordinals give a sense of place or ordering. The non-negative integers $0, 1, 2, \cdots$ can be thought of as being examples of ordinal numbers, and so can the set $\{0, 1, 2, 3, \cdots\}$. However, these sets are countable in their cardinality. A special set called "the set of all countable ordinals", and abbreviated Ω, is also a set of ordinal numbers. The purpose of this chapter is to construct that set; however, before moving directly to its proof, I want to consider more about what an ordinal really is. Sheldon Davis, in *Topology*, gives the following definition of an ordinal:

Definition 2.1 *A set α is an* ***ordinal*** *provided the following are true:*

1. *if $x \in y \in \alpha$, then $x \in \alpha$ (i.e. α is $\in$ transitive) and*
2. *if $x \in y \in z \in \alpha$, then $x \in z$ (i.e. each element of α is $\in$ transitive).*

The empty set satisifies Definition 2.1, so that $\emptyset$ is an ordinal. Professor Davis shows that each of the following is an ordinal: the union of an ordinal with the set including that ordinal is an ordinal, the union of subsets of an ordinal is an ordinal, and the intersection between nonempty subsets of an ordinal is an ordinal. Davis goes on to show how ordinals can be recursively constructed with the following example:

Example 2.2 *Here are some ordinals:*

- *define* $0 = \emptyset$
- $1 = 0 + 1 = \emptyset \cup \{\emptyset\} = \{0\}$
- $2 = 1 + 1 = 1 \cup \{1\} = \{0, 1\}$
- $3 = 2 + 1 = 2 \cup \{2\} = \{0, 1, 2\}$ *and so on...*
- $n + 1 = n \cup \{n\} = \{0, 1, 2, ..., n - 1, n\}$. [1]

An important consquence of the transitive set method of constructing ordinals comes in the form of the ordering of the ordinals. Since I am borrowing directly from Professor Davis, I will repeat his ordering below as an unproven lemma.

Lemma 2.3 *The ordinals are well-ordered by "$\in$ or $=$", that is $\alpha \leq \beta$ if and only if $\alpha = \beta$ or $\alpha \in \beta$.* [2]

There are a few things to note about the given definition and the above examples about ordinals. Ordinals generated by this transitive set method can be generated from unions of smaller ordinals, where a smaller ordinal is an ordinal less than another. The proof given below will show that there is no such thing as the largest ordinal, or an ordinal without any ordinal greater than it, in part because of this construction. This is in some ways analogous to the power set method of constructing ever larger sets in set theory. As there is no largest set, so too is there no greatest ordinal.

Theorem 2.4 *There is no largest ordinal.*

Proof: Assume that there is a largest ordinal in order to get a contradiction, call it γ. Consider $\gamma \cup \{\gamma\}$, call it γ^+. I will show that γ^+ follows the transitive definition of an ordinal given above.

- Case 1: Consider $x \in y \in \gamma^+$. Since $y \in \gamma^+$ then $y \in \gamma$ or $y \in \{\gamma\}$. If $y \in \{\gamma\}$ then $y = \gamma$ and since $x \in y$ then $x \in \gamma$ which means that $x \in \gamma^+$ by deifnition of union. If $y \in \gamma$ then $x \in \gamma$ which means $x \in \gamma^+$ by the definition of union.
- Case 2: Consider $x \in y \in z \in \gamma^+$. Since $z \in \gamma^+$ then $z \in \gamma$ or $z \in \{\gamma\}$. If $z \in \gamma$ since γ is an ordinal, then it already has the property that if $x \in y \in z \in \gamma$ then $x \in z$. If $z \in \{\gamma\}$ then $z = \gamma$. Since γ already has the property that its elements are transitive, this completes the proof.

I have shown that γ^+ fulfills the definition of being an ordinal; however, γ^+ is larger than γ, which contradicts the hypothesis that γ is the largest ordinal. Therefore, there is no largest ordinal. $\square$

[1] *Topology*, Sheldon Davis, 2005, pp 97-98.
[2] *Topology*, Sheldon Davis, 2005, pp 97.

Definition 2.5 *If α is an ordinal, then the **successor** of α is $\alpha + 1 = \alpha \cup \{\alpha\}$.*

From the definition and the proof given, it should be obvious that every ordinal has a successor, simply take the given ordinal and union it with the set consisting of itself. The ordinal that comes immediately after a given ordinal is called that ordinal's **immediate successor**. The above definition gives the method for creating any ordinal's immediate successor. Any ordinal that comes before a given ordinal is called its **predecessor** and the ordinal that comes immediately before it is called its **immediate predecessor**. Now, every ordinal has an immediate successor, simply by use of the definition given above and Theorem 2.4. However, not every ordinal has an immediate predecessor, a trivial example being the ordinal 0. There also exist non-trivial examples. For a proof concerning the existence of nontrivial ordinals that lack an immediate predecessor see Chapter 2 Section 1 of this paper.

While the above theorem shows that there is no such thing as the largest ordinal, it is not clear that the construction given above ever produces ordinals that are uncountable in cardinality. Every ordinal has a successor, as follows immediately from the definition given below, but it is not immediately clear that a method of transitive construction will allow for an ordinal to ever pass outside of the countable. I will show in this chapter a separate means of constructing ordinals that will lead to a set that is uncountable in its total size.

3 Well-Ordering Theorems

The following theorems involve properties of well-ordering. A well-ordered set is a linearly ordered set X with the additional property that any nonempty subset of X, including the set itself, has a first element. [3]

From the definition given, a well-ordered set has a first element, and every subset of the well-ordered set has a first element, and thus is also well-ordered. However, this does not mean that every well-ordered set will have the same first element. For a given well-ordered set, W, the first element is often described as 0_W. For example, the set $\{0, 1, 2\}$ with the usual ordering as the first element of 0, while the set $\{1, 2, 3\}$ has the first element of 1.

Important to this paper are well-ordered subsets of the real numbers, $\mathbb{R}$, with the usual ordering. The set of real numbers is not itself well-ordered, but does contain many well-ordered subsets, for example:

a) $\{n : n \in \mathbb{Z}, 0 \leqslant n\}$ (where $\mathbb{Z}$ is the set of integers)

b) $\{1 - \frac{1}{n} : n \geqslant 1\} \bigcup \{1\}$

c) If X_n is a well-ordered subset of $\mathbb{R}$, then $\forall n \geqslant 1$ there is a set Y_n that is an order-preserving copy of X_n inside $[n, n + 1)$. Let $Y = \bigcup \{ Y_n : \text{n} \geqslant 1 \}$. Then Y is well-ordered.

d) Any further iteration of (c)

[3] To be even more explicit, X is a linearly ordered set if the order, $\leq$, is transitive and antisymmetric, and for any $a, b \in X$ either $a \leq b$ or $b \leq a$ holds true.

Technically, (a) given above can be a well-ordered set starting from any value in the integers. Likewise, (a) and (b) are obviously well-ordered. Assertion (d) is obvious after (c) is shown to be true. Thus, the first theorem of this section will seek to prove (c).

Theorem 3.1 *If X_n is a well-ordered subset of $\mathbb{R}$, then $\forall n \geqslant 1$ there is a set Y_n that is an order-preserving copy of X_n inside $[n, n+1)$. Let $Y = \bigcup \{ Y_n : n \geqslant 1 \}$. Then Y is well-ordered.*

Proof: Consider the function $f(x) = \arctan(x)$. This function generates an order-preserving copy of $\mathbb{R}$ into $(\frac{-\pi}{2}, \frac{\pi}{2})$. Then $g_n(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2} + n$ will map $\mathbb{R}$ and any subset of $\mathbb{R}$ into an order-preserving copy within the interval $[n, n+1)$. Let each $Y_n = g_n[X_n]$ where $g_n(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2} + n$.

It will first be shown that Y is linearly ordered. Let $a, b \in Y$. Then a and b each come from some original Y_n. If a and b come from different Y_n, then each comes from a different $[n, n+1)$ and inherits the ordering of the index. If a and b come from the same Y_n, then the elements inherit the ordering accordingly. Therefore, Y is linearly ordered.

Now that Y is linearly ordered, it remains to be shown that it and every nonempty subset of it has a first element. Therefore, let $A \subseteq Y$ where $A \neq \emptyset$. Obviously, the elements of A come from Y. Therefore, the elements of A fall into the various intervals of $[n, n+1)$. Pick the first such interval from which elements of A come. Consider the elements of A from this interval, they come from some initial Y_n. Since Y_n is well-ordered, the subset $A \cap Y_n$ has a first element. This element is the first element of A. Since A could have been any subset of Y, Y has the property that every subset has a first element. □

The above examples are all countable in size, which can easily be seen as they are all constructed in countably many steps from sets of integers, which itself is countable. However, there is another reason too.

Theorem 3.2 *Any subset of $\mathbb{R}$ that is well-ordered by the usual ordering of $\mathbb{R}$ must be countable.*

Proof: Let S be a well-ordered subset of $\mathbb{R}$ under the usual ordering. Consider any element of S, say α, and the interval $[\alpha, \alpha^+)$ where α^+ is the immediate successor of α in S, when and where it exists. Otherwise, α is the largest element of S, and let $\alpha^+ = \alpha$. If $\alpha = \alpha^+$ then $[\alpha, \alpha^+) = \emptyset$, otherwise $\alpha < \alpha^+$ so $\exists \beta \in \mathbb{Q}$ such that $\alpha < \beta < \alpha^+ \in \mathbb{R}$. This happens becuase $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, each half open interval of elements of S can be indexed by a unique rational number, since the intervals are disjoint. Since $\mathbb{Q}$ is countable, then S is therefore countable. Furthermore, since $[\alpha, \alpha^+) = \emptyset$ can only happen once in the entire subset S, this case does not defeat the countability of S. □

Other than the integers and the rational numbers, the reals are the most familiar family of numbers. It seems reasonable to ask whether or not, following from the previous thereom, any uncountable well-ordered set exists at all. As the next theorems show, the Axiom of Choice gurantees the existence of the set of all countable ordinals, Ω. Such a set is not only well-ordered but uncountable in size. However, it also has the additional property that the set of all predecessors of any element of Ω will be at most countable in size.

Theorem 3.3 *The Axiom of Choice is equivalent to the statement "Any set can be well-ordered".*[4]

Since an equivalent formulation of the Axiom of Choice states that any set can be well-ordered, it follows that taking any set of uncountable size and well-ordering it will allow for the immediate construction of Ω given the following theorem.

Theorem 3.4 *If W is an uncountable, well-ordered set, then there exists $\Omega \subseteq W$ such that for every $\alpha \in \Omega$, the set of all predecessors of α is countable and Ω is uncountable and well-ordered.*

Proof: Let (*) denote the property that the set of all precessors of any element of W is countable.

- Case 1: W has (*). Then $W = \Omega$ and the proof is complete.
- Case 2: W does not have (*). Then the set of all elements of W with uncountably many predecessors is non-empty. Call this set Λ. Clearly, $\Lambda \subseteq W$ and since W is well-ordered, Λ has a first element, call it ω_1. Consider the set $[0_W, \omega_1) \subseteq W$. Since this set is a subset of W, it is well-ordered, and since $\omega_1 \in \Lambda$, the set is uncountable. Since ω_1 is the first element with uncountably many predecessors and $[0_W, \omega_1)$ only includes predecessors of ω_1, then every element of $[0_W, \omega_1)$ can have only countably many predecessors. Therefore $[0_W, \omega_1) = \Omega$. □

So given any set, uncountable in size, the Axiom of Choice may then be used, by an equivalent formulation, in conjunction with Theorem 3.4 to well-order such a set and then derive Ω from that set. The purpose of this chapter is to attempt to construct Ω using far weaker axioms than the Axiom of Choice. [5]

Definition 3.5 *The Countable Axiom of Choice (**CAC**): For every sequence of nonempty sets (A_n) there exists a sequence (x_n) such that $x_n \in A_n$ for all $n \in \mathbb{N}$.*

Definition 3.6 *The Axiom of Dependent Choice (**DC**): If $\boldsymbol{r}$ is a relation on the set A such that $\forall x \in A\ \exists y \in A$ such that $x\boldsymbol{r}y$, then for any $a \in A$ there exists a sequence (x_n) in A such that $x_0 = a$ and $x_n \boldsymbol{r} x_{n+1}$ for all $n \in \mathbb{N}$.* [6]

The Axiom of Dependent Choice is provable from the Axiom of Choice, and the Countable Axiom of Choice is provable from the Axiom of Dependent Choice. However, the implications do not reverse. The Countable Axiom of Choice cannot prove the Axiom of Dependent Choice, nor can the Axiom of Dependent Choice prove the Axiom of Choice.[7] In the rest of this chapter, the Axiom of Dependent Choice is the strongest axiom used.

[4]For proof see: Potter Michael, *Set Thoery and Its Philosophy*, p 244, Oxford Univesrity Press, 2004.

[5]Sheldon Davis in his book *Topology* claims that Ω can be constructed without the Axiom of Choice using the transitive set approach to ordinals outlined in the previous section. While strictly true that the Axiom of Choice is not necessary, a few other theorems proved indispensable.

[6]Both of these definitions are taken from Potter Michael, *Set Thoery and Its Philosophy*, p 161 and p 238, Oxford Univesrity Press, 2004.

[7]Potter, Michael. *Set Theory and Its Philosophy*, Oxford University Press, 2004, pp 239.

Theorem 3.7 *Assume DC. A linearly ordered set $(X, \leq)$ is well-ordered if and only if it does not contain any strictly decreasing sequences.*

Proof: Beginning in the forward direction, I will show that a well-ordered set does not contain any strictly decreasing sequences. For the purposes of a contradiction, suppose not. Let P be a well-ordered set that contains a strictly decreasing sequence. Consider the set of all elements of the strictly decreasing sequence, call it P'. Clearly, $P' \subseteq P$, which means that P' is well-ordered. By the definition of a well-ordered set, P' has a first element, call it p. Since p is an element of P', and P' is strictly decreasing, there is some other element $p^- \in P'$ such that $p^- < p$. But p is the first element of the strictly decreasing sequence, and so this is a contradiction. Therefore, a well-ordered set cannot contain any strictly decreasing sequences.

It remains to be shown now that a linearly ordered set without a strictly decreasing sequence is well-ordered. This will require the DC axiom. Let the linearly ordered set without any infinitely decreasing subsequence be called X. Let S be any non-empty subset of X. Consider any element of S, say a_0. If a_0 is the smallest element of S, i.e., $\min(S)$, then the proof is complete, and a_0 is then the first element of S. However, if a_0 is not the first element of S, then there is some element that precedes it. Chose any such element in S, and call it a_1. Again, a_1 can either be the first element of S, which then completes the proof, or it might have elements previous to it. Given a_n, either a_n is the first element of S or else we may choose $a_{n+1} \in S$ with $a_{n+1} < a_n$. This process must stop after a finite number of steps, otherwise it produces a strictly decreasing sequence in X, which is contrary to the hypothesis. □

It should be noted that the forward direction of the above theorem did not rely upon the DC axiom. The reverse direction of the theorem did rely upon the DC axiom.[8] Therefore, it would be possible to take the forward direction as an independent theorem about well-ordered sets regardless of the truth of the DC axiom. Nonetheless, the reverse direction of the theorem will be necessary for the purposes of this paper.

4 The Order Isomorphism Theorems

The following section makes use of the definition of an order isomorphism, defined below along with a strictly increasing function, of which an order isomorphism will be seen to be an example. Combining order isomorphisms with well-ordered sets results in several interesting properties. It will be shown, for example, that no well-ordered set is order-isomorphic to a proper initial segment of itself. Again, the application of these theorems will be confined to proving the existence of Ω although their consequences range further than the narrow scope of this paper.

Definition 4.1 *Let X and Y be linearly ordered sets. A function f from X to Y is* ***strictly increasing*** *if $x < y$ in X implies $f(x) < f(y)$ in Y.*

[8] I am indebted to Professor Herman Rubin of Purdue University for pointing out the need for the Dependent Axiom of Choice.

Definition 4.2 *Let $(U, <_U)$ and $(V, <_V)$ be linearly ordered sets. An **order isomorphism** is a function from one linearly ordered set onto another, say $f : U \to V$, such that if $u_1 <_U u_2$ in U, then $f(u_1) <_V f(u_2)$ in V.*

Along with order isomorphisms, the following definition of a cofinal subset will prove useful.

Definition 4.3 *Let W be a well-ordered set. A cofinal subset of W is any set $C \subseteq W$ with the property that for each $x \in W$, there is some $y \in C$ such that $x \leq y$.*

The tools are now sufficient to prove the following theorem:

Theorem 4.4 *If $(U, <)$ is a well-ordered set and f: $U \to U$ is strictly increasing, then the set f[U] is a cofinal subset of U.*

Proof: For the purposes of a contradiction, suppose not. Then U is a well-ordered set, f: U $\to$ U is strictly increasing and f[U] is not cofinal with U. Therefore $\{v \in U | \forall u \in U, f(u) < v\}$ is nonempty. Call this set S. Then S is a non-empty subset of U, which is well-ordered, and therefore S has a first element. Call that first element y_1. Consider $f(y_1)$. It is known that $f(y_1) \in$ U and $f(y_1) \in$ f[U], which by hypothesis means $f(y_1) < y_1$. Let $f(y_1) = y_2$. Then $y_2 < y_1$. Consider $f(y_2)$. Since f is an order isomorphism and $y_2 < y_1$ then $f(y_2) < f(y_1)$ and since $f(y_1) = y_2$ then $f(y_2) < y_2$. Let $f(y_2) = y_3$. For the purposes of induction, suppose y_1 through y_n exist with $f(y_1) = y_2$, $f(y_2)$ $= y_3$, ... , $f(y_{n-1}) = y_n$ and $y_n < y_{n-1} < ... < y_2 < y_1$. Then let $y_{n+1} = f(y_n)$. Then $y_{n+1} < y_n$ because $y_{n+1} = f(y_n)$ and $y_n = f(y_{n-1})$. Since $y_n < y_{n-1}$ and f is an order isomorphism, then $f(y_n)$ $< f(y_{n-1})$, which by the identities just given, proves that $y_{n+1} < y_n$.

Therefore, f can be used to create a strictly decreasing sequence of elements of U. However, by Theorem 3.7 this is impossible. A contradiction is thereby attained, and the hypothesis thus rejected, completing the proof for the theorem. □

Continuing on with the consequences of joining order isomorphisms with well-ordered sets, I will now show that order isomorphisms divide well-ordered sets into equivalence classes.

Definition 4.5 *For well-ordered sets C_1 and C_2, $C_1 \sim C_2$ means there is an order isomorphism from C_1 onto C_2.*

Theorem 4.6 *The relation $\sim$ is an equivalence relation.*

Proof: It must be shown that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: It must be shown that $C_1 \sim C_1$. Let f be the identity function. Then if $c_1 < c_2$ with c_1,$c_2 \in C_1$, then $f(c_1) < f(c_2)$ since $f(c_1) = c_1 < c_2 = f(c_2)$. Clearly, f is onto.

Symmetry: Suppose $f : C_1 \to C_2$ is an order isomorphism. Then f is 1-1 onto, so $f^{-1} : C_2 \to C_1$ exists and is 1-1 onto. I will show that if $y_1 < y_2$ in C_2 and if $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ then $x_1 < x_2$ in C_1. Since C_1 is well-ordered, then either $x_1 < x_2$, $x_1 = x_2$, or $x_1 > x_2$. If $x_1 = x_2$, then $f(x_1) = y_1$, $f(x_2) = y_2$, and since f is a function, then $y_1 = y_2$ which contradicts the assumption that $y_1 < y_2$, so equality cannot occur. If $x_1 > x_2$, then $f(x_1) = y_1$, $f(x_2) = y_2$, and $y_1 > y_2$ because

f is order-preserving, but the hypothesis states that $y_1 < y_2$, so this cannot occur. Therefore, the only possibe outcome is that $x_1 < x_2$.

Transitivity: It must be shown that if $C_1 \sim C_2$ and $C_2 \sim C_3$ then $C_1 \sim C_3$. Let f be the order isomorphism from C_1 into C_2 and let g be the order isomorphism from C_2 into C_3. It is already known that $g \circ f$ is one-to-one and onto. If $c_1 < c_2$ with c_1, $c_2 \in C_1$, then $f(c_1) < f(c_2)$ with $f(c_1)$, $f(c_2) \in C_2$. Since $f(c_1) < f(c_2)$ with $f(c_1)$, $f(c_2) \in C_2$ then $g(f(c_1)) < g(f(c_2))$ by hypothesis. Therefore $g \circ f$ is an order isomorphism from C_1 onto C_3. □

The above proof holds for all well-ordered sets with order isomorphisms. This section will be concerned with a smaller collection of these sets, namely those equivalence classes that come from well-ordered subsets of the real numbers, $\mathbb{R}$. For the purposes of further notation, these equivalence classes of $\sim$ will be written with small Greek letters, and Ψ will stand for the collection of all of these equivalence classes of $\sim$.

Two new definitions will be introduced into the mix, that of an initial segment and a special ordering property of certain signficance. They lead directly into a proof concerning proper initial segments and the ordering of the previously defined equivalence classes.

Definition 4.7 *An initial segment of a well-ordered set W is a set $I \subseteq W$ such that if x,y $\in W$ satisfy $x \le y \in I$, then $x \in I$. A proper initial segement is one in which I is a proper subset of W.*

Definition 4.8 *Suppose α, $\beta \in \Psi$. Then $\alpha \preceq \beta$ if either $\alpha = \beta$ or if there exist some $A \in \alpha$ and $B \in \beta$ and some order isomorphism from A onto a proper initial segment of B.*

Theorem 4.9 *Suppose α,$\beta \in \Psi$ and $\alpha \prec \beta$. Then $\forall A \in \alpha$ and $\forall B \in \beta$ there is an order isomorphism from A onto a proper initial segment of B*

Proof: Since $\alpha \prec \beta$, there is some $A' \in \alpha$ and $B' \in \beta$ and some order isomorphism f from A' onto a proper initial segment of B'. Let $A'' \in \alpha$ and $B'' \in \beta$. Then there is an order isomorphism $g : A'' \to A'$ and an order isomorphism $h : B' \to B''$. As in Theorem 4.6, the composite mapping $h \circ f \circ g : A'' \to B''$ is an order isomorphism as required. □

I am now ready to show how strictly increasing functions can be used to relate arbitrary well-ordered sets. The next proof shows that any two non-empty well-ordered sets can be mapped order isomorphically to each other or one onto a proper initial segment of another.

Theorem 4.10 *If U and V are two non-empty well-ordered sets, then either U is order isomorphic to V, or U is order isomorphic to some proper initial segment of V, or V is order isomorphic to some proper initial segment of U.*

Proof: Assume that the previous theorem is false. Then there exist two well-ordered sets U and V such that U cannot be order isomorphic to V, and U cannot be order isomorphic to some proper initial segment of V, and V cannot be order isomorphic to some proper initial segment of U.
Let f_0 be a function from U to V such that $f_0 : [0_u, 0_u] \to [0_v, 0_v]$. Let f_1 be a function from U to V such that $f_1 : [0_u, 1_u] \to [0_v, 1_v]$, such that $f(0_u) = 0_v$ and $f(1_u) = 1_v$. Let f_2 be a function from U to V such that $f_2 : [0_u, 2_u] \to [0_v, 2_v]$, such that $f(0_u) = 0_v$, $f(1_u) = 1_v$ and $f(2_u) = 2_v$. Let

$\beta \in U$. Assume that for every $\alpha \in U$ where $\alpha < \beta$ there is a unique $f_\alpha : [0_u, \alpha] \to V$ such that f_α is strictly increasing and $\{f_\alpha(\gamma) | 0 \leq \gamma \leq \alpha\}$ is an initial segment of V.

Note that if $\gamma < \alpha < \beta$ then f_α extends f_γ since $f_\alpha = f_\gamma$ when f_α is restricted to the domain of f_γ. This is made possible by the uniqueness assumption in the hypothesis.

Define $g = \bigcup\{f_\alpha | \alpha < \beta\}$. Notice the g is well defined, strictly increasing, and onto an initial segment of V. The $dom(g) = [0_u, \beta)$.

The function g must now be extended to include $\beta \in U$. Consider the output, or the image of g, that is $\{g(\alpha)|\alpha < \beta\}$. From the original hypothesis, this set cannot be all of V, otherwise, $g^{-1} : V \to [0, \alpha)$ would be a strictly increasing one-to-one mapping of V onto an initial segment of U, but this is denied by the hypothesis. Therefore $V - \{g(\alpha)|\alpha < \beta\} \neq \emptyset$. Since this forms a non-empty subset of V, the set has a first element, v_β. Extend g by defining $g(\beta) = v_\beta$. Obviously, g is a strictly increasing and onto an initial segment of V, namely $[0_v, v_\beta]$. Now let $f_\beta = g$

I showed that there is a unique $f_\alpha : [0_u, \alpha) \to V$ for each $\alpha \in U$. Let $h = \bigcup\{f_\alpha|\alpha \in U\}$. Because f_β extends f_α whenever $\alpha < \beta$, h is well defined, strictly increasing, onto an initial segment of the well-ordered set V, and the domain is all of U. This contradicts the original hypothesis. □

It is important to note that the above theorem does not make use of the Axiom of Choice. At each stage of construction, the elements chosen are given or defined by and from the well-ordering. It has now been proven that any two well-ordered sets, regardless of the particular meaning of the ordering for either one, can be functionally placed in an order isomorphic relation either with each other or with some proper initial segment from one to the other. I will now apply that result to Ψ.

5 Obtaining Ω

The next few theorems deal directly with obtaining Ω from Ψ. Previous theorems showed how well-ordered subsets of $\mathbb{R}$ can be placed into different equivalence classes and how an ordering relationship between the equivalence classes can be defined using the order isomorphic property. I will subsequently show how Ψ is well-ordered under the relation $\preceq$.

Theorem 5.1 *The relation $\preceq$ is a linear ordering on the set Ψ.*

Proof: Consider $\alpha, \beta \in \Psi$. Assume $\alpha \neq \beta$. Then there is a well-ordered set $A \in \alpha$ and a well-ordered set $B \in \beta$, which by their respective equivalence class are order isomorphic to all other elements of α and β. The sets A and B are well-ordered sets, so by Theorem 4.10 one of three things must happen: either A is order isomorphic to B, which cannot happen because $\alpha \neq \beta$, or A is order isomorphic to some proper initial segment of B, in which case $\alpha \prec \beta$, or B is order isomorphic to some proper initial segment of A, in which case $\beta \prec \alpha$.

Furthermore, it cannot happen that $\alpha \prec \beta$ and $\beta \prec \alpha$. Otherwise, A would be order isomorphic to some proper initial segment of B, which would be order isomorphic to some proper initial segment of A, which by transitivity, would mean that A is order isomorphic to some proper initial segment of itself, which is impossible by Theorem 4.4. □

Recall that the DC axiom shows that a well-ordered set is linearly ordered set that contains no strictly decreasing sequences.

Theorem 5.2 *Assume DC. The relation $\preceq$ is a well-ordering of Ψ*

Proof: For the purposes of contradiction, the theorem will be assumed false. According to Theorem 3.7, Ψ contains an infinite decreasing sequence. Therefore, there exists some infinite sequence $\alpha_n \in \Psi$ with $\alpha_{n+1} < \alpha_n$, for each n. Fix $A_1 \in \alpha_1$. It will be shown that there is a unique proper initial segment of $A_n \subseteq A_1$ with $A_n \in \alpha_n$. Choose any B,C $\in \alpha_n$. Because $\alpha_n \prec \alpha_1$ there are order isomorphisms f: B $\rightarrow A_1$ and g: C $\rightarrow A_1$ where $f[B]$ and $g[C]$ are both proper initial segments of A_1. I claim $f[B] = g[C]$. If $f[B] \neq g[C]$ then either $f[B] \subset g[C]$ or $g[C] \subset f[B]$. Without loss of generality, assume that $f[B] \subset g[C]$. Then $f[B]$ is a proper initial segment of $g[C]$ and the composite function $g^{-1} \circ f$: B $\rightarrow$ C is an order isomorphism of B onto a proper initial segment of C. This result is impossible since B,C $\in \alpha_n$ by Theorem 4.4. Therefore $f[B] = g[C]$, thus showing that there is a unique proper initial segment of A_1 that is a member of α_n. Therefore, every equivalence class less than α_1 contains a unique element that is a proper initial segment of A_1.

Now consider A_n and A_{n+1}. The set $A_{n+1} \subset A_n$ since A_{n+1} is the unique representative from α_{n+1} that is a proper initial segment of A_1 and A_n is the unique representative from α_n that maps to a proper initial segment of A_1, and since $\alpha_{n+1} \prec \alpha_n$. Define x_n as the first element of $A_n - A_{n+1}$. Clearly $x_{n+1} < x_n$ since, x_n is the first element from A_n not in A_{n+1}, and x_{n+1} is the first element of A_{n+1} not in A_{n+2}, and $A_{n+1} \in \alpha_{n+1}$, and $\alpha_{n+1} \prec \alpha_n$. The sequence (x_n) contradicts Theorem 3.7 since it is a strictly decreasing sequence contained within the well-ordered set A_1. □

The final paragraph of Theorem 5.2 seems to involve making countably many simultaneous choices, something that normally requires CAC. In this case, CAC is not needed because once A_1 is chosen, the other A_n are uniquely defined. The DC axiom is needed in the first paragraph. Now, Ψ has been shown to be well-ordered. I will now show that Ψ is also uncountable.

Theorem 5.3 Ψ *is uncountable.*

Proof: [9] Suppose that Ψ is not uncountable. Index $\Psi = \{\alpha_n : n \geq 1\}$ and using CAC, choose $A_n \in \alpha_n$. Let B_n be an order isomorphic copy of A_n inside the interval $[n, n+1)$ and let Y = $\bigcup\{B_n : n \geq 1\}$. Since Ψ is countable and $\mathbb{N}$ is countable, then by Theorem 3.1, Y is a well-ordered subset of $\mathbb{R}$ so the equivalence class of Y, which I denote by α_Y, is an element of Ψ. Then Y is order isomorphic to some B_n. But then Y is order isomorphic to B_n which is not cofinal in Y, which by Theorem 4.4 is impossible. Therefore, Ψ is uncountable. □

The above proof makes use of the Countable Axiom of Choice in order to pick the sets $B_n \in \alpha_n$. It should be noted that now Ψ is an uncountable well-ordered set, which by Theorem 3.4, means that the existence of Ω follows immediately from it. However, a more fascinating result can be shown, namely, that $\Psi = \Omega$.

Theorem 5.4 *For each $\alpha \in \Psi$ the set $\{\beta \in \Psi : \beta \prec \alpha\}$ is countable.*

[9]This proof uses the Countable Axiom of Choice.

Proof: The elements of Ψ are equivalence classes constructed from well-ordered subsets of the real numbers, $\mathbb{R}$. By Theorem 3.1, any well-ordered subset of the reals can be at most countably infinite. Fix $A \in \alpha$. For each $\beta \in \Psi$ with $\beta \prec \alpha$, there is a unique initial segment A_β of A with $A_\beta \in \beta$, and if $\beta_1 \neq \beta_2$ are predecessors of α, then $A_{\beta_1} \neq A_{\beta_2}$. But A is countable, so there can be at most a countable number of initial segments of A. Hence $\{\beta \in \Psi | \beta \prec \alpha\}$ is countable. □

I have thus proven the existence of Ω given the usual axioms of set theory, the set $\mathbb{R}$, the Countable Axiom of Choice, and the Axiom of Dependent Choice.

6 The Rationals Suffice for Ω

This section represents an appendix to the rest of the chapter. From the real numbers, the axioms of set theory, and the DC axiom, Ω was constructed. For a quick summary, well-ordered subsets of the reals were broken into various equivalence classes according to order isomorphisms. Then, by using the order ismorphism property, these classes were ordered according to their ability to be mapped to proper initial segments of other well-ordered subests in separate classes. The result of this process gave an uncountable, well-ordered set with the property that for every element chosen from that set, the number of predecessor elements could at most be countable.

By an early theorem in this chapter, Theorem 3.2, it was shown that all well-ordered subsets at of the reals are at most countable in size. That motivated a question as to whether or not the real numbers were really necessary to the construction of Ω. The reals are an important set of numbers in analysis and other fields, but the construction of the reals has certain uncomfortable qualities. In his *Principles of Mathematical Analysis*, Walter Rudin uses Dedekind cuts to construct the reals.[10] At other times, the real numbers are defined as the power set of the natural numbers. Both constructions have led many mathematicians to see how far in mathematics they can go without making use of the real numbers.

It turns out, that the reals are entirely unnecessary for the construction of Ω. Instead, it is possible to map the rational numbers, $\mathbb{Q} \cap [1, \infty)$, into an order-preserving copy in the half open interval from $\mathbb{Q} \cap [n, n+1)$, which then allows for the construction of Ω to proceed as before. The only difference comes in that Theorem 3.2 is unnecessary. It showed that all the well-ordered subsets of the reals are at most countable. However, since $\mathbb{Q}$ is already countable, then any well-ordered subset cannot help but to be countable.

Because this section seeks to show that the rational numbers suffice to construct Ω, along with the usual axioms of set thoery and the DC axiom, Theorem 3.1 must be redone in a manner that will use a function that does not involve the reals or irrational numbers.

Theorem 6.1 *Suppose Y is a well-ordered subset of $\mathbb{Q} \cap [1, \infty)$. For each $n \geq 1$, the interval $[n, n+1) \cap \mathbb{Q}$ contains an order isomorphic copy of Y.*

Proof: Consider the function $f(x) = \frac{1}{x}$. This function maps elements from $\mathbb{Q} \cap [1, \infty)$ into the interval $(0, 1]$. However, rather than be order-preserving, it is order-reversing. Consider $x, y \in \mathbb{Q} \cap [1, \infty)$

[10] Rudin, Walter. *Principles of Mathematical Analysis.* McGraw Hill. 1976. pp 17.

such that $x < y$. Since x, y are rational, they can be given as the ratio between two positive integers. Let $x = \frac{a}{b}$ and let $y = \frac{c}{d}$. Then, $\frac{a}{b} < \frac{c}{d}$, which by the rules of algebra gives that $\frac{d}{c} < \frac{b}{a}$ which is really just $f(y) < f(x)$.

The order-reversing can be solved by multiplying the function by -1. Let $g(x) = \frac{-1}{x}$. Then consider any $x, y \in \mathbb{Q} \cap [1, \infty)$ such that $x < y$. Since x and y are both rational numbers, they can be represented as the ratio between two positive integers. Let $x = \frac{a}{b}$ and let $y = \frac{c}{d}$. Then $\frac{a}{b} < \frac{c}{d}$. Following algebra, $\frac{d}{c} < \frac{b}{a}$ multiplying by -1 gives $\frac{-d}{c} > \frac{-b}{a}$ which gives $f(y) > f(x)$ thus preserving order. Furthermore, since $x, y \in \mathbb{Q} \cap [1, \infty)$ then the numerator is greater than the denominator for both x and y. Therefore, $g(x)$ and $g(y)$ have denominators that are bigger than their numerators, and they are negative. Thus, they are elements in $\mathbb{Q} \cap [-1, 0)$.

Note that while $g(x)$ maps rationals from $\mathbb{Q} \cap [1, \infty)$ in an order-preserving way into $\mathbb{Q} \cap [-1, 0)$, the inverse of $g(x)$ maps rationals from $\mathbb{Q} \cap [-1, 0)$ into rationals in $\mathbb{Q} \cap [1, \infty)$. Let $y = g(x)$. Then, $y = \frac{-1}{x}$. Inverting $g(x)$ gives $x = \frac{-1}{y}$. Solving for y once again gives $y = \frac{-1}{x}$. Since $y \in \mathbb{Q} \cap [-1, 0)$ then y is negative, and by multiplying by a negative it becomes positive. Furthermore, since $y \in \mathbb{Q}$ then it can be represented as a rational number between two integers, but since it comes from $[-1, 0)$ the numerator must be less than the denominator, which means that x will have a numerator greater than the denominator. Thus $x \in \mathbb{Q} \cap [1, \infty)$.

The last part is to place Y, a well-ordered subset of $\mathbb{Q} \cap [1, \infty)$, into an interval $[n, n+1)$. Let $h(x) = \frac{-1}{x} + (n + 1)$. Then for all $n \geq 1$, $h(x)$ maps elements from Y order-preserving into an interval $[n, n + 1)$. $\square$

Thus, I have shown that for the entirety of the construction of Ω, while the reals work, the rationals suffice.

Chapter 2: Topological Properties of the Set of All Countable Ordinals

1 Introduction

This chapter looks at several properties of the set Ω along with its subsets, functions from Ω to the reals, and functions from Ω to itself. These properties will help to familiarize Ω and make its apparently strange nature less so. Thus, this chapter builds the machinery that will help to explore the results of this thesis in Chapter 3.

2 Sets, Topology, and Basic Defintions

How to give a short definition of what is topology and what it studies remains difficult for me even today. Therefore, I will quote from the introduction of John McCleary's book, *A First Course in Topology* [1] :

> The central concept in topology is continuity, defined for functions between sets equipped with a notion (topological spaces) which is preserved by a continuous function. Topology is a kind of geometry in which important properties of a figure are those that are preserved under continous motions (homeomorphisms...). The popular image of topology as *rubber sheet geometry* is captured in this characterization. Topology provides a language of continuity that is general enough to include a vast array of phenomenon while being precise enough to be developed in new ways.

Before launching into a series of proofs, it will be helpful to establish some groundwork and definitions. Important to the concepts of continuity are also the concepts of openness and closedness for a set and its subsets. Traditionally, concepts like openness and closedness are introduced in analysis under concepts of metric spaces. However, metric spaces are actually a small subset of topological spaces, which requires a definition of these concepts outside of the general metric space notion. Furthermore, it will be even more important to the development of this paper, as Ω will have open and closed sets, but Ω itself, under the usual ordering of its elements, is not a metric space, as will be proved in Theorem 6.4.

[1] McCleary, John. *A First Course in Topology: Continuity and Dimension*, Student Mathematical Library, Volume 31, American Mathematical Society, 2006

Definition 2.1 [2]*Suppose X is a set. A **topology** on X is a collection τ of subsets of X such that the following are true:*

1. $\emptyset \in \tau$
2. $X \in \tau$
3. *If* $U \in \tau$ *and* $V \in \tau$, *then* $U \cap V \in \tau$
4. *If* $U_i \in \tau$, *then* $\bigcup_{i \in I} U_i \in \tau$.

Thus can a topological space be represented by an ordered pair containing the set and the topology, i.e., (X, τ). It can be shown that nearly every set has a topological space associated with it, actually at least two: the discrete and indiscrete topology. The exception to this is with a set with at most one element, these two topologies are then the same. This is similar to how every set has at least one metric, the discrete metric. It can also be shown that metric spaces also create topological spaces, but the reverse does not hold. However, these facts and others will be brought in and proved only insofar as they serve to further the purpose of this paper.

The following definitions will provide the basic framework for working within a topological space. I provide the definition of what it means for a set to be open, for a set to be closed, and for an element of a set to be a limit point.

Definition 2.2 [3]*Let X be a set and let τ be a topology on X:*

(a) *A set is called **open** if and only if it is a member of* τ

(b) *A set is called **closed** if and only if it is the complement is a member of* τ

(c) *A point p is a **limit point** of a set $S \subseteq X$ if and only if for each $U \in \tau$, if $p \in U$ then $U \cap S$ has more than one point.*

The sets in τ can be generated in a number of different ways. Sometimes, the manner of their generation is a given and τ is akin to a gift from God. However, removing platonism from the picture, or perhaps seeking logical priority to the construction of τ, the generation of τ can come in familar and unfamiliar forms. Metric spaces and linear orderings all give topological spaces. [4] For example, let X be a set and let d be a metric on the set. I define a special collection μ by saying "$U \in \mu$ if and only if for each $p \in U$ there is some $\epsilon > 0$ with $Ball(p, \epsilon) \subset U$". Beginning with a linear ordering on X, I define a special collection λ of subsets of X by the rule "$U \in \lambda$ if and only if for each $p \in U$ either there are points $a < p < b$ with interval $(a, b) \subseteq U$ or p is the left endpoint of X and there is some $b \in X$ with $p < b$ and $[p, b) \subseteq U$ or else p is the right endpoint of X and there is some $a < p$ with $(a, p] \subseteq U$". I will show that both of these satisfy the definition of a topology in the following Lemmas. Because I make use of the notion of a metric space in the proof below, I also provide a brief definition of what it means for a set to have a metric defined on it.

[2]This definition comes from *Topology*, by Sheldon Davis, 2005, p 41.

[3]These definitions were provided to me by David Lutzer. I owe him a great deal of thanks for showing me how these work within the wider scope of topology, rather than the narrower scope of a metric space.

[4]I once again owe David Lutzer for showing me these examples and others.

Definition 2.3 [5]*Suppose X is a set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ such that the following conditions are satisfied:*

(1) $d(x, y) = 0$ if and only if $x = y$

(2) $d(x, y) = d(y, x), \forall x, y \in X$

(3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

Lemma 2.4 *Metric spaces give topologies.*

Proof: Let X be a set and let d be a metric on the set. I define a special collection μ by saying "$U \in \mu$ if and only if for each $U \subseteq X$ and for each $p \in U$ there is some $\epsilon > 0$ with $Ball(p, \epsilon) \subset U$". Obviously, $\emptyset \subseteq X$ and trivially satisfies the requirement to be in μ. The entire set X will also satisfy the require since any ϵ will give a $Ball(p, \epsilon)$ fully contained in X. Suppose $U \in \mu$ and $V \in \mu$. Then conider any $p \in U \cap V$. By the definition of the intersection $p \in U$ and $p \in V$. There exists $Ball(p, \epsilon_U) \subseteq U$ and $Ball(p, \epsilon_V) \subseteq V$. Now either $\epsilon_U < \epsilon_V$, $\epsilon_V < \epsilon_U$, $\epsilon_U = \epsilon_V$. Without loss of generality $\epsilon_U \leq \epsilon_V$. Then obviously $Ball(p, \epsilon_U) \subseteq U$ since that was given at the beginning. And since $\epsilon_U \leq \epsilon_V$ then $Ball(p, \epsilon_U) \subseteq Ball(p, \epsilon_V) \subseteq V$, so by the transitivity of the subsets $Ball(p, \epsilon_U) \subseteq V$. Finally, if $U_i \in \mu$ then $\bigcup_{i \in I} U_i \in \mu$ since given any $p \in U_i$ there is some $Ball(p, \epsilon_{U_i}) \subseteq U_i$. Then that $Ball(p, \epsilon_{U_i}) \subseteq \bigcup_{j \in I} U_j$. Thus, every metric space gives a topology. □

Lemma 2.5 *Linear orderings give topologies.*

Proof: Beginning with a linear ordering on X, I define a special collection λ of subsets of X by the rule "$U \in \lambda$ if and only if for each $p \in U$ either there are points $a < p < b$ with interval $(a, b) \subseteq U$ or p is the left endpoint of X and there is some $b \in X$ with $p < b$ and $[p, b) \subseteq U$ or else p is the right endpoint of X and there is some $a < p$ with $(a, p] \subseteq U$". Obviously, $\emptyset \in \lambda$ since it fulfills the requirements trivially. Since, X is linearly ordered, then every element of X except an endpoint can be put into an ordered interval $a < p < b$ with interval $(a, b) \subseteq X$. Otherwise, if the p chosen is the maximal or minimal element of X, then it will fulfill the second or third requirements with $[p, b)$ or $(a, p] \subseteq X$. So, $X \in \lambda$.

I consider the case where p is not an endpoint of X Given any two sets $U, V \in \lambda$, then for any $p \in U \cap V$ consider the interval surrounding p in U, call it (a_U, b_U), and the interval surrounding p in V, call it (a_V, b_V). Since the elements are linearly ordered in X they can be put into an order relation with one another. Consider the maximum element between a_U and a_V and consider the minimal element b_U and b_V. Whichever is the maximal element between a_U and a_V, call it a'. Whichever is the minimal element between b_U and b_V, call it b'. Consider the interval (a', b'). Obviously, $p \in (a', b')$ and $(a', b') \subseteq (a_U, b_U)$ and $(a', b') \subseteq (a_V, b_V)$ which means that $(a', b') \subseteq U \cap V$ which means $U \cap V \in \lambda$. Given $\bigcup_{j \in I} U_j$ consider any $p \in U_i$. There is some $(a, b) \subseteq U_i$. Therefore, for any $p \in \bigcup_{j \in I} U_j$ $(a, b) \subseteq \bigcup_{j \in I} U_j$. Therefore, $\bigcup_{j \in I} U_j \in \lambda$. Thus, linear orderings give topologies. □

The above proofs show that metrics and linear orderings give topologies on sets. By Lemma 2.4, I have shown that metric spaces are a subclass of topological spaces. It bears repeating though, that sometimes a topology has no involvement with any metric or linear ordering. [6] Let $X = \mathbb{N}$,

[5]This definition was taken from *Topology*, by Sheldon Davis, 2005, p15.

[6]Once again I must thank David Lutzer for this information and this example.

the set of all positive integers. Define τ by the rule that $U \in \tau$ if and only if either $U = \emptyset$ or else $U \subseteq X$ and $X - U$ is finite. I will show that this τ satisfies the requirements for being a topology, but there is not metric on X that defines τ and no linear ordering on X that defines τ either.

In order to simplify this proof, I will make use of the concept of a Hausdorff space.

Definition 2.6 *A topological space X is said to be **Hausdorff** if and only if for any two distinct points x and y of X there exist disjoint open subsets U and V of X such that $x \in U$ and $y \in V$.* [7]

It so happens that all metric spaces and all spaces defined by linear orderings are Hausdorff. This will be given in the lemma below.

Lemma 2.7 *All metric spaces and linear orderings give Hausdorff topologies.*

Proof: I will begin with metric spaces. Given any metric space (X, d), consider any two points x, y such that $x \neq y$. Consider the distance between x, y that is $d(x, y)$. Call that number, ϵ, which by the definition of a metric must be greater than 0. Now consider the set $Ball(x, \frac{\epsilon}{3})$, call this set U, and $Ball(y, \frac{\epsilon}{3})$, call this set V. Obviously, these two sets are disjoint, otherwise, the distance between $x, y \leq \frac{2\epsilon}{3}$ but the distance was already set at ϵ, so ϵ would have to equal $\frac{2\epsilon}{3}$, which means $\epsilon = 0$ which is impossible. Therefore, metric spaces are Hausdorff.

Now I will show that linear orderings are also Hausdorff. Given an linear ordering of a set $(Y, <)$, consider any two points s, t such that $s \neq t$. Therefore, $s < t$ or $t < s$. Without loss of generality, say $s < t$. Then there are two cases to consider:

Case 1 : Consider the case where $(s, t) = \emptyset$. Consider any element less than s. Call it a. Consider any element greater than t, call it b. Then the two sets (a, t) and (s, b) are disjoint open subsets of Y, with $s \in (a, t)$ and $t \in (s, b)$. The cases where s is the first point in X or t si the last point of X are similar.

Case 2 : Consider the case where $(s, t) \neq \emptyset$. Consider any element in (s, t) call it c. Then consider any element less than s, call it a, and consider any element greater than t, call it b. Then (a, c) and (c, b) are disjoint subsets of Y with $s \in (a, c)$ and $t \in (c, b)$. If s is the first element of Y or t is the maximal element, then the sets $[s, c)$ or $(c, t]$ are mutually disjoint open subsets of Y. $\square$

Now, I return to the set described before the definition of Hausdorff was given. This set will be shown to satisfy the definition of a topology. Furthermore, it will be proven not to be given by any metric or linear ordering. This will be done by a proof by contradiction involving the use of the Lemma 2.7.

Lemma 2.8 *Let $X = \mathbb{N}$, the set of all positive integers. Define τ by the rule that $U \in \tau$ if and only if either $U = \emptyset$ or else $U \subseteq X$ and $X - U$ is finite. The topology, τ, is not given by any metric nor linear ordering.*

[7] This definition was taken from Sheldon Davis, *Topology*, McGraw Hill, 2005, pg 71.

Proof: By definition, $\emptyset \in \tau$. Also $X \in \tau$ trivially since $X - X = \emptyset$ is finite. If $U \in \tau$ and $V \in \tau$, then $U \cap V$ is either U, which is in τ, or V, which is in τ, or is a subset of $U \cap V$. Let $U \cap V = W$. Since $U \neq \emptyset$ and $V \neq \emptyset$, $X - W = X - (U \cap V) = (X - U) \cup (X - V)$ which is the union of two finite sets. Therefore, $W \in \tau$. Also, since $U, V \subseteq X$, then $W \subseteq X$. Finally, let $U_i \in \tau$. Then $\bigcup_{i \in I} U_i \in \tau$ is a collection of infinite sets whose set difference with X is finite. Because $X - U_i$ for any i is finite given that $U \in \tau$, then $\bigcup_{i \in I} U_i$ is a union of infinite subsets of X with finite set differences. Therefore, $X - \bigcup_{i \in I} U_i$ must be finite, otherwise, $X - U_j$ for some j must be infinite, which is contrary to $U_j \in \tau$.

Now, I will show that this topology, τ, is not given by any metric or linear ordering. This proof will be accomplished by contradiction. Assume that τ is given by either a metric space or a linear ordering. Then τ is also Hausdorff by Lemma 2.7. Consider any two points in X, which would be any two numbers in $\mathbb{N}$. Call one s and the other t and let $s \neq t$. Since τ is Hausdorff, the two points are contained in disjoint sets, say $s \in S$ and $t \in T$, and $S \in \tau$ and $T \in \tau$. However, since $S \in \tau$ and $S \cap T = \emptyset$ then $T \subseteq X - S$, which means that T is finite, but then $T \in \tau$ and $T \notin \tau$. Furthermore, since $T \in \tau$ and $T \cap S = \emptyset$ then $S \subseteq X - T$, which means that S is finite, but then $S \in \tau$ and $S \notin \tau$. A contradiction is reached for either possibility. Therefore, τ cannot be Hausdorff, which means that τ cannot be given by any metric space or linear ordering. □

The proofs given above show just a few of the ways in which a topology can be generated on a set.

3 Limit Ordinals

In order to discuss the properties of Ω at length and the consequences of its construction, I will make use of an important property of Ω. The set of all countable ordinals has elements which lack immediate predecessors. These elements are extremely important to future proofs concerning Ω.

Theorem 3.1 *There are non-trivial elements of Ω without any immediate predecessor.*

Proof: The first element of Ω has no immediate predecessor; however, this example is trivial and not interesting. More interesting is that there are elements of Ω that have no immediate predecessor that are not the first element of Ω. Let 0 be the first element of Ω. Let 1 be the immediate successor of 0. Let 2 be the immediate successor of 1. Continue this operation for all numbers n, and consider the set of all n numbers obtained in this fashion, $\{0, 1, 2, \cdots, n, \cdots\}$. This set is a countable. For simplicity, let's call this set C. Consider $\Omega - C$. Since Ω is uncountable and C is countable, this set is nonempty. Consider the first element of $\Omega - C$. This element has no immediate predecessor. □

Ordinals without immediate predecessors are called **limit ordinals**. This creates an interesting situation, because there are ordinals in Ω without any immediate predecessor; however, from the construction of Ω, the set including any element of Ω and its predecessors is countable, which then leads into an interesting theorem.

Theorem 3.2 *The set of all limit ordinals in Ω is uncountable.*

Proof: This proof will proceed in two parts. First, it will be shown that for any given limit ordinal, there is a limit ordinal that exists after it. Second, it will be shown that the set of all limit ordinals in Ω must be uncountable.

I must show that for any limit ordinal in Ω there must be a limit ordinal that comes after it. I will proceed by contradiction. Assume that there is a maximum limit ordinal in Ω. Let Λ be the set of all elements preceeding and including this ordinal. By the properties of Ω, it follows that Λ must be countable. Therefore, $\Omega - \Lambda$ must be uncountable. Let α_0 be the first element of $\Omega - \Lambda$, and let α_1 be its immediate successor. Let α_2 be the immediate successor of α_1, and so on until α_n. Consider the union of all sets of elements reached in this manner, $\bigcup\{[0, \alpha_n] : \alpha_n \geq \alpha_0\}$. This is a countable collection of countable sets. Consider $(\Omega - \Lambda) - \bigcup\{[0.\alpha_n] : \alpha_n \geq \alpha_0\}$. This set is uncountably large and contains a first element. This first element cannot have any immediate predecessors, otherwise, it would have been captured by the construction of $\bigcup\{[0, \alpha_n] : \alpha_n \geq \alpha_0\}$, but this means that this first element is a limit ordinal, which means I have obtained a contradiction. Therefore, for any limit ordinal in Ω there is some limit ordinal beyond it.

Now I must complete the proof by showing that the set of all limit ordinals in Ω is uncountable. I will proceed by contradiction. Assume that the set of all limit ordinals in Ω is countable. Since the set of all predecessors elements of any element in Ω is countable, then $[0, \lambda]$ is countable given that 0 is the first element of Ω and λ is any limit ordinal of Ω. Let $P = \bigcup\{[0, \lambda] : \lambda \in \Omega$ and λ is a limit ordinal$\}$. Since P is the union of a countable collection of countable sets, then P itself is countable. However, since P includes all limit ordinals of Ω and all their predecessors, and since there is no largest limit ordinal in Ω, then P must include all elements of Ω. However, that would mean that $P = \Omega$, but this is impossible since P is countable and Ω is uncountable. I have reached a contradiction, therefore, the set of all limit ordinals in Ω must be uncountable. □

The above proof can also be repeated for the collection of any successor ordinals of limit ordinals. For example, the set of all immediate successors of limit ordinals will also be uncountable, and the set of all successors of successors of limit ordinals will also be uncountable, and so on for the same reasons given above.

Theorem 3.3 *Any uncountable subset of Ω will have the same cardinality as Ω.*

Proof: Assume this is not the case. Then there are two possibilities:

- Case 1: The set will have a greater cardinality than Ω. This would mean that Ω contains more elements than it contains, which is a clear contradiction, so
- Case 2: The set will have a smaller cardinality than Ω. Take any subset of Ω, say Λ. This set will be well-ordered, from the well-ordering of Ω. Now, either Λ is order isomorphic to Ω (in which case $|\Lambda| = |\Omega|$) or Ω is order isomorphic to a proper initial segment of Λ or Λ is order isomorphic to some proper initial segment of Ω. If Ω were order isomorphic to some proper initial semgent of Λ then some element of Λ must have uncountably many predecessors, which is impossible since $\Lambda \subseteq \Omega$. If Λ were order isomorphic to some initial segment of Ω, then there exists a one-to-one function from Ω onto Λ and then from Λ onto a proper initial semgent of Ω, which means that there is some function that maps Ω onto a proper initial segment of itself, but this is impossible. Therefore, $|\Lambda| = |\Omega|$.

Therefore, there is a contradiction; therefore, any uncountable subset of Ω must be the same cardinality as Ω. $\square$

This section concerned itself with basic properties of Ω and some of the basic properties of its elements. The following section will build upon these properties and look at the properties of subsets of Ω. Such subsets will have several counter-intuitive properties that will play out later in other counter-intuitive theorems about Ω.

4 Subsets of Ω and Their Properties

As a result of the existence of ordinals and limit ordinals in Ω, subsets and sequences of Ω carry several interesting properties. These properties will later be important to mappings between Ω and the set of real numbers. First, I will look at theorems dealing with sequences of elements in Ω.

Theorem 4.1 *Given any increasing sequence, $\alpha_1 \leq \alpha_2 \leq \cdots$, in Ω, there exists an element of Ω that is the supremum of the sequence.*

Proof: Suppose $\alpha_n \in \Omega$ with $\alpha_1 \leq \alpha_2 \leq \cdots$. If there is some k such that $\alpha_k = \alpha_{k+1} = \alpha_{k+2} = \cdots$, then $\alpha_k = \sup \alpha_n | n \geq 1$. So assume that for all k, some $j > k$ has $\alpha_k < \alpha_j$. Each set $A_n = \{\beta \in \Omega | \beta \leq \alpha_n\}$ is countable. Hence so is the set $A = \bigcup\{A_n | n \geq 1\}$. But Ω is uncountable, so $\Omega - A \neq \emptyset$. Let γ be the first element of $\Omega - A$. Clearly $\alpha_n \leq \gamma$. If $\delta < \gamma$, then $\delta \notin \Omega - A$ because γ is the first element of $\Omega - A$, so $\delta \in A$. Then some $j > n$ has $\delta \leq \alpha_n < \alpha_j$ so δ is not an upper bound for the sequence. This shows that $\gamma = sup\{\alpha_n | n \geq 1\}$. $\square$

To give this theorem even greater importance comes the following theorem:

Theorem 4.2 *Every sequence of elements in Ω has a convergent subsequence.*

Proof: Assume, in order to get a contradiction, that it is not the case that every sequence of elements in Ω has a convergent subsequence. Then there is some sequence of elements in Ω that does not contain a convergent subsequence. Let $< \alpha_n >$ be such a sequence where $n \geq 1$. Now consider the set $P = \{n : \forall m > n, \alpha_m < \alpha_n\}$. Then there are two cases to consider:

- Case 1: If P is infinite. Let n_1 be the first element of P, n_2 the second, n_3 the third, and so on. Then $P = \{n_1, n_2, n_3, \cdots\}$. Consider n_k. Because $n_k \in P$, for each $m > n_k$ I have $\alpha_{n_{k+1}} < \alpha_{n_k}$. In particular, $n_{k+1} > n_k$ so $\alpha_{n_{k+1}} < \alpha_{n_k}$. This is impossible because the well-ordered set Ω cannot contain any strictly decreasing sequences.

- Case 2: If P is finite, there is some positive integer K wth $P \subseteq \{1, 2, 3 \cdots, K\}$. Let $n_1 = K+1$. Suppose I have $n_1 < n_2 < n_3 < \cdots < n_k$ with $\alpha_{n_1} \leq \alpha_{n_2} \leq \cdots \leq \alpha_{n_k}$. Because $K < n_1 < n_k$, I know that $n_k \notin P$ so $\exists n_{k+1} > n_k$ with $\alpha_{n_k} \leq \alpha_{n_{k+1}}$. Hence $< \alpha_{n_k} >$ is a subsequence of $< \alpha_n >$ and $< \alpha_{n_k} >$ converges to $\sup\{\alpha_{n_k} | n \geq 1\}$. Therefore, a sequence of elements taken from Ω has a convergent subsequence, but this is contrary to the hypothesis.

Therefore, I have reached a contradiction out of the only two possibilities, which means the assumption must be false. Therefore, every sequence of elements in Ω has a convergent subsequence. □

Finally, these two theorems will aid in developing the theorem below, which will subsequently show interesting results stemming from uncountable subsets of Ω.

Theorem 4.3 *If C is a non-empty countable subset of Ω, then there is some element of Ω that is the supremum of C.*

Proof: For each $\alpha \in C$ look at $B_\alpha = \{\beta \in \Omega | \beta \leq \alpha\}$. By the properties of Ω, each B_α is countable. Therefore, so is $D = \bigcup\{B_\alpha | \alpha \in C\}$. Let γ be the first element of $\Omega - D$. If γ is a limit ordinal, then consider any $\delta < \gamma$. Since γ is the first element of $\Omega - D$, then $\delta \in D$, which means that $\delta \in C$ or $\delta < \alpha$ for some $\alpha \in C$. Thus, γ is the supremum of C. If γ is not a limit ordinal, then there is some $\delta \in D$ such that γ is the immediate predecessor of δ. Consider such a δ, since γ is the first element of $\Omega - D$, then $\delta \in D$ and since D is composed of only elements from C and their predecessors, then δ must be the largest element in C. If δ where not in C, then it would have to be a predecessor of some element of C, which would mean that it could not be the immediate predecessor of γ, contrary to the hypothesis. Therefore, γ is the supremum of C. □

The previous theorems will now be employed in exploring the properties of uncountable closed subsets of Ω. Recall that a closed set is defined, in topology, as the complement of an open set. The following thoerems will make use of the concept of interlacing in order to show that two uncountable closed sets must share uncountably many common elements.

Definition 4.4 *The elements of two sets **interlace** if there exists $\alpha_n \in C_1$ and $\beta_n \in C_2$ that have the property that $\alpha_n < \beta_n < \alpha_{n+1}$ for each $n \geq 1$.*

Definition 4.5 *Two sets C_1 and C_2 **eventually interlace** if $\forall\gamma\ \exists\alpha_n \in C_1$, $\beta_n \in C_2$ with $\gamma < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$.*

Theorem 4.6 *If $C_1, C_2 \subseteq \Omega$ are uncountable and closed, then the elements of C_1 and C_2 eventually interlace.*

Proof: Let $\gamma \in \Omega$. Then $[0, \gamma]$ is countable so some $\alpha_1 \in C_1$ has $\gamma < \alpha_1$. I will proceed by induction. From the construction of Ω, the set of all predecessors of any element of Ω is countable. Consider the set $[0, \alpha_1]$. This set is countable. Therefore, the set $C_2 - [0, \alpha_1]$ is nonempty. Choose the first element, call it β_1. Note that $\alpha_1 < \beta_1$ Consider the set $[0, \beta_1]$. This set is countable, therefore $C_1 - [0, \beta_1]$ is non-empty. Pick the first element. Call it α_2. Note that $\beta_1 < \alpha_2$ Suppose $n \geq 2$ and suppose $\alpha_1 < \beta_1 < ... < \alpha_n < \beta_n$ are already defined, with each $\alpha_i \in C_1$ and each $\beta_i \in C_2$. Consider the set $[0, \beta_n]$. This is a subset of Ω containing β and all of its predecessors. Therefore, the set is countable. Therefore, the set $C_1 - [0, \beta_n]$ is nonempty. Pick the first element and call it α_{n+1}. Clearly, $\beta_n < \alpha_{n+1}$. It remains to be shown that a β_{n+1} can be chosen from C_2 such that

$\alpha_{n+1} < \beta_{n+1}$. Consider $[0, \alpha_{n+1}]$ which is a subset of Ω containing α_{n+1} and all of its predecessors. Therefore, the set is countable. Therefore, $C_2 - [0, \alpha_{n+1}]$ is nonempty. Pick the first element of this set and call it β_{n+1}. Clearly, $\alpha_{n+1} < \beta_{n+1}$. Thus the induction is completed and I have proven that given any two uncountable closed sets in Ω, their elements eventually interlace. □

Theorem 4.7 *If $C_1, C_2 \in \Omega$ are uncountable and closed, then $C_1 \cap C_2$ is a nonempty closed uncountable subset of Ω.*

Proof: Fix $\gamma \in \Omega$. Then $C_1 - [0, \gamma]$ and $C_2 - [0, \gamma]$ are uncountable closed sets so there are interlaced sequences $< \alpha_n >$ in $C_1 - [0, \gamma]$ and $< \beta_n >$ in $C_2 - [0, \gamma]$. Then $\delta = \sup < \alpha_n >= \sup < \beta_n >$ is in $C_1 \cap C_2$ and $\delta > \gamma$. Hence $C_1 \cap C_2$ is uncountable. □

The results of this theorem then allow for an inductive proof to show that any number of finite intersections of uncountable closed sets again produces closed uncountable subsets of Ω.

Theorem 4.8 *Finite intersections of closed uncountable subsets of Ω produce uncountable closed subsets.*

Proof: Theorem 4.7 guaruntees that uncountable closed sets C_1 and C_2 have an uncountable closed intersection. Let $C_1, C_2, C_3, ..., C_n, C_{n+1}$ be uncountable closed sets. For the purposes of induction, assume $(C_1 \cap C_2 \cap C_3 \cap ... \cap C_n)$ is an uncountable closed set, call it C'. The set $C' \cap C_{n+1}$ is an uncountable closed set, by Theorem 4.7. □

Theorem 4.9 *Given an infinite sequence, C_n, of uncountable closed subsets of Ω, then the set $\bigcap\{C_n | n \geq 1\}$ is an uncountable closed subset of Ω.*

Proof: Countable intersections of closed sets are again closed. This holds since the union of open sets is open. Thus, it needs only be shown that the countable intersection of closed sets of Ω is nonempty and uncountable.

It is already known that each set $C_1, \ C_1 \cap C_2, \ C_1 \cap C_2 \cap C_3, \ \cdots$ is uncountable and closed by Theorem 4.8. Fix $\gamma \in \Omega$ and choose $\alpha_1 \in C_1$ with $\gamma < \alpha_1$. Choose any $\alpha_2 \in C_1 \cap C_2$ with $\alpha_1 < \alpha_2$. This is possible because $[0, \alpha_1]$ is countable while $C_1 \cap C_2$ is uncountable. In general, if $\alpha_n \in C_1 \cap C_2 \cap C_3 \cap ... \cap C_n$, choose $\alpha_{n+1} \in C_1 \cap C_2 \cap C_3 \cap ... \cap C_n \cap C_{n+1}$ with $\alpha_n < \alpha_{n+1}$. By Theorem 4.2, some $\beta \in \Omega$ has $\beta = \sup < \alpha_n >$. Since $\alpha_k \in C_n$ for every $k \geq n$ then each of the C_n has a sequence which converges to β making β a limit point for each C_n, and since each C_n is closed, $\beta \in C_n$. But then $\beta \in \bigcap_1^\infty C_n$. Since $\beta > \gamma$, $\bigcap_1^\infty C_n$ is uncountable. □

Thus it has been shown that intersections of countably many uncountable closed subsets of Ω always produce uncountable closed subsets of Ω. This result is quite counter-intuitive, since intersections tend to decrease the total number of elements in a set, which would seem to reduce the cardinality of a set. Thus, this result concludes the section regarding subsets of Ω. The next section deals with functions from Ω to the real number line.

5 Ω and the Real Number Line

This section will explore relationships between Ω and the set of reals, $\mathbb{R}$. Because $|\Omega| \leq |\mathbb{R}|$, there are many 1-1 functions from Ω into $\mathbb{R}$. George Cantor sought, and failed, to prove that Ω and $\mathbb{R}$ have the same cardinality. This is known as the continuum hypothesis, i.e., $|\Omega| = |\mathbb{R}|$. The reason he failed was that the continuum hypothesis and its negation are each consistent with the axioms of set theory called **ZFC**, and neither one can be proven from those axioms, nor can either one be disproved (which of course would constitute a method of proof of the other). The next few theorems will build up relationships between continuous functions going from Ω to $\mathbb{R}$.

The following Theorem will require a definition and a lemma from real analysis. These are taken from Walter Rudin's *Principles of Mathematical Analysis.*

Definition 5.1 *A subset of $S \subset \mathbb{R}$ is compact if and only if whenever $\mathcal{U}$ is a collection of open sets in $\mathbb{R}$ that covers S (meaning $S \subseteq \bigcup \mathcal{U}$), then there is a finite subcollection of $\mathcal{U}$ that also covers S.*

Lemma 5.2 *A subset $S \subset \mathbb{R}$ is compact if and only if it is closed and bounded.* [8]

A Note about Notation: Ω is also represented by the half open interval $[0, \omega_1)$ where ω_1 is understood to be the first ordinal with uncounably many predecessors. Also ω is understood to be the first limit ordinal in Ω, $\omega + \omega$ or 2ω is the second limit ordinal in Ω, and so on. The limit ordinal of these ordinals is represented as $\omega\omega$ or ω^2, and so on. Notation becomes ever more complicated with the need to show ever more ordinals of different limits.

Theorem 5.3 *Given that f: $[0, \omega_1) \to \mathbb{R}$ is a continuous function, the set $f[[0, \omega_1)]$ is compact.*

Proof: Write $\Omega = [0, \omega_1)$. For sets of real numbers being compact is equivalent to being closed and bounded. Write $S = f[\Omega]$. Thus, it must be shown that S is closed and bounded.

- **CLOSED**: Assume S is not closed. Then there is some limit point of S not in S. Let b be such point. Let s_n be a sequence of elements in S that converges to b. Since $s_n \in S$, then for every s_n there is an α_n such that $f(\alpha_n) = s_n$. The sequence of $\alpha_n \in \Omega$ has a convergent subsequence, say α_{n_k}, by Theorem 4.2 of this chapter. Let its limit point be denoted by β. Well, $\beta < \omega_1$ so $f(\beta) \in S$, and since continuous functions preserve limits, the convergent subsequence of elements α_{n_k} will produce a convergent sequence of elements $f(\alpha_{n_k})$ that will converge to $f(\beta)$. Since α_{n_k} was taken as a subsequence of α_n, then $f(\alpha_{n_k})$ forms a convergent subseqence of elements from the sequence s_n. But from theorems of analysis, a sequence is convergent if and only if all of its subsequences converge to the same limit. Since s_n is convergent to b by hypothesis, then $f(\beta) = b$ since $f(\alpha_{n_k})$ converges to $f(\beta)$. But this is a contradiction since $f(\beta) \in S$ and $b \notin S$. Thus, the hypothesis must be rejected and S is therefore closed.
- **BOUNDED**: Suppose that S is not bounded. Then $\forall n \in \mathbb{R}\ \exists \alpha_n \in \Omega$ such that $|f(\alpha_n)| > n$. Consider a convergent subsequence of $< \alpha_n >$ in Ω, say $< \alpha_{n_k} >$. The limit of the subsequence will be an element of Ω, say β, so that $\lim_{n\to\infty} < \alpha_{n_k} >= \beta$. However, every $n_k < |f(\alpha_{n_k})|$ which means the sequence of $f(\alpha_{n_k})$ is unbounded. However, convergent sequences are bounded and $f(\alpha_{n_k})$ is convergent. Therefore, S is bounded.

[8]Rudin, Walter. *Principles of Mathematical Analysis.* McGraw Hill. 1976. pp 36-40.

Therefore, S is closed and bounded, which is equivalent to being compact. Thus the image of Ω under f is compact. $\Box$

Theorem 5.4 *Suppose that for every $\alpha \in \Omega$ the set C_α is a nonempty compact subset of $\mathbb{R}$. Suppose that whenever $\alpha < \beta$ then $C_\alpha \supset C_\beta$. Thus, larger elements of Ω give smaller subsets of $\mathbb{R}$. Then $\bigcap\{C_\alpha | 0 \leq \alpha < \omega_1\} \neq \emptyset$.*

Proof: Let $U_\alpha = \mathbb{R} - C_\alpha$. Then each U_α is open in $\mathbb{R}$ and $\alpha < \beta$ implies $U_\alpha \subseteq U_\beta$.

For the purposes of a contradiction suppose that the intersection of all C_α is empty. The union of all U_α must be $\mathbb{R}$, since the intersection of all C_α is empty, and U_α is defined as the set difference between $\mathbb{R}$ and C_α, then the union of all U_α contains every number in the real number line. Because C_0 is compact, finitely many of the sets of U_α must cover C_0, for the purposes of this proof let that be $U_{\alpha_1}, ..., U_{\alpha_n}$. If necessary, renumber the subscripts so that $\alpha_1 < \alpha_2 < ... < \alpha_n$ so that U_{α_n} is the largest of the sets U_{α_i} and $C_0 \subseteq U_{\alpha_n} = \mathbb{R} - C_{\alpha_n}$. However, this is impossible. Consider U_{α_n}. That means that there is a corresponding C_{α_n} such that $U_{\alpha_n} = \mathbb{R} - C_{\alpha_n}$, which means that there is some element of C_{α_n} that is not a member of U_{α_n}, but $C_{\alpha_n} \subset C_0$ which means that the elements of C_{α_n} do not belong to the union of U_{α_i} which means that $C_0 \not\subset U_{\alpha_n}$. $\Box$

Theorem 5.5 *Given that $f : \Omega \rightarrow \mathbb{R}$ is a continuous function and given that $\forall \alpha \in \Omega$ $C_\alpha = f[[\alpha, \omega_1)]$, then the sets C_α have $\bigcap\{C_\alpha | \alpha < \omega_1\} \neq \emptyset$.*

Proof: Let $C_\alpha = f[[\alpha, \omega_1)]$. The same proof used for Theorem 5.3 shows that each C_α is a compact non-empty subset of $\mathbb{R}$ and clearly $\alpha < \beta$ gives $C_\beta \subseteq C_\alpha$. Applying Theorem 5.4 gives the result. $\Box$

Theorem 5.6 *With C_α defined as in Theorem 5.5, $\bigcap\{C_\alpha | \alpha < \omega_1\}$ contains exactly one real number.*

Proof: Suppose not. Then there are at least two real numbers, say x and y, which are members of $\bigcap\{C_\alpha | \alpha < \omega_1\}$ and $x \neq y$. Since x, y are elements in every set of the intersection, by definition of the intersection, then there is some $\alpha_1 \in \Omega$ with $f(\alpha_1) = x$. Because $y \in \bigcap\{C_\alpha | \alpha_1 < \alpha < \omega_1\}$, there is another $\beta_1 \in \Omega$ such that $f(\beta_1) = y$ with $\alpha_1 < \beta_1$. Since x is a member of every set of the intersection, there is some $\alpha_2 \in \Omega$ and $f(\alpha_2) = x$ and $\alpha_1 < \beta_1 < \alpha_2$. Assume that this has been done up to the n^{th} element for both α_n and β_n. Because $x \in \bigcap\{C_\alpha | \beta_n < \alpha < \omega_1\}$, there is some $\alpha_{n+1} \in \Omega$ with $f(\alpha_{n+1}) = x$ and with $\alpha_1 < \beta_1 < ... < \alpha_n < \beta_n < \alpha_{n+1}$. There is some $\beta_{n+1} \in \Omega$ with $f(\beta_{n+1}) = y$ and with $\alpha_1 < \beta_1 < ... < \alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1}$. This produces a countable sequence of interlaced elements in Ω which means that there must be a supremum in Ω that the interlaced sequence approaches, say γ. Then $f(\gamma) = f(\lim \alpha_n) = \lim f(\alpha_n) = x$ and similarly $f(\gamma) = f(\lim \beta_n) = \lim f(\beta_n) = y$. But this violates the definition of a function. Therefore, the assumption is false and the theorem true. $\Box$

Theorem 5.7 *Given C_α as described in Theorem 5.5, let y_0 be the unique point in $\bigcap\{C_\alpha | \alpha < \omega_1\}$. For each $n \geq 1$, the set $f^{-1}[\mathbb{R} - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n})]$ is countable.*

Proof: First, it will be proven that for any open interval $(y_0 - \frac{1}{n}, y_0 + \frac{1}{n})$, there is some C_α fully contained within that open interval. Assume that this is not the case. Then there is some n for which $(y_0 - \frac{1}{n}, y_0 + \frac{1}{n})$ does not contain any of the compact sets C_α. Therefore, let $\mathcal{C}$ be the set $\{C_\alpha - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n}) | \alpha < \omega_1\}$. By Theorem 5.4, $\emptyset \neq \bigcap\{C_\alpha - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n}) | \alpha < \omega_1\}$ which is a subset of $\bigcap\{C_\alpha | \alpha < \omega\} = \{y_0\}$. Therefore, the point $y_0 \in \bigcap\{C_\alpha - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n}) | \alpha < \omega\}$. This is a contradiction. □

This result will then lead to the proof of the theorem. Fix n. Using the first part of the proof, find α with $C_\alpha \subseteq (y_0 - \frac{1}{n}, y_0 + \frac{1}{n})$. Note that if $\beta \in \Omega$ and $f(\beta) \in \mathbb{R} - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n})$, then $f(\beta) \notin C_\alpha$ so that $\beta \notin [\alpha, \omega_1)$. Hence $\beta < \alpha$ and there are only countably many such β. □

Theorem 5.8 *With f and y_0 as in Theorem 5.7 the set $f^{-1}[\mathbb{R} - \{y_0\}]$ is countable.*

Proof: Let $S_n = f^{-1}[\mathbb{R} - (y_0 - \frac{1}{n}, y_0 + \frac{1}{n})]$. Then each S_n is countable by Theorem 5.7. Consider the set which is the union of all S_n. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then the union of all S_n will be $f^{-1}[\mathbb{R} - \{y_0\}]$. Since a countable union of countable sets is countable, then the union of all S_n is countable. □

Theorem 5.9 *Let $f : \Omega \to \mathbb{R}$ be continuous. There is some $y_0 \in \mathbb{R}$ and some $\alpha_0 \in \Omega$ with the property that whenever $\alpha_0 \leq \beta < \omega_1$, then $f(\beta) = y_0$.*

Proof: By Theorem 5.8, $f^{-1}[\mathbb{R} - \{y_0\}]$ is countable. Every countable subset of Ω has a supremum in Ω. Let α_0 be this supremum. Since α_0 is the supremum of all elements not mapped to y_0, then for all $\beta \in \Omega$ such that $\beta > \alpha_0$, $f(\beta) = y_0$. □

6 Functions from Ω to Ω

This section will look at functions from Ω into itself. Note that the functions considered in this chapter might not be continuous. For this section a new definition is needed of a pressing down function.

Definition 6.1 *A **pressing down function** is a function $f : \Omega \to \Omega$ such that if $\alpha \in [1, \omega_1)$ then $f(\alpha) < \alpha$.*

The point of this section will be to explore the following theorem:

Theorem 6.2 *Given a pressing down function that maps Ω into itself, then for some $\beta \in \Omega$, the set $f^{-1}(\beta) = \{\alpha \in \Omega | f(\alpha) = \beta\}$ must be uncountable.*

Proof: In order to obtain a contradiction, it will be assumed that for every $\beta \in \Omega$ the preimage $f^{-1}[\beta]$ is countable. Then $f^{-1}[0, \gamma]$ for each $\gamma \in \Omega$ is a countable union of countable sets, and, therefore, countable.

A recursive construction of elements of Ω will now be made. This construction will lead to the eventual contradiction. Since $f(1) < 1$ this means that $f(1) = 0$. Therefore, let $\gamma_1 = 1$. Then $f^{-1}[[0, \gamma_1]]$ is countable, which means that there is some γ_2 with $\gamma_2 > \gamma_1$ and $f^{-1}[[0, \gamma_1]] \subseteq [0, \gamma_2)$.

Therefore, if $\gamma_2 < \alpha$, then $\gamma_1 < f(\alpha)$. Generalizing the construction, if $\gamma_n \in \Omega$ is defined, the set $f^{-1}[[0, \gamma_n]]$ is countable so that there is a γ_{n+1} with $\gamma_n < \gamma_{n+1}$ with $f^{-1}[[0, \gamma_n]] \subseteq [0, \gamma_{n+1})$. Thus, if $\gamma_{n+1} < \alpha$ then $\gamma_n < f(\alpha)$. Thus, there is a sequence of γ_n such that $\gamma_1 < \gamma_2 < ... < \gamma_n < \gamma_{n+1} < ...$. By previous theorems, any increasing sequence of elements of Ω has a supremum, thus there is some $\delta \in \Omega$ such that $\delta = \sup\{\gamma_n | n \geq 1\}$.

Since f is a pressing down function, $f(\delta) < \delta$ which means that there is some n such that $f(\delta) < \gamma_n < \delta$. But this means that $f(\delta) \in [0, \gamma_n)$. Thus $\delta \in f^{-1}(f(\delta)) \subseteq f^{-1}[0, \gamma_n) \subseteq [0, \gamma_{n+1})$, which means that $\delta < \gamma_{n+1}$. But then $\delta < \gamma_{n+1} < \gamma_{n+2} < ... < \delta$, which means $\gamma < \gamma$, which is a contradiction. Therefore, for some $\beta \in \Omega$, the preimage $f^{-1}[\beta]$ must be uncountable. □

It should be noted that the β in Theorem 6.2 is not unique. Consider, for example, the function $f(\alpha) = 0$ if α is not a limit ordinal and $f(\alpha) = 1$ if α is not a limit ordinal. Both $f^{-1}(0)$ and $f^{-1}(1)$ are uncountable.

Along with a pressing down function, there could also be a function which pushes upward, in the same sense as the original function pushed down. Since every element of Ω has a successor, then consider a function which maps every element to its immediate successor. That would be a function, $f\colon \Omega \to \Omega$ such that for every $\alpha \in \Omega$ $f(\alpha) > \alpha$. Obviously, such a function could not map Ω 1-1 onto itself since given any pushing up function, there would be nothing to map to the first element of Ω, just like with the pressing down function, there would be nothing to map the first element to.

The next proof of this section will explore functions that do map Ω into itself in a one-to-one fashion.

Theorem 6.3 *If $f : \Omega \to \Omega$ is 1-1 and $f(\alpha) \leq \alpha$ for each $\alpha \in \Omega$, then f has a fixed point, i.e., $\exists \alpha \in \Omega, f(\alpha) = \alpha$.*

Proof: In order to get a contradiction, assume this is not the case. Then $f : \Omega \to \Omega$ 1-1 and has $f(\alpha) \leq \alpha$ $\forall \alpha \in \Omega$ and $\forall \alpha \in \Omega, f(\alpha) \neq \alpha$. Since $f(\alpha) \leq \alpha$ but $f(\alpha) \neq \alpha$, that means that for every $\alpha \in \Omega$ $f(\alpha) < \alpha$, which means that f is a pressing down function, which means that the preimage of at least one of the elements of Ω is uncountably large given Theorem 6.2, but this contradicts the given premise that f is one-to-one. Therefore, the assumption must be rejected and there must exist a fixed point. □

There is another application of these pressing down functions, which is to show that Ω is not a metric space. It was shown earlier that metric spaces and linear orderings give topologies. Because Ω is linearly ordered, since it is well-ordered, then it follows that the ordering gives a topology. However, as will be shown, while metric spaces and linear orderings often give similar topologies, that does not follow in the case of Ω.

Theorem 6.4 *The topology of Ω is not given by a metric.*

Proof: Suppose that this is not the case in order to get a contradiction. So the topology of Ω is given by some metric $d(\alpha, \beta)$. Fix $n \geq 1$ such that for each $\alpha > 0$, consider the open ball $Ball(\alpha, \frac{1}{n})$. There is some point $f_n(\alpha) < \alpha$ with $(f_n(\alpha), \alpha] \subseteq Ball(\alpha, \frac{1}{n})$. Then f_n is a pressing down function so there is some β_n for which the set $S_n = \{\alpha \in \Omega | f_n(\alpha) = \beta_n\}$ is uncountable. Now consider $\delta = \sup\{\beta_n | n \geq 1\} + 1$. I have $\beta_n < \delta$ for each n. I can choose $\alpha_n \in S_n$ with $\delta + 1 < \alpha_n$ because $[0, \delta + 1]$ is countable, but S_n is uncountable, so $S_n - [0, \delta + 1] \neq \emptyset$. Therefore, I have that $f_n(\alpha_n) = \beta_n < \delta < \delta + 1 < \alpha_n$, so that $\delta \in (f_n(\alpha_n), \alpha_n] \subseteq Ball(\alpha_n, \frac{1}{n})$ for each $n \geq 1$. Hence $d(\alpha_n, \delta) < \frac{1}{n}$ giving that the sequence $< \alpha_n >$ converges to δ. But that is impossible since $\delta + 1 < \alpha_n$ for each n. □

7 Stationary Sets

Definition 7.1 *A* ***stationary set*** *is some set S, which is a subset of Ω, such that $S \cap C \neq \emptyset$ for every uncountable closed set $C \subseteq [0, \omega_1)$.*

Theorem 7.2 *Any stationary set must be uncountable.*

Proof: Assume that a stationary set S is only countable in size. I will show there is some uncountable closed set C that S would not intersect, which would be a contradiction in the very definition of a stationary set. Because S is assumed to be countable, S has a supremum in Ω, call it β. Since $\beta \in \Omega$, then the set of β and its predecessors is at most countable. Consider some uncountable closed set whose first element is beyond β, say $C' = [\beta + 1, \omega_1)$. Clearly such a set exists since Ω is uncountable, simply construct such a set as the set difference between Ω and the set including β and its predecessors, call it C'. The set C' is closed and uncountable. However, S does not intersect C' , since S contains no elements beyond β. Since this is a contradiction, it cannot obtain, and therefore, stationary sets are uncountable. □

Theorem 7.3 *If a set $S \subseteq \Omega$ contains a closed unbounded set, then S is stationary.*

Proof: Suppose $C \subseteq S$ is closed and unbounded. Let D be any closed unbounded set. By Theorem 4.7, $C \cap D \neq \emptyset$. Therefore $S \cap D \neq \emptyset$. Therefore, S is stationary. □

The above theorem shows that any set containing a closed unbounded set will be stationary. However, there is a question as to whether or not there is any stationary set that does not contain a closed unbounded set. Much of this section describes research done by Mary Ellen Rudin and her paper, "A Subset of the Countable Ordinals" [9]. She showed that there are sets which intersect every uncountable closed set of Ω but that are not themselves uncountable closed sets and do not contain any uncountable closed set. The theorem below will show that the complement of the set is also stationary. Sets that are stationary and whose complements are also stationary are called **bistationary sets**.

Theorem 7.4 *There is a subset S of Ω such that neither S nor $\Omega - S$ contains a closed uncountable set and both S and $\Omega - S$ are stationary.*

[9] Rudin, Mary Ellen. "A Subset of the Countable Ordinals". *American Mathematical Monthly*, 54(1957), p 351

Proof: For a contradiction assume $(*)$ for each $S \subseteq \Omega$ either S or $\Omega - S$ contains a closed uncountable set, and both S and $\Omega - S$ are stationary.

For each n fix any collection $\mathcal{G}_n$ satisfying the following:

- each memeber of $\mathcal{G}_n$ is an open interval of real numbers with diameter less than $\frac{1}{n}$;
- the collection $\mathcal{G}_n$ is countable
- every real number belongs to at least one member of $\mathcal{G}_n$, i.e., $\mathcal{G}_n$ covers $\mathbb{R}$.

Fix any $1-1$ function $f : \Omega \to \mathbb{R}$. Such a function exists since $|\Omega| \leq |\mathbb{R}|$.

Claim 1: Fix n. There is some member $G_n \in \mathcal{G}_n$ such that $f^{-1}[G_n]$ contains a closed uncountable set. For suppose Claim 1 is false. Then $(*)$ implies that for each $G \in \mathcal{G}_n$, the set $\Omega - f^{-1}[G]$ contains some closed uncountable set. Choose one such closed uncountable set and call it $C(G)$. The intersection $\bigcap\{C(G) : G \in \mathcal{G}_n\}$ is a closed uncountable set by Theorem 4.9. Now choose any $\alpha \in \bigcap\{C(G) : G \in \mathcal{G}_n\}$. Since α is common to all the closed uncountable sets, then $f(\alpha) \notin G$ for each $G \in \mathcal{G}_n$ since $C(G) \subseteq \Omega - f^{-1}[G]$. However, this is impossible because every real number belongs to some $G \in \mathcal{G}_n$. Therefore, Claim 1 is true.

With $G_n \in \mathcal{G}_n$ as in Claim 1, let D_n be a closed uncountable set with $D_n \subseteq f^{-1}[G_n]$. Then there exist two points $\beta \neq \gamma$ that both belong to the set $\bigcap\{D_n | n \geq 1\}$ by Theorem 4.9. For each n, the distance $|f(\beta) - f(\gamma)| < \frac{1}{n}$ since each G_n cannot have a diameter greater than $\frac{1}{n}$ and $D_n \subseteq f^{-1}[G_n]$. Thus, since $\lim_{n\to\infty} \frac{1}{n} = 0$, $f(\beta) = f(\gamma)$. But this is imposssible since $\beta \neq \gamma$ and f is $1-1$. Therefore, $(*)$ leads to a contradiction, therefore $(*)$ must be rejected.

To see that S is staionary suppose C is a closed uncountable subset of Ω. If $S \cap C = \emptyset$, then $C \subseteq \Omega - S$, which is false. To see that $\Omega - S$ is stationary suppose C' is a closed uncountable subset of Ω. If $(\Omega - S) \cap C' = \emptyset$, then $C' \subseteq S$, which is false. □

The final two theorems of this section and this chapter prove facts concerning stationary sets and unions of sets composing stationary sets.

Theorem 7.5 *Suppose that $S = S_1 \cup S_2$. If S is stationary, then so is at least one of S_1 and S_2.*

Proof: Suppose that this is not the case in order to get a contradiction. Then, neither S_1 nor S_2 is stationary, meaning there is at least one uncountable closed set that has an empty intersection for each of them. Let $C_1 \cap S_1$ be empty and let $C_2 \cap S_2$ be empty, where C_1 and C_2 are such uncountable closed sets. Since S is stationary then $C_1 \cap S \cap C_2$ is clearly uncountable, becuase S is stationary and $C_1 \cap C_2$ is closed and uncountable by Theorem 4.7. However, by substitute $S_1 \cup S_2$ for S, then $C_1 \cap S \cap C_2 = C_1 \cap (S_1 \cup S_2) \cap C_2$. Distributing from the left, $C_1 \cap S \cap C_2 = C_1 \cap (S_1 \cup S_2) \cap C_2 = [(C_1 \cap S_1) \cup (C_1 \cap S_2)] \cap C_2$. Distributing from the right this time results in $[(C_1 \cap S_1) \cup (C_1 \cap S_2)] \cap C_2 = [(C_1 \cap S_1 \cap C_2) \cup (C_1 \cap S_2 \cap C_2)]$. Since $S_1 \cap C_1 = \emptyset$ and $S_2 \cap C_2 = \emptyset$, then $[(C_1 \cap S_1 \cap C_2) \cup (C_1 \cap S_2 \cap C_2)] = [(\emptyset \cap C_2) \cup (C_1 \cap \emptyset)] = [\emptyset \cup \emptyset] = \emptyset$. Finally, by the transitivity of equality, $C_1 \cap S \cap C_2 = \emptyset$, but this is not possible since S is stationary. Therefore, a contradiction has been reached. □

Theorem 7.6 *Suppose $S = \bigcup\{S_n | n \geq 1\}$. If S is stationary, then so is at least one of the sets S_n.*

Proof: Suppose, in order to get a contradiction, that this is not the case. Then S is a stationary set and each of the S_n has at least one corresponding uncountable closed set with which S_n has an empty intersection. Now consider for each S_n a corresponding C_n for which the given S_n has an empty intersection. Now consider $\bigcap\{C_n | n \geq 1\}$. This set is an uncountable closed subset of Ω by Theorem 4.9, call this set B. However, it shares no common points with any of the S_n. Because S is stationary. $B \cap S \neq \emptyset$; however, by construction $B \cap \bigcup\{S_n | n \geq 1\} = \emptyset$. This is a contradiction. Therefore, at least one of the sets of the union must be a stationary set.

8 The Ulam Matrix

An **Ulam matrix** is a collection $\{A(n, \alpha) | n < \omega, \alpha < \omega_1\}$ of subsets of Ω such that:

(1.) if $\alpha \neq \beta$ then $A(n, \alpha) \cap A(n, \beta) = \emptyset$

(2.) $\bigcup\{A(n, \alpha) | n < \omega\} = (\alpha, \omega_1)$

To constuct such a matrix, for each γ chose any onto function $f_\gamma : [0, \omega) \to [0, \gamma)$. Note that this requires the Axiom of Choice. Now define $A(n, \alpha) = \{\gamma < \omega_1 | f_\gamma(n) = \alpha\}$. Now, I will prove several lemmas concerning this construction to then move to a larger theorem concerning the Ulam matrix.

Lemma 8.1 *For each n $A(n, \alpha) \subseteq (\alpha, \omega_1)$.*

Proof: Each of the $A(n, \alpha)$ is a set of elements from Ω that contains all the elements of Ω that index a function f_γ that maps an ordinal less than the first non-trivial limit ordinal in Ω, denoted ω, to α. Since every element of Ω is less than ω_1 then it need only be shown that the elments of $A(n, \alpha)$ are all greater than α for each n.

Fix n. Assume, in order to get a contraction, that $A(n, \alpha)$ contains some element, γ, less than α. Then there would have be some f_γ with $\gamma \in A(n, \alpha)$ such that $f_\gamma(n) = \alpha$. However, $f_\gamma : [0, \omega) \to [0, \gamma)$, that is f_γ maps elements that come before ω to elements that come before γ. Since, $\gamma < \alpha$, f_γ cannot map any element in its domain to an element greater than γ, but this contradicts the hypothesis that $f_\gamma(n) = \alpha$. $\square$

Lemma 8.2 $\bigcup\{A(n, \alpha) | n < \omega\} = (\alpha, \omega_1)$

Proof: By Lemma 8.1, $\bigcup\{A(n, \alpha) | \alpha < \omega_1\} \subseteq (\alpha, \omega_1)$. To complete the proof, it must be shown that every element of (α, ω_1) can be found in at least one of the sets of the union $\bigcup\{A(n, \alpha) | n < \omega\}$. Fix some element from (α, ω_1). Call it β. It must be shown that $\beta \in \bigcup\{A(n, \alpha) | n < \omega\}$.

Consider the function f_β. This function maps the domain $[0, \omega)$ onto the codomain $[0, \beta)$. Since $\alpha < \beta$, $\alpha \in [0, \beta)$. And since f is onto, there is some $n \in [0, \omega)$ such that $f_\beta(n) = \alpha$. Therefore, $\beta \in A(n, \alpha)$. Therefore, $\beta \in \bigcup\{A(n, \alpha) | n < \omega\}$. $\square$

Lemma 8.3 *Fix n. If $\alpha \neq \beta$, then $A(n,\alpha) \cap A(n,\beta) = \emptyset$.*

Proof: In order to get a contradiction, assume that $A(n,\alpha) \cap A(n,\beta) \neq \emptyset$. Let γ be a common element between them. That means that there is some f_γ such that $f_\gamma(n) = \alpha$ and $f_\gamma(n) = \beta$. However, since $\alpha \neq \beta$ this violates the definition of a function. Therefore, this is impossible. □

Rudin's theorem produced two disjoint stationary sets. An older theorem of Ulam produces many more.

Theorem 8.4 *There is a family $\{S(\alpha) | \alpha < \omega_1\}$ of pairwise disjoint stationary subsets of $[0,\omega_1)$.*

Proof: Consider any Ulam matrix $\{A(n,\alpha) | n < \omega, \alpha < \omega_1\}$ of subsets of $[0,\omega_1)$. For each $\alpha < \omega_1$, $\bigcup\{A(n,\alpha) | n < \omega\} = (\alpha,\omega_1)$ by Lemma 8.2. There are three things to prove:

Fix α. Then, for some n_α, $A(n_\alpha,\alpha)$ is stationary. Clearly, (α,ω_1) is stationary, then by Theorem 7.6, there is some set in the union $\bigcup\{A(n,\alpha) | n < \omega\}$ that is stationary.

Second, let $B(k) = \{\alpha < \omega_1 | n_\alpha = k\}$. Then for some k_0, the set $B(k_0)$ is uncountable. Since $[0,\omega_1)$ is uncountable and $[0,\omega_1) = \bigcup\{B(k) | 0 \leq k < \omega\}$ then some $B(k)$ must be uncountable, otherwise, there would be a $[0,\omega_1)$ would be equal to a countable union of countable sets, which would be countable. But since $[0,\omega_1)$ is uncountable, that would produce a contradiction. Therefore, there is some k_0, such that $B(k_0)$ is uncountable.

Third, for each $\alpha \in B(k_0)$, $A(k_0,\alpha)$ is stationary. Take any $\alpha \in B(k_0)$. Then $A(n_\alpha,\alpha)$ is stationary. But $n_\alpha = k_0$ because $\alpha \in B(k_0)$ so $A(k_0,\alpha)$ is stationary.

Then $\{A(k_0,\alpha) | \alpha \in B(k_0)\}$ is the required collection of stationary sets. □

9 The Long Line

The set Ω is sometimes called "the long sequence". By filling in the holes between adjacent members of Ω, we get "the long line". The long line is technically defined as the set $[0,\omega_1) \times [0,1)$ with the lexicographic order $(\alpha,s) < (\beta,t)$ if $\alpha < \beta$ or if $\alpha = \beta$ and $s < t$.

Definition 9.1 *Two topological spaces are* ***homeomorphic*** *if and only if there is a continuous, open, one-to-one, and onto mapping from one set into another.*

Theorem 9.2 *For each $\alpha \in \Omega$, the subspace $[(0,0),(\alpha,0))$ of the long line is homeomorphic to the subspace of $[0,1)$ of $\mathbb{R}$, where the homeomorphism is also a strictly increasing function.*

Proof: Suppose the opposite in order to get a contradiction. Let β be the first element of Ω so that $[(0,0),(\beta,0))$ is not homeomorphic to $[0,1)$ under a strictly increasing mapping. There are two cases to consider.

Case 1 : If $\beta = \alpha + 1$. Then $[(0,0),(\alpha,0))$ is homeomorphic to $[0,1)$ under a strictly increasing mapping f. Extend f by defining f^* at each point of $[(0,0),(\alpha+1,0)]$ by the rule:

(a) If $(\gamma, s) \in [(\alpha,0),(\alpha+1,0))$ then $f^*(\gamma,s) = s+1$.

(b) If $(\gamma, s) \in [(0,0),(\alpha,0)]$ then $f^*(\gamma,s) = f(\gamma,s)$.

Then f^* is an increasing homeomorphism from $[(0,0),(\alpha+1,0))$ into $[0,2)$. Now consider $g(\gamma,s) = \frac{1}{2}f^*(\gamma,s)$. This function is a strictly increasing map of $[(0,0),(\alpha+1,0))$ into $[0,1)$. Therefore, Case 1 cannot occur.

Case 2 : If β is a limit ordinal, say $\beta = \sup\{\alpha_n | n \geq 1\}$ where $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$. There are increasing homeomorphisms $f_1 : [(0,0),(\alpha_1,0)) \to [0,1)$, $f_2 : [(\alpha_1,0),(\alpha_2,0)) \to [1,2)$, $f_3 : [(\alpha_2,0),(\alpha_3,0)) \to [2,3)$, $\cdots$, and so on. Let $f = \bigcup_{n=1}^{\infty} f_n$. Then f is an increasing homeomorphism from $[(0,0),(\beta,0))$ onto $[0,\infty)$. Consider $g(x) = \frac{2}{\pi}\arctan(x)$. Let $h = g \circ f$. Then h is an increasing homeomorphism from $[(0,0),(\beta,0))$ onto $[0,1)$. This contradicts the hypothesis and exhausts all possibilities.

Therefore, a contradiction has been made. □

Chapter 3: The Set of All Countable Ordinals is Hereditarily Subcompact

1 Introduction

This chapter will construct a basis of Ω to show that the set of all countable ordinals is hereditarily subcompact.

2 Definitions

The point of this chapter is to prove that Ω, the set of all countable ordinals, is hereditarily subcompact. This section will lay out the relevant definitions with the second section giving the relevant proof. The important definitions will be that of a basis, filterbase, regular filterbase, and subcompact.

Definition 2.1 *A* ***basis*** *of a topological space X with a given topology τ is a collection of open sets of the topology from which any open set of the topology can be constructed using set unions.*

For example: In the usual space $\mathbb{R}$, the countable collection $\{(a, b)|a < b, a, b \in \mathbb{Q}\}$ is a base.

Definition 2.2 *A* ***filterbase*** *on a set X is a nonempty collection $\boldsymbol{C}$ of nonempty subsets of X such that if $U_1, U_2 \in \boldsymbol{C}$, then there exists $U_3 \in \boldsymbol{C}$ such that $U_3 \subset U_1 \cap U_2$.*

Definition 2.3 *A* ***regular filterbase*** *is a filterbase, $\boldsymbol{F}$ such that if $F_1, F_2 \in \boldsymbol{F}$ then some $F_3 \in \boldsymbol{F}$ has $cl(F_3) \subseteq F_1 \cap F_2$.*

A space (X, τ) is **subcompact** if and only if there is a basis $\mathbf{B}$ for τ such that $\bigcap \mathbf{F} \neq \emptyset$ whenever $\mathbf{F} \subseteq \mathbf{B}$ is a regular filterbase. The basis $\mathbf{B}$ is called a **subcompact base** for X.

Now with the appropriate definitions, the proof that Ω is subcompact can commence.

3 Ω is Hereditarily Subcompact

The proof that Ω is hereditarily subcompact will involve two parts. First it will be shown that Ω has a subcompact base. It will then be proved for any arbitrary subspace of Ω has such a base.

Theorem 3.1 *Ω has a subcompact base.*

Proof: Let $\mathbf{B}$ be a basis of Ω such that:

1. $\forall \alpha \in \Omega$, then $\{\alpha\} \in \mathbf{B}$ if α is not a limit ordinal.
2. $\forall \lambda \in \Omega$ such that λ is a limit ordinal of Ω, then $(\beta, \lambda] \in \mathbf{B}$ where $\beta < \lambda$ and $\beta \in \Omega$.

The above basis is composed in such a way that every non-limit ordinal of Ω has a corresponding singleton in the basis and every limit ordinal produces an open set including itself and sets of every length to the left of the limit ordinal. This is an obvious basis.

Given a regular filterbase, $\mathbf{F} \subseteq \boldsymbol{B}$, it remains to be shown that $\bigcap \mathbf{F} \neq \emptyset$. There are two cases to consider:

1. There is a singleton element, $F_1 = \{\alpha\}$, that is a member of the filterbase. This means that for any element of the filterbase, $F_2 \in \mathbf{F}$, the intersection between F_1 and F_2 must include $\{\alpha\}$, which means that the intersection over the entire family of elements of the filterbase, $\bigcap \mathbf{F}$, is nonempty, for it must always include $\{\alpha\}$.
2. There are no singleton elements in the filterbase. This means that every set, F of the filterbase is composed of some open interval $(\beta_F, \lambda_F]$ where $\beta_F, \lambda_F \in \Omega$ and where λ_F is a limit ordinal of Ω. Now, consider the set of all limit ordinals λ_F for $F \in \mathbf{F}$, call it Λ. Such a set is a subset of Ω and therefore is well-ordered and has a first element, call that first element λ_0. Every set of the filterbase must include λ_0 otherwise, an intersection between two elements of the filterbase would fail to produce a third element of the filterbase, which would violate the definition of a filterbase. Thus, the intersection over the entire family of sets must include the element λ_0, which means $\bigcap \mathbf{F}$ is nonempty. $\square$

It has therefore been proven that Ω has a subcompact base. The same construction will yield a subcompact base for any subset of Ω.

Theorem 3.2 *Let $X \subseteq \Omega$, then X has a subcompact base.*

Proof: Allow $\boldsymbol{B}$ to be a basis of X such that:

1. $\forall \alpha \in X$, then $\{\alpha\} \in \mathbf{B}$ if $\{\alpha\}$ is a relatively open set in the subspace X.
2. $\forall \lambda \in X$ such that λ is a limit point of X, then $(\beta, \lambda] \cap X \in \mathbf{B}$ for all $\beta < \lambda$ with $\beta \in X$.

Given a regular filterbase, $\mathbf{F} \subseteq \mathbf{B}$, it remains to be shown that $\bigcap \mathbf{F} \neq \emptyset$. There are two cases to consider:

1. There is a singleton element, $\{\alpha\}$, that is a member of the filterbase. This means that for any elements of the filterbase, F_1, F_2, the intersection between them must include $\{\alpha\}$, which means that the intersection over the entire family of elements of the filterbase, $\bigcap \mathbf{F}$, is nonempty, for it must always include $\{\alpha\}$.

2. There are no singleton elements in the filterbase. This means that every set of the filterbase is composed of some open interval $(\beta_F, \lambda_F] \cap X$ where $\beta_F, \lambda_F \in X$ and where λ_F is a limit ordinal of X. Now, consider the set of all limit ordinals inside the filterbase that are right endpoints of some $F \in \mathbf{F}$, call it Λ. Such a set is a subset of X and therefore is well-ordered and has a first element, since X itself is well-ordered, call that first element λ_0. Every set of the filterbase must include λ_0 otherwise, an intersection between two elements of the filterbase would fail to produce a third element of the filterbase, which would violate the definition of a filterbase. Thus, the intersection over the entire family of sets must include the element λ_0, which means $\bigcap \mathbf{F}$ is nonempty. □

Part of the reason why this proof works so well and is practically identical to Theorem 3.1 is because of the behavoir of points in Ω upon being made to compose a subset of Ω. A subset of Ω can be empty, in which case the above proof is not interesting, finite, in which case the basis would be composed of nothing but singletons, or infinite, in which case the set would be composed either of infinitely many singletons or would have a handful of limit ordinals, or would be uncountable, in which case it could have both singletons and the larger open intervals. In any case, a subset of Ω may or may not contain limit ordinals, but those limit ordinals of Ω can only either continue being limit ordinals in the subset or become mere successor ordinals. As such, every subset of Ω looks either like an initial subset of Ω or looks like Ω itself.

4 Why It Matters

Professor Lutzer explained that the reason one wants to know about subcompactness of subspaces of Ω is because of the relation between subcompactness and the more technical property in the literature called **domain representability**. It is known that every subcompact space is domain representable, but whether the converse holds has yet to be answered.[1] The space Ω and its subspaces are known to be domain representable. [2] It was not known, however, whether or not Ω or one of its subspaces could be domain representable and not subcompact. I have shown that this cannot be the case.

[1]For a technical definition of domain representability see Martin, K. Mislove, M. and Reed, G. "Topology and Domain Theory" pp 371-393 in *Recent Progress in General Topology II*, Elsevier, Amsterdam, 2002.

[2]Bennett, H., and Lutzer, D. Domain Representable Spaces, Fundamenta Mathematicae, 189(2006), 255-268.

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