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The classification of edges and the change in multiplicity of an eigenvalue of a real symmetric matrix resulting from the change in an edge value

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The classification of edges and the change in multiplicity of an eigenvalue of a real symmetric matrix resulting from the change in an edge value

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Abstract: We take as given a real symmetric matrix A, whose graph is a tree T, and the eigenvalues of A, with their multiplicities. Each edge of T may then be classified in one of four categories, based upon the change in multiplicity of a particular eigenvalue, when the edge is removed (i.e. the corresponding entry of A is replaced by 0). We show a necessary and sufficient condition for each possible classification of an edge. A special relationship is observed among 2-Parter edges, Parter edges and singly Parter vertices. Then, we investigate the change in multiplicity of an eigenvalue based upon a change in an edge value. We show how the multiplicity of the eigenvalue changes depending upon the status of the edge and the edge value. This work explains why, in some cases, edge values have no effect on multiplicities. We also characterize, more precisely, how multiplicity changes with the removal of two adjacent vertices.

Keywords: Edges, Eigenvalues, Graph, Matrix entries, Multiplicity, Real symmetric matrix, Tree

MSC: 15A18, 05C50, 15B57, 13H15, 05C05

1 Introduction

If T is a simple, undirected tree on n vertices, denote by $\mathcal{S}(T)$ the set of all n-by-n real symmetric matrices, the graph of whose off-diagonal entries is T. There are many papers that relate the structure of T to the multiplicities of the eigenvalues of the matrices in $\mathcal{S}(T)$. Among these, we make use of [2, 3, 5, 7, 8], and [4] is particularly relevant to this work.

We denote the multiplicity of an eigenvalue λ of $A \in \mathcal{S}(T)$ by $m_A(\lambda)$, and the set of eigenvalues of A by $\sigma(A)$. When we remove a vertex u from T, the remaining graph is denoted by $T(u)$. Then we denote by $A(u)$ the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathcal{S}(T)$, resulting from deletion of the row and column corresponding to u. So, $A(u) \in \mathcal{S}(T(u))$. When an edge $\{i, j\}$ is removed from T, we denote the remaining graph by $T(e_{ij})$. Further, when two vertices i and j are removed from T, we denote the remaining graph by $T(\{i, j\})$. $A[U]$ denotes the principal submatrix of A corresponding to the subgraph U of T. For an identified $A \in \mathcal{S}(T)$, we often speak interchangeably about the graph and the matrix, for convenience.

In past work on multiplicities, it has been noticed that often an off-diagonal entry in $A \in \mathcal{S}(T)$, an *edge value*, has little influence on the multiplicity of some eigenvalues. Some of our results explain how this happens. In particular, we do two main, related things. Among the known possibilities for the change in a mul-

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tiplicity when an edge is removed from a tree, we characterize exactly which occurs, and we determine what happens to multiplicity when an edge value is changed. These questions have been studied in the case of a vertex (instead of an edge), and there are some strong analogies to that case, both in terms of status (there is some correlation between the status of the two vertices of an edge and that of the edge) and the consequences of change.

In a tree T, an edge can be classified in four ways based upon the change in the multiplicity of a particular eigenvalue, when the edge is removed from T. We call the classification the *status* of the edge for a given eigenvalue. In [4], the status of an edge for an eigenvalue has been investigated in detail from the status of vertices incident to the edge. In this paper, in section 2, we investigate the necessary and sufficient conditions for each possible classification of an edge. Furthermore, we observe some properties concerning 2-Parter edges and Parter edges.

In [2], when two adjacent vertices are removed from T, the change in the multiplicity of an eigenvalue is also investigated for all cases. In section 3, we clarify the situations in which the multiplicity of the eigenvalue increases or not, when adjacent vertices are removed.

In section 4, we investigate the change in the multiplicity of an eigenvalue, when the edge value is changed. And we show how the change depends upon the status of the edge and the edge value. We show that if the status of the edge is neutral and the status of the incident vertices are both neutral for the eigenvalue, then the multiplicity of the eigenvalue can increase with the change in the edge value.

Because of the interlacing inequalities for an Hermitian matrix and a principal submatrix of it, the multiplicity of the eigenvalue may change by at most 1, when a particular vertex is deleted [1]. A vertex u of T is called "$Parter$" (respectively "$neutral$" or "$downer$") for an eigenvalue λ of $A \in \mathcal{S}(T)$ if

$$m_{A(u)}(\lambda) = m_A(\lambda) + 1 \text{ (resp. } m_A(\lambda),\ m_A(\lambda) - 1).$$

We call these the *status* of the vertex for the eigenvalue λ relative to A. Let T_0 be a branch at vertex v in T that contains the vertex u_0 adjacent to v. If $m_{A[T_0(u_0)]}(\lambda) = m_{A[T_0]}(\lambda) - 1$, then T_0 is called a downer branch at v for the eigenvalue λ. A downer branch plays an important role in identifying a Parter vertex in lemma 2.

When a vertex is removed from a tree, the change in the multiplicity of an eigenvalue of an Hermitian matrix whose graph is the tree, has been investigated. There is a very important theorem coming from previous work in [5, 7, 8], the Parter, Wiener etc. theorem.

Theorem 1. *Let A be a Hermitian matrix whose graph is a tree T, and suppose that there exists a vertex v of T and a real number λ such that λ is an eigenvalue of both A and $A(v)$. Then*
1. there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;
2. if $m_A(\lambda) \geq 2$, then v' may be chosen so that $\deg v' \geq 3$ and so that there are at least three components T_1, T_2, and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$;
3. if $m_A(\lambda) = 1$, then v' may be chosen so that $\deg v' \geq 2$ and so that there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.

To identify a Parter vertex for λ in T, it is known that the next lemma is very useful.

Lemma 2. [5] *Let T be a tree, v a vertex of T, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. Then v is a Parter vertex for λ if and only if there is a downer branch for λ at v.*

2 The classification of edges in a tree

Let T be a tree, and $A \in \mathcal{S}(T)$. Let v be a Parter vertex for an eigenvalue λ of $A \in \mathcal{S}(T)$. If there is only one downer branch at v for λ, v is called a *singly Parter vertex* for λ, and if there is more than one downer branch at v, v is a *multiply Parter vertex* for λ [4]. We denote a Parter vertex, a multiply Parter vertex, a singly Parter vertex, a neutral vertex and a downer vertex for λ by P, P_m, P_s, N, D, respectively. In Theorem 2.5 in [4], it is shown how the multiplicity of an eigenvalue λ of A can change, when we remove one edge from a tree T.

Table 1

i	j	classifications for edge$\{i,j\}$
P_m	P_m	neutral
P_m	P_s	neutral
P_m	N	neutral
P_m	D	neutral
P_s	P_s	2 – Parter, neutral
P_s	N	Parter, neutral
P_s	D	i,j cannot be adjacent
N	N	neutral
N	D	i,j cannot be adjacent
D	D	downer

P_m : multiply Parter vertex, P_s : singly Parter vertex
N : neutral vertex, D : downer vertex

Theorem 3. [4] *Let T be a tree, $A \in \mathcal{S}(T)$, and $\lambda \in \sigma(A)$. When an edge $\{i,j\}$ is removed from T, then*

$$m_A(\lambda) - 1 \leq m_{A(e_{ij})}(\lambda) \leq m_A(\lambda) + 2.$$

Note that, for a tree, $m_A(\lambda) - 2$ cannot occur. Then, we can classify edges in T as 2-Parter (resp. Parter, neutral or downer) for λ in $A \in \mathcal{S}(T)$ when $m_{A(e_{ij})}(\lambda) = m_A(\lambda)+2$ (resp. $m_A(\lambda)+1$, $m_A(\lambda)$ or $m_A(\lambda)-1$) in the same way we classify vertices. In [4], possible classifications for an edge $\{i,j\}$ are investigated, when the classifications of vertices i and j are known in T. The results are in Table 1.

There, it is shown that if vertices i and j are adjacent and both P_s, then edge$\{i,j\}$ is either 2-Parter or neutral. Further, if i is a P_s and j is N, then the edge $\{i,j\}$ is either Parter or neutral. In this paper, we give necessary and sufficient conditions for an edge $\{i,j\}$ in a tree T to be 2-Parter, Parter or neutral, respectively. For the case that $\{i,j\}$ is downer, the characterization is already known as indicated in the next lemma from [4, Theorem 3.7], or from Table 1.

Lemma 4. [4] *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. The edge $\{i,j\}$ is downer for λ in $A \in \mathcal{S}(T)$ if and only if vertices i and j are both downer vertices for λ in A.*

For a 2-Parter edge, a necessary condition on the vertices is known [4, Theorem 3.5].

Lemma 5. [4] *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. If an edge $\{i,j\}$ is 2-Parter for λ, then i and j are both P_s for λ in T.*

The above lemma shows only a necessary condition for a 2-Parter edge in T. When the adjacent vertices are both P_s, it may also happen that the edge between them is neutral, as shown in the Table 1. So, we need to investigate a necessary and sufficient condition for an edge $\{i,j\}$ in a tree T to be 2-Parter to distingwish these two cases.

Theorem 6. *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$ and $\{i,j\}$ an edge of T. Then $\{i,j\}$ is 2-Parter for λ if and only if i and j are both P_s for λ in T, and each is the downer neighbor for the other.*

Proof. First we show necessity of the stated condition for a 2-Parter edge. From lemma 5 (theorem 3.5 in [4]), i and j must be P_s in T. Then, the downer branch at i contains j, because if there is a downer branch that does not contain j, i is still Parter in $T(e_{ij})$. Since $m_{A(e_{ij})}(\lambda) = m_A(\lambda) + 2$, $m_{A(e_{ij},i)}(\lambda) = m_{A(i)}(\lambda) = m_A(\lambda) + 3$. This is contradictory to the interlacing inequalities. So, the downer branch at i must contain j. Similarly, the downer branch at j must contain i.

To show sufficiency, assume that i and j are P_s with each being the downer for the other. Now, $m_{A(i)}(\lambda) - m_A(\lambda) = (m_{A(e_{ij})}(\lambda) - m_A(\lambda)) + (m_{A(i)}(\lambda) - m_{A(e_{ij})}(\lambda))$. Since i is Parter in T, the left-hand side is 1. On the other hand, because i is a downer in its branch at j, $m_{A(i)}(\lambda) - m_{A(e_{ij})}(\lambda) = -1$. We conclude that $m_{A(e_{ij})}(\lambda) - m_A(\lambda) = 2$, which means that $\{i, j\}$ is 2-Parter. □

Though part of the necessity portion of theorem 6 appears as theorem 3.5 of [4], for completeness, we have chosen to give a full, self-contained proof here.

Next we investigate the conditions for an edge to be Parter or neutral. Let T be a tree with an edge $\{i, j\}$. In [4], it is indicated that if i is P_s and j is N, then edge $\{i, j\}$ is either Parter or neutral, as in Table 1. Now we consider these cases more precisely. In the next theorem, we give a necessary and sufficient condition for an edge to be Parter for λ in $A \in \mathcal{S}(T)$.

Theorem 7. *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$. The edge $\{i, j\}$ of T is Parter for λ if and only if vertex i is P_s for λ such that j is the downer neighbor for i, and j is N for λ, or vice versa for i and j.*

Proof. We suppose edge $\{i, j\}$ is Parter for λ in A. Then it follows that vertex i is P_s, and j is N for λ in A, or vice versa (see Theorem 3.6, [4]). If the downer branch at i does not contain j, then i is P in $T(e_{ij})$ since there is a downer branch at i in $T(e_{ij})$. Thus, $m_{A(e_{ij},i)}(\lambda) = m_{A(e_{ij})}(\lambda) + 1$. Since i is P in T, $m_{A(i)}(\lambda) = m_A(\lambda) + 1$. From $T(e_{ij}, i) = T(i)$, $m_{A(e_{ij})}(\lambda) = m_A(\lambda)$. So, edge $\{i, j\}$ is neutral. This is contradictory to the assumption that edge $\{i, j\}$ is Parter. So the downer branch at i contains j.

Next we consider sufficiency for an edge $\{i, j\}$ to be Parter. We suppose that vertex i is P for λ such that a downer branch at i contains j, and j is N for λ in A. Then $m_{A(j)}(\lambda) = m_A(\lambda)$. Since j is D in $T(e_{ij})$, $m_{A(e_{ij},j)}(\lambda) = m_{A(e_{ij})}(\lambda) - 1$. From $T(j) = T(e_{ij}, j)$, $m_A(\lambda) = m_{A(j)}(\lambda) = m_{A(e_{ij})}(\lambda) - 1$. Thus the edge $\{i, j\}$ is Parter in T. □

Next we give a necessary and sufficient condition for an edge to be neutral for λ in $A \in \mathcal{S}(T)$. When an edge $\{i, j\}$ is neutral for λ in $A \in \mathcal{S}(T)$, then there are multiple cases for the classification of vertices i and j, as seen in Table 1. There we can observe that for an edge $\{i, j\}$, if i is P_m, then edge $\{i, j\}$ is neutral, and if i is P_s, then there are three possibilities for the classification of an edge $\{i, j\}$ as 2-Parter, Parter or neutral. Furtermore if i and j are both neutral vertices, then the edge $\{i, j\}$ is neutral.

Now, for all these cases, there is a necessary and sufficient condition for an edge $\{i, j\}$ to be neutral for λ in $A \in \mathcal{S}(T)$.

Theorem 8. *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. The edge $\{i, j\}$ is neutral for λ if and only if vertex i is P for λ such that there is a downer branch at i that does not contain j, or both i and j are N for λ in A. Here i and j are interchangeable.*

Proof. We suppose edge $\{i, j\}$ is neutral. From Table 1, we see that for a neutral edge, there are two cases so that i(or j) is P or i and j are both neutral in T. When i is P_m, there is a downer branch at i that does not contain j. Next we consider the case that i is P_s. We suppose the case that i is P_s and j is P_m, then there is a downer branch at j that does not contain i. Next we consider the case in which i is P_s and j is P_s or N as seen at Table 1. If the downer branch at i contains j, then from Theorem 6 and Theorem 7, then the edge $\{i, j\}$ will be 2-Parter or Parter. It is contradiction since $\{i, j\}$ is neutral. So there is a downer branch at i that does not contain j.

Further, from Table 1, we see that when the edge $\{i, j\}$ is neutral, there is a case such that i and j are both neutral vertices.

Next we give the sufficiency for a neutral edge. If i and j are both neutral vertices, then it is shown that edge $\{i, j\}$ is neutral from [4, corollary 3.8] as seen at Table 1. Next if vertex i is P such that there is a downer branch at i that does not contain j, then i is also P in $T(e_{ij})$. So the relation $m_{A(e_{ij},i)}(\lambda) = m_{A(e_{ij})}(\lambda) + 1 = m_{A(i)}(\lambda) = m_A(\lambda) + 1$ holds. Thus, $m_{A(e_{ij})}(\lambda) = m_A(\lambda)$, then the edge $\{i, j\}$ is neutral. □

We may observe that neither do two 2-Parter edges, nor a 2-Parter edge and a Parter edge, share a vertex.

Corollary 9. *Let T be a tree. $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$. Two 2-Parter edges for λ in A cannot share a vertex, nor can a Parter edge and a 2-Parter edge share a vertex in T.*

Proof. If edge $\{i, j\}$, and $\{j, k\}$ are 2-Parter edges adjacent to each other, then from Theorem 6, j is D in $A(e_{ij})$. But, since the edge $\{j, k\}$ is a 2-Parter edge, there is a downer branch at j containing k. Then j is P in $A(e_{ij})$, a contradiction. Thus 2-Parter edges are not adjacent.

Next we suppose an edge $\{i, j\}$ is a 2-Parter edge and an edge $\{j, k\}$ is a Parter edge. Then i and j are Parter vertices and k is a netutral vertex. Then from Theorem 6, since the downer branch at i contains j, j is D in $A(e_{ij})$. But, since the edge $\{j, k\}$ is a Parter edge, from Theorem 7 there is a downer branch at j containing k. Then j is P in $A(e_{ij})$, a contradiction. So, a 2-Parter edge and a Parter edge cannot be adjacent. □

The next theorem shows that the number of Parter edges or 2-Parter edges for λ in $A \in \mathcal{S}(T)$ are bounded by the number of singly Parter vertices for λ in $A \in \mathcal{S}(T)$.

Theorem 10. *Let T be a tree. $A \in \mathcal{S}(T)$, and $\lambda \in \sigma(A)$. Let vertex i be P_s for λ in $A \in \mathcal{S}(T)$. Then, exactly one edge incident to i is either 2-Parter or Parter, and any other edges incident to i are neutral in T.*

Proof. Let i be P_s in T, and we denote by B the downer branch at i. Let j be a vertex in B, which is adjacent to i. For edge $\{i, j\}$, since the classification of vertices i and j in T cannot be (P_s, D) as seen at Table 1 (cf.[4]), classification of j is one of P_m, P_s and N.

If j is P_m in T, then there is a downer branch at j in B. Then j is P in B. Since j is D in B, this is contradiction. Thus j cannot be P_m in T. If j is P_s, since the downer branch at i contains j, edge $\{i, j\}$ is a 2-Parter edge from Theorem 6. If j is N, then the edge $\{i, j\}$ is a Parter edge from Theorem 7. From Corollary 9, there is not the case such that edge $\{i, j\}$ is 2-Parter and another edge $\{i, k\}$ is 2-Parter or Parter.

If edge $\{i, j\}$ is Parter, then edge $\{i, k\}$ is not 2-Parter from Corollary 9. Furthermore there is not the case such that edge $\{i, j\}$ is Parter and $\{i, k\}$ is also Parter, because if both edges $\{i, j\}$ and $\{i, k\}$ are Parter edges adjacent to i, then the classification of j and k is both neutral, then there are two downer branches at i from Theorem 7. Since i is P_s, a contradiction.

From the above arguments, one 2-Parter edge or Parter edge is incident to i, and if there are other edges incident to i other than the 2-Parter or Parter edge, then they are all neutral edges from Theorem 8. □

Since the incident vertices of a 2-Parter edge are both P_s from Theorem 6, we have the next corollary from Theorem 10.

Corollary 11. *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$. All edges adjacent to a 2-Parter edge are neutral for λ in A.*

From the previous corollaries and Theorem 10, we can show a strong relationship between the number of 2-Parter edges, Parter edges and singly Parter vertices. From Theorem 6 and Theorem 7, every 2-Parter edge is incident to two singly Parter vertices, and every Parter edge is incident to one singly Parter vertex. From corollary 9, 2-Parter edges cannot be ajacent to each other, and a 2-Parter edge and a Parter edge are also not adjacent. If two Parter edges $\{i, j\}$, $\{j, k\}$ are adjacent, then j cannot be P_s from Theorem 10. Thus i and k will be P_s and j is N. So in this case, there are two singly Parter vertices. Thus we can deduce a relationship among the number of 2-Parter edges, Parter edges and singly Parter vertices.

Theorem 12. *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$. Let s, t and u be the number of 2-Parter edges, Parter edges and singly Parter vertices in T, respectively. Then, we have*

$$2s + t = u.$$

If there are no singly Parter vertices, Theorem 12 makes it clear that a tree may have only neutral or downer edges. Since lemma 4 makes it clear when an edge is downer, in all other circumstances an edge is neutral. This seems to be a common occurrence.

Table 2

i	j	$m_A(\lambda) - m_{A(\{i,j\})}(\lambda)$
P	P	$-2, 0$
P	N	$-1, 0$
P	D	0
N	N	0
N	D	Not possible
D	D	1

P : Parter vertex, N : Neutral vertex, D : Downer vertex

3 The classification of vertex pairs

When adjacent vertices i and j are removed from the tree T, the possible change in multiplicity of an eigenvalue λ of $A \in \mathcal{S}(T)$ is investigated. Table 2 below, in [2] or [4], shows the possibilities. There, it is noted that if the classification of vertices i and j is (P, P), then there are two possible cases: $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -2$ or 0. (cf.Table 1 [4]). Furthermore if the classification of vertices i and j is (P, N), then there are two possible cases: $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ or 0. We analyze what determines which of these two actually occurs.

Lemma 13. *Let T be a tree with adjacent vertices i and j, $A = (a_{ij}) \in \mathcal{S}(T)$, and $\lambda \in \sigma(A)$. We suppose that i and j are both Parter vertices for λ, then*
(i) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -2$ if and only if there is a downer branch at i that does not contain j in T, or vice versa for i and j; and
(ii) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$ if and only if there is a downer branch at i that contains j in T, or vice versa for i and j.

Proof. (i) Let $\widetilde{T} = T(e_{ij})$. Let $\widetilde{A} = (\widetilde{a}_{ij})$ be a matrix corresponding to $\widetilde{T}$ obtained from A, i.e, $\widetilde{a}_{ij} = \widetilde{a}_{ji} = 0$ and other elements are same as A.

We suppose there is a downer branch at i in T for λ in A that does not contain j. Then the edge $\{i, j\}$ is neutral in T from Theorem 8.

If there is a downer branch at j that contains i, then from Theorem 6, the edge $\{i, j\}$ becomes 2-Parter, a contradiction. So, downer branch at j does not contain i. Then j is P in $\widetilde{T}$ for λ in $\widetilde{A}$. So, i and j are Parter vertices in $\widetilde{T}$, then $m_{A(\{i,j\})}(\lambda) - m_A(\lambda) = m_{\widetilde{A}(\{i,j\})}(\lambda) - m_A(\lambda) = 2$.

Conversely, we suppose $m_{A(\{i,j\})}(\lambda) - m_A(\lambda) = 2$. Then j must be P in $T(i)$. If there is a downer branch at i in T that contains j, then j is D in $T(i)$. This is contradiction. Thus downer branches at i does not contain j.

(ii) If there is a downer branch at i that contains j, then j is D in $T(i)$. Thus $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$.

If $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$, since i is P in T, j must be D in $T(i)$. thus j is contained in a downer branch at i. □

By similar arguments, when the status of adjacent vertices i and j is P and N respectively, we can determine the difference between the two possibilities, $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ or 0.

Lemma 14. *Let T be a tree with adjacent vertices i and j. $A \in \mathcal{S}(T)$, and $\lambda \in \sigma(A)$. We suppose that vertex i is P and j is N in T, then*
(i) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ if and only if there is a downer branch at i that does not contain j in T.
(ii) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$ if and only if there is a downer branch at i that contains j in T.

Proof. (i) If there is a downer branch at i that does not contain j, the edge $\{i, j\}$ is neutral from Theorem 8. Let $\widetilde{T} = T(e_{ij})$. Then i is P in $\widetilde{T}$ and j is N in $\widetilde{T}$ since the edge $\{i, j\}$ is neutral in T. Thus $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$.

Conversely, we suppose $m_{A(\{i,j\})}(\lambda) - m_A(\lambda) = -1$. Since vertex j is N in T, vertex i must be P in $T(j)$. Thus there is a downer branch at i that does not contain j.

(ii). If there is a downer branch at i that contains j, then j is D in $T(i)$. Since i is P in T, it holds $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$.

Conversely when $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$, j must be D in $T(i)$. Then j is contained in a downer branch at i in T. □

For adjacent vertices i and j in a tree T, when $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -2$ for an eigenvalue λ of $A \in \mathcal{S}(T)$, it is clear that i and j are both P, from Table 2. Furthermore when $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ for an eigenvalue λ of $A \in \mathcal{S}(T)$, i is P and j is N in T. So we can deduce a necessary and sufficient condition for $m_{A(\{i,j\})}(\lambda) - m_A(\lambda) = -2$ and $m_{A(\{i,j\})}(\lambda) - m_A(\lambda) = -1$ for adjacent vertices i and j, as in the following corollary.

Corollary 15. *Let T be a tree with adjacent vertices i and j, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$.*
(i) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -2$ if and only if i and j are both Parter vertices for λ in A and there is a downer branch at i that does not contain j, or vice versa for i and j.
(ii) $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ if and only if i is P and j is N for λ in A and there is a downer branch at i that does not contain j, or vice versa for i and j.

4 The change in multiplicity due to a change in an edge value

In [6, Theorem 5], when a diagonal entry is changed, the possible change in the multiplicitiy of a given eigenvalue was determined, as in the next theorem, in which E_{ii} denotes a matrix the same size as A with (i, i) entry 1 and zeros elsewhere.

Theorem 16. [6] *Let G be a graph, and i a vertex in G. For $A \in \mathcal{H}(G)$, let $A' = A + tE_{ii}$, $t \neq 0$. Then*
(1) $m_{A'}(\lambda) = m_A(\lambda)$ *if and only if i is Parter in A or i is neutral in A and t is not a unique value t_0;*
(2) $m_{A'}(\lambda) = m_A(\lambda) + 1$ *if and only if i is neutral in A, and $t = t_0$; and*
(3) $m_{A'}(\lambda) = m_A(\lambda) - 1$ *if and only if i is downer in A.*

In this section, we investigate how the multiplicity of an eigenvalue λ of $A \in \mathcal{S}(T)$ changes as a result of change in an *edge* value. We consider the change in multiplcity of an eigenvalue due to the change in an edge value for a 2-Parter, Parter, neutral and downer edge, respectively in Theorem 18. Berfore that, we give a necessary lemma. We define two functions $f_1(x)$ and $f_2(x)$ as follows, $f_1(x) = \prod_{\substack{i \\ \mu_i \neq \lambda}} (x - \mu_i)$, $\mu_i \in \sigma(A(e_{uv}))$ and

$f_2(x) = \prod_{\substack{j \\ \nu_j \neq \lambda}} (x - \nu_j)$, $\nu_j \in \sigma(A(\{u, v\}))$.

Lemma 17. *Let T be a tree and $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. Let an edge $\{u, v\}$ be neutral for λ in T. We suppose u is P and v is D in T. Then $\frac{f_1(\lambda)}{f_2(\lambda)} < 0$.*

Proof. Let $m_A(\lambda) = m$. Then $m_{A(e_{uv})}(\lambda) = m$. Let l be the number of eigenvalues greater than λ in $A(e_{uv})$ including multiplicity. Since u is still P in $A(e_{uv})$, $m_{A(e_{uv},u)}(\lambda) = m + 1$. From the interlacing inequality, the number of eigenvalues greater than λ in $A(e_{uv}, u)$ including multiplicity is $l - 1$. Since v is downer in $A(e_{uv})$, v is still downer in $A(e_{uv}, u)$. Then, the number of eigenvalues greater than λ including multiplicity in $A(e_{uv}, u, v) = A(u, v)$ is $l - 1$ from the interlacing inequality. So the difference of the number $\mu_i^{'}s$ and $\nu_j^{'}s$ greater than λ is 1. So, $\frac{f_1(\lambda)}{f_2(\lambda)} = \frac{\prod(\lambda-\mu_i)}{\prod(\lambda-\nu_j)} < 0$. □

Theorem 18. *Let T be a tree and $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. Let $\widetilde{A} \in \mathcal{S}(T)$ be a matrix obtained from a change in the edge value on $\{u, v\}$. Then the multiplicity of λ in $\widetilde{A}$ is as follows,*

(i) If the edge $\{u,v\}$ is 2-Parter or Parter for λ in A, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.
(ii) If the edge $\{u,v\}$ is downer for λ in A, and if $\widetilde{a}_{uv} = \pm a_{uv}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$; otherwise, $m_{\widetilde{A}}(\lambda) = m_A(\lambda) - 1$.
(iii) If the edge $\{u,v\}$ is neutral for λ in A, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$, except for the case in which the status of the vertices incident to the edge is (N,N) in A and $|\widetilde{a}_{uv}|^2 = \frac{f_1(\lambda)}{f_2(\lambda)}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$.

Proof. If we focus upon the edge $\{u,v\}$, then the characteristic polynomial of A can be represented as follows,

$$p_A(x) = p_{A(e_{uv})}(x) - |a_{uv}|^2 p_{A(\{u,v\})}(x). \tag{1}$$

(i) When the value of a 2-Parter edge is changed, we investigate the change in the multiplicity of λ. If we put $m_A(\lambda) = m$, then $m_{A(\{u,v\})}(\lambda) = m$, since u and v are downer vertices in $A(e_{uv})$. Then, from (1)

$$\begin{aligned} p_A(x) &= (x-\lambda)^{m+2} f_1(x) - |a_{uv}|^2 (x-\lambda)^m f_2(x) \\ &= (x-\lambda)^m \{(x-\lambda)^2 f_1(x) - |a_{uv}|^2 f_2(x)\}, \end{aligned}$$

in which $f_1(x) = \prod\limits_{\substack{i \\ \mu_i \neq \lambda}} (x-\mu_i)$, $\mu_i \in \sigma(A(e_{uv}))$, $f_2(x) = \prod\limits_{\substack{j \\ \nu_j \neq \lambda}} (x-\nu_j)$, $\nu_j \in \sigma(A(\{u,v\}))$. If we set $g(x) = (x-\lambda)^2 f_1(x) - |a_{uv}|^2 f_2(x)$, even if the edge value a_{uv} is changed, $g(\lambda) \neq 0$, unless $a_{uv} = 0$. Since A and $\widetilde{A}$ are in $\mathcal{S}(T)$, $a_{uv} \neq 0$ and $\widetilde{a}_{uv} \neq 0$. Thus, $m_{\widetilde{A}}(\lambda) = m$.

Next we consider the change in the edge value on a Parter edge in $A \in \mathcal{S}(T)$. We suppose an edge $\{u,v\}$ is Parter, then we note $m_{A(\{u,v\})}(\lambda) = m_A(\lambda)$, since u is neutral and v is downer in $A(e_{uv})$. So, from (1)

$$\begin{aligned} p_A(x) &= (x-\lambda)^{m+1} f_1(x) - |a_{uv}|^2 (x-\lambda)^m f_2(x) \\ &= (x-\lambda)^m \{(x-\lambda) f_1(x) - |a_{uv}|^2 f_2(x)\}, \end{aligned}$$

in which $f_1(x), f_2(x)$ are similar definitions as before. From a similar argument, for the matrix $\widetilde{A} \in \mathcal{S}(T)$ obtained by change of a_{uv} in A, $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$ holds.

(ii) We consider the change in the edge value on a downer edge in $A \in \mathcal{S}(T)$. We suppose an edge $\{u,v\}$ is downer, then $m_{A(\{u,v\})}(\lambda) = m_A(\lambda) - 1$, since u and v are neutral vertices in $A(e_{uv})$. Then

$$\begin{aligned} p_A(x) &= (x-\lambda)^{m-1} f_1(x) - |a_{uv}|^2 (x-\lambda)^{m-1} f_2(x) \\ &= (x-\lambda)^{m-1} \{f_1(x) - |a_{uv}|^2 f_2(x)\} \end{aligned}$$

Since $m_A(\lambda) = m$, if we set $h(x) = f_1(x) - |a_{uv}|^2 f_2(x)$, then it must be $h(\lambda) = 0$. Thus, $|a_{uv}|^2 = \frac{f_1(\lambda)}{f_2(\lambda)}$ must hold. The characteristic polynomial of $\widetilde{A}$ is $(x-\lambda)^{m-1}\{f_1(x) - |\widetilde{a}_{uv}|^2 f_2(x)\}$. So, if $\widetilde{a}_{uv} \neq \pm a_{uv}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda) - 1$. If $\widetilde{a}_{uv} = \pm a_{uv}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

(iii) We change the edge value on a neutral edge in $A \in \mathcal{S}(T)$. For two vertices to which a neutral edge is incident, there are four cases (P,P), (P,N), (P,D) and (N,N) as the status of u and v with reference with Table 1 before.

In the case of (P,P), if we set $m_A(\lambda) = m$, from (1) and corollary 15

$$\begin{aligned} p_A(x) &= (x-\lambda)^m f_1(x) - |a_{uv}|^2 (x-\lambda)^{m+2} f_2(x) \\ &= (x-\lambda)^m \{f_1(x) - |a_{uv}|^2 (x-\lambda)^2 f_2(x)\}. \end{aligned}$$

For changed matrix $\widetilde{A}$, $p_{\widetilde{A}}(x) = (x-\lambda)^m\{f_1(x) - |\widetilde{a}_{uv}|^2 (x-\lambda)^2 f_2(x)\}$. If we set $q(x) = f_1(x) - |\widetilde{a}_{uv}|^2 (x-\lambda)^2 f_2(x)$, then $q(\lambda) \neq 0$. Thus, $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

In case (P,N), from (1) and corollary 15

$$\begin{aligned} p_A(x) &= (x-\lambda)^m f_1(x) - |a_{uv}|^2 (x-\lambda)^{m+1} f_2(x) \\ &= (x-\lambda)^m \{f_1(x) - |a_{uv}|^2 (x-\lambda) f_2(x)\}. \end{aligned}$$

By similar arguments as before, we have $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

Next we consider the case (P, D). Since u is Parter and v is downer in $A(e_{uv})$, $m_{A(\{u,v\})}(\lambda) = m$, then

$$\begin{aligned} p_A(x) &= (x-\lambda)^m f_1(x) - |a_{uv}|^2 (x-\lambda)^m f_2(x) \\ &= (x-\lambda)^m \{f_1(x) - |a_{uv}|^2 f_2(x)\}. \end{aligned}$$

If $|\widetilde{a}_{uv}|^2 = \frac{f_1(\lambda)}{f_2(\lambda)}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$. But, this does not happen, because $\frac{f_1(\lambda)}{f_2(\lambda)} < 0$ from lemma 17. Thus $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

Lastly, in case (N, N),

$$\begin{aligned} p_A(x) &= (x-\lambda)^m f_1(x) - |a_{uv}|^2 (x-\lambda)^m f_2(x) \\ &= (x-\lambda)^m \{f_1(x) - |a_{uv}|^2 f_2(x)\}. \end{aligned}$$

Then, if $|\widetilde{a}_{uv}|^2 = \frac{f_1(\lambda)}{f_2(\lambda)}$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$, otherwise, $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$. □

From the above results, we can observe that when an edge value is changed, the multiplicity of an eigenvalue can be changed only in the case in which the status of the vertices incident to the edge for the eigenvalue is (N, N) or (D, D) and the edge value takes a particular value. In other cases, the multiplicity of the eigenvalue is not affected by change in the edge value. This explains an often observed phenomenon that edge values matter surprisingly little in the determination of multiplicity.

From the above theorem, it follows immediately that the status of the edge for λ in A changes to that in $\widetilde{A}$ as follows.

Corollary 19. *Let T be a tree and $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$. Let $\widetilde{A} \in \mathcal{S}(T)$ be a matrix obtained from a change in the edge value of $\{u, v\}$ in A. Then the status of the edge $\{u, v\}$ for λ changes to that in $\widetilde{A}$ as follows.*

(i) In case $\{u, v\}$ is 2-Parter or Parter for λ in A, the status of $\{u, v\}$ for λ in $\widetilde{A}$ is same as in A.

(ii) In case $\{u, v\}$ is downer for λ in A, if $\widetilde{a}_{uv} = \pm a_{uv}$, then the status of $\{u, v\}$ for λ in $\widetilde{A}$ is downer, otherwise, neutral.

(iii) In case $\{u, v\}$ is neutral for λ in A, if the status of u, v is (N, N) and $|\widetilde{a}_{uv}|^2 = \frac{f_1(\lambda)}{f_2(\lambda)}$, then the status of $\{u, v\}$ for λ in $\widetilde{A}$ is downer, otherwise, neutral.

From Theorem 18, we can deduce the well known fact that even if the signs of off-diagonal entries $a_{ij}(= a_{ji}) \neq 0, (i \neq j)$ are changed, the spectra of the matrix does not change as shown in the next corollary.

Corollary 20. *Let T be a tree and $A \in \mathcal{S}(T)$. Let $\widetilde{A} \in \mathcal{S}(T)$ be a matrix resulting from changing the signs of some entries $a_{ij}(= a_{ji}) \neq 0, (i \neq j)$, then $\sigma(A) = \sigma(\widetilde{A})$.*

In this paper, we clarified the conditions for the classifications of edges in a tree, based upon the change in multiplicity of a particular eigenvalue, then we investigated the change in multiplicity of an eigenvalue of a real symmetric matrix whose graph is a tree, when an edge value is changed.

Two observations are worthy of note. First, in comparison with [6], there are parallels between the change in edge value and the change in vertex value. Second, there seems to be a rough correlation between the two: the "higher" the statuses of the two vertices, the "higher" the status of the edge tends to be. We also note that much of what we have said is equally valid in $\mathcal{H}(T)$, the Hermitian matrices with graph T, as an Hermitian matrix whose graph is a tree is similar to a real symmetric one by a diagonal unitary similarity, but, as some results are clearer to state for $\mathcal{S}(T)$, we decided to present results in $\mathcal{S}(T)$.

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