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A STUDY OF MULTIPLICATIVE PRESERVERS

A thesis submitted in partial fulfillment of the
requirements for the degree of Bachelor of Arts with Honors in
Mathematics from the College of William and Mary in Virginia,

by

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Accepted for Highest Honors

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Abstract

Let $A*B$ be a product on matrices such as the usual matrix product $A*B = AB$, the entrywise product $A*B = [a_{ij}b_{ij}]$, and the Jordan triple product $A*B = ABA$. Characterizations of multiplicative maps with respect to $*$ which leave certain functions of matrices invariant are given. These functions include the rank, eigenvalues, and higher rank numerical ranges of matrices.

Keywords: Multiplicative preserver, matrix products, spectral radius, matrix rank, nonnegative matrices, numerical range.

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Contents

Chapter 1

Introduction

The central topic of this thesis comes from the study of preserver problems in matrix theory. This chapter begins with an overview of this area of research, some discussion of matrix products, followed by a description of our research and ends with an overview of the notation which will be used.

1.1 Preserver Problems

The study of preserver problems in matrix theory has been an active area of research. The statements of such a problems may be quite diverse but their essential question is typically of the following form:

Let $S \subseteq \mathbf{M}_{m,n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries from a field $\mathbb{F}$. Then given a function $\psi : S \to X$ for an arbitrary set X, can we characterize functions $\phi : S \to S$ such that $\psi(\phi(A)) = \psi(A)$ for all A in S?

In addition to this premise, one usually assumes that the function ϕ additionally satisfies some algebraic relation (for example, $\phi(A \circ B) = \phi(A) \circ \phi(B)$ for all $A, B \in S$ and some closed binary operation $\circ$) on S. This provides a rich structure which precludes many otherwise pathological functions. Requiring the function be linear is a common restraint for preservers. This class of problems has a long history of study (i.e. one example comes from a paper published in 1897) and continues to be an area of research interest both due to the elegance of its results as well as its ability to capture features of the matrices and the given function.

The following chapters will consider multiplicative preservers of the spectral radius, rank, and higher rank numerical ranges and radii. As an introduction to typical problems and results in preservers, we present examples including some linear preserver results that in part motivate our study. We use $\mathbf{M}_{m,n}$ ($\mathbf{M}_n$) to denote the set of $m \times n$ ($n \times n$) complex matrices.

Determinant Preserver (Frobenius [16]) A linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $\det(A) = \det(\phi(A))$ if and only if there exist $M, N \in \mathbf{M}_n$ with $\det(MN) = 1$ such that either

$$\phi(A) = MAN \text{ for all } A \in \mathbf{M}_n, \text{ or } \phi(A) = MA^tN \text{ for all } A \in \mathbf{M}_n$$

Spectrum Preserver (Marcus and Purves [38]) A linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $\sigma(\phi(A)) = \sigma(A)$ if and only if there exists an invertible $S \in \mathbf{M}_n$ such that either

$$\phi(A) = SAS^{-1} \text{ for all } A \in \mathbf{M}_n, \text{ or } \phi(A) = SA^tS^{-1} \text{ for all } A \in \mathbf{M}_n$$

Numerical Range Preserver (Pellegrini [43]) Let $W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$ be the numerical range of $A \in \mathbf{M}_n$. A linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $W(\phi(A)) = W(A)$ if and only if there exists an unitary $U \in \mathbf{M}_n$ such that either

$$\phi(A) = UAU^* \text{ for all } A \in \mathbf{M}_n, \text{ or } \phi(A) = UA^tU^* \text{ for all } A \in \mathbf{M}_n$$

Rank Preserver (Beasley [2]) A linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $\operatorname{rank}(\phi(A)) = k$ whenever $\operatorname{rank}(A) = k$ if and only if there exist invertible $M \in \mathbf{M}_m$, $N \in \mathbf{M}_n$ such that either

$$\phi(A) = MAN \text{ for all } A \in \mathbf{M}_n, \text{ or } m = n \text{ and } \phi(A) = MA^tN \text{ for all } A \in \mathbf{M}_n$$

Each of these results exemplifies a standard form of preserving functions. See [27] for further discussion and a survey of results in linear preservers. Furthermore, in the case of the spectrum and numerical range preservers, one may note that the linear preservers are also multiplicative or antimultiplicative maps. Indeed we will obtain similar standard forms for multiplicative maps.

1.2 Matrix Products

Motivated by theory and applications, researchers consider different kinds of matrix products in addition to the usual product. Some of these products may seem artificial with respect to the usual multiplication, but they may have settings which make them the natural choice. In our study, we will consider two alternative products besides the usual matrix multiplication.

The first product is the Jordan Triple Product on $n \times n$ matrices defined as $A * B = ABA$. This product is considered in the study of Jordan algebras, but one can certainly identify contexts where such a product would be convenient. For example, this product is closed on the set of $n \times n$ Hermitian matrices and the set of $n \times n$ positive definite matrices.

The second product is the entrywise product on $m \times n$ matrices, also referred to as the Hadamard or Schur product. Under this product the i, j entry of a matrix product is the product of the i, j entries of each factor. More explicitly, for $A = [a_{ij}], B = [b_{ij}]$ we define $A \circ B = [c_{ij}]$ with $c_{ij} = a_{ij}b_{ij}$. This product is closed on the set of positive definite matrices and has been used to prove some inequalities in the study of those matrices.

1.3 Our Study

In this thesis we study some multiplicative preserver problems with respect to different matrix products. Such problems have not recieved as much attention as linear preserver problems but remarkably product preservers also tend to be of certain standard forms.

The following topics are studied:

In Chapter 2, we consider preservers of the spectral radius of the usual product or the Jordan Triple Product of nonnegative matrices.

In Chapter 3, we characterize injective maps on $\mathbf{M}_{m,n}$ preserving the entrywise product. We then apply this result in the particular case of preserving rank k matrices.

In Chapter 4, we consider multiplicative preservers of the higher rank numerical ranges and radii, which are generalizations of the numerical range and radius. The study of these objects is motivated by the theory of quantum computing.

1.4 Notations

The following notations will be used in our discussion:

$\mathbf{M}_{m,n}(\mathbb{F})$: the set of m by n matrices with entries from a field.

$J_{m,n}$: the matrix in $\mathbf{M}_{m,n}(\mathbb{F})$ with all entries equal to 1.

$0_{m,n}$: the matrix in $\mathbf{M}_{m,n}(\mathbb{F})$ with all entries equal to 0.

E_{ij}: the matrix in $\mathbf{M}_{m,n}(\mathbb{F})$ with 1 in the (i,j)th position and zeros everywhere else.

$\mathcal{B}$: the set $\{E_{ij} : 1 \leq\ i \leq m,\ 1 \leq j \leq n\} \subseteq M_{m,n}(\mathbb{F})$.

When $m = n$, we simplify the notation to $\mathbf{M}_n(\mathbb{F}), J_n, 0_n$, etc.

When $\mathbb{F} = \mathbb{C}$, we supress the field argument (i.e. $\mathbf{M}_{m,n} = \mathbf{M}_{m,n}(\mathbb{C})$).

$\mathbf{GL}_n$: the group of invertible matrices in $\mathbf{M}_n$;

$\mathbf{SL}_n$: the group of matrices in $\mathbf{GL}_n$ of determinant 1;

$\mathbf{U}_n$: the group of unitary matrices in $\mathbf{M}_n$;

$\mathbf{SU}_n$: the group of matrices in $\mathbf{U}_n$ of determinant 1;

$\mathbf{M}_n^{(m)}$: the semigroup of matrices in $\mathbf{M}_n$ with rank at most m.

$\mathbf{M}_n^+$: the set of $n \times n$ real matrices with nonnegative entries.

$\mathbf{P} \subseteq \mathbf{M}_n^+$: the group of permutation matrices.

$\mathbf{PD} \subseteq \mathbf{M}_n^+$ the group of matrices of the form PD where P is a permutation matrix and D is a diagonal matrix with positive entries on the diagonal.

We adopt the convention that unless otherwise defined a_{ij} denotes the matrix entry in the ith row and jth column of the matrix A and likewise for other letters.

$\sigma(A)$: the spectrum (the set of eigenvalues) of a matrix A.

$r(A)$: the spectral radius of a matrix A ($r(A) = \max\limits_{\lambda \in \sigma(A)} |\lambda|$)

$\sigma_p(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(A)\}$ the peripheral spectrum of A.

$\mathrm{tr}\,(A)$: the trace of A.

$p_A(t) = \det(A - tI)$: the characteristic polynomial of A .

A^t: the transpose of A.

$\oplus$: the direct sum of matrices (i.e. $X \oplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$)

$\mathrm{diag}\,(x_1, \ldots, x_n)$: the diagonal matrix D with $d_{ii} = x_i$. (in that order)

Chapter 2

Spectral Radius Preservers

The purpose of this chapter is to characterize preservers of the spectral radius, spectrum and peripheral spectrum which also preserve either the usual product or the Jordan triple product on the set of entrywise nonnegative matrices. We note that the literature on preservers in the context of entrywise nonnegative matrices is meager; see [39]. The content of this chapter is based on the paper [12].

2.1 Introduction

Recall we define $\mathbf{M}_n^+$ to be the set of matrices such that $A \in \mathbf{M}_n^+$ satisfies $a_{ij} \geq 0$ for all $i, j \in \{1, 2, \ldots n\}$. It is easy to see that this set is closed under matrix multiplication, and so multiplicative maps are well-defined. These matrices also have nice properties (see [21], Chapter 8), and particularly the spectral radius of a nonnegative matrix happens to be an eigenvalue of the matrix. These facts make multiplicative spectral radius maps a natural preserver to study on this set. Moreover, [37, 46] serve as a motivation for our study of peripheral spectrum preservers, which happen to be exactly the spectral radius preservers. Note that $A \in \mathbf{M}_n^+$ has the property that A is invertible and $A^{-1} \in \mathbf{M}_n^+$ if and only if $A \in \mathbf{PD}$ (see, e.g., [34] for a proof).

Here is our main theorem.

Theorem 2.1.1. *Let $n \geq 2$. For $A, B \in \mathbf{M}_n^+$, let $A * B$ denote the usual product $A * B = AB$ or the Jordan triple product $A * B = ABA$. Then the following statements* (1) - (4) *are equivalent for a surjective function $f : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$:*

(1)

$$r(A * B) = r(f(A) * f(B)), \qquad \forall \;\; A, B \in \mathbf{M}_n^+. \tag{2.1.1}$$

(2)
$$\sigma_p(A * B) = \sigma_p(f(A) * f(B)), \qquad \forall \;\; A, B \in \mathbf{M}_n^+. \tag{2.1.2}$$

(3)
$$\sigma(A * B) = \sigma(f(A) * f(B)), \qquad \forall \;\; A, B \in \mathbf{M}_n^+. \tag{2.1.3}$$

(4) *There exists a matrix* $Q \in \mathbf{PD}$ *such that either*

$$f(A) = Q^{-1}AQ, \qquad \forall \;\; A \in \mathbf{M}_n^+,$$

or

$$f(A) = Q^{-1}A^tQ, \qquad \forall \;\; A \in \mathbf{M}_n^+.$$

Note that in Theorem 2.1.1 the function f is not assumed to be linear or multiplicative *a priori.*

The result of Theorem 2.1.1 for $A * B = A + B$ was obtained in [34] without the surjective assumption. It would be interesting to remove the surjective assumption in Theorem 2.1.1. We are not able to do that at present.

Since for $A \in \mathbf{M}_n^+$ we always have $r(A) \in \sigma_p(A)$, the implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are trivial. Also, (4) $\Longrightarrow$ (3) is easy to verify. It remains to show that (1) implies (4). We first present some preliminary and auxiliary results in Section 2. In particular, we prove a function $f : \mathbf{M}_n^+ \to \mathbf{M}_n^+$ having some special properties on matrix units will have the nice form described in Theorem 2.1.1 (4). Then we show that a function $f : \mathbf{M}_n^+ \to \mathbf{M}_n^+$ satisfying Theorem 2.1.1 (1) will possess the special properties on matrix units, and hence f has the form in Theorem 2.1.1 (4). This is done in Sections 3 and 4 for the usual product and Jordan triple product, respectively.

2.2 Preliminaries

In this section we present some known results and easy observations that will be often used, sometimes without explicit reference, throughout the chapter. We list several well-known properties of nonnegative matrices and their spectral radii (see, for example, [21, Theorem 8.4.5] or [3]).

The following two observations are useful when considering the triple product. Let $\sqrt{E_{ij}} = \begin{cases} E_{ik} + E_{kj} : k \neq i, j & \text{if } i \neq j \\ E_{ii} & \text{if } i = j \end{cases}$ for which a trivial calculation shows $\sqrt{E_{ij}}^2 = E_{ij}$. Clearly our choice of the specific k in the above definition does not matter so long as it respects our constraint in each case. Note that this construction requires $n \geq 3$, so $n = 2$ will be covered separately.

Since $r((BA)B) = r(B(BA)) = r(B^2A) = r(AB^2)$, we will use the following three equivalent conditions for the triple product interchangeably:

$$r(BAB) = r(f(B)f(A)f(B)) \tag{2.2.1}$$

$$r(B^2A) = r(f(B)^2f(A)) \tag{2.2.2}$$

$$r(AB^2) = r(f(A)f(B)^2). \tag{2.2.3}$$

Lemma 2.2.1. *Let $f : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$ be a map that satisfies* (2.1.1) *and is surjective. Assume further $n \geq 3$ if $A * B$ is the Jordan triple product. Then f is bijective.*

Proof. Since we assume surjectivity, we will prove injectivity. Suppose $A, B \in \mathbf{M}_n^+$ satisfy $f(A) = f(B)$. For any (i, j) pair with $1 \leq i, j \leq n$, since AE_{ij} has all columns zero except for the jth column, and the jth column of AE_{ij} is just the ith column of A, we have $r(A\sqrt{E_{ij}}^2) = r(AE_{ij}) = a_{ji}$. Similarly $r(B\sqrt{E_{ij}}^2) = r(BE_{ij}) = b_{ji}$. Then by our spectral radius conditions,

$$a_{ji} = r(AE_{ij}) = r(f(A)f(E_{ij})) = r(f(B)f(E_{ij})) = r(BE_{ij}) = b_{ji},$$

or

$$a_{ji} = r(f(A)f(\sqrt{E_{ij}}^2)) = r(f(B)f(\sqrt{E_{ij}})^2) = r(B\sqrt{E_{ij}}^2) = b_{ji}.$$

Thus, $A = B$. ■

Remark 2.2.2. Since f is a bijection, it is simple to observe that its inverse f^{-1} fulfills (2.1.1) if f does, i.e.,

$$r(f(A) * f(B)) = r(A * B) = r(f^{-1}(f(A)) * f^{-1}(f(B))).$$

The following observations will be used throughout our discussion.

Lemma 2.2.3. *Assume that the function $f : \mathbf{M}_n^+ \to \mathbf{M}_n^+$ satisfies condition* (1) *in Theorem* 2.1.1. *Then:*

(a) *For any $A \in \mathbf{M}_n^+$ we have $r(A) = r(f(A))$.*

(b) *$A \in \mathbf{M}_n^+$ is nilpotent if and only if $f(A)$ is nilpotent.*

(c) *If $A \in \mathbf{M}_n^+$ is nilpotent, i.e., $r(A) = 0$, then all diagonal elements of A and $f(A)$ are zeros.*

(d) *If in addition the range of f contains a matrix with positive entries, then A is nonzero if and only if $f(A)$ is nonzero.*

Proof. Condition (a) follows from setting $A = B$ in (2.1.1). Condition (b) follows trivially from (a).

Condition (c) follows from nilpotency and nonnegativity. A nilpotent matrix has all zero eigenvalues, so the sum of all eigenvalues, and equivalently the trace, is zero. Since the trace is the sum of the diagonal entries, all of which are nonnegative, we finally obtain that the diagonal entries must all be zero. By (b), we get the conclusion on $f(A)$.

For condition (d), let $A \in \mathbf{M}_n^+$, and let $X \in \mathbf{M}_n^+$ be the matrix with all entries equal to $1/n$. Then $X^2 = X$, and $\operatorname{tr}(A * X) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}/n$. Hence, if $A \neq 0$, then $\operatorname{tr}(A * X) \neq 0$ and hence $0 \neq r(A * X) = r(f(A) * f(X))$. It follows that $f(A) \neq 0$. Similarly, if $f(A) \neq 0$ and if $f(Y) = Z$, where Z is a matrix with all entries positive in the range of f, then $\operatorname{tr}(f(A) * Z) \neq 0$. So $0 \neq r(f(A) * Z) = r(A * Y)$ and hence $A \neq 0$. ■

Theorem 2.2.4. *Let $f : \mathbf{M}_n^+ \to \mathbf{M}_n^+$, $n \geq 3$, be such that*

$$r(A) = r(f(A)) \quad \forall\ A \in \mathbf{M}_n^+, \tag{2.2.4}$$

and let $\mathcal{N} = \{(i,j) : 1 \leq i,j \leq n\}$. Assume that there exist a permutation τ on the set $\mathcal{N}$ and a collection of positive numbers $\{\gamma_{ij} : (i,j) \in \mathcal{N}\}$ that satisfies

(a) $\gamma_{ij} = 1/\gamma_{ji}$,

(b) $\tau(i,j) = (p,q) \Rightarrow \tau(j,i) = (q,p)$,

and f has the property that

$$f(\sum_{(i,j)\in S} E_{ij}) = \sum_{(i,j)\in S} \gamma_{ij} E_{\tau(i,j)}, \quad \forall\ S \subseteq \mathcal{N},\ S \neq \emptyset.$$

*Then there exists a matrix $Q \in$ **PD** such that either*

$$f(E_{ij}) = Q^{-1} E_{ij} Q, \qquad \forall\ (i,j) \in \mathcal{N}, \tag{2.2.5}$$

or

$$f(E_{ij}) = Q^{-1} E_{ij}^t Q, \qquad \forall\ (i,j) \in \mathcal{N}. \tag{2.2.6}$$

Proof. Let $F_{ij} = f(E_{ij})$ for $1 \leq i, j \leq n$. We adjust our map f via $X \mapsto P^t f(X) P$ for a suitable permutation matrix P so that $F_{jj} = E_{jj}$ for $j = 1, \ldots, n$. We proceed in 4 steps:

Step 1. We show that $F_{ij} = \gamma_{ij} E_{ij}$ or $F_{ij} = E_{ji}/\gamma_{ij}$. We may assume $i \neq j$. Let $F_{ij} = \gamma_{ij} E_{pq}$, and assume $p \neq i, j$. Consider $A = E_{ij} + E_{ji} + E_{pp}$. Then clearly $r(A) = 1$. But $f(A) = \gamma_{ij} E_{pq} + E_{qp}/\gamma_{ij} + E_{pp}$, and $p_{f(A)}(t) = t^n - t^{n-1} - t^{n-2} = t^{n-2}(t^2 - t - 1)$, so $r(f(A)) = \frac{1+\sqrt{5}}{2} > 1 = r(A)$, a contradiction. Similarly, assume $q \neq i, j$, and we reach the same contradiction. In view of (2.2.4), we cannot have $p = q$. Therefore $F_{ij} = \gamma_{ij} E_{ij}$ or $F_{ij} = E_{ji}/\gamma_{ij}$.

Step 2. We show that $F_{1j} = \gamma_{1j} E_{1j}$ and $F_{j1} = E_{j1}/\gamma_{1j}$, after possible replacement of f by a map of the form $X \mapsto f(X)^t$.

We may assume that $F_{12} = \gamma_{12} E_{12}$ and $F_{21} = E_{21}/\gamma_{21}$; otherwise, use the map $X \mapsto f(X)^t$. Now consider $A = E_{12} + E_{2j} + E_{j1}$ for $3 \leq j \leq n$. Then $r(A) = 1 = r(f(A))$. But

$$f(A) = F_{12} + F_{2j} + F_{j1} = \gamma_{12} E_{12} + F_{2j} + F_{j1},$$

and if $F_{2j} = \gamma_{j2} E_{j2}$ or $F_{j1} = \gamma_{1j} E_{1j}$, then $f(A)$ is nilpotent, and $r(f(A)) = 0$, a contradiction. So $F_{j1} = E_{j1}/\gamma_{1j}$ which gives us $F_{1j} = \gamma_{1j} E_{1j}$.

Step 3. We show that $F_{ij} = \gamma_{ij} E_{ij}$ and $F_{ji} = E_{ji}/\gamma_{ij}$.

We only need to consider the case $1 < i < j \leq n$. So let $A = E_{1i} + E_{ij} + E_{j1}$, and we have

$$f(A) = F_{1i} + F_{ij} + F_{j1} = \gamma_{1i} E_{1i} + F_{ij} + E_{j1}/\gamma_{1j}.$$

But again, if $F_{ij} = E_{ji}/\gamma_{ij}$ then $f(A)$ is nilpotent. But then $r(A) = 1 > 0 = r(f(A))$, a contradiction. Therefore, $f(E_{ij}) = \gamma_{ij}E_{ij}$ and $f(E_{ji}) = E_{ji}/\gamma_{ij}$.

Step 4. We show $\gamma_{ij} = \gamma_{1j}/\gamma_{1i}$.

Consider the same matrix $A = E_{1i} + E_{ij} + E_{j1}$, with $r(A) = 1 = r(f(A))$. But $f(A) = \gamma_{1i}E_{1i} + \gamma_{ij}E_{ij} + E_{j1}/\gamma_{1j}$, so

$$p_{f(A)}(t) = t^n - \gamma_{1i}\gamma_{ij}/\gamma_{1j}t^{n-3} = t^{n-3}(t^3 - \gamma_{1i}\gamma_{ij}/\gamma_{1j}).$$

Then

$$\sigma(f(A)) = \{0, \sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}, \omega\sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}, \omega^2\sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}\}$$

(zero is present only if $n > 3$), where ω is the primitive cubic root of 1. Since $r(f(A)) = 1$, we have $\sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}} = 1$, so $\gamma_{1i}\gamma_{ij}/\gamma_{1j} = 1$. Our conclusion follows.

Now replace f by the map $X \mapsto Df(X)D^{-1}$ with $D = \operatorname{diag}(1, \gamma_{12}, \ldots, \gamma_{1n})$. Then we have $f(E_{ij}) = F_{ij} = E_{ij}$ for all $1 \leq i, j \leq n$. Therefore, reversing our modifications, for $Q = PD \in \mathbf{PD}$, then f must be of the form (2.2.5) or of the form (2.2.6). ■

2.3 The Usual Product

This section concerns the proof of "(1) ⇒ (4)" of Theorem 2.1.1 for the usual product $A * B = AB$. For the rest of this section, we always assume that f is a bijective (cf. Lemma 2.2.1) map on $\mathbf{M}_n^+$ that satisfies

$$r(AB) = r(f(A)f(B)), \qquad \forall \;\; A, B \in \mathbf{M}_n^+. \tag{2.3.1}$$

Let us now define a set of matrices useful for our proof.

Definition 2.3.1. For every $A \in \mathbf{M}_n^+$ define $\mathcal{F}(A) = \{X \in \mathbf{M}_n^+ : r(AX) > 0\}$.

The next few results examine and exploit the relationships between these sets and the bijectivity of our function to extract relationships between a matrix and its image.

Lemma 2.3.2. *Suppose* $A = [a_{ij}] \in \mathbf{M}_n^+$, $B = [b_{ij}] \in \mathbf{M}_n^+$.

(a) *If there is (i,j) pair such that $b_{ij} > 0 = a_{ij}$ then $\mathcal{F}(B) \setminus \mathcal{F}(A)$ is non-empty.*

(b) *The inclusion $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ holds if and only if $b_{ij} > 0$ whenever $a_{ij} > 0$, i.e., there is $\gamma > 0$ such that $\gamma B - A \in \mathbf{M}_n^+$.*

(c) *The following two conditions are equivalent:*

(c.1) $\mathcal{F}(A) = \mathcal{F}(B)$.

(c.2) $a_{ij} = 0$ *if and only if* $b_{ij} = 0$.

Proof. (a) Let $a_{ij} = 0$, and $b_{ij} > 0$. Let $X = E_{ji}$. Then

$$r(BX) = b_{ij} > 0 = a_{ij} = r(AX).$$

So $X \in \mathcal{F}(B)$ and $X \notin \mathcal{F}(A)$. The result follows.

(b) The necessity follows from (a): if $a_{ij} > 0 = b_{ij}$, then $\exists X \in \mathcal{F}(A) \setminus \mathcal{F}(B)$.

To prove the sufficiency, assume that $b_{ij} > 0$ whenever $a_{ij} > 0$. Then there is $\gamma > 0$ such that $\gamma B - A \in \mathbf{M}_n^+$. So $\gamma B \geq A$ (entrywise inequality), and then $\gamma BX \geq AX$ and $r(\gamma BX) \geq r(AX)$ for all $X \in \mathbf{M}_n^+$. (We use here the well known monotonicity property of the spectral radius, see, e.g., [21, Theorem 8.1.18] or [3].) Thus, for any $X \in \mathcal{F}(A)$,

$$r(\gamma BX) = \gamma r(BX) \geq r(AX) > 0.$$

So $r(BX) > 0$, thus $X \in \mathcal{F}(B)$. It follows that $\mathcal{F}(A) \subseteq \mathcal{F}(B)$.

(c) Note that $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $\mathcal{F}(A) \subseteq \mathcal{F}(B) \subseteq \mathcal{F}(A)$. By (b), this is equivalent to any one of the following conditions:

(i) $b_{ij} > 0$ if and only if $a_{ij} > 0$. (ii) $b_{ij} = 0$ if and only if $a_{ij} = 0$. ∎

Corollary 2.3.3. *A matrix $X \in \mathbf{M}_n^+$ has exactly k nonzero entries if and only if there is a sequence of matrices $X_1, \ldots, X_k, \ldots X_{n^2}$ in $\mathbf{M}_n^+$ with $X_k = X$ such that $\mathcal{F}(X_i)$ is proper non-empty subset of $\mathcal{F}(X_{i+1})$ for $i = 1, \ldots, n^2 - 1$.*

Proof. If $X_k = X$ has exactly k nonzero entries, we can replace zero entries with nonzero entries one at a time to get $X_{k+1}, \ldots, X_{n^2}$. Similarly, we can replace nonzero entries with zeros one at a time to get the required $X_{k-1}, X_{k-2}, \ldots, X_1$. Observe that since we only replaced $k-1$ nonzero entries with zeros, $X_1 \neq 0$. So this construction yields the desired sequence.

Conversely, if $X = X_1, X_2, \ldots, X_{n^2}$ have the described property, then $X_1 \neq 0$ because $\mathcal{F}(X_1)$ is non-empty. Moreover, by Lemma 2.3.2, X_{i+1} has at least one more nonzero entry than X_i. It follows that X_i must have exactly i nonzero entries for each i, so the result follows. ∎

Note that for $A \in \mathbf{M}_n^+$, we have

$$\begin{aligned}\mathcal{F}(f(A)) &= \{X \in \mathbf{M}_n^+ : r(f(A)X) > 0\} = \{X \in \mathbf{M}_n^+ : r(Af^{-1}(X)) > 0\} \\ &= \{f(Y) \in \mathbf{M}_n^+ : r(AY) > 0\} = f(\mathcal{F}(A)).\end{aligned}$$

Thus, we have the following.

Lemma 2.3.4. *If $A \in \mathbf{M}_n^+$, then $\mathcal{F}(f(A)) = f(\mathcal{F}(A))$.*

Corollary 2.3.5. *A matrix $X \in \mathbf{M}_n^+$ has exactly k nonzero entries if and only if $f(X)$ has exactly k nonzero entries.*

Proof. Let $X \in \mathbf{M}_n^+$ such that X has exactly k nonzero entries. By Corollary 2.3.3 there exist $X_1, \ldots, X_{n^2}$ in $\mathbf{M}_n^+$ with $X_k = X$ such that $\mathcal{F}(X_i)$ is proper non-empty subset of $\mathcal{F}(X_{i+1})$ for $i = 1, \ldots, n^2 - 1$. By Lemma 2.3.4, we have

$$\mathcal{F}(f(X_i)) = f(\mathcal{F}(X_i)) \subseteq f(\mathcal{F}(X_{i+1})) = \mathcal{F}(f(X_{i+1})),$$

and the inclusion is strict in view of bijectivity of f. Thus, $f(X_1), \ldots, f(X_{n^2})$ is a sequence satisfying Corollary 2.3.3. So, $f(X_k) = f(X)$ has k nonzero entries. Applying the above proof to f^{-1} in place of f (see Remark 2.2.2) we see that $f^{-1}(X)$ has k nonzero entries. ∎

This concludes our direct involvement with our (2.3.1) sets. Now we will use our obtained results to characterize the image of another set of useful matrices: the matrix units.

Lemma 2.3.6. *Let $f(E_{ij}) = F_{ij}$ for $1 \le i, j \le n$. Then:*

(a) *For $i \neq j$, we have $F_{ij} = \gamma_{ij} E_{pq}$ for some $1 \le p, q \le n$, $p \neq q$, where $\gamma_{ij} > 0$.*

(b) $\{F_{11}, \ldots, F_{nn}\} = \{E_{11}, \ldots, E_{nn}\}$.

(c) *If $i \neq j$, then*

$$F_{ij} = \gamma_{ij} E_{pq} \qquad \Longrightarrow \qquad F_{ji} = \gamma_{ij}^{-1} E_{qp};$$

thus $\gamma_{ji} = \gamma_{ij}^{-1}$.

(d) *There is a permutation τ of $\{(i, j) : 1 \le i, j \le n\}$ with the properties that*

$$\tau(i, j) = (p, q) \qquad \Longrightarrow \qquad \tau(j, i) = (q, p) \tag{2.3.2}$$

and $F_{ij} = \gamma_{ij} E_{\tau(i,j)}$ for all pairs (i, j), $1 \le i, j \le n$ (we take $\gamma_{ii} = 1$ for $i = 1, 2, \ldots, n$).

(e) *For $f([a_{kl}]) = [x_{kl}]$, we have $\gamma_{ij} a_{ij} = x_{\tau(i,j)}$.*

Proof. For (a) let $i \neq j$. From Corollary 2.3.5, F_{ij} has exactly one nonzero entry. But $r(F_{ij}) = r(E_{ij}) = 0$, so this nonzero entry is not on the diagonal, thus $F_{ij} = \gamma_{ij} E_{pq}$ for some positive γ_{ij}, $p \neq q$.

For (b), by Corollary 2.3.5 F_{ij} has one nonzero entry for all i, j. For $i = j$, since $r(F_{ii}) = r(E_{ii}) = 1$, this nonzero entry must be on the diagonal, and it must be 1. So $F_{ii} = E_{pp}$ for some p. Furthermore, since $r(F_{ii} F_{kk}) = r(E_{ii} E_{kk}) = 0$ for $i \neq k$, no two F_{ii}'s can have the same nonzero position, so we get the desired result.

For (c), let $i \neq j$. So $F_{ij} = \gamma_{ij} E_{pq}$ for some (p, q), $p \neq q$. Since $r(F_{ij} F_{ji}) = r(E_{ij} E_{ji}) = 1$, then the nonzero entry of F_{ji} must be in the transposed position to the nonzero entry of F_{ij} to get a nonzero entry on the diagonal. Furthermore, these entries therefore must be inverse to each other. Thus $F_{ji} = E_{qp} / \gamma_{ij}$.

Since for $(p, q) \notin \{(i, j), (j, i)\}$ we have

$$r(F_{ij} F_{pq}) = r(E_{ij} E_{pq}) = 0 = r(E_{ji} E_{pq}) = r(F_{ji} F_{pq}),$$

it is clear that no other F_{pq} shares its nonzero position with F_{ij} or F_{ji}. Then our τ can be defined by $\tau(i, j) = (p, q)$ if $F_{ij} = \gamma_{ij} E_{pq}$, and is bijective by our above discussion, so it is the permutation required by (d). Property (2.3.2) holds in view of (c).

Finally for (e), note that

$$\begin{aligned} a_{ij} &= r(AE_{ji}) = r([x_{ij}]f(E_{ji})) = r\left([x_{ij}]\gamma_{ji}E_{\tau(j,i)}\right) \\ &= r\left([x_{ij}]\gamma_{ij}^{-1}E^t_{\tau(i,j)}\right) = \gamma_{ij}^{-1}x_{\tau(i,j)} \end{aligned}$$

by (d). It follows that $x_{\tau(i,j)} = \gamma_{ij}a_{ij}$. ■

Corollary 2.3.7. *Let $\mathcal{N} = \{(i,j) : 1 \leq i,j \leq n\}$, and let $S \subseteq \mathcal{N}$, $S \neq \emptyset$. Then*

$$f(\sum_{(i,j)\in S} E_{ij}) = \sum_{(i,j)\in S} F_{ij}.$$

Proof. Let $A = \sum\limits_{(i,j)\in S} E_{ij} = [a_{ij}]$, and $f(A) = [x_{ij}]$.

First observe that $a_{ij} = 1$ if $(i,j) \in S$ and $a_{ij} = 0$ otherwise. From (e) we have that $\gamma_{ij}a_{ij} = x_{\tau(i,j)}$. Then $x_{\tau(i,j)} = \gamma_{ij}$ if $(i,j) \in S$ and $x_{\tau(i,j)} = 0$ otherwise. Representing $f(A)$ as a combination of matrix units, we get

$$f(A) = \sum_{(i,j)\in\mathcal{N}} x_{ij}E_{ij} = \sum_{\tau(i,j)\in S} \gamma_{ij}E_{\tau(i,j)} = \sum_{(i,j)\in S} F_{ij}.$$

■

We are now ready to present the proof of Theorem 2.1.1 for the usual product.

Proof. Recall that the bijective map $f : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$ has the property (2.3.1). If $n = 2$, then Theorem 2.1.1 follows easily from Lemma 2.3.6 and Corollary 2.3.7. Thus suppose $n > 2$. By Lemma 2.3.6 and Corollary 2.3.7, the hypotheses of Theorem 2.2.4 are satisfied. Thus, either (2.2.5) or (2.2.6) holds. For $f(E_{ij}) = Q^{-1}E_{ij}Q$, define $\hat{f}_1(A) = Qf(A)Q^{-1}$, and for $f(E_{ij}) = Q^{-1}E^t_{ij}Q$, define $\hat{f}_2(A) = \hat{f}_1(A)^t$. Then $\hat{f}_1, \hat{f}_2 : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$ since $Q, f(A), Q^{-1} \in \mathbf{M}_n^+$. Also, for all $A, B \in \mathbf{M}_n^+$,

$$r(AB) = r(f(A)f(B)) = r(Qf(A)(Q^{-1}Q)f(B)Q^{-1}) = r(\hat{f}_1(A)\hat{f}_1(B)),$$

and

$$r(AB) = r(\hat{f}_1(A)\hat{f}_1(B)) = r(\hat{f}_1(B)^t\hat{f}_1(A)^t) = r(\hat{f}_2(B)\hat{f}_2(A)) = r(\hat{f}_2(A)\hat{f}_2(B)),$$

thus we may use our previous machinery on these functions.

Trivially, $\hat{f}_1(E_{ij}) = \hat{f}_2(E_{ij}) = E_{ij}$. Applying Lemma 2.3.6 to $\hat{f}_k$, we have $\gamma_{ij} = 1$ and τ the identity permutation. Then for $A = [a_{ij}] \in \mathbf{M}_n^+$ and $\hat{f}_k(A) = [x_{ij}]$, we apply part (e) of that lemma to get $x_{ij} = x_{\tau(i,j)} = \gamma_{ij}a_{ij} = a_{ij}$, and so $\hat{f}_k(A) = A$. Thus, $f(A) = Q^{-1}\hat{f}_1(A)Q = Q^{-1}AQ$ or $f(A) = Q^{-1}\hat{f}_2(A)^tQ = Q^{-1}A^tQ$. ■

2.4 The Triple Product

This section concerns the proof of "(1) $\Rightarrow$ (4)" of Theorem 2.1.1 for the Jordan triple product $A * B = ABA$. For the rest of this section, we always assume that the surjective map f on $\mathbf{M}_n^+$ satisfies

$$r(ABA) = r(f(A)f(B)f(A)), \qquad \forall \;\; A, B \in \mathbf{M}_n^+. \tag{2.4.1}$$

Note that by Lemma 2.2.1 f is automatically bijective if $n > 2$.

We first treat the special case when $n = 2$.

Lemma 2.4.1. *Suppose $n = 2$ and $A * B = ABA$. Then the implication of "(1) $\Rightarrow$ (4)" of Theorem 2.1.1 holds.*

Proof. We divide the proof into several assertions. We will use the observation that $A \in \mathbf{M}_2^+$ is a non-zero nilpotent if and only if it has exactly one non-zero entry at the off-diagonal position.

Assertion 1 $\{f(E_{11}), f(E_{22})\} = \{E_{11}, E_{22}\}$.

To see this, let $f(E_{ii}) = [x_{pq}]$. Observe that since all diagonal entries of a nilpotent matrix are 0, $r(E_{ii}^2N) = r(E_{ii}N) = 0$ for any nilpotent matrix N. So for N such that $f(N) = E_{12}$, we have $r(f(E_{ii})^2E_{12}) = r(E_{ii}^2N) = 0$. But

$$f(E_{ii})^2E_{12} = \begin{pmatrix} 0 & x_{11}^2 + x_{12}x_{21} \\ 0 & x_{21}(x_{11} + x_{22}) \end{pmatrix},$$

so $x_{21}(x_{11}+x_{22})=0$. By similar argument, we see that $x_{12}(x_{11}+x_{22})=0$.

Assume that $x_{11}+x_{22}=0$. Then

$$f(E_{ii})^2 = \operatorname{diag}(x_{12}x_{21}, x_{12}x_{21}) = x_{12}x_{21}I.$$

But for $i \neq j$,

$$0 = r(E_{ii}E_{jj}) = r(f(E_{ii})^2 f(E_{jj})) = r(x_{12}x_{21}f(E_{jj})) = x_{12}x_{21}r(f(E_{jj})) = x_{12}x_{21}.$$

This is impossible, for then $r(f(E_{ii})) = 0 \neq 1 = r(E_{ii})$ which is a contradiction. So we must have $x_{21}=x_{12}=0$.

Thus, $f(E_{11}) = \operatorname{diag}(x_{11}, x_{22})$ and $f(E_{22}) = \operatorname{diag}(y_{11}, y_{22})$ for some nonnegative numbers $x_{11}, x_{22}, y_{11}, y_{22}$. But then

$$f(E_{11})^2 f(E_{22}) = \begin{pmatrix} x_{11}^2 y_{11} & 0 \\ 0 & x_{22}^2 y_{22} \end{pmatrix}.$$

Since $0 = r(E_{ii}E_{jj}) = r(f(E_{ii})^2 f(E_{jj}))$, we have $x_{11}^2 y_{11} + x_{22}^2 y_{22} = 0$. Now, since $x_{11}+x_{22}=1$ and $y_{11}+y_{22}=1$, $f(E_{11})$ and $f(E_{22})$ must have exactly one nonzero entry on the diagonal in different positions, and that nonzero entry must be 1. This completes the proof of Assertion 1.

Replacing f by the map $X \mapsto P^t f(X) P$ for a suitable permutation matrix P, we may assume that $f(E_{ii}) = E_{ii}$. Additionally, up to transposition, $f(E_{12}) = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ for $\gamma \geq 0$. Observe that since E_{ij} is nonzero, we have $\gamma > 0$. We will assume this is the case since if it is not we can instead consider the map $X \to f(X)^{\mathrm{t}}$.

After these modifications, we can proceed to prove the following.

Assertion 2 Let $A = [a_{ij}]$ and $f(A) = [f_{ij}]$. Then $f_{ii} = a_{ii}$.

To see this, simply consider $f_{ii} = r(E_{ii}^2 f(A)) = r(E_{ii}^2 A) = a_{ii}$.

Assertion 3

Let $X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$f(X) = E_{11} + \gamma E_{12} \qquad \text{and} \qquad f(Y) = \gamma^{-1}E_{21} + E_{22}. \tag{2.4.2}$$

Let $f(X) = [x_{ij}]$, $f(Y) = [y_{ij}]$. Then $x_{11} = 1 = y_{22}$, $x_{22} = 0 = y_{11}$ by the previous assertion. But then

$$f(X)^2 = \begin{pmatrix} 1 + x_{12}x_{21} & x_{12} \\ x_{21} & x_{12}x_{21} \end{pmatrix}, \qquad f(Y)^2 = \begin{pmatrix} y_{12}y_{21} & y_{12} \\ y_{21} & 1 + y_{12}y_{21} \end{pmatrix},$$

which gives us

$$0 = r(X^2E_{22}) = r(f(X)^2E_{22}) = x_{12}x_{21}$$

and

$$0 = r(Y^2E_{11}) = r(f(Y)^2E_{11}) = y_{12}y_{21}.$$

Now

$$0 = r(X^2E_{12}) = r(f(X)^2\gamma E_{12}) = \gamma x_{21},$$

so $x_{21} = 0$. Similarly,

$$1 = r(Y^2E_{12}) = r(f(Y)^2\gamma E_{12}) = \gamma y_{21},$$

so $y_{21} = \gamma^{-1}$ and $y_{12} = 0$.

Finally, observe $X^2Y = X$, so

$$1 = r(X^2Y) = r(f(X)^2f(Y)) = r(f(X)f(Y)).$$

But $f(X)f(Y) = \begin{pmatrix} \gamma^{-1}x_{12} & x_{12} \\ 0 & 0 \end{pmatrix}$, therefore, $x_{12} = \gamma$, giving us the desired result (2.4.2).

We can now modify our function by $X \to Df(X)D^{-1}$ where $D = \operatorname{diag}(\gamma^{-1}, 1)$ so $f(X) = X$ and $f(Y) = Y$. With this additional modification, we can complete the proof of our lemma by proving the following.

Assertion 4 $f(A) = A$ for all $A \in \mathbf{M}_2^+$.

Let $A = [a_{ij}]$ and $f(A) = [x_{ij}]$. Then by Assertion 2 $x_{ii} = a_{ii}$. Furthermore,

$$X^2A = XA = \begin{pmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ 0 & 0 \end{pmatrix}$$

and

$$f(X)^2f(A) = f(X)f(A) = \begin{pmatrix} a_{11} + x_{21} & x_{12} + a_{22} \\ 0 & 0 \end{pmatrix},$$

so

$$a_{11} + x_{21} = r(X^2 f(A)) = r(X^2 A) = a_{11} + a_{21},$$

thus $a_{21} = x_{21}$.

Repeating the same calculation with Y yields $a_{12} = x_{12}$. Thus $a_{ij} = x_{ij}$, so $f(A) = A$. ∎

Now, we turn to the case when $n > 2$.

Remark 2.4.2. It is clear from our consideration of $n = 2$ that the arguments in Section 3 are not directly extendable to the triple product. However, we shall adopt the same approach and modify it as needed to obtain our new result for $n \geq 3$. For those results having similar proofs exhibited in Section 3, we shall restate the results but often suppress the proof.

For every $A \in \mathbf{M}_n^+$ define

$$\widetilde{\mathcal{F}}(A) = \{X \in \mathbf{M}_n^+ : r(AX^2) > 0\}.$$

Lemma 2.4.3. *Suppose $A = [a_{ij}], B = [b_{ij}] \in \mathbf{M}_n^+$.*

(a) *If there is (i, j) pair such that $b_{ij} > 0 = a_{ij}$ then $\widetilde{\mathcal{F}}(B) \setminus \widetilde{\mathcal{F}}(A) \neq \emptyset$.*

(b) *The inclusion $\widetilde{\mathcal{F}}(A) \subseteq \widetilde{\mathcal{F}}(B)$ holds if and only if $b_{ij} > 0$ whenever $a_{ij} > 0$, i.e., there is $\gamma > 0$ such that $\gamma B - A \in \mathbf{M}_n^+$.*

(c) *The following two conditions are equivalent:*

(c.1) *$\widetilde{\mathcal{F}}(A) = \widetilde{\mathcal{F}}(B)$.*

(c.2) *$a_{ij} = 0$ if and only if $b_{ij} = 0$.*

Proof. (a) Let $a_{ij} = 0$, and $b_{ij} > 0$. Let $X = \sqrt{E_{ji}}$. Then $r(BX^2) = b_{ij} > 0 = a_{ij} = r(AX^2)$, so $X \in \widetilde{\mathcal{F}}(B)$ and $X \notin \widetilde{\mathcal{F}}(A)$. The result follows.

(b) The necessity follows from (a): if $a_{ij} > 0 = b_{ij}$, then $\exists X \in \widetilde{\mathcal{F}}(A) \setminus \widetilde{\mathcal{F}}(B)$.

To prove the sufficiency, assume that $b_{ij} > 0$ whenever $a_{ij} > 0$. Then there is $\gamma > 0$ such that $\gamma B - A \in \mathbf{M}_n^+$. So $\gamma B \geq A$, so $r(\gamma BX) \geq r(AX)$ for all $X \in \mathbf{M}_n^+$. Thus, for any $X \in \widetilde{\mathcal{F}}(A)$,

$$r(\gamma BX^2) = \gamma r(BX^2) \geq r(AX^2) > 0,$$

thus $X \in \widetilde{\mathcal{F}}(B)$. It follows that $\widetilde{\mathcal{F}}(A) \subseteq \widetilde{\mathcal{F}}(B)$.

Finally, (c) follows easily from (b). ■

Corollary 2.4.4. *A matrix $X \in \mathbf{M}_n^+$ has exactly k nonzero entries if and only if there is a sequence of matrices $X_1, X_2, \dots X = X_k, \dots, X_{n^2}$ in $\mathbf{M}_n^+$ such that $\widetilde{\mathcal{F}}(X_i)$ is proper non-empty subset of $\widetilde{\mathcal{F}}(X_{i+1})$ for $i = 1, \dots, n^2 - 1$.*

Proof. Similar to that of Corollary 2.3.3. ■

Note that for $A \in \mathbf{M}_n^+$, we have

$$\begin{aligned} \widetilde{\mathcal{F}}(f(A)) &= \{X \in \mathbf{M}_n^+ : r(f(A)X^2) > 0\} = \{X \in \mathbf{M}_n^+ : r(A(f^{-1}(X))^2) > 0\} \\ &= \{f(Y) \in \mathbf{M}_n^+ : r(AY^2) > 0\} = f(\widetilde{\mathcal{F}}(A)). \end{aligned}$$

So, we have the following.

Lemma 2.4.5. *If $A \in \mathbf{M}_n^+$, then $\widetilde{\mathcal{F}}(f(A)) = f(\widetilde{\mathcal{F}}(A))$.*

Corollary 2.4.6. *A matrix $X \in \mathbf{M}_n^+$ has exactly k nonzero entries if and only if $f(X)$ has exactly k nonzero entries.*

Proof. Similar to that of Corollary 2.3.5. ■

Lemma 2.4.7. *Let $f(E_{ij}) = F_{ij}$ for $1 \leq i, j \leq n$.*

(a) *If $i \neq j$, then $F_{ij} = \gamma_{ij} E_{pq}$ for some $1 \leq p, q \leq n$, with $p \neq q$ and $\gamma_{ij} > 0$.*

(b) *We have $\{F_{11}, \dots, F_{nn}\} = \{E_{11}, \dots, E_{nn}\}$.*

(c) *There is a permutation τ of $\{(i, j) : 1 \leq i, j \leq n\}$ such that $F_{ij} = \gamma_{ij} E_{\tau(i,j)}$ and $F_{ji} = \gamma_{ji} E_{\tau(i,j)}^t$, for all pairs (i, j); moreover, τ satisfies the property:*

$$\tau(i, j) = (p, q) \qquad \Longrightarrow \qquad \tau(j, i) = (q, p).$$

(d) *For $f([a_{ij}]) = [x_{ij}]$, we have $\gamma_{ij} a_{ij} = x_{\tau(i,j)}$.*

Proof. For (a), let $i \neq j$. By Corollary 2.4.6, F_{ij} has exactly one nonzero entry. But $r(F_{ij}) = r(E_{ij}) = 0$, so this nonzero entry is not on the diagonal, thus $F_{ij} = \gamma_{ij} E_{pq}$ for some $\gamma_{ij} > 0$, $p \neq q$.

For (b), again by Corollary 2.4.6 F_{ii} has one nonzero entry, and since $r(F_{ii}) = r(E_{ii}) = 1$, this nonzero entry must be on the diagonal, and it must be equal 1. So $F_{ii} = E_{pp}$ for some p. Furthermore, since $r(F_{ii}F_{kk}) = r(E_{ii}E_{kk}) = 0$ for $i \neq k$, no two F_{ii}'s can have the same nonzero position, so we get the desired result.

For (c), let $i \neq j$. By Lemma 2.4.3 (a) we have that $\widetilde{\mathcal{F}}(E_{ij}) \not\subseteq \widetilde{\mathcal{F}}(E_{ji})$ and vice versa, so by Lemma 2.3.4 we have $\widetilde{\mathcal{F}}(F_{ij}) \not\subseteq \widetilde{\mathcal{F}}(F_{ji})$. It is clear then that F_{ij} and F_{ji} do not have the same nonzero position. However, consider $A = E_{ij} + E_{ji}$. Then by our lemmas, we know that $\widetilde{\mathcal{F}}(E_{ij}) \subseteq \widetilde{\mathcal{F}}(A)$ and $\widetilde{\mathcal{F}}(E_{ji}) \subseteq \widetilde{\mathcal{F}}(A)$, so it follows that $\widetilde{\mathcal{F}}(F_{ij}) \subseteq \widetilde{\mathcal{F}}(f(A))$ and $\widetilde{\mathcal{F}}(F_{ji}) \subseteq \widetilde{\mathcal{F}}(f(A))$. By (2.4.6) $f(A)$ must have two nonzero entries. But by Lemma 2.4.3 (b), the nonzero positions of F_{ij} and F_{ji} must lie in the nonzero positions of $f(A)$, so each must occupy a distinct nonzero position of $f(A)$. (It is not possible for F_{ij} and F_{ji} to have the same nonzero position; indeed, if they did, then we would have $\widetilde{\mathcal{F}}(F_{ij}) = \widetilde{\mathcal{F}}(F_{ji})$ and

$$f(\widetilde{\mathcal{F}}(E_{ij})) = \widetilde{\mathcal{F}}(f(E_{ij})) = \widetilde{\mathcal{F}}(F_{ij}) = \widetilde{\mathcal{F}}(F_{ji}) = \widetilde{\mathcal{F}}(f(E_{ji})) = f(\widetilde{\mathcal{F}}(E_{ji})),$$

which gives a contradiction because f is bijective and $\widetilde{\mathcal{F}}(E_{ij}) \neq \widetilde{\mathcal{F}}(E_{ji})$.) Furthermore, since $r(f(A)) = r(A) = 1$, the nonzero positions of A must form a cycle, and so must be transposed of each other.

Thus if $F_{ij} = \gamma_{ij} E_{pq}$ then it must be the case that $F_{ji} = \gamma_{ji} E_{qp} = \gamma_{ji} E_{pq}^t$. By our previous considerations it is clear that each F_{ij} has a unique nonzero position (p, q) with respect to one another so we may define a bijection $\tau(i, j) = (p, q)$ accordingly, giving us the required permutation.

Finally for (d), we temporarily return to our square roots. By Corollary 2.4.6, for $i \neq j$ we have that $f(\sqrt{E_{ij}})$ has exactly two nonzero entries, so either $f(\sqrt{E_{ij}})^2 = 0$ or $f(\sqrt{E_{ij}})^2$ has exactly 1 nonzero entry (note that $f(\sqrt{E_{ij}})$ is nilpotent by Lemma 2.2.3). Since

$$r(F_{ji} f(\sqrt{E_{ij}})^2) = r(E_{ji} \sqrt{E_{ij}}^2) = 1,$$

it must be the latter case. Moreover, if $F_{ji} = \gamma_{ji} E_{\tau(j,i)}$, then it is clear that $f(\sqrt{E_{ij}})^2 = E_{\tau(i,j)}/\gamma_{ji}$. Now let $A = [a_{ij}]$ and $f(A) = [x_{ij}]$. Fix $i \neq j$, and consider $r(A\sqrt{E_{ji}}^2) = r(AE_{ji}) = a_{ij}$ and

$$r(A\sqrt{E_{ji}}^2) = r(f(A)f(\sqrt{E_{ji}})^2) = r([x_{ij}]E_{\tau(j,i)}/\gamma_{ij}) = x_{\tau(i,j)}/\gamma_{ij}.$$

Therefore, $x_{\tau(i,j)} = \gamma_{ij} a_{ij}$. In the case $i = j$, $\sqrt{E_{ii}} = E_{ii}$, so

$$a_{ii} = r(AE_{ii}) = r(AE_{ii}^2) = r([x_{ij}]F_{ii}) = r([x_{ij}]E_{\tau(i,i)}) = x_{\tau(i,i)}.$$

■

Corollary 2.4.8. *Let $\mathcal{N} = \{(i,j) : 1 \leq i,j \leq n\}$, and let $S \subseteq \mathcal{N}$, $S \neq \emptyset$. Then*

$$f(\sum_{(i,j)\in S} E_{ij}) = \sum_{(i,j)\in S} F_{ij}.$$

The proof is completely analogous to that of Corollary 2.3.7.

We note the following equalities:

$$\gamma_{ji} = 1/\gamma_{ij}, \qquad \forall \;\; i,j, \quad 1 \leq i,j \leq n. \tag{2.4.3}$$

Indeed, $f(E_{ij} + E_{ji}) = F_{ij} + F_{ji}$, so

$$\gamma_{ij}\gamma_{ji} = r(F_{ij} + F_{ji}) = r(E_{ij} + E_{ji}) = 1.$$

We are now ready to prove the implication "(1) ⇒ (4)" in Theorem 2.1.1 for the Jordan triple product.

Proof. Recall that $f : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$ has the property (2.4.1). Using Corollary 2.4.8 and Theorem 2.2.4, we see that f must satisfy either (2.2.5) or (2.2.6). For $f(E_{ij}) = Q^{-1}E_{ij}Q$, define $\hat{f}_1(A) = Qf(A)Q^{-1}$, and for $f(E_{ij}) = Q^{-1}E_{ij}^tQ$, define $\hat{f}_2(A) = \hat{f}_1(A)^t$. Then $\hat{f}_1, \hat{f}_2 : \mathbf{M}_n^+ \longrightarrow \mathbf{M}_n^+$ since $Q, f(A), Q^{-1} \in \mathbf{M}_n^+$. Also, for all $A, B \in \mathbf{M}_n^+$,

$$\begin{aligned} r(AB^2) &= r(f(A)f(B)^2) = r(Qf(A)f(B)^2Q^{-1}) = r(Qf(A)(Q^{-1}Q)f(B)^2Q^{-1}) \\ &= r((Qf(A)Q^{-1})(Qf(B)Q^{-1})^2) = r(\hat{f}_1(A)\hat{f}_1(B)^2), \end{aligned}$$

and

$$\begin{aligned} r(AB^2) &= r(\hat{f}_1(A)\hat{f}_1(B)^2) = r((\hat{f}_1(B)^t)^2\hat{f}_1(A)^t) \\ &= r(\hat{f}_2(B)^2\hat{f}_2(A)) = r(\hat{f}_2(A)\hat{f}_2(B)^2). \end{aligned}$$

Trivially, $\hat{f}_1(E_{ij}) = \hat{f}_2(E_{ij}) = E_{ij}$. Applying Lemma 2.4.7 to $\hat{f}_k$ we have $\gamma_{ij} = 1$ and $\tau(i,j) = (i,j)$. Now the proof is completed exactly as in the case of the usual product. ■

Chapter 3

Schur Multiplicative Rank Preservers

The purpose of this chapter is to obtain characterizations of Schur multiplicative maps which we then apply to characterize rank-preserving Schur multiplicative maps. The content of this chapter is based on the paper [11].

3.1 Introduction

Let $\mathbf{M}_{m,n}(\mathbb{F})$ be the set of $m \times n$ matrices over a field $\mathbb{F}$ with at least three elements. In this chapter we will shorten the notation so that $\mathbf{M}_{m,n} = \mathbf{M}_{m,n}(\mathbb{F})$. Define the *Schur product* (also known as *Hadamard product* or *entrywise product*) of $A = [a_{ij}], B = [b_{ij}] \in \mathbf{M}_{m,n}$ by $A \circ B = [a_{ij}b_{ij}]$. A map $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is *Schur multiplicative* if

$$f(A \circ B) = f(A) \circ f(B) \qquad \text{for all } A, B \in \mathbf{M}_{m,n}.$$

The study of Schur product is related to many pure and applied areas; see [20].

We first consider general Schur multiplicative maps $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$. In particular, it is shown that under some mild assumptions on the Schur multiplicative map f has the form

(†) $[a_{ij}] \mapsto \mathcal{P}[f_{ij}(a_{ij})]$, *where* $f_{ij} : \mathbb{F} \to \mathbb{F}$ *satisfies* $f_{ij}(0) = 0$ *for each* (i, j) *pair, and* $\mathcal{P}(X) \in \mathbf{M}_{m,n}$ *is obtained from* X *by permuting its entries in a fixed pattern.*

The result is then used to study Schur multiplicative maps which map rank k matrices to rank k matrices for a given value k. In particular, our results include the characterization of those Schur multiplicative maps that preserve the rank function, and those Schur multiplicative maps that map the set of singular (respectively, invertible) square matrices to itself.

3.2 Schur Multiplicative Maps

The structure of a Schur multiplicative map $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ can be quite arbitrary if one does not impose any additional assumptions on f. In general, one can define $f(A) = [f_{ij}(A)]$, where $f_{ij} : \mathbf{M}_{m,n} \to \mathbb{F}$ is any Schur multiplicative map. For example, one can define $f(A) = B$ for a fixed matrix B satisfying $B \circ B = B$; another example is to define $f(A) = J_{m,n}$ if $a_{11} \neq 0$ and $f(A) = E_{11}$ otherwise. On the other hand, if one imposes some mild conditions on a Schur multiplicative map, then its structure will be more tractable as shown in the following.

Theorem 3.2.1. *Let $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$. The following conditions are equivalent.*

(A1) *f is Schur multiplicative, $f(0_{m,n}) = 0_{m,n}$, and $f(E_{ij}) \neq 0_{m,n}$ for each (i,j) pair.*

(A2) *f is Schur multiplicative and $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$.*

(A3) *There is a mapping $\mathcal{P} : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ such that $\mathcal{P}(A)$ is obtained from A by permuting its entries in a fixed pattern, and a family of multiplicative maps $f_{ij} : \mathbb{F} \to \mathbb{F}$ satisfying $f_{ij}^{-1}[\{0\}] = \{0\}$ such that*

$$f([a_{ij}]) = \mathcal{P}\left([f_{ij}(a_{ij})]\right).$$

Proof. Note that a matrix $X \in \mathbf{M}_{m,n}$ satisfies $X \circ X = X$ if and only if all the entries of X belong to $\{0, 1\}$.

Assume that (A1) holds. Suppose there is X with nonzero (i,j) entry such that $f(X) = 0_{m,n}$. Then $f(E_{ij}) = f(E_{ij} \circ X/x_{ij}) = f(E_{ij}/x_{ij}) \circ f(X) = 0_{m,n}$, which is a contradiction. Thus, $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$. We see that (A2) holds.

Suppose (A2) holds. Consider $X \in \mathcal{B} = \{E_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Then $f(X) = f(X) \circ f(X)$. So, all entries of $f(X)$ lie in $\{0, 1\}$, and $f(X) \neq 0$ by assumption (A2). For any $X, Y \in \mathcal{B}$ with $X \neq Y$, we have $f(X) \circ f(Y) = f(0_{m,n}) =$

$0_{m,n}$. Thus, $f(X)$ and $f(Y)$ have nonzero entries in different positions. As a result, for each $X \in \mathcal{B}$, $f(X)$ has exactly one non-zero entry. Thus, $f(\mathcal{B}) = \mathcal{B}$.

We can apply a map $\mathcal{P} : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ such that $\mathcal{P}(A)$ is obtained from A by a fixed permutation of the entries of A so that $\mathcal{P}(f(E_{ij})) = E_{ij}$ for all (i,j). It remains to show that there are $f_{ij} : \mathbb{F} \to \mathbb{F}$ such that $\mathcal{P}(f(A)) = [f_{ij}(a_{ij})]$ for any $A = [a_{ij}]$.

Replace f by the map $A \mapsto \mathcal{P}^{-1}(f(A))$, where $\mathcal{P}^{-1}(\mathcal{P}(X)) = X$ for all matrices X. If we can prove the conclusion for the modified map, then the same conclusion will be valid for the original map. So, we assume that $\mathcal{P}$ is the identity map, i.e., $f(E_{ij}) = E_{ij}$ for all (i,j) pairs. Now, fix an (i,j) pair. For any $a \in \mathbb{F}$, $f(aE_{ij}) = f(aE_{ij}) \circ f(E_{ij}) = bE_{ij}$ for some $b \in \mathbb{F}$. Define $f_{ij} : \mathbb{F} \to \mathbb{F}$ such that $f(aE_{ij}) = f_{ij}(a)E_{ij}$. Since $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$, $f_{ij}(x) = 0$ if and only if $x = 0$. Also, for any $a, b \in \mathbb{F}$,

$$f_{ij}(ab)E_{ij} = f(abE_{ij}) = f(aE_{ij}) \circ f(bE_{ij}) = f_{ij}(a)f_{ij}(b)E_{ij}.$$

Suppose $A = [a_{ij}]$ and $f(A) = [b_{ij}]$. Then

$$b_{ij}E_{ij} = E_{ij} \circ f(A) = f(E_{ij} \circ A) = f_{ij}(a_{ij})E_{ij}.$$

Thus, we see that $f(A) = [f_{ij}(a_{ij})]$, and the conclusion holds.

The implication (A3) $\Rightarrow$ (A1) is clear. ■

Corollary 3.2.2. *Let $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$. The following are equivalent.*

(A4) *f is Schur multiplicative and injective.*

(A5) *Condition* (A3) *in Theorem 3.2.1 holds with the additional assumption that f_{ij} is injective for each (i,j) pair.*

Proof. Suppose f is Schur multiplicative and injective. Since $f(0_{m,n}) = f(0_{m,n}) \circ f(0_{m,n})$, all entries of $f(0_{m,n})$ lie in $\{0, 1\}$. Let S be the set of (i,j) pairs such that the (i,j) entry of $f(0_{m,n})$ equals 1. Then for any $X \in \mathbf{M}_{m,n}$, we have

$$f(0_{m,n}) = f(X \circ 0_{m,n}) = f(X) \circ f(0_{m,n}).$$

Hence the (i,j) entry of $f(X)$ equals 1 for each $(i,j) \in S$.

For $(i,j) \neq (p,q)$, we have $f(E_{ij}) \neq f(E_{pq})$ and $f(E_{ij} \circ E_{pq}) = f(0_{m,n})$. Thus, $f(E_{ij})$ and $f(E_{pq})$ cannot have a common nonzero entry at the (r,s) position if $(r,s) \notin S$. Because f is injective, $f(E_{ij}) \neq f(0_{m,n})$. Thus, every $f(E_{ij})$ has at least one nonzero entry at a position $(r,s) \notin S$. Since E_{ij} and $E_{p,q}$ cannot have nonzero entry at any (r,s) position with $(r,s) \notin S$, we need at least mn pairs of $(r,s) \notin S$ to accommodate the nonzero entries of $f(E_{ij})$. Hence, we conclude that $S = \emptyset$, i.e., $f(0_{m,n}) = 0_{m,n}$, and each $f(E_{ij})$ has exactly one nonzero entry equal to 1. So, condition (A1) of Theorem 3.2.1 holds and f has the form described in (A3). Since f is injective, for $x \neq y$ in $\mathbb{F}$ we have seen that $f(xE_{ij}) \neq f(yE_{ij})$ and hence $f_{ij}(x) \neq f_{ij}(y)$. So, f_{ij} is injective for each (i,j) pair.

The implication (A5) $\Rightarrow$ (A4) is clear. ■

Remark 3.2.3. As we will see in the subsequent discussion, in the study of preserver problems we can sometimes assume only

(A0) $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is Schur multiplicative and $f(0_{m,n}) = 0_{m,n}$,

together with some preserving property to conclude that f has the form (†) with some additional nice structure. In some problems, we believe that one can even remove the assumption that $f(0_{m,n}) = 0_{m,n}$ in (A0). On the other hand, we will see that the assumption (A1) or (A0) are indispensable in certain problems.

Note also that our result and proof are valid if $\mathbb{F}$ is replaced by an integral domain **D**.

3.3 Rank preservers

Linear maps, additive maps, and multiplicative maps on matrices mapping the set of rank-k matrices to itself have been studied by many researchers; e.g., see [2, 5, 44, 47] and their references. In this section, we characterize Schur multiplicative maps that map the set of rank-k matrices to itself. We begin with rank one preservers.

Theorem 3.3.1. *Suppose $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a Schur multiplicative map. Then $f(0_{m,n}) = 0_{m,n}$ and f maps rank one matrices to rank one matrices if and only if there exist permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$, and a multiplicative map $\tau : \mathbb{F} \to \mathbb{F}$ satisfying $\tau(\mathbb{F}^*) \subseteq \mathbb{F}^*$ such that*

(a) *f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$, or*

(b) *$m = n$ and f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]^tQ$.*

Proof. First we consider the implication ($\Leftarrow$). Note that $A \in \mathbf{M}_{m,n}$ has rank one if and only if there are $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathbb{F}$ such that $A = [a_{ij}] = [x_1, \ldots, x_m]^t[y_1, \ldots, y_n]$. Thus, for any injective multiplicative map $\tau : \mathbb{F} \to \mathbb{F}$, we have

$$[\tau(a_{ij})] = [\tau(x_i)\tau(y_j)] = [\tau(x_1), \ldots, \tau(x_m)]^t[\tau(y_1), \ldots, \tau(y_n)]$$

with rank one. By this observation, the implication ($\Leftarrow$) is clear.

Next, we consider the converse. By the given assumption, f satisfies condition (A1) in Theorem 3.2.1 and hence its conclusion. Thus, f has the form (†). Without loss of generality, we may assume that $m \leq n$. The case $n < m$ can be proved by similar arguments. Since $f(X)$ has rank one for $X = \sum_{j=1}^{n} E_{1j}$, we see that the nonzero entries of $f(X)$ lie in the same row, or in the same column if $m = n$. We may assume that the former case holds. Otherwise, replace f by a map of the form $A \mapsto f(A)^t$. Note that if we can prove the result for the modified map, the conclusion will be valid for the original map. Then there exist permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ so that $f(E_{1j}) = PE_{1j}Q$ for $j = 1, \ldots, n$. Replace f by the map $A \mapsto P^tf(A)Q^t$ so that we have $f(E_{1j}) = E_{1j}$ for $j = 1, \ldots, n$. Now, consider $f(X)$ for $X = \sum_{i=1}^{m} E_{i1}$. Since $f(E_{11}) = E_{11}$ and f maps rank one matrices to rank one matrices, we see that $f(X) = X$. There exists a permutation matrix $R \in \mathbf{M}_m$ such that $f(E_{i1}) = RE_{i1}$ for $i = 1, \ldots, m$. We may replace f by the map $A \mapsto R^tf(A)$, and assume that $f(E_{i1}) = E_{i1}$ for $i = 1, \ldots, m$. For any (i, j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$, we see that $f(E_{ij}) = E_{ij}$. Furthermore, for any $1 \leq j \leq n$ and $a \in \mathbb{F}$ the matrix $f(X)$ has rank one for $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$, we see that $f_{11}(a) = f_{1j}(a)$. Similarly, we can show that $f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbb{F}$. Finally, for any (i, j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$ with $a \in \mathbb{F}$, $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbb{F}$. Our conclusion follows. ■

The conclusion of Theorem 3.3.1 may fail if the Schur multiplicative map does not map $0_{m,n}$ to itself. For example, we can choose a fixed rank one matrix B

satisfying $B \circ B = B$ and define $f(A) = B$ for all $A \in \mathbf{M}_{m,n}$. Then f is Schur multiplicative and maps rank one matrices to rank one matrices.

Next, we show that one can get a similar conclusion for maps on matrices of the form (†) even though f_{ij} is not assumed to be multiplicative *a priori*. Note that a monomial matrix is defined to be the product of an nonsingular diagonal matrix and a permutation matrix.

Theorem 3.3.2. *Suppose $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ has the form (†). Then f maps rank one matrices to rank one matrices if and only if there exist invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ and a multiplicative map $\tau : \mathbb{F} \to \mathbb{F}$ satisfying $\tau(\mathbb{F}^*) \subseteq \mathbb{F}^*$ such that*

(a) *f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$, or*

(b) *$m = n$ and f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]^t Q$.*

Proof. The implication ($\Leftarrow$) can be verified as in the proof of Theorem 3.3.1.

We consider the converse. Assume that f has the form (†) and maps rank one matrices to rank one matrices. Without loss of generality, we may assume that $m \leq n$. Since $f(X)$ has rank one for $X = \sum_{j=1}^{n} E_{1j}$, we see that the nonzero entries of $f(X)$ lie in the same row, or in the same column if $m = n$. We may assume that the former case holds. Otherwise, replace f by a map of the form $A \mapsto f(A)^t$. Then there exist permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ so that $f(E_{1j}) = Pf_{1j}(1)E_{1j}Q$ for $j = 1, \ldots, n$. Let $D = \operatorname{diag}(f_{11}(1), f_{12}(1), \ldots, f_{1n}(1))$. Since $f(E_{1j})$ has rank 1, we see that $f_{1j}(1) \neq 0$ for $j = 1, \ldots, n$. Replace f by the map $A \mapsto P^{-1}f(A)Q^{-1}D^{-1}$ so that we have $f(E_{1j}) = E_{1j}$ for $j = 1, \ldots, n$. Now, consider $f(X)$ for $X = \sum_{i=1}^{m} E_{i1}$. Since $f(E_{11}) = E_{11}$ and f maps rank one matrices to rank one matrices, there exists an invertible monomial matrix $R \in \mathbf{M}_m$ such that $f(E_{i1}) = RE_{i1}$ for $i = 1, \ldots, m$. We may replace f by the map $A \mapsto R^{-1}f(A)$, and assume that $f(E_{i1}) = E_{i1}$ for $i = 1, \ldots, m$. For any (i,j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$, we see that $f(E_{ij}) = E_{ij}$.

Note that for any (i,j) pair and any nonzero $a \in \mathbb{F}$, $f(aE_{ij}) = f_{ij}(a)E_{ij}$ has rank one, and thus $f_{ij}(a) \neq 0$. Furthermore, for any $1 \leq j \leq n$ and any $a \in \mathbb{F}$ the matrix $f(X)$ has rank one for $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$, we see that $f_{11}(a) = f_{1j}(a)$. Similarly, we can show that $f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbb{F}$. Finally, for any (i,j) with

$i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$ with $a \in \mathbb{F}$, $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbb{F}$.

Let $f_{11} = \tau$. For any $a, b \in \mathbb{F}$, let $X = E_{11} + aE_{12} + bE_{21} + abE_{22}$. Since $f(X)$ has rank one, we see that $\tau(ab) = \tau(a)\tau(b)$. So, τ is multiplicative. ∎

Next, we characterize maps $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ of the form (†) which map the set of rank k matrices to itself for $1 < k < \min\{m, n\}$. It turns out that such maps will preserve the ranks of all matrices, and have very nice structure. The result will be used to characterize Schur multiplicative maps which preserve rank k matrices in Corollary 3.3.4

Theorem 3.3.3. *Let $1 < k < \min\{m, n\}$. Suppose $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ has the form (†). The following are equivalent.*

(a) $\text{rank}\,(f(A)) = \text{rank}\,(A)$ *for all $A \in \mathbf{M}_{m,n}$.*

(b) *f maps rank k matrices to rank k matrices.*

(c) *There are invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$, and a field monomorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that one of the following holds.*

  (c.i) *f has the form $A \mapsto P[\tau(a_{i,j})]Q$.*

  (c.ii) *$m = n$ and f has the form $A \mapsto P[\tau(a_{i,j})]^t Q$.*

Note that for rank preservers f, we have $f(0_{m,n}) = 0_{m,n}$. Thus, one may further relax the assumption that $f_{ij}(0) = 0$ for all (i, j) pairs in (†), and conclude that conditions (b) and (c) are equivalent.

Proof. The implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear. We focus on the proof of (b) $\Rightarrow$ (c). Without loss of generality, we may assume that $m \leq n$. The proof for the case $n < m$ is similar. We divide the proof into several assertions.

Assertion 1 There is a diagonal matrix $D \in \mathbf{M}_m$ and permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ such that $f(E_{jj}) = PDE_{jj}Q$ for $j = 1, \ldots, m$.

Consider $\mathcal{D} = \{E_{jj} : 1 \leq j \leq m\}$. If X is a sum of k matrices in $\mathcal{D}$, then $k = \text{rank}\,(X) = \text{rank}\,(f(X))$. So, $f(X)$ must have k nonzero entries lying on k distinct rows and k distinct columns. Thus, the m non-zero entries of $f(\sum_{j=1}^{m} E_{jj})$

lie on m different rows and m different columns. Hence, there are permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ such that $f(E_{jj}) = Pf_{jj}(1)E_{jj}Q$ for $j = 1, \ldots, m$. Let $D = \operatorname{diag}(f_{11}(1), \ldots, f_{mm}(1))$. Then we get the desired conclusion.

By Assertion 1, we may replace f by the map $A \mapsto D^{-1}P^t f(A)Q^t$ and assume that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$. *We will make this assumption in the rest of the proof.*

Assertion 2 For any (i, j) pair, $f_{ij}(\mathbb{F}^*) \subseteq \mathbb{F}^*$.

Let $a \in \mathbb{F}^*$, and let $X = aE_{ij} + \sum_{s \in S} E_{ss}$ for a subset S of $\{1, \ldots, m\} \setminus \{i, j\}$ with $k - 1$ elements. Since $f(X)$ has rank k, we see that $f_{ij}(a) \neq 0$.

Assertion 3 For any $1 \leq i < j \leq m$, we have $f(E_{ij} + E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ji}$ for some $b_{ij} \in \mathbb{F}^*$.

For simplicity, assume that $(i, j) = (1, 2)$, and $X = E_{12} + E_{21}$. If $Y = X + \sum_{j=3}^{k} E_{jj}$, then $f(Y)$ has rank k. So, $f(X) = f_{12}(1)E_{pq} + f_{21}(1)E_{rs}$ for some $p \neq q$ and $r \neq s$. If $Y = \sum_{j=1}^{k+1} E_{jj} + X$, then $f(Y)$ has rank k. Thus, $p, q, r, s, \in \{1, \ldots, k+1\}$; otherwise, the leading $(k+1) \times (k+1)$ matrix of $f(Y)$ will be invertible so that $f(Y)$ has rank larger than k. Furthermore, we must have $(p, q) = (s, r)$ and $f(X) = bE_{pq} + b^{-1}E_{qp}$ for some $b \in \mathbb{F}$ with $1 \leq p < q \leq k + 1$; otherwise, $f(Y)$ has rank larger than k. Now, for any $s \in \{3, \ldots, k+1\}$, we have $k = \operatorname{rank}(Z) = \operatorname{rank}(f(Z))$ for any $Z \in \{Y - E_{ss} - E_{11}, Y - E_{ss} - E_{22}\}$. It follows that $p, q \notin \{3, \ldots, k + 1\}$, i.e., $\{p, q\} = \{1, 2\}$. So, $f(X) = bE_{12} + b^{-1}E_{21}$ as asserted.

Assertion 4 There is an invertible diagonal matrix $D \in \mathbf{M}_m$ such that one of the following holds.

(i) $f(E_{ij}) = D^{-1}E_{ij}(D \oplus I_{n-m})$ for all $1 \leq i \leq m$ and $1 \leq j \leq m$.

(ii) $f(E_{ij}) = D^{-1}E_{ji}(D \oplus I_{n-m})$ for all $1 \leq i, j \leq m$.

By Assertion 3, $f(E_{ij} + E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ij}$ for all (i, j) pairs with $1 \leq i, j \leq m$. Let $D^{-1} = \operatorname{diag}(1, b_{21}, b_{31}, \ldots, b_{m1})$. Then $f(X) = D^{-1}X(D \oplus I_{n-m})$ for $X = E_{1j} + E_{j1}$ for $j = 2, \ldots, m$. Replace f by the map $A \mapsto Df(A)(D^{-1} \oplus I_{n-m})$. Then

$$\text{(i)'} \quad f(E_{12}) = E_{12}, \qquad \text{or} \qquad \text{(ii)'} \quad f(E_{12}) = E_{21}.$$

Assume (i)' holds. We prove that (i) holds accordingly as follows. First, consider $f(E_{1j})$ for $j = 3, \ldots, m$. Consider $A = E_{11} + E_{12} + E_{1j} + E_{22} + E_{2j} + E_{j2} + E_{jj} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \ldots, n\} \setminus \{j\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$. If $f(E_{1j}) = E_{j1}$, then $f(A)$ have rank $k+1$, which is a contradiction. Thus, $f(X) = X$ for $X \in \{E_{1j}, E_{j1}\}$.

Now, suppose $1 \notin \{i, j\}$. Let $A = E_{11}+E_{1i}+E_{1j}+E_{i1}+E_{ii}+E_{ij}+E_{jj}+\sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i, j\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A)$. If $f(E_{ij}) = b_{ij}^{-1} E_{ji}$, then $\operatorname{rank}(f(A)) = k+1$, which is a contradiction. So, $f(E_{ij}) = b_{ij}E_{ij}$. If $b_{ij} \neq 1$, consider $B = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij} + E_{j1} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i, j\}$ with $k-2$ elements. Then $\operatorname{rank}(B) = k < k+1 = \operatorname{rank}(f(B))$, which is a contradiction. So, we conclude that $f(X) = X$ for $X \in \{E_{ij}, E_{ji}\}$. Our proof of (i) is complete.

If condition (ii)' holds, we can prove (ii) by a similar argument.

Assertion 5 Suppose $m < n$. Then condition (ii) in Assertion 4 cannot hold, and there is an invertible monomial matrix $Q \in \mathbf{M}_n$ such that $f(E_{ij}) = E_{ij}Q$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$.

To prove the above assertion, note that if $r > m$ then $f(E_{1r}) = E_{pq}$ for some $q > m$ because $f(E_{ij}) = E_{ij}$ for $1 \leq i, j \leq m$. If $p \neq 1$, then for $A = E_{1r}+E_{11}+\sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{p\}$ with $k-1$ elements, we see that $f(A)$ has $k+1$ linear independent rows and thus $\operatorname{rank}(f(A)) = k+1$ and $\operatorname{rank}(A) = k$, which is a contradiction. So, there are $b_{1j} \in \mathbb{F}^*$ for $j = m+1, \ldots, n$ such that $\{f(E_{1r}) : m < r \leq n\} = \{b_{1r}E_{1r} : m < r \leq n\}$. We may assume that $f(E_{1r}) = E_{1r}$ for all $m < r \leq n$. Otherwise, replace f by a map of the form $A \mapsto f(A)Q$, where $Q \in \mathbf{M}_n$ is a monomial matrix of the form $I_m \oplus \tilde{Q}$ with $\tilde{Q} \in \mathbf{M}_{n-m}$ is an invertible monomial matrix.

To see that condition (ii) cannot hold, consider $A = E_{1,m+1} + E_{12} + E_{23} + \cdots + E_{k,k+1}$. Then there is $b \in \mathbb{F}^*$ such that $f(A) = bE_{1,m+1} + E_{21} + E_{32} + \cdots + E_{k+1,k}$ has rank $k+1$ while $\operatorname{rank}(A) = k$, which is a contradiction. So, at this point, we have $f(X) = X$ for $X = E_{ij}$ for $1 \leq i, j \leq m$ and $X \in \{E_{1r} : m < r \leq n\}$.

Now, for E_{ij} with $i > 1$ and $j > m$, consider $A = E_{1i}+E_{1j}+E_{i1}+E_{ij}+\sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i\}$ with $k-2$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we conclude that $f(E_{ij}) = E_{ij}$.

By the above discussion, we may further replace f by a map of the form $A \mapsto Pf(A)Q$ for some suitable invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ so that the resulting map satisfies

(1) $f(E_{ij}) = E_{ij}$ for all (i,j) pairs, or (2) $m = n$ and $f(E_{ij}) = E_{ji}$ for all (i,j) pairs.

Assertion 6 There is a field monomorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that $f_{ij} = \tau$ for every (i,j) pair.

First, for any $a \in \mathbb{F}$, consider $A = aE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \dots, m\}$ with $k-1$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we have $f_{11}(a) = f_{1j}(a)$. Hence $f_{1j} = f_{11}$ for $j = 2, \dots, n$. Similarly, we can show that $f_{ii} = f_{ij}$ for any $j \in \{1, \dots, n\} \setminus \{i\}$.

Next, for any $a \in \mathbb{F}$ consider $A = aE_{11} + aE_{j1} + E_{21} + E_{j2} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \dots, m\}$ with $k-1$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we have $f_{11}(a) = f_{j1}(a)$. Hence $f_{j1} = f_{11}$ for $j = 2, \dots, m$. Similarly, we can show that $f_{ii} = f_{ri}$ for any $r \in \{1, \dots, m\} \setminus \{i\}$.

By the arguments in the above two paragraphs, we conclude that there is $\tau : \mathbb{F} \to \mathbb{F}$ such that $f_{ij} = \tau$ for all (i,j) pairs.

Suppose $\tau(a) = \tau(b)$ for some $a \neq b$ in $\mathbb{F}$. Let $A = E_{11} + aE_{12} + E_{21} + bE_{22} + \sum_{j=3}^{k+1} E_{jj}$. Then $\operatorname{rank}(A) = k > k-1 = \operatorname{rank}(f(A))$, which is a contradiction. So, τ is injective. Now, let $A = E_{11} + aE_{12} + bE_{21} + abE_{22} + \sum_{j=3}^{k+1} E_{jj}$. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$ implies that $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in \mathbb{F}$.

Finally, let

$$A = E_{11} + aE_{12} + (a+b)E_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s\in S} E_{ss}$$

for some subset S of $\{4, \dots, n\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$. Since

$$f(A) = E_{11} + \tau(a)E_{12} + \tau(a+b)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s\in S} E_{ss},$$

this implies $\tau(a+b) - \tau(b) = \tau(a)$, or equivalently $\tau(a+b) = \tau(a) + \tau(b)$. Thus, τ is also additive, and the result follows. ∎

Corollary 3.3.4. *Let* $2 < k < \min\{m,n\}$ *and* $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$.

(1) *If* f *is Schur multiplicative, then* (b) *and* (c) *in Theorem* 3.3.3 *are equivalent with the additional requirement in condition* (c) *that* P *and* Q *are permutation matrices.*

(2) *If* f *is Schur multiplicative and has the form* (†) (*or satisfies any of the conditions* (A1) – (A3) *in Theorem* 3.2.1), *then conditions* (a) – (c) *in Theorem* 3.3.3 *are equivalent with the additional requirement in condition* (c) *that* P *and* Q *are permutation matrices.*

Proof. Suppose f is Schur multiplicative. Clearly, (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

If (b) holds, then condition (A1) in Theorem 3.2.1 holds, and hence f has the form (†). We can then apply Theorem 3.3.3 to get condition (c) for some invertible monomial matrices P and Q. Now, if $X \circ X = X$, i.e., X has entries in $\{0,1\}$, then so is $f(X)$. Thus, we see that P and Q can be chosen to be permutation matrices in condition (c).

If f is Schur multiplicative and has the form (†), we can apply Theorem 3.3.3 and the argument in the last paragraph to get the conclusion. ∎

The conclusion in Corollary 3.3.4 (2) is not valid if we just assume that f is Schur multiplicative and $f(0_{m,n}) = 0_{m,n}$. For instance, one can define f by $f(0_{m,n}) = 0_{m,n}$ and $f(A) = B$ for all other A, where $B \in \mathbf{M}_{m,n}$ is any rank k satisfying $B \circ B = B$. Then f maps all rank k matrices to a rank k matrix, but f does not have the structure described in Theorem 3.3.3 (c).

One can examine the proof and see that condition (b) in Theorem 3.3.3 (and also Corollary 3.3.4 can be replaced by any one of the following conditions.

(b.1) $f(A)$ *has rank at most* k *whenever* $A \in \mathbf{M}_{m,n}$ *has rank* k.

(b.2) $f(A)$ *has rank at most* k *whenever* $A \in \mathbf{M}_{m,n}$ *has rank at most* k.

In particular, the conclusion holds for those functions which map singular matrices to singular matrices when $m = n$. This can be viewed as an analog of the linear preserver result of Dieudonné [14].

Next, we consider preservers of full rank matrices.

Theorem 3.3.5. *Suppose $2 \le m \le n$ and $f : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ has the form (†). If f maps rank m matrices to rank m matrices, then there exist invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$, and maps $f_{ij} : \mathbb{F} \to \mathbb{F}$ such that $f_{ij}(\mathbb{F}^*) \subseteq \mathbb{F}^*$ for all (i, j) pairs and one of the following holds:*

(a) *f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$.*

(b) *$m = n$ and f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^t Q$.*

If one of the f_{ij} is surjective, then there is an injective multiplicative map $\tau : \mathbb{F} \to \mathbb{F}$ such that $f_{ij} = \tau$ for all (i, j) pairs; furthermore, if $m \ge 3$, then τ is a field automorphism.

Proof. We divide the proof into several assertions.

Assertion 1 There are invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ such that $f(X) = PXQ$ for $X \in \{E_{jj} : 1 \le j \le m\}$.

To prove the assertion, let $X = \sum_{j=1}^m E_{jj}$. Since $\operatorname{rank}(f(X)) = \operatorname{rank}(X) = m$, we see that $f(X)$ has nonzero entries on m distinct rows and columns. So, there are permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ such that $f(E_{jj}) = P f_{jj}(1) E_{jj} Q$ for $j = 1, \ldots, m$. We may replace f by the map $A \mapsto P^{-1} f(A) Q^{-1}$ for suitable invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ so that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$.

Assertion 2 Assume $m < n$. There are invertible monomial matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ such that $f(X) = PXQ$ for $X \in \{E_{jj} : 1 \le j \le m\} \cup \{E_{1j} : m < j \le n\}$. Moreover, $f(E_{ij}) = f_{ij}(1) P E_{ij} Q$.

By Assertion 1, we may assume that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$. For any $r > m$, consider $X = E_{1r} + \sum_{s=2}^m E_{ss}$. Assume that $f(E_{1r}) = f_{1r}(1) E_{pq}$. Since $\operatorname{rank}(f(X)) = \operatorname{rank}(X) = m$ and $f(E_{jj}) = E_{jj}$, it is impossible to have $p > 1$ or $q \le m$. It follows that $f(E_{1r}) = f_{1r}(1) E_{1q}$ for some $q > m$. Thus, we may further modify f by a map of the form $A \mapsto f(A)(I_m \oplus Q)$ for a suitable invertible monomial matrix $Q \in \mathbf{M}_{n-m}$ so that

$$f(X) = X \quad \text{for} \quad X \in \{E_{jj} : 1 \le j \le m\} \cup \{E_{1j} : m < j \le n\}. \tag{3.3.1}$$

If $n = m + 1$, consider $X = E_{i,m+1} + \sum_{j \ne i} E_{jj}$. Since $\operatorname{rank}(f(X)) = \operatorname{rank}(X) = m$, we see that $f(E_{i,m+1}) = f_{i,m+1}(1) E_{i,m+1}$ for all $i > 1$. Now, suppose $n > m + 1$.

Consider $f(E_{2j})$ for $j > m$. Let $X = E_{1r} + E_{2j} + \sum_{s=3}^{m} E_{ss}$ with $r > m$ and $r \neq j$. Assume that $f(E_{2j}) = f_{2j}(1)E_{pq}$. Since $\mathrm{rank}\,(f(X)) = \mathrm{rank}\,(X) = m$ and and condition (3.3.1) holds, we see that $p = 2$, $q \neq r$, and $q > m$. Because the argument holds for all $r > m$ with $r \neq j$, we conclude that $f(E_{2j}) = f_{2j}(1)E_{2j}$. Using the same argument, we can prove that $f(E_{ij}) = f_{ij}(1)E_{ij}$ for all $i > 1$ and $j > m$ as asserted.

Next, we turn to $f(E_{ij})$ for $1 < i \leq m$, $1 \leq j \leq m$ with $i \neq j$. The result is clear if $m = 2$. Assume $m \geq 3$ and $f(E_{ij}) = f_{ij}(1)E_{pq}$. Let $X = E_{ij} + E_{j,m+1} + \sum_{s \in S} E_{ss}$, where $S = \{1, 2, ...m\} \setminus \{i, j\}$. By the conclusion above $f(E_{j,m+1}) = f_{j,m+1}(1)E_{j,m+1}$. Since $m = \mathrm{rank}\,(f(X)) = \mathrm{rank}\,(X)$ and (3.3.1) holds, we see that $(p, q) = (i, j)$.

Assertion 3 Assume that $m = n$. Then there are invertible monomial matrices $P, Q \in \mathbf{M}_n$ such that (i) $f(E_{ij}) = Pf_{ij}(1)E_{ij}Q$ for all (i, j) pairs, or (ii) $f(E_{ij}) = Pf_{ij}(1)E_{ji}Q$ for all (i, j) pairs.

By Assertion 1, we may assume that $f(E_{jj}) = E_{jj}$ for all j. Consider $X = E_{ij} + E_{ji} + \sum_{s \notin \{i,j\}} E_{ss}$. Since $m = \mathrm{rank}\,(X) = \mathrm{rank}\,(f(X))$, we see that either

(i)' $f(E_{ij}) = f_{ij}(1)E_{ij}$ and $f(E_{ji}) = f_{ji}(1)E_{ji}$, or
(ii)' $f(E_{ij}) = f_{ij}(1)E_{ji}$ and $f(E_{ji}) = f_{ji}(1)E_{ij}$.

Assume $f(E_{12}) = f_{12}(1)E_{12}$; otherwise replace f by the map $A \mapsto f(A)^t$. We will prove that conclusion (i) holds. To this end, let $X = E_{12} + E_{2j} + E_{j1} + \sum_{s \in S} E_{ss}$, where $S = \{3, \ldots, m\} \setminus \{j\}$. Then $m = \mathrm{rank}\,(A) = \mathrm{rank}\,(f(A))$. If $f(E_{j1}) = f_{j1}(1)E_{1j}$, then $f(A)$ will only have $m - 1$ nonzero columns so $\mathrm{rank}\,(f(A)) < m$, which is a contradiction. Thus, condition (i) holds for (i, j) pairs with $i = 1$ or $j = 1$. Now, for $X = E_{ij}$ with $1 \notin \{i, j\}$ and $i \neq j$, consider $X = E_{1i} + E_{ij} + E_{j1} + \sum_{s \in S} E_{ss}$, where $S = \{2, \ldots, m\} \setminus \{i, j\}$ with $m-3$ elements. Since $m = \mathrm{rank}\,(A) = \mathrm{rank}\,(f(A))$, we see that condition (i) holds.

By Assertions 2 and 3, we get the first conclusion of the theorem, namely, f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$ or $m = n$ and f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^tQ$, We finish the proof by establishing the following.

Assertion 4 Suppose there is (p, q) such that f_{pq} is surjective. Then $f_{ij} = f_{pq}$ for each (i, j) pair, and f_{pq} is injective multiplicative. Furthermore, if $m \geq 3$ then f_{pq} is a field isomorphism.

Assume condition (a) holds. (If (b) holds, replace f by the map $A \mapsto f(A)^t$ and apply a similar argument.) We may further assume that $P = I_m$ and $Q = I_n$ in condition (a); otherwise, replace f by the map $A \mapsto P^{-1}f(A)Q^{-1}$. Moreover, we assume that $(p,q) = (1,1)$, i.e., f_{11} is a surjective map. Otherwise, we may find a pair of permutation matrices $R \in \mathbf{M}_m$ and $S \in \mathbf{M}_n$ such that $RE_{11}S = E_{pq}$, and replace the map f by the map $A \mapsto R^t f(RAS)S^t$. Furthermore, we may replace f by the map $A \mapsto f(A)/f_{22}(1)$ and assume that $f_{22}(1) = 1$. Let $D_1 = \operatorname{diag}(f_{12}(1), 1, f_{32}(1), \ldots, f_{m2}(1))$ and $D_2 = \operatorname{diag}(f_{21}(1), 1, f_{23}(1), \ldots, f_{2n}(1))$. We may replace f by the map $A \mapsto D_1^{-1}f(A)D_2^{-1}$ and assume that

$$f(X) = X \qquad \text{for} \qquad X \in \{E_{i2} : 1 \leq i \leq m\} \cup \{E_{2j} : 1 \leq j \leq n\}.$$

We claim that $f_{i1} = f_{11}$ for all $i > 1$. To see this, let $a \in \mathbb{F}$ and let $\{s_3, \ldots, s_m\} = \{1, \ldots, m\} \setminus \{1, i\}$. If $b \neq a$, then $Y = bE_{11} + aE_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m E_{s_k,k}$ has rank m and so has $f(Y) = f_{11}(b)E_{11} + f_{i1}(a)E_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m f_{s_k,k}(1)E_{s_k,k}$. It follows that $f_{11}(b) \neq f_{i1}(a)$ whenever $b \neq a$. Since f_{11} is surjective, $f_{i1}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{i1}(a)$.

Next, we show that $f_{1j} = f_{11}$ for all $j > 1$. To see this, let $a \in \mathbb{F}$ and let $\{s_3, \ldots, s_m\}$ be an $m-2$ element subset of $\{1, \ldots, n\} \setminus \{1, j\}$. If $b \neq a$, then $Y = bE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{j=3}^m E_{j,s_j}$ has rank m and so has $f(Y) = f_{11}(b)E_{11} + f_{1j}(a)E_{1j} + E_{21} + E_{2j} + \sum_{k=3}^m f_{k,s_k}(1)E_{k,s_k}$. It follows that $f_{11}(b) \neq f_{1j}(a)$ whenever $b \neq a$. Since f_{11} is surjective, $f_{1j}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{1j}(a)$.

Now, consider f_{ij} with $i, j > 1$. Let $a \in \mathbb{F}$, $\{r_3, \ldots, r_m\} = \{1, \ldots, m\} \setminus \{1, i\}$, and $\{s_3, \ldots, s_m\}$ be an $m-2$ element subset of $\{1, \ldots, n\} \setminus \{1, j\}$. If $b \neq a$, then $Z = bE_{11} + bE_{i1} + bE_{1j} + aE_{ij} + \sum_{k=3}^m E_{r_k,s_k}$ has rank m and so has $f(Z) = f_{11}(b)E_{11} + f_{11}(b)E_{i1} + f_{11}(b)E_{1j} + f_{ij}(a)E_{ij} + \sum_{k=3}^m f_{r_k,s_k}(1)E_{r_k,s_k}$. It follows that

$$f_{11}(b) \neq f_{ij}(a) \qquad \text{whenever } b \neq a. \tag{3.3.2}$$

Since f_{11} is surjective, $f_{ij}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{ij}(a)$.

At this point, we may assume that $f_{ij} = f_{11} = \tau$ for all (i,j) pairs, with $\tau(0) = 0$ and $\tau(1) = f_{11}(1) = 1$.

Now, we show that τ is multiplicative. Let $a, b \in \mathbb{F}$. If $a = 0$ or $b = 0$, then $\tau(ab) = 0 = \tau(a)\tau(b)$. If $ab \neq 0$, then for any $c \neq ab$, the matrix $X = cE_{11} + aE_{12} +$

$bE_{21}+\sum_{j=2}^m E_{jj}$ has rank m, and so is $f(X) = \tau(c)E_{11}+\tau(a)E_{12}+\tau(b)E_{21}+\sum_{j=2}^n E_{jj}$. Thus, $\tau(c) \neq \tau(a)\tau(b)$. Since τ is surjective, $\tau(a)\tau(b)$ is in the range of τ. Thus, $\tau(ab) = \tau(a)\tau(b)$. Note that for $b \neq a$, we have $\tau(b) = f_{11}(b) \neq f_{ij}(a) = \tau(a)$ by (3.3.2). Thus, τ is injective.

Finally, suppose $m \geq 3$. Let $a, b \in \mathbb{F}$. If $a = 0$ or $b = 0$, then $\tau(a + b) = \tau(a) + \tau(b)$. Suppose $ab \neq 0$. Let $c \neq a + b$. $X = E_{11} + aE_{12} + cE_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s=4}^m E_{ss}$. Then X has rank m and so is $f(X) = E_{11} + \tau(a)E_{12} + \tau(c)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s=4}^m E_{ss}$. Thus, $\tau(c) - \tau(b) \neq \tau(a)$, or equivalently $\tau(c) \neq \tau(a) + \tau(b)$. Since τ is surjective, $\tau(a) + \tau(b)$ is in the range of τ. Thus, $\tau(a + b) = \tau(a) + \tau(b)$. ■

Corollary 3.3.6. *Suppose f is Schur multiplicative and has the form (†). Then the conclusion of Theorem* 3.3.5 *holds with the additional restriction that f_{ij} is multiplicative for each (i, j) pair, P and Q are permutation matrices.*

Clearly, the conclusion of Corollary 3.3.6 holds if f is Schur multiplicative and satisfies any of the conditions (A1) – (A3) in Theorem 3.2.1. However, the conclusion is no longer valid if we just assume that f is Schur multiplicative. For instance, one can define f such that $f(0_{m,n}) = 0_{m,n}$ and $f(A) = B$ for all other A, where B is any rank m matrix satisfying $B \circ B = B$.

Chapter 4

Higher Rank Numerical Range and Radii Preservers

The purpose of this chapter is to obtain characterizations of multiplicative maps which also preserve the higher rank numerical ranges and radii. The content of this chapter is based on the paper [13].

4.1 Introduction

In the context of quantum information theory, if the quantum states are represented as matrices in $\mathbf{M}_n$, then a *quantum channel* is a trace preserving completely positive linear map $L : \mathbf{M}_n \to \mathbf{M}_n$. We can consider the operator sum representation

$$L(A) = \sum_{j=1}^{r} E_j A E_j^*, \tag{4.1.1}$$

where $E_1, \ldots, E_r \in \mathbf{M}_n$ satisfy $\sum_{j=1}^{r} E_j^* E_j = I_n$. The matrices $E_1, \ldots, E_r$ are known as the *error operators* of the quantum channel L. A subspace V of $\mathbb{C}^n$ is a *quantum error correction code* for the channel L if and only if the orthogonal projection $P \in \mathbf{M}_n$ with range space V satisfies $PE_i^*E_jP = \gamma_{ij}P$ for all $i, j \in \{1, \ldots, r\}$; for example, see [24, 25]. In this connection, for $1 \leq k < n$ researchers define the *rank-k numerical range* of $A \in \mathbf{M}_n$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k\text{ orthogonal projection } P\},$$

and the *joint rank-k numerical range* of $A_1, \ldots, A_m \in \mathbf{M}_n$ by $\Lambda_k(A_1, \ldots, A_m)$ to be the collection of complex vectors $(a_1, \ldots, a_m) \in \mathbb{C}^{1\times m}$ such that $PA_jP = a_jP$ for a rank-k orthogonal projection $P \in \mathbf{M}_n$. Evidently, there is a quantum error correction code V of dimension k for the quantum channel L described in (4.1.1) if and only if $\Lambda_k(A_1, \dots, A_m)$ is non-empty for $(A_1, \dots, A_m) = (E_1^*E_1, E_1^*E_2, \dots, E_r^*E_r)$, where $m = r(r+1)/2$. Also, it is easy to see that if $(a_1, \dots, a_m) \in \Lambda_k(A_1, \dots, A_m)$ then $a_j \in \Lambda_k(A_j)$ for $j = 1, \dots, m$.

One readily checks that $\mu \in \Lambda_k(A)$ if and only if there is an $n \times k$ matrix X such that $X^*X = I_k$ and $X^*AX = \mu I_k$. When $k = 1$, $\Lambda_k(A)$ reduces to the *classical numerical range* defined and denoted by

$$W(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1\},$$

which is a useful concept in studying matrices and operators; see [21]. Recently, interesting results have been obtained for the rank-k numerical range and the joint rank-k numerical range; see [6, 7, 8, 9, 17, 30, 31, 35, 49]. In particular, an explicit description of the rank-k numerical range of $A \in \mathbf{M}_n$ is given in [35], namely,

$$\Lambda_k(A) = \bigcap_{\xi \in [0,2\pi)} \{\mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\overline{\mu} \leq \lambda_k(e^{-i\xi}A + e^{i\xi}A^*)\}, \tag{4.1.2}$$

where $\lambda_k(X)$ is the kth largest eigenvalue of a Hermitian matrix X. For a normal matrix $A \in \mathbf{M}_n$ with eigenvalues $a_1, \dots, a_n$, we have

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \operatorname{conv}\{a_{j_1}, \dots, a_{j_{n-k+1}}\}, \tag{4.1.3}$$

where "$\operatorname{conv} S$" denotes the convex hull of the set S. In [32], complete description of $\Lambda_k(A)$ for quadratic operators A is given.

In the study of numerical range and its generalizations, researchers are interested in studying their preservers; see [5, 18, 26]. For example, a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $W(\phi(A)) = W(A)$ for all $A \in \mathbf{M}_n$ if and only if there is a unitary $U \in \mathbf{M}_n$ such that ϕ has the form

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU. \tag{4.1.4}$$

Define the *numerical radius* of $A \in \mathbf{M}_n$ by

$$w(A) = \max\{|\mu| : \mu \in W(A)\}.$$

It is known that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $w(\phi(A)) = w(A)$ for all $A \in \mathbf{M}_n$ if and only if there is $\xi \in \mathbb{C}$ with $|\xi| = 1$ and a unitary $U \in \mathbf{M}_n$ such that ϕ has the form

$$A \mapsto \xi U^* A U \qquad \text{or} \qquad A \mapsto \xi U^* A^t U. \tag{4.1.5}$$

In particular, a linear preserver of the numerical radius must be a scalar multiple of a linear preserver of the numerical range.

In [10], linear preservers of the rank-k numerical range are characterized. In particular, it is shown that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies

$$\Lambda_k(\phi(A)) = \Lambda_k(A) \qquad \text{for all } A \in \mathbf{M}_n$$

if and only if there is a unitary $U \in \mathbf{M}_n$ such that ϕ has the form (4.1.4). Define the *rank-k numerical radius* of $A \in \mathbf{M}_n$ by

$$r_k(A) = \max\{|\mu| : \mu \in \Lambda_k(A)\}.$$

It is also shown in [10] that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies

$$r_k(\phi(A)) = r_k(A) \qquad \text{for all } A \in \mathbf{M}_n$$

if and only if there is $\xi \in \mathbb{C}$ with $|\xi| = 1$ and a unitary $U \in \mathbf{M}_n$ such that ϕ has the form (4.1.5). Once again, a linear preserver of the rank-k numerical radius must be a scalar multiple of a linear preserver of the rank-k numerical range.

Let $\mathcal{S}$ be a semigroup of matrices in M_n. A map $\phi : \mathcal{S} \to \mathbf{M}_n$ is *multiplicative* if

$$\phi(AB) = \phi(A)\phi(B) \quad \text{for all } A, B \in \mathcal{S}.$$

In this chapter, we determine the structure of multiplicative preservers of the rank-k numerical range and the structure of multiplicative preservers of the rank-k numerical radius. In the context of quantum error correction, one needs to consider the rank-k numerical range of matrices of the form $A = E_i^* E_j$. Moreover, in some quantum channels such as the randomized unitary channels and the Pauli channels, the error operators $E_1, \ldots, E_r$ actually come from a certain (semi)group of matrices in $\mathbf{M}_n$; see [41]. Moreover, if the quantum states go through two channels with operator sum representations $L(A) = \sum_{j=1}^{r} E_j A E_j^*$ and $\tilde{L}(A) = \sum_{j=1}^{\tilde{r}} \tilde{E}_j A \tilde{E}_j^*$, then the combined effect will lead to the consideration of the rank-k numerical range of the matrices $\tilde{E}_p E_q E_r^* \tilde{E}_s^*$. Thus, it is natural to consider multiplicative maps $\phi : \mathcal{S} \to \mathbf{M}_n$ which preserve the rank-k numerical radius or the rank-k numerical range. In the following, we consider functions on $\mathbf{GL}_n$, $\mathbf{SL}_n$, $\mathbf{U}_n$, $\mathbf{SU}_n$, and $\mathbf{M}_n^{(m)}$ where $n \geq 3$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Here are our main theorems.

Theorem 4.1.1. *Let $n \geq 3$, $k \in \{1, \ldots, n-1\}$ and $(\mathcal{S}, \mathcal{T}) = (\boldsymbol{U}_n, \partial\mathbb{D})$, $(\boldsymbol{SU}_n, \{1\})$, $(\boldsymbol{GL}_n, \mathbb{C})$, $(\boldsymbol{SL}_n, \{1\})$, $(\mathbf{M}_n^{(m)}, \mathbb{C})$ with $m \in \{k, \ldots, n\}$. A multiplicative map $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfies $r_k(\phi(A)) = r_k(A)$ for all $A \in \mathcal{S}$ if and only if there exists a multiplicative map $f : \mathcal{T} \to \partial\mathbb{D}$ such that one of the following holds.*

(a) *$k = 1$, $\mathcal{S} \in \{\boldsymbol{SU}_n, \boldsymbol{U}_n\}$, and there is a non-zero Hermitian idempotent $P \in M_n$ such that ϕ has the form*

$$A \mapsto f(\det A)P.$$

(b) *There exists $U \in \boldsymbol{U}_n$ such that ϕ has the form*

$$A \mapsto f(\det A)U^* AU \qquad \text{or} \qquad A \mapsto f(\det A)U^* \overline{A} U.$$

Theorem 4.1.2. *Let $n \geq 3$, $k \in \{1, \ldots, n-1\}$ and $\mathcal{S} = \boldsymbol{U}_n$, $\boldsymbol{SU}_n, \boldsymbol{GL}_n, \boldsymbol{SL}_n$, or $\mathbf{M}_n^{(m)}$ with $m \in \{k, \ldots, n\}$. A multiplicative map $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfies $\Lambda_k(A) = \Lambda_k(\phi(A))$ for all $A \in \mathcal{S}$ if and only if there exists a unitary $U \in \mathbf{M}_n$ such that ϕ has the form $A \mapsto U^* AU$.*

Note that $\Lambda_k(A) = \{0\}$ if A has rank smaller than k. Thus, we assume $m \in \{k, \ldots, n\}$ if $\mathcal{S} = \mathbf{M}_n^{(m)}$ to avoid trivial consideration in the above theorems.

It is easy to deduce from Theorem 4.1.2 that an anti-multiplicative map $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfies $\Lambda_k(A) = \Lambda_k(\phi(A))$ if and only if there exists a unitary matrix U such that ϕ has the form $A \mapsto U^* A^{\mathrm{t}} U$.

It is clear that a linear preserver of the rank-k numerical range (radius) on $\mathbf{M}_n$ is either a multiplicative preserver or an anti-multiplicative preserver of the rank-k numerical range (radius).

4.2 Proof of the results for $\mathcal{S} = \mathbf{U}_n$ and $\mathbf{SU}_n$

Let $A_\tau = [\tau(a_{ij})]$. In [50] the authors define an almost homomorphism to be a map $\tau : \mathbb{D} \to \mathbb{C}$ such that it is a nonzero map, $\tau(a+b) = \tau(a) + \tau(b)$ for all $a, b \in \mathbb{D}$ with $a+b \in \mathbb{D}$, and $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in \mathbb{D}$. We make the following observation.

Lemma 4.2.1. *An almost homomorphism $g : \mathbb{D} \to \mathbb{C}$ can be extended to a field homomorphism on $\mathbb{C}$.*

Proof. Suppose $g : \mathbb{D} \to \mathbb{C}$ is an almost homomorphism. Notice that $g(1) = 1$ and it can be checked that $g(r) = r$ for all $r \in \mathbb{Q} \cap \mathbb{D}$.

For any $z \in \mathbb{C}$, there is $r \in \mathbb{Q} \cap \mathbb{D}$ such that $rz \in \mathbb{D}$. Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = r^{-1} g(rz).$$

We claim that the map h is well defined. To see this, suppose there are $r, s \in \mathbb{Q}$ such that $rz, sz \in \mathbb{D}$. Without loss of generality, we assume $|r| \leq |s|$. Then $r/s \in \mathbb{Q} \cap \mathbb{D}$ and $g(r/s) = r/s$. Thus,

$$(r/s)g(sz) = g(r/s)g(sz) = g(rz) \quad \Rightarrow \quad s^{-1}g(sz) = r^{-1}g(rz).$$

Now for any $z_1, z_2 \in \mathbb{C}$, there is $r \in \mathbb{Q} \cap \mathbb{D}$ such that $rz_1, rz_2, r(z_1 + z_2) \in \mathbb{D}$. Then

$$h(z_1+z_2) = r^{-1}g(r(z_1+z_2)) = r^{-1}g(rz_1+rz_2) = r^{-1}g(rz_1)+r^{-1}g(rz_2) = h(z_1)+h(z_2)$$

and as $r^2 z_1 z_2 = (rz_1)(rz_2) \in \mathbb{D}$,

$$h(z_1 z_2) = r^{-2}g(r^2 z_1 z_2) = r^{-2}g((rz_1)(rz_2)) = (r^{-1}g(rz_1))(r^{-1}g(rz_2)) = h(z_1)h(z_2).$$

Thus, h is a homomorphism on $\mathbb{C}$. Furthermore, we see that $h(z) = g(z)$ for all $z \in \mathbb{D}$. ∎

In consideration of this lemma, we may restate [50, Theorem 3].

Theorem 4.2.2. *Suppose $n \geq 3$. A multiplicative map $\phi : \boldsymbol{U}_n \to \mathbf{M}_n$ has one of the following forms:*

(a) *There are $S \in \boldsymbol{GL}_n$ and a multiplicative homomorphism ρ from $\partial\mathbb{D}$ to $\boldsymbol{GL}_r$ for some $r \in \{0, \ldots, n\}$ such that ϕ has the form*

$$A \mapsto S(\rho(\det A) \oplus 0_{n-r})S^{-1}.$$

(b) *There are $S \in \boldsymbol{GL}_n$, a multiplicative homomorphism f from $\partial\mathbb{D}$ to $\mathbb{C}$, and a field monomorphism τ on $\mathbb{C}$ such that ϕ has the form*

$$A \mapsto f(\det A) S A_\tau S^{-1}.$$

This result can be extended to show that multiplicative maps on $\mathbf{SU}$ are simply the restrictions of multiplicative maps on $\mathbf{U}$.

Theorem 4.2.3. *Suppose $n \geq 3$. A multiplicative map $\phi : \boldsymbol{SU}_n \to \mathbf{M}_n$ has one of the following forms:*

(a) *There are $S \in \boldsymbol{GL}_n$ and $r \in \{0, \ldots, n\}$ such that $\phi(A) = S(I_r \oplus 0_{n-r})S^{-1}$ for all $A \in \boldsymbol{SU}_n$.*

(b) *There are $S \in \boldsymbol{GL}_n$ and a field monomorphism τ on $\mathbb{C}$ such that ϕ has the form*

$$A \mapsto S A_\tau S^{-1}.$$

Proof. Let $\omega = e^{2\pi i/n}$. Since $(\phi(\omega I_n))^{n+1} = (\phi(\omega I_n))$, the minimal polynomial $p(\lambda)$ of the matrix $\phi(\omega I_n)$ is a factor of $\lambda^{n+1} - \lambda$. Thus, there exist an invertible $S \in \mathbf{M}_n$ and positive integers $p_1, \ldots, p_{r-1}$ and $n_1, \ldots, n_r$ with $n_1 + \cdots + n_r = n$ such that

$$\phi(\omega I_n) = S(\omega^{p_1} I_{n_1} \oplus \cdots \oplus \omega^{p_{r-1}} I_{n_{r-1}} \oplus 0_{n_r})S^{-1}$$

For any $A \in \mathbf{SU}_n$, $\phi(A)$ and $\phi(\omega I_n)$ commute and therefore $\phi(A)$ must have the form

$$S(A_1 \oplus \cdots \oplus A_r)S^{-1}$$

with $A_j \in \mathbf{M}_{n_j}$. We define a map $\psi : \mathbf{U}_n \to \mathbf{M}_n$ as follows. For any $\mu \in \partial\mathbb{D}$, take

$$\psi(\mu I_n) = S(\mu^{p_1} I_{n_1} \oplus \cdots \oplus \mu^{p_{r-1}} I_{n_{r-1}} \oplus 0_{n_r})S^{-1}.$$

Also for each non-scalar matrix $A \in \mathbf{U}_n$, there exists $\mu \in \partial\mathbb{D}$ such that $\mu A \in \mathbf{SU}_n$. we define

$$\psi(A) = \psi(\mu^{-1} I_n)\phi(\mu A).$$

Clearly, $\psi(\mu\nu I_n) = \psi(\mu I_n)\psi(\nu I_n)$ for all $\mu, \nu \in \partial\mathbb{D}$ and $\psi(\mu I_n)\phi(A) = \phi(A)\psi(\mu I_n)$ for all $\mu \in \partial\mathbb{D}$ and $A \in \mathbf{SU}_n$. Now suppose there are $\mu, \nu \in \partial\mathbb{D}$ such that both μA

and νA are in $\mathbf{SU}_n$. Then $\mu\nu^{-1}I_n \in \mathbf{SU}_n$ and

$$\begin{aligned}\phi(\mu^{-1}I_n)\phi(\mu A) &= \psi(\mu^{-1}I_n)\phi(\mu\nu^{-1}I_n)\phi(\nu A)\\ &= \psi(\mu^{-1}I_n)\psi(\mu\nu^{-1}I_n)\phi(\nu A) = \psi(\nu^{-1}I_n)\phi(\nu A).\end{aligned}$$

Thus, ψ is well-defined. In particular, we have $\psi(A) = \phi(A)$ for all $A \in \mathbf{SU}_n$. Now for any $A, B \in \mathbf{U}_n$, there are $\mu, \nu \in \partial\mathbb{D}$ such that $\mu A, \nu B \in \mathbf{SU}_n$. Then $\mu\nu AB \in \mathbf{SU}_n$ and

$$\begin{aligned}\psi(AB) &= \psi(\mu^{-1}\nu^{-1}I_n)\phi(\mu\nu AB) = \psi(\mu^{-1})\psi(\nu^{-1}I_n)\phi(\mu A)\phi(\nu B)\\ &= \phi(\mu^{-1})\phi(\mu A)\psi(\nu^{-1}I_n)\phi(\nu B) = \psi(A)\psi(B).\end{aligned}$$

Therefore, ψ is a multiplicative map form $\mathbf{U}_n$ to $\mathbf{M}_n$ and $\psi(A) = \phi(A)$ for all $A \in \mathbf{SU}_n$. Then the result follows from Theorem 4.2.2. ∎

Proof of Theorem 4.1.1 when $\mathcal{S} = \mathbf{U}_n$ or $\mathbf{SU}_n$.

The sufficiency condition is clear. We focus on the necessity condition.

Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ is a multiplicative map satisfying $r_k(\phi(A)) = r_k(A)$ for all $A \in \mathcal{S}$.

Case 1 Assume that $k > 1$. Then ϕ has the form (a) or (b) in Theorem 4.2.2 or 4.2.3. First, we show that a map of the form (a) in Theorem 4.2.2 or 4.2.3 cannot preserve the rank-k numerical radius. Assume that it is not true and ϕ has the form (a) and preserves the rank-k numerical radius. Consider the identity matrix I_n and the special unitary diagonal matrix $W = \operatorname{diag}(w, \ldots, w^n)$, where w is the $\frac{n(n+1)}{2}$th root of unity. Then $\Lambda_k(W)$ belongs to the interior of $\mathbb{D}$ by (4.1.3), and hence $r_k(I_k) > r_k(W)$. However, we have $\phi(I_n) = \phi(W)$ so that $r_k(\phi(I_n)) = r_k(\phi(W))$, which is a contradiction.

Suppose ϕ has the form (b) in Theorem 4.2.2 or 4.2.3, i.e., $\phi(A) = f(\det A)SA_\tau S^{-1}$ for all $A \in \mathcal{S}$. when $\mathcal{S} = \mathbf{U}_n$, or $\phi(A) = SA_\tau S^{-1}$ for all $A \in \mathbf{SU}_n$ when $\mathcal{S} = \mathbf{SU}_n$. Notice that in the former case, $f(\det A) = f(1) = 1$ for all $A \in \mathbf{SU}_n$.

Write $S = QR$ with unitary Q and upper triangular R. Now for each $\mu \in \partial\mathbb{D}$, take $X = [\mu^{1-n}] \oplus \mu I_{n-1} \in \mathbf{SU}_n$. Then

$$\phi(X) = QR\begin{bmatrix}\tau(\mu^{1-n}) & 0\\ 0 & \tau(\mu)I_{n-1}\end{bmatrix}R^{-1}Q^* = Q\begin{bmatrix}\tau(\mu^{1-n}) & *\\ 0 & \tau(\mu)I_{n-1}\end{bmatrix}Q^*.$$

Notice that when $k > 1$, $\Lambda_k(X) = \{\mu\}$ and $\Lambda_k(\phi(X)) = \{\tau(\mu)\}$. Then

$$|\tau(\mu)| = r_k(\phi(X)) = r_k(X) = 1.$$

Therefore, $|\tau(\mu)| = 1$ for all $\mu \in \partial\mathbb{D}$. As τ is an field homomorphism, it follows that $|\tau(z)| = |z|$ for all $z \in \{z \in \mathbb{D} : |z| \in \mathbb{Q}\}$. Furthermore, for any $z \in \mathbb{D}$ and $\mu \in \partial\mathbb{D}$, $\tau(\mu z) = \tau(\mu)\tau(z)$. Thus, $|\tau(z_1)| = |\tau(z_2)|$ whenever $z_1, z_2 \in \mathbb{D}$ and $|z_1| = |z_2|$.

Now fixed $z \in \mathbb{D}$ with $|z| \notin \mathbb{Q}$. We may assume that $|z| < 1/2$ as $\tau(z) = 2\tau(z/2)$. For any $\epsilon > 0$, there are $r_1, r_2 \in \mathbb{Q}$ such that $|z| - \epsilon < r_1 < |z| < r_2 < |z| + \epsilon$. First, there exist $z_1, z_2 \in \mathbb{D}$ with $|z_1| = 2r_1$ and $|z_2| = |z|$ such that $z_1 = z + z_2$. Then

$$2r_1 = |z_1| = |\tau(z_1)| = |\tau(z) + \tau(z_2)| \leq |\tau(z)| + |\tau(z_2)| = 2|\tau(z)|$$

and so $r_1 \leq |\tau(z)|$. On the other hand, there exist $z_3, z_4 \in \mathbb{D}$ with $|z_3| = |z_4| = r_2/2$ such that $z = z_3 + z_4$. Then

$$|\tau(z)| = |\tau(z_3) + \tau(z_4)| \leq |\tau(z_3)| + |\tau(z_4)| = |z_3| + |z_4| = r_2.$$

It follows that

$$|z| - \epsilon < r_1 \leq |\tau(z)| \leq r_2 < |z| + \epsilon$$

and hence $|\tau(z)| = |z|$ for all $z \in \mathbb{D}$. This implies τ is either an identity map or a conjugate map on $\mathbb{D}$. By replacing ϕ with $A \mapsto \phi(\bar{A})$, if necessary, we may assume that the former case holds.

Now write $S = UDV$ for unitary U and V and diagonal $D = \operatorname{diag}(d_1, \ldots, d_n)$ with positive diagonal entires. We claim that D is a scalar matrix. Suppose not, without loss of generality, we assume that $d_1 \neq d_2$. Let $B = \begin{bmatrix} 0 & d_1/d_2 \\ d_2/d_1 & 0 \end{bmatrix}$. Then $\Lambda_1(B)$ is an non-degenerate elliptical disk with foci 1 and -1, and hence $\Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1, -1\})$ is nonempty. Take $w \in \Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1, -1\})$ and choose distinct $w_{k+2}, \ldots, w_n \in \partial\mathbb{D} \setminus \{1, -1, w\}$ so that $-w^{n-1}w_{k+2} \cdots w_n = 1$. Let

$$X = V^* \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus wI_{k-1} \oplus W \right) V \quad \text{ with } W = \operatorname{diag}(w_{k+2}, \ldots, w_n).$$

Then $X \in \mathbf{SU}_n$. It can be easily checked that $\Lambda_k(X)$ lies in the interior of $\mathbb{D}$ and hence $r_k(X) < 1$. On the other hand,

$$\phi(X) = U\left(B \oplus wI_{k-1} \oplus W\right)U^*.$$

Then $w \in \Lambda_k(\phi(X))$ and hence $r_k(\phi(X)) \geq |w| = 1$, which is a contradiction. Therefore, S is a multiple of unitary matrix. Replace (S, S^{-1}) by $(\gamma S, (\gamma S)^{-1})$ for a suitable $\gamma > 0$, we may assume that S is unitary. Thus the necessity condition of Theorem 4.1.1 follows for $\mathcal{S} = \mathbf{SU}_n$.

In the case when $\mathcal{S} = \mathbf{U}_n$, for any $A \in \mathbf{U}_n$,

$$r_k(A) = r_k(f(\det A)SAS^{-1}) = |f(\det A)| r_k(A).$$

Thus, f is a multiplicative map on $\partial\mathbb{D}$ and the necessity condition holds for $\mathcal{S} = \mathbf{U}_n$.

Case 2 Assume that $k = 1$. Recall we use the notation $W(A)$ and $w(A)$ for the numerical range and the numerical radius.

Suppose $\mathcal{S} = \mathbf{SU}_n$. If Theorem 4.2.3 (a) holds, then $\phi(I)$ is unitarily similar to $Y = \begin{bmatrix} I_r & Y_{12} \\ 0 & 0_{n-r} \end{bmatrix}$. If Y_{12} is nonzero, we may replace Y by V^*YV for some suitable $V = V_1 \oplus V_2 \in \mathbf{U}_r \oplus \mathbf{U}_{n-r}$ and assume that the $(1, 1)$ entry of Y_{12} equal to a positive number γ. But then Y will have a principal submatrix $B = \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix}$ so that $W(B)$ is an elliptical disk with 1 as an interior point and hence $w(Y) \geq w(B) > 1$, which is a contradiction. So, Y_{12} is zero and hence $\phi(I)$ is a Hermitian idempotent. Thus, Theorem 4.1.1 (a) holds.

Next, suppose Theorem 4.2.3 (b) holds. Then for any $\mu \in \partial\mathbb{D}$ and $X = \mu I_{n-1} + \mu^{1-n}$, we have $\phi(X) = SX_\tau S^{-1}$. Denote by $r(Y)$ the spectral radius of $Y \in M_n$. Then

$$w(X) = w(\phi(X)) \geq r(\phi(X)) = \max\{|\tau(\mu)|, |\tau(\mu)|^{1-n}\}.$$

Thus, $|\mu| = |\tau(\mu)| = 1$ for all $\mu \in \partial\mathbb{D}$. Using argument similar to those in Case 1, we see that τ has the form $\mu \mapsto \mu$ or $\mu \mapsto \bar{\mu}$. Then we can show that S is unitary. Hence Theorem 4.1.1 (b) holds.

Suppose $\mathcal{S} = \mathbf{U}_n$. Consider the restriction of ϕ on $\mathbf{SU}_n$, we get the desired conclusion.

Proof of Theorem 4.1.2 when $\mathcal{S} = \mathbf{U}_n$ or $\mathbf{SU}_n$.

Let $\mathcal{S} = \mathbf{SU}_n, \mathbf{U}_n$ and $\phi : \mathcal{S} \to \mathbf{M}_n$ be a multiplicative map satisfying $\Lambda_k(\phi(A)) = \Lambda_k(A)$ for all $A \in \mathcal{S}$. Then $r_k(\phi(A)) = r_k(A)$, so by Theorem 4.1.1 ϕ is of the prescribed form. Suppose ϕ is of the form 4.1.1 (a). Then in particular $\phi(A) = \phi(B)$

and so $\Lambda_k(A) = \Lambda_k(B)$ for all $A, B \in \mathbf{SU}_n$. However, if $A = I_n, B = (-I_2) \oplus I_{n-2}$ then $\Lambda_k(A) \neq \Lambda_k(B)$. This is a contradiction, so ϕ must be of the form in 4.1.1 (b).

Suppose there exists $U \in \mathbf{U}_n$ such that $\phi(A) = f(\det A)U^*\bar{A}U$ for all $A \in \mathcal{S}$. If $k \geq 3$, choose $A = \omega_k I_k \oplus I_{n-k} \in \mathbf{SU}_n$ where $\omega_k = e^{2\pi i/k}$. Then $\Lambda_k(A) = \{(1-t) + t\omega_k : t \in [0,1]\} \neq \{\bar{z} : z \in \Lambda_k(A)\} = \Lambda_k(\bar{A}) = \Lambda_k(\phi(A))$.

If $k = 1, 2$ then choose $A = iI_2 \oplus [-1] \oplus I_{n-3} \in \mathbf{SU}_n$. Then $-i \notin \Lambda_k(A)$ but $-i \in \Lambda_k(\bar{A}) = \Lambda_k(\phi(A))$. So in either case, we have a contradiction

Finally suppose there exists $U \in \mathbf{U}_n$ such that $\phi(A) = f(\det A)U^*AU$ for all $A \in \mathcal{S}$. Then $\Lambda_k(A) = \Lambda_k(\phi(A)) = f(\det A)\Lambda_k(A)$. For any $A \in \mathbf{U}$, $\det A = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then consider $A = e^{i\theta/n}I_n$, so $\{e^{i\theta/n}\} = \Lambda_k(A) = \{f(e^{i\theta})e^{i\theta/n}\}$. Then $f(e^{i\theta}) = 1$ for all $\theta \in [0, 2\pi)$ and the result follows. The converse is clear. ■

4.3 Proof of the results for $\mathcal{S} = \mathbf{GL}_n, \mathbf{SL}_n$ or $\mathbf{M}_n^{(m)}$

The study of multiplicative maps of matrices have been studied by many authors. We have the following result.

Theorem 4.3.1. *Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ is a multiplicative map, where one of the following holds.*

(1) *$n \geq 2$ and $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n$.* (2) *$n \geq 3$ and $\mathcal{S} \in \{\boldsymbol{GL}_n, \boldsymbol{SL}_n, \mathbf{M}_n\}$.*

Then there exist $S \in \boldsymbol{GL}_n$, a multiplicative map $f : \mathbb{C} \to \mathbb{C}$, and a field endomorphism $\delta : \mathbb{C} \to \mathbb{C}$ such that ϕ has one of the following forms.

(a) $A \mapsto f(\det A)SA_\delta S^{-1}$.

(b) $A \mapsto f(\det A)S((\operatorname{adj} A)^t)_\delta S^{-1}$, *where* $\operatorname{adj} A$ *denotes the adjoint matrix of A.*

(c) $A \mapsto S(I_r \oplus \rho(\det A) \oplus 0_{n-r-s})S^{-1}$, *where $r \in \{0, \ldots, n\}$, $s \in \{0, \ldots, n-r\}$, and $\rho : \mathbb{C} \to \mathbf{M}_s$ is a multiplicative map such that $(\rho(0), \rho(1)) = (0_s, I_s)$.*

Note that if condition (1) holds, then either the maps in (a) is the zero map or $f(0) = 1$. If $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n - 1$, then the map in (b) reduces to the zero map. If $\mathcal{S} = \mathbf{M}_n^{(n-1)}$, then either the map in (b) is the zero map or $f(0) = 1$. If

$\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m \le n-1$, then the map ρ in (c) reduces to the zero map, and hence is vacuous. If $\mathcal{S} = \mathbf{SL}_n$, then the map f in (b) and (c) reduces to a constant map, and the map ρ is vacuous.

Proof. For $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m \in \{1, \ldots, n\}$, the theorem follows from [51, Theorems 1 & 2]. For $\mathcal{S} = \mathbf{SL}_n$, we can extend a multiplicative map $\phi : \mathbf{SL}_n \to \mathbf{M}_n$ to a multiplicative map $\psi : \mathbf{GL}_n \to \mathbf{M}_n$ as done in the last section. Furthermore, we can always extend a multiplicative map $\phi : \mathbf{GL}_n \to \mathbf{M}_n$ to a multiplicative map $\psi : \mathbf{M}_n \to \mathbf{M}_n$ by defining $\psi(A) = \phi(A)$ if $A \in \mathbf{GL}_n$ and let $\psi(A) = 0$ if A is singular. One can then apply the results on $\mathcal{S} = \mathbf{M}_n$ to deduce the conclusion. ∎

Proof of Theorem 4.1.1.

The sufficiency condition is clear. We focus on the necessity condition. Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ such that $r_k(\phi(A)) = r_k(A)$ for all $A \in \mathcal{S}$.

Case 1 Assume $k > 1$. By Theorem 4.3.1, ϕ has one of the form (a) – (c). Since there is $A \in \mathcal{S}_0$ such that $0 < r_k(A) = r_k(\phi(A))$, we see that ϕ is not the zero map. Thus, $f(0) = 1$ if condition (1) holds.

First, we show that ϕ cannot have the form in (c). In case (1), let $X = I_k \oplus 0_{n-k}$ and $Y = \mathrm{diag}\,(1, w, \ldots, w^{k-1}) \oplus 0_{n-k}$ such that $w = e^{2\pi i/k}$; in case (2), let $X = I_n n$ and $Y = \mathrm{diag}\,(1, w, \ldots, w^{n-1})$ such that $w = e^{2\pi i/n}$. By (4.1.3), $1 = r_k(X) > r_k(Y)$. If ϕ has the form (c), then $\phi(X) = \phi(Y)$ so that $r_k(X) = r_k(\phi(X)) = r_k(\phi(Y)) = r_k(Y)$, which is a constradiction.

Second, we show that ϕ cannot have the form in (b). If (1) holds with $m < n-1$ and ϕ has the form in (b), then ϕ is the zero map, which is impossible. Suppose $\mathcal{S}_0 = \mathbf{M}_n^{(m)}$ with $m \ge n-1$, then for $A = I_{n-1} \oplus 0$, we have $r_k(\phi(A)) = 0$ and $r_k(A) = 1$, which is a contradiction. Suppose $\mathcal{S}_0 \in \{\mathbf{GL}_n, \mathbf{SL}_n\}$, and ϕ has the form (b). Since $f(1)^p = f(1)$ for all positive integer p, we have $f(1) \in \{0, 1\}$. Since ϕ is not the zero map, we have $f(1) = 1$. Let $A = (1/2)I_{n-1} \oplus [2^{n-1}]$. Then $r_k(A) = 1/2$ and $r_k(\phi(A)) = 2$, which is a contradiction.

Now, suppose ϕ has the form (a). If (1) holds, then $f(0) = 1$. For $A_\mu = \mu I_k \oplus 0_{n-k}$ with $\mu \in \mathbb{C}$ such that $|\mu| = 1$, we have

$$r_k(A_\mu) = r_k(\phi(A_\mu)) = r_k(\delta(\mu)\phi(A_1)) = r_k(A_1) = 1.$$

Thus, $|\delta(\mu)| = 1$. Hence, δ is the identity map or the conjugation map. Next, we show that all the singular values of S are the same. If it is not true, assume that $S = UDV$ such that U, V are unitary, and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ such that $d_1/d_2 = d > 1$. Let $\mu = (d + 1/d)/2$

$$A = V^* \left(\begin{bmatrix} \mu & 1 \\ 1 & \mu \end{bmatrix} \oplus I_{k-2} \oplus 0_{n-k} \right) V.$$

Then $r_k(A) = \mu - 1 > 0$ and $r_k(\phi(A)) = 0$, which is a contradiction. If (2) holds, we can consider the restriction of ϕ on $\mathbf{SU}_n$ and conclude that S is unitary using the arguments in the last section.

Case 2 Suppose $k = 1$. If (1) holds, the result is proved in [5]. If If (2) holds, one can consider the restriction of ϕ on $\mathbf{SU}_n$ and conclude that it is of the form (a) or (b) in Theorem 4.1.1. Consider $\phi(A)$ for $A = \operatorname{diag}(1/2, 2, 1, \ldots, 1)$, we see that condition (a) cannot hold. ∎

Proof of Theorem 4.1.2.

Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ preserves the rank-k numerical range. Then it also preserves the rank-k numerical radius, and has the form described in Theorem 4.1.1. By considering the restriction to $\mathbf{SU}_n$, it is clear that $\phi(A) = f(\det A)U^*AU$. For $\mathcal{S} = \mathbf{SL}_n$, the result follows. Otherwise, suppose $z = re^{i\theta}$ with $r > 0, \theta \in [0, 2\pi)$. Then let $A = r^{1/n}e^{i\theta/n}I_n$ with $r^{1/n}$ the positive real nth root of r. Then

$$\{r^{1/n}e^{i\theta/n}\} = \Lambda_k(A) = \Lambda_k(\phi(A)) = \Lambda_k(f(z)A) = \{f(z)r^{1/n}e^{i\theta/n}\}.$$

hence $f(z) = 1$ and the result follows. The converse is clear ∎

4.4 Further Study

We believe that Theorems 4.1.1 and 4.1.2 can be expanded to cover some additional cases. The $n = 2, k = 1$ case is nontrivial and is interesting to study. The theorem still holds for the case $M_2^{(1)}$ and the range preserving condition has yielded a result for $\mathbf{SU}_2$, but the radius preserving condition only weakly restricts the map. One extension would be to characterize these maps on arbitrary subsemigroups, or subsemigroups $\mathcal{S}$ containing $\mathcal{S}_0 = \mathbf{SU}_n, \mathbf{U}_n, \mathbf{SL}_n, \mathbf{GL}_n$, or $\mathbf{M}_n^{(m)}$. Indeed, since

$\mathcal{S}_0 \subset \mathcal{S} \subset \mathbf{M}_n$ and the class of preservers ϕ has a standard form on $\mathcal{S}_0$ and on $\mathbf{M}_n$, it would seem likely that the map must also be of the standard form. It is simple to generate examples of such subsemigroups where this question is trivial: $\mathcal{S} = \mathcal{S}_0 \cup \mathbf{M}_n^{(m)}$ is a subsemigroup of $\mathbf{M}_n$, and restricting the map to both sets yields the standard form for each which clearly must be the same form for multiplicativity to hold.

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