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PRE-IMAGES OF EXTREME POINTS OF THE NUMERICAL RANGE, AND APPLICATIONS

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Abstract. We extend the pre-image representation of exposed points of the numerical range of a matrix to all extreme points. With that we characterize extreme points which are multiply generated, having at least two linearly independent pre-images, as the extreme points which are Hausdorff limits of flat boundary portions on numerical ranges of a sequence converging to the given matrix. These studies address the inverse numerical range map and the maximum-entropy inference map which are continuous functions on the numerical range except possibly at certain multiply generated extreme points. This work also allows us to describe closures of subsets of 3-by-3 matrices having the same shape of the numerical range.

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