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Investigating Computational Aspects of the Coincidence Condition for Substitutions of Pisot Type

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Contents

3

Chapter 1

Introduction

We call mappings from an alphabet $\mathcal{A} = \{a_1, a_2, ..., a_d\}$ to the set $\mathcal{A}^* = \{W = w_1 w_2 ... w_r | r \in \mathbb{N}, w_i \in \mathcal{A}\}$ of all finite ordered sequences comprised of letters in $\mathcal{A}$, *substitutions* over $\mathcal{A}$. By definition, a substitution φ maps $\mathcal{A} \to \mathcal{A}^*$, but it can be extended to map $\mathcal{A}^* \to \mathcal{A}^*$ by $\varphi(w_1 w_2 ... w_r) = \varphi(w_1)\varphi(w_2)...\varphi(w_r)$. Substitutions have many interesting qualities, and arise in the mathematical fields of geometry, combinatorics, and dynamics, and also in various fields of physical sciences (notably, the study of quasicrystals).

There is a still-unproven conjecture in the study of substitutions, known as the *Coincidence Conjecture*. The Coincidence Conjecture states that every substitution that fulfills the criteria for being of *Pisot type* (see Definition 3.2.7 achieves the following combinatorial condition: for every distinct $i, j \in \mathcal{A}$, there exist integers k, n such that $\varphi^n(i)$ and $\varphi^n(j)$ have the same kth letter, and the prefixes of length $k - 1$ of $\varphi^n(i)$ and $\varphi^n(j)$ are the same up to the reordering of the letters. This coincidence condition has important implications for the properties of the substitution, and so the Coincidence Conjecture is of particular significance in the study of substitutions. The Coincidence Conjecture has been proven for substitutions defined on two-letter alphabets, but is still open for alphabets of higher order.

This paper attempts to do several things. First, it provides a thorough survey of the background of the study of substitutions, providing a good starting reference for further study of literature in this area. It also expands on the background, providing several new results. Finally, it explores several computational aspects of the Coincidence Conjecture that have not been previously investigated, resulting in some interesting new observations.

Chapter 2

Basic Ideas and Terminology

We begin by exploring some basic ideas and terminology in the field of symbolic dynamics, and then move on to define and discuss the main object of study in this paper, substitutions.

2.1 Alphabets, Words, and Other Terminology

We begin with an overview of some essential terminology in the field of symbolic dynamics. Many of these definitions are standard in this field, and we use the notation of [3] unless indicated otherwise.

We start by introducing the notion of an *alphabet*, the basic set on which the rest of the field of symbolic dynamics is built.

Definition 2.1.1. An *alphabet* $\mathcal{A}$ is a finite set of *symbols* $\{a_1, a_2, ...a_d\}$, where each a_i is usually an integer or a letter. The order of an alphabet $|\mathcal{A}| = |\{a_1, a_2, ...a_d\}| = d$ is more commonly referred to as its *size*.

It is natural to combine letters of the alphabet into words, defined formally below, along with an assortment of other useful terms and notations.

Definition 2.1.2. A *word* is a finite ordered sequence of symbols in $\mathcal{A}$. The set of all words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. Given a word $W = w_1 w_2 ... w_r$, $w_i \in \mathcal{A}$, the *length* r of W is denoted by $|W|$. The number of occurences of the letter w_i in W is given by $|W|_{w_i}$. W_i or $[W]_i$ denotes the ith letter of W. Finally, for $i < j$, $W_{[i,j]} = W_i W_{i+1} ... W_j$ denotes the new word obtained by only considering the symbols from position i to position j in W.

We now define a natural operation on words, joining them together to create a larger word. This operation is called *concatenation*, and is formally defined as follows.

Definition 2.1.3. Define the *concatenation* of two words $V = v_1 v_2 ... v_r$ and $W = w_1 w_2 ... w_s$ by $VW = v_1 v_2 ... v_r w_1 w_2 ... w_s$. The *empty word* ϵ is the word of no letters so that $W\epsilon = \epsilon W = W$.

We next extend the notion of finite words of letters to (right) infinite sequences of letters.

Definition 2.1.4. Let $\mathcal{A}^{\mathbb{N}} = \{u = (u_n)_{n \in \mathbb{N}} = u_0 u_1 u_2 ... | u_n \in \mathcal{A}\}$. An element of $\mathcal{A}^{\mathbb{N}}$ is called a *right-infinite sequence* of symbols of $\mathcal{A}$.

We similarly define sequences indexed by $\mathbb{Z}$.

Definition 2.1.5. Let $\mathcal{A}^{\mathbb{Z}} = \{u = (u_n)_{n \in \mathbb{Z}} = ... u_{-2} u_{-1}.u_0 u_1 u_2 ... | u_n \in \mathcal{A}\}$, where the decimal indicates the divide between negative and nonnegative indices of positions of the sequence. An element of $\mathcal{A}^{\mathbb{Z}}$ is called a *bi-infinite sequence* of symbols of $\mathcal{A}$.

The choice to work with either $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ is largely based on context. It is usually sufficient to work in $\mathcal{A}^{\mathbb{Z}}$, since the set $\mathcal{A}^{\mathbb{N}}$ can be viewed as the subset of $\mathcal{A}^{\mathbb{Z}}$ with some neutral symbol occupying all the negative indices. When necessary, we will clarify the specific differences between the two sets with regards to definitions or propositions.

We often use exponents to indicate repeated blocks of symbols in words or sequences. For example, rather than write $W = 1112$, we may write $W = 1^3 2$. Instead of $W = 1212$, we may write $W = (12)^2$. We may use ∞ as an exponent to indicate countably infinite repetition in a sequence; for example, the right-infinite sequence of alternating 1 and 2 may be written $u = (12)^{\infty}$.

2.2   Substitutions: Definitions and Basic Properties

We now define the main object of study in this paper, substitutions.

Definition 2.2.1. A *substitution* φ is a mapping from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^*$ of finite words on $\mathcal{A}$. That is, for all $a_i \in \mathcal{A}$, $\varphi(a_i) = a_{i_1} a_{i_2} ... a_{i_m}$, where $a_{i_j} \in \mathcal{A}$. It can be extended to a map on $\mathcal{A}^*$ naturally by concatenation; for a word $W = w_1 w_2 ... w_r$, $\varphi(W) = \varphi(w_1)\varphi(w_2)...\varphi(w_r)$. It can be similarly extended to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$, with the decimal preserving the center position in $\mathcal{A}^{\mathbb{Z}}$; that is, for a bi-infinite sequence $u = ...u_{-2}u{-1}.u_0 u_1 u_2...$, $\varphi(u) = ...\varphi(u_{-2})\varphi(u{-1}).\varphi(u_0)\varphi(u_1)\varphi(u_2)....$

For reasons that shall become clear later, it is usually required that, for at least one letter $a_i \in \mathcal{A}$, $\varphi(a_i)$ begins with a_i and $|\varphi(a_i)| \geq 2$. We shall assume this is the case unless specified otherwise.

We now define two well-known substitutions that we shall return to many times over the course of this paper.

Definition 2.2.2. On the alphabet $\mathcal{A} = \{1, 2\}$, define the *Morse substitution* by $\varphi(1) = 12$, $\varphi(2) = 21$. Define the *Fibonacci substitution* by $\varphi(1) = 12$, $\varphi(2) = 1$.

Note that the choice of alphabet is usually obvious in the definition of a substitution, since the substitution must be defined for all letters in the alphabet. *I.e.*, by defining the Fibonacci substitution by $\varphi(1) = 12$, $\varphi(2) = 1$, it is understood that the alphabet the substitution is defined on is $\{1, 2\}$.

As with many broad categories of mathematical objects, it is useful to distinguish between substitutions of different qualities or properties. One of the simplest properties of a substitution has to do with the sizes of the images of the alphabet's letters under the substitution; other distinctions will be considered later.

Definition 2.2.3. A substitution is of *constant length* k if, for all $a_i \in \mathcal{A}$, $|\varphi(a_i)| = k$.

Remark. The Morse subsitution defined in Definition 2.2.2 ($\varphi(1) = 12$, $\varphi(2) = 21$) is of constant length 2, since $|\varphi(1)| = |\varphi(2)| = 2$.

We now formally define the concept of applying a substitution multiple times in succession.

Definition 2.2.4. A substitution can be *iterated*, defined recursively by $\varphi^n(a_i) = \varphi(\varphi^{n-1}(a_i))$, for $a_i \in \mathcal{A}$, $n \in \mathbb{N}$. In the case of $n = 0$, $\varphi^0(a_i) = a_i$.

Example 2.2.5. For the Fibonacci substitution defined in Definition 2.2.2 ($\varphi(1) = 12$, $\varphi(2) = 1$), $\varphi^2(1) = \varphi(12) = \varphi(1)\varphi(2) = 121$, and $\varphi^2(2) = \varphi(1) = 12$.

Finally, we may also consider φ^m, for $m \in \mathbb{N}$, where for each $a_i \in \mathcal{A}$, φ^m maps a_i to $\varphi^m(a_i)$.

Chapter 3

Substitutions and Matrices

Each substitution has a matrix associated with it that captures several important properties of the associated substitution. We first define these matrices, and then discuss several of their properties. Again, we use the notation of [3] unless indicated otherwise.

3.1   The Incidence Matrix

Definition 3.1.1. Let φ be a substitution defined over the alphabet $\mathcal{A} = \{a_1, ..., a_d\}$ of order d. The *incidence matrix* $A_\varphi = (a_{ij})$ is the $d \times d$ matrix with entries determined by $a_{ij} = |\varphi(a_j)|_{a_i}$.

That is, the i,jth entry of A_φ is the number of times the symbol a_i appears in the image of a_j. For the incidence matrix A_φ, we say φ is the *parent substitution* of A_φ.

Example 3.1.2. The incidence matrix of the Morse substitution defined in Definition 2.2.2 ($\varphi(1) = 12$, $\varphi(2) = 21$) is $A_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The incidence matrix of the Fibonacci substitution defined in Definition 2.2.2 ($\varphi(1) = 12$, $\varphi(2) = 1$) is $A_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Several remarks can be made about the incidence matrix. The incidence matrix provides motivation to use the specific alphabet $\mathcal{A} = \{1, 2, ..., d\}$, since then there is a more direct correspondence between the letters of the alphabet and the entries of the incidence matrix.

It is also important to note that a substitution is not uniquely determined by its incidence matrix. While the incidence matrix does capture many fundamental properties of the parent substitution (some described below), it does not capture the specific ordering of the letters. Below is a simple example showing that two different substitutions may have the same incidence matrix.

Example 3.1.3. Suppose $A_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then the parent substitution could be the Fibonacci substitution, or it could be the following substitution: $\varphi(1) = 21$, $\varphi(2) = 1$.

We may define a similar process for words in $\mathcal{A}^*$ instead of substitutions, which will map words to vectors in $\mathbb{N}^d$.

Definition 3.1.4. Let φ be a substitution defined over the alphabet $\mathcal{A} = \{a_1, ..., a_d\}$ of order d. For all $W \in \mathcal{A}^*$, define the *canonical homomorphism* (or *abelianization map*) $l : \mathcal{A}^* \to \mathbb{N}^d$ as follows:

$$l(W) = (|W|_{a_i})_{1 \leq i \leq d}.$$

3.2   Categorizing Substitutions by Incidence Matrices

Qualities of the incidence matrix are often used to categorize substitutions. With this in mind, we first define the notion of a *primitive matrix*, a standard class of matrices in linear algebra.

Definition 3.2.1. ([4]) A matrix A is *primitive* if there exists a positive integer k such that $A^k > 0$ entrywise.

Example 3.2.2. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Since $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, which is entrywise positive, A is primitive. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity 2×2 matrix. Therefore, every even power of A is the identity, and every odd power of A is A itself. Consequently, there is no power of A that is entrywise positive, so A is not primitive.

There is also a notion of primitive substitutions, defined below.

Definition 3.2.3. A substitution φ is *primitive* if there exists a positive integer k so that, for every a and b in $\mathcal{A}$, the letter a occurs in $\varphi^k(b)$; that is, $|\varphi^k(b)|_a > 0$.

Example 3.2.4. The Fibonacci substitution ($\varphi(1) = 12$, $\varphi(2) = 1$) is primitive, as $\varphi^2(1) = 121$ and $\varphi^2(2) = 12$.

Unsurprisingly, a substition is primitive exactly when its incidence matrix is primitive. This follows directly from the definitions; the kth power of the incidence matrix being entrywise positive means that the kth iterate of each letter contains every letter of alphabet.

Different types of substitutions are characterized by the eigenvalues of the substitution's incidence matrix. The eigenvalues of the incidence matrix capture some fundamental behaviors or properties of the substitution, as we will see in section 1.4. We first distinguish a type of substitution based on the determinant of its incidence matrix.

Definition 3.2.5. A substitution φ is called *unimodular* if its incidence matrix A_φ is unimodular; that is, if its determinant is ± 1.

Example 3.2.6. Example: The incidence matrix of the Fibonacci substitution is $A_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, with $\det A_\varphi = -1$; therefore, the Fibonacci substitution is unimodular.

We now define another type of substitution determined by the eigenvalues of the incidence matrix, and the one this paper is most concerned with.

Definition 3.2.7. A substitution φ is called *Pisot* (or *of Pisot type*) if there exists a dominant eigenvalue α of A_φ such that for every other eigenvalue λ of A_φ, $\alpha > 1 > |\lambda| > 0$.

Example 3.2.8. Since the eigenvalues of the incidence matrix of the Fibonacci substitution are (approximately) -0.618, 1.618, it fits the criteria for being of Pisot type.

Example 3.2.9. The eigenvalues of the incidence matrix of the Morse substitution are 0 and 2, so it is not of Pisot type.

One reason Pisot substitutions are of interest is their link to something called the Coincidence Conjecture, which will be discussed in Section 5. We will first state a few basic properties of the incidence matrices of Pisot substitutions.

Proposition 3.2.10. ([3]) *Let φ be a substitution of Pisot type. Then the characteristic polynomial p_φ of the incidence matrix A_φ is irreducible over $\mathbb{Q}$. Hence, the matrix A_φ is diagonalizable (over $\mathbb{C}$), the eigenvalues being simple. Furthermore, φ cannot be of constant length.*

We end this section with a final result linking the notions of Pisot type and primitivity.

Proposition 3.2.11. ([3]) *Any substitution of Pisot type is primitive.*

As will be explained in Section 4, the primitivity of Pisot substitutions will have several ramifications for the substitutive dynamical system associated with the substitution.

3.3   Iterative Behavior of Substitutions

First, we prove a direct link between iterating a substitution and taking its incidence matrix to a higher power.

Proposition 3.3.1. *Given a substitution φ and its incidence matrix A_φ, the incidence matrix of φ^m is A_φ^m; that is, $A_{\varphi^m} = A_\varphi{}^m$.*

Proof. Let φ be a substitution on alphabet $\mathcal{A} = \{1, ..., d\}$ of order d. We proceed by induction. For $m = 1$, the incidence matrix A_φ is found from φ as described above, and the proposition is true. Let the proposition hold for all m; that is, the incidence matrix of φ^m is A_φ^m. We must show that the incidence matrix of φ^{m+1} is A_φ^{m+1}. By Definition 3.1.1, $(A_\varphi^{m+1})_{i,j} = |\varphi^{m+1}(j)|_i$. That is, the i,jth entry of A_φ^{m+1} is the number of times i appears in $\varphi^{m+1}(j)$. Remembering that $\varphi^{m+1}(j) = \varphi(\varphi^m(j))$, to count $|\varphi^{m+1}(j)|_i$, we count the number of times a letter k appears in $\varphi(j)$, then multiply by the number of times i appears in $\varphi^m(k)$, for all $k \in \mathcal{A}$. That is,

$$|\varphi^{m+1}(j)|_i = \sum_{k=1}^{d} |\varphi(j)|_k \cdot |\varphi^m(k)|_i.$$

By our definition of the incidence matrix and the mechanics of matrix multiplication, we have

$$\sum_{k=1}^{d} |\varphi(j)|_k \cdot |\varphi^m(k)|_i = \sum_{k=1}^{d} |\varphi^m(k)|_i \cdot |\varphi(j)|_k = \sum_{k=1}^{d} (A_{\varphi}^m)_{i,k} \cdot (A_{\varphi})_{k,j} =$$

$$(A_{\varphi}^m \cdot A_{\varphi})_{i,j} = (A_{\varphi}^{m+1})_{i,j}.$$

Hence, $(A_{\varphi}^{m+1})_{i,j} = |\varphi^{m+1}(j)|_i$, and therefore the incidence matrix of φ^{m+1} is A_{φ}^{m+1}. $\square$

We now discuss some properties of incidence matrices as they are taken to higher and higher powers, using those statements to say things about their parent substitutions. We focus on the incidence matrices of primitive substitutions (Definition 3.2.3), which are themselves primitive (Definition 3.2.1). We have the following results from linear algebra:

Proposition 3.3.2. *([4]) Let an $n \times n$ matrix A be primitive. Then the spectral radius of A, $\rho(A)$, is an eigenvalue of A, and no other eigenvalue achieves $\rho(A)$. Furthermore,*

$$\lim_{m \to \infty} [\rho(A)^{-1} A]^m = L > 0,$$

where $L = xy^T$, $x, y \in \mathbb{R}^n$ so that $Ax = \rho(A)x$, $A^T y = \rho(A)y$, with $x, y > 0$, and $x^T y = 1$.

Essentially, the above proposition states that a primitive matrix divided entrywise by its spectral radius converges to a rank 1 matrix when taken to higher powers. The specific implications for incidence matrices require more discussion, however, as dividing the incidence matrix entrywise by its spectral radius destroys its necessarily integer entries. We can, however, use the above result to prove the following, more relevant proposition.

Proposition 3.3.3. *Let an $n \times n$ matrix A be primitive. Then*

$$\lim_{m \to \infty} \frac{(A^m)_{i,j}}{(A^m)_{k,l}}$$

exists for all $i, j, k, l \in \{1, 2, ..., n\}$.

Proof. We begin with

$$\lim_{m \to \infty} \frac{(A^m)_{i,j}}{(A^m)_{k,l}} = \lim_{m \to \infty} \frac{\rho(A)^{-m}(A^m)_{i,j}}{\rho(A)^{-m}(A^m)_{k,l}} = \lim_{m \to \infty} \frac{(\rho(A)^{-m}A^m)_{i,j}}{(\rho(A)^{-m}A^m)_{k,l}}.$$

Multiplying a single element of a matrix by a scalar value is equivalent to multiplying a matrix by a scalar and then considering a single element, so

$$\lim_{m\to\infty} \frac{\rho(A)^{-m}(A^m)_{i,j}}{\rho(A)^{-m}(A^m)_{k,l}} = \lim_{m\to\infty} \frac{(\rho(A)^{-m}A^m)_{i,j}}{(\rho(A)^{-m}A^m)_{k,l}} = \lim_{m\to\infty} \frac{((\rho(A)^{-1}A)^m)_{i,j}}{((\rho(A)^{-1}A)^m)_{k,l}}$$

(The last equality above is justified by rules of matrix-scalar multiplaction). Then by the rules of limits,

$$\lim_{m\to\infty} \frac{((\rho(A)^{-1}A)^m)_{i,j}}{((\rho(A)^{-1}A)^m)_{k,l}} = \frac{\lim_{m\to\infty}((\rho(A)^{-1}A)^m)_{i,j}}{\lim_{m\to\infty}((\rho(A)^{-1}A)^m)_{k,l}},$$

as long both numerator and denominator have limits, which we will show shortly. To justify the next equality, observe that considering the i,jth entry of a matrix being taken to a limit is equivalent to taking a matrix to a limit and then considering the i,jth entry, since i,j remain the same. Then

$$\frac{\lim_{m\to\infty}((\rho(A)^{-1}A)^m)_{i,j}}{\lim_{m\to\infty}((\rho(A)^{-1}A)^m)_{k,l}} = \frac{(\lim_{m\to\infty}(\rho(A)^{-1}A)^m)_{i,j}}{(\lim_{m\to\infty}(\rho(A)^{-1}A)^m)_{k,l}} = \frac{L_{i,j}}{L_{k,l}}$$

from Proposition 3.3.2, and we are done. □

If we interpret A as the incidence matrix of a primitive substitution, we then get the following result immediately:

Corollary 3.3.4. *Let φ be a primitive substitution defined on an alphabet $\mathcal{A}$. Then*

$$\lim_{m\to\infty} \frac{|\varphi^m(j)|_i}{|\varphi^m(l)|_k}$$

exists for all $i, j, k, l \in \mathcal{A}$.

That is, as a substitution is iterated, the ratios of the occurances of different letters eventually stabilizes to a constant value. Moreover, we can see from the last line of the proof of Proposition 3.3.3 that the ratio depends solely on the left and right eigenvectors of the incidence matrix. We can improve on Corollary 3.3.4 by calculating that ratio exactly.

Corollary 3.3.5. *Let φ be a primitive substitution defined on an alphabet $\mathcal{A}$ with order d. Let A_φ be the $d \times d$ incidence matrix of φ, with spectral radius $\rho(A_\varphi)$, and vectors $x = [x_1\ x_2\ \cdots\ x_d]^T$, $y = [y_1\ y_2\ \cdots\ y_d]^T$ in $\mathbb{R}^d$ so that $A_\varphi x = \rho(A_\varphi)x$, $A_\varphi{}^T y = \rho(A_\varphi)y$, with $x, y > 0$, and $x^T y = 1$. Then*

$$\lim_{m \to \infty} \frac{|\varphi^m(j)|_i}{|\varphi^m(l)|_k} = \frac{x_i y_j}{x_k y_l}$$

for all $i, j, k, l \in \mathcal{A}$.

Proof. From Proposition 3.3.3 and Corollary 3.3.4, we know

$$\lim_{m \to \infty} \frac{|\varphi^m(j)|_i}{|\varphi^m(l)|_k} = \frac{L_{i,j}}{L_{k,l}},$$

where $L = xy^T$. Computing L, we see that

$$L = xy^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_d \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_d \\ \vdots & \vdots & \ddots & \vdots \\ x_d y_1 & x_d y_2 & \cdots & x_d y_d \end{bmatrix},$$

so $L_{i,j} = x_i y_j$. Therefore,

$$\lim_{m \to \infty} \frac{|\varphi^m(j)|_i}{|\varphi^m(l)|_k} = \frac{x_i y_j}{x_k y_l}.$$

$\square$

Chapter 4

Dynamical Properties of Substitutions

One of the main uses of substitutions is to build what are called *substitutive dynamical systems*. Working towards this notion, we begin by taking a step back from the subject of substitutions to define some basic terminology regarding metric spaces and dynamical systems. In this section we use the notation found in [5] unless indicated otherwise.

4.1   Metric Spaces

Definition 4.1.1. A *metric space* (X, d) consists of a set X together with a *metric* (or *distance function*) $d : X \times X \to [0, \infty)$ such that, for all points $x, y, z \in X$,

 1. $d(x, y) = 0$ if and only if $x = y$,

 2. $d(x, y) = d(y, x)$,

 3. $d(x, z) \leq d(x, y) + d(y, z)$.

   With the proper distance function, $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ are metric spaces. We define such a metric on $\mathcal{A}^{\mathbb{Z}}$ below, and prove that then we have a metric space.

Proposition 4.1.2. ([5]) *Define* $d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \to [0, \infty)$ *as follows. For* $x, y \in \mathcal{A}^{\mathbb{Z}}$, $x = (x_i)_{i \in \mathbb{Z}}$, $y = (y_i)_{i \in \mathbb{Z}}$,

$$d(x, y) = \begin{cases} 2^{-\min\{|k|:\ k\in\mathbb{Z},\ x_k \neq y_k\}} & \textit{if } x \neq y, \\ 0 & \textit{if } x = y. \end{cases}$$

Then $(\mathcal{A}^{\mathbb{Z}}, d)$ is a metric space.

Proof. Conditions (1) and (2) are clearly satisfied by d, so to show that d is a metric, we need to show that it satisfies (3), the triangle inequality. If $d(x, y) = 2^{-k}$, then k is minimal so that $x_{[-k,k]} \neq y_{[-k,k]}$; that is, $x_{[-k+1,k-1]} = y_{[-k+1,k-1]}$. Likewise, if $d(y, z) = 2^{-l}$, then l is minimal so that $y_{[-l,l]} \neq z_{[-l,l]}$, and so $y_{[-l+1,l-1]} = z_{[-l+1,l-1]}$. Let $m = \min\{k, l\}$. Then $x_{[-m+1,m-1]} = y_{[-m+1,m-1]} = z_{[-m+1,m-1]}$, so $d(x, z) \leq 2^{-m}$.

$$d(x, z) \leq 2^{-m} \leq 2^{-k} + 2^{-l} = d(x, y) + d(y, z)$$

Hence d is a metric on $\mathcal{A}^{\mathbb{Z}}$. □

If we observe as before that $\mathcal{A}^{\mathbb{N}}$ can be viewed as a subset of $\mathcal{A}^{\mathbb{Z}}$ with a single neutral letter occupying all the negative indices, the distance function simplifies to $d(x, y) = 2^{-\min\{k:\ k\in\mathbb{N},\ x_k \neq y_k\}}$, for $x, y \in \mathcal{A}^{\mathbb{N}}$, $x \neq y$, and consequently, $\mathcal{A}^{\mathbb{N}}$ is also a metric space.

Henceforth we shall understand $(\mathcal{A}^{\mathbb{N}}, d)$ and $(\mathcal{A}^{\mathbb{Z}}, d)$ to be metric spaces with the distance d defined as above. We next define the usual notion of convergence in the setting of metric spaces.

Definition 4.1.3. Let X be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X *converges to* x in X if $d(x_n, x) \to 0$ as $n \to \infty$. In this case we write $x_n \to x$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = x$.

We use the notion of convergence to define closed sets, a common notion in analysis.

Definition 4.1.4. A *limit point* of a set X is a point that is the limit of a sequence of points in X. The *closure* of X is the union of X and all of its limit points, denoted $\overline{X}$. The set X is *closed* if $X = \overline{X}$.

We also use the notion of convergence to define the notion of a compact metric space.

Definition 4.1.5. A metric space X is *compact* if every sequence in X has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, where $n_1 < n_2 < \dots$.

The metric spaces we are concerned with, $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$, are compact.

Proposition 4.1.6. ([5]) *The metric spaces $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ are compact.*

We make one final claim on the topic of compactness, a basic result from analysis stating exactly when a subset of a compact set is also compact.

Proposition 4.1.7. ([5]) *Let X be a compact metric space. A subset $E \subseteq X$ is compact if and only if E is closed.*

4.2 Substitutive Dynamical Systems

We first introduce the terminology necessary to define a dynamical system, and then use those notions to define *substitutive dynamical systems*.

First, we define the common notion of a continuous function.

Definition 4.2.1. A function $f : X \to Y$ from one metric space to another is *continuous* if, whenever $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.

We now introduce a new function on $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$, the *shift map*. Though this paper focuses on substitutions, the shift map is a significant notion in the field of symbolic dynamics. We define it below for $\mathcal{A}^{\mathbb{Z}}$.

Definition 4.2.2. Define the *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\sigma((u_n)_{n \in \mathbb{Z}} = ...u_{-2}u_{-1}.u_0u_1u_2...) = ...u_{-1}u_{-0}.u_1u_2u_3... = (u_{n+1})_{n \in \mathbb{Z}}.$$

In $\mathcal{A}^{\mathbb{Z}}$, the shift map is particularly well-behaved.

Proposition 4.2.3. ([5]) *The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is continuous, one-to-one, and onto.*

The definition remains fundamentally the same in $\mathcal{A}^{\mathbb{N}}$.

Definition 4.2.4. Define the *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by

$$\sigma((u_n)_{n \in \mathbb{N}} = u_0u_1u_2...) = u_1u_2u_3... = (u_{n+1})_{n \in \mathbb{N}}.$$

In $\mathcal{A}^{\mathbb{N}}$, the shift map moves all the letters in a sequence to the left, discarding the first letter; in $\mathcal{A}^{\mathbb{Z}}$, the shift map moves all the letters one position to the left, respective to the central decimal. Information is lost under the action of the shift map in $\mathcal{A}^{\mathbb{N}}$, but this is not the case in $\mathcal{A}^{\mathbb{Z}}$. This offers one reason for working in $\mathcal{A}^{\mathbb{Z}}$ rather than $\mathcal{A}^{\mathbb{N}}$.

We define a final notion for the shift map, its orbit.

Definition 4.2.5. ([3]) Given a sequence $u \in \mathcal{A}^{\mathbb{Z}}$, the *orbit of u* under σ is defined by $O(u) = \{\sigma^n(u) : n \in \mathbb{Z}\}$.

We now define a general dynamical system, using the terminology we have already defined.

Definition 4.2.6. A *(discrete) dynamical system* (X, f) consists of a compact metric space X together with a continuous map $f : X \to X$.

We now return to substitutions for the final terminology necessary to define a substitutive dynamical system. We also limit ourselves to considering $\mathcal{A}^{\mathbb{N}}$ for the time being. A parallel theory and similar definitions can be constructed in $\mathcal{A}^{\mathbb{Z}}$, usually intuitively; when it is useful or necessary, we shall work through that case specifically.

We define two special types of sequences now; fixed and periodic sequences, under the action of the substitution.

Definition 4.2.7. ([3]) A *fixed point* of a substitution φ is a point $u \in \mathcal{A}^{\mathbb{N}}$ such that $\varphi(u) = u$.

Definition 4.2.8. ([3]) A *periodic point* of a substitution φ is a point $u \in \mathcal{A}^{\mathbb{N}}$ such that $\varphi^k(u) = u$ for some positive integer k.

Note that the above definitions work equally well in $\mathcal{A}^{\mathbb{Z}}$. We now state that every substitution that fulfills a specific condition will have at least one periodic point.

Proposition 4.2.9. ([3]) *Every substitution φ such that $|\varphi^n(a_i)|$ tends to infinity with n for all $a_i \in \mathcal{A}$ has at least one periodic point.*

We now give a sufficient condition for a substitution to always have a fixed point; we first present this condition in the context of $\mathcal{A}^{\mathbb{N}}$.

Proposition 4.2.10. *Every substitution φ such that there exists at least one letter $a_i \in \mathcal{A}$ so that $\varphi(a_i)$ begins with a_i and $|\varphi(a_i)| \geq 2$ has at least one fixed point associated with it, denoted u_{a_i}.*

Proof. Define u_{a_i} by $[u_{a_i}]_k = [\varphi^k(a_i)]_k$, observing that since $|\varphi(a_i)| \geq 2$, $[\varphi^k(a_i)]_k$ always exists for all $k \in \mathbb{N}$. Then $[\varphi(u_{a_i})]_k = [\varphi(\varphi^k(a_i))]_k = [\varphi^{k+1}(a_i)]_k = [\varphi^k(a_i)\varphi^k(a_{i_2})...\varphi^k(a_{i_j})]_k$. But since $|\varphi^k(a_i)| \geq k$, the kth entry of $[\varphi^k(a_i)\varphi^k(a_{i_2})...\varphi^k(a_{i_j})]_k$ is $[\varphi^k(a_i)]_k$; therefore $[\varphi(u_{a_i})]_k = [\varphi^k(a_i)]_k = [u_{a_i}]_k$, proving that u_{a_i} is a fixed point. $\square$

Example 4.2.11. For the Morse substitution ($\varphi(1) = 12$, $\varphi(2) = 21$), the images of both letters fulfill the above criteria. The fixed points generated by these letters are

$$u_1 = 1221211221121221...,$$
$$u_2 = 2112122112212112....$$

We will now develop the parallel proposition for $\mathcal{A}^{\mathbb{Z}}$ While the two sets are usually interchangeable, this is an example of where $\mathcal{A}^{\mathbb{Z}}$ requires some extra work to yield the same sort of result.

Proposition 4.2.12. *Every substitution φ such that there exists at least one letter $a_i \in \mathcal{A}$ so that $\varphi(a_i)$ begins with a_i and $|\varphi(a_i)| \geq 2$, and also such that there exists at least one letter $a_j \in \mathcal{A}$ so that $\varphi(a_j)$ ends with a_j and $|\varphi(a_j)| \geq 2$, has at least one fixed point associated with it, denoted $u_{a_j.a_i}$.*

Proof. First, given a general substitution $\varphi(a_i) = a_{i_1} a_{i_2} ... a_{i_m}$ for all $a_i \in \mathcal{A}$, define the *reverse substitution* $\varphi'(a_i) = a_{i_m} a_{i_{m-1}} ... a_{i_1}$. Then, given the above assumptions, we define $u_{a_j.a_i}$ by

$$[u_{a_j.a_i}]_p = \begin{cases} [\varphi^p(a_i)]_p & p \geq 0 \\ [\varphi'^{|p|}(a_j)]_{|p|} & p < 0 \end{cases}$$

Note that we now essentially have two right-infinite sequences; the one obtained for positive indices generated by $\varphi(a_i)$, and the one obtained for negative indices generated by $\varphi'(a_j)$. We then essentially repeat the proof for the right-infinite case; for any $p \in \mathbb{Z}$, we have

$$[\varphi(u_{a_j.a_i})]_p = \begin{cases} [\varphi(\varphi^p(a_i))]_p = [\varphi^p(a_i a_{i_2} ... a_{i_m})]_p = [\varphi^p(a_i)]_p & p \geq 0 \\ [\varphi'(\varphi'^{|p|}(a_j))]_{|p|} = [\varphi'^{|p|}(a_j a_{j-1} ... a_{j_1})]_{|p|} = [\varphi'^{|p|}(a_j)]_{|p|} & p < 0 \end{cases}$$

Hence, $[\varphi(u_{a_j.a_i})]_p = [u_{a_j.a_i}]_p$, and $u_{a_j.a_i}$ is a fixed point in $\mathcal{A}^{\mathbb{Z}}$. $\qquad \square$

Example 4.2.13. Let $\varphi(1) = 122$, $\varphi(2) = 12$. Then from the criteria above, we have the fixed point

$$u_{2.1} = ...122121212212.122121212212....$$

We define two terms related to sequences of symbols, the idea of a sequence's language, and the notion of minimality.

Definition 4.2.14. ([3]) The *language* of the sequence u, denoted by $\mathcal{L}(u)$, is the set of all words in $\mathcal{A}^*$ which occur in u.

Definition 4.2.15. ([3]) A sequence $u = (u_n)$ is *minimal* if every word ocurring in u occurs in an infinite number of positions with bounded gaps.

The notion of minimality is an important one, especially for the purpose of building substitutive dynamical systems.

Proposition 4.2.16. ([3]) *Let φ is primitive. Then for any periodic points u, v of φ, u and v are minimal and $\mathcal{L}(u) = \mathcal{L}(v)$.*

Finally, we define *substitutive dynamical systems*.

Definition 4.2.17. ([3]) Given a primitive substitution φ, the *symbolic dynamical system associated with φ*, also called the *substitutive dynamical system of φ*, is the dynamical system $X_\varphi = (\overline{O(u)}, \sigma)$, where u is any periodic point of φ.

Because φ is primitive, the above construction is well-defined: all its fixed and periodic points have the same language, and therefore all its possible substitutive dynamical systems coincide.

Much of the research in the field of symbolic dynamics has to do specifically with studying the substitutive dynamical systems that arise from certain types of substitutions.

Chapter 5

Coincidence

We now turn to the focus of this research, widely known as the Coincidence Conjecture. We start by defining the Coincidence Conjecture and related terminology, then we proceed to investigate it numerically.

5.1 The Coincidence Conjecture

We begin by defining the *coincidence condition*, a combinatorial condition on words that has far-reaching implications for the properties of the substitution and its associated dynamical system.

Definition 5.1.1. ([3]) A substitution φ on the alphabet $\mathcal{A} = \{1, 2, ..., d\}$ is said to satisfy the *coincidence condition* if for every $i, j \in \mathcal{A}$, there exist integers k, n such that $[\varphi^n(i)]_k = [\varphi^n(j)]_k$ and $(\varphi^n(i))_{[1,k-1]}$ and $(\varphi^n(j))_{[1,k-1]}$ have the same image under the abelianization map l of Definition 3.1.4.

Example 5.1.2. The Fibonacci substitution $(\varphi(1) = 12, \varphi(2) = 1)$ is immediately seen to satisfy the coincidence condition for $k = 1$, $n = 1$.

Example 5.1.3. The substitution $\varphi(1) = 112$, $\varphi(2) = 21$ does not immediately satisfy the coincidence condition. Its second iterate, however, is $\varphi^2(1) = 11211221$, $\varphi^2(2) = 21112$. They agree in the fourth position with letter 1, and the prefixes 112 and 211 are the same up to a reordering of the letters. Therefore, this substitution satisfies the coincidence condition for $k = 4$, $n = 2$.

The coincidence condition was developed and studied because of its link to the properties of substitutive dynamical systems. In particular, satisfying the coincidence condition is equivalent to the dynamical system having what is known as a *discrete spectrum.*

Having discrete spectrum is a property from spectral theory; in brief, a dynamical system has discrete spectrum if the operator associated with the system has at most a countable number of eigenvalues. This paper does not delve into the theory of spectral analysis enough to fully explain or justify the above notions; for more information, see [6]. It turns out that a substitution satisfying the coincidence condition is equivalent to its substitutive dynamical system having discrete spectrum.

Proposition 5.1.4. ([1]) *A substitution φ satisfies the coincidence condition if and only if the substitutive dynamical system associated with φ has discrete spectrum.*

It is valuable to know what sorts of substitutions always satisfy the coincidence condition, leading to the following *Coincidence Conjecture.*

Conjecture 5.1.5 (The Coincidence Conjecture)**.** ([3]) *Any substitution of Pisot type satisfies the coincidence condition.*

Example 5.1.6. Both previous examples are substitutions of Pisot type. We already showed that the Fibonacci substitution is of Pisot type in Example 3.2.8; the incidence matrix of the second substitution ($\varphi(1) = 112$, $\varphi(2) = 21$) is $A_\varphi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, which has eigenvalues approximately 0.3821, 2.6180. Therefore, φ is Pisot.

As mentioned above, the Coincidence Conjecture is still open, though there is a sizable body of evidence in its support. Currently, the strongest result relating to the Coincidence Conjecture is as follows:

Theorem 5.1.7. (Barge and Diamond, [1]) *Let φ be a Pisot substitution on an alphabet $\mathcal{A} = \{1, 2, ..., d\}$. There are distinct letters $i, j \in \mathcal{A}$ for which there are integers k, n such that $\varphi^n(i)$ and $\varphi^n(j)$ have the same kth letter, and the prefixes of length $k-1$ of $\varphi^n(i)$ and $\varphi^n(j)$ have the same image under the abelianization map.*

The following corollary is a direct result of the above theorem:

Corollary 5.1.8. *Any Pisot substitution over two letters satisfies the Coincidence Conjecture.*

5.2 Computing Coincidence

5.2.1 Generating and Checking Substitutions for Coincidence

We now investigate several issues related to the idea of computing coincidence; that is, determining computationally whether a given substitution satisfies the coincidence condition.

It is relatively straightforward to turn Definition 5.1.1 into an algorithm for checking coincidence; fix $i, j \in \mathcal{A}$, then proceed through $\varphi^n(i)$ and $\varphi^n(j)$ for $n = 1, 2, ...$, looking for some k to satisfy $[\varphi^n(i)]_k = [\varphi^n(j)]_k$ and $l((\varphi^n(i))_{[1,k-1]}) = l((\varphi^n(j))_{[1,k-1]})$. When those conditions are met, continue for all other i, j pairs in $\mathcal{A}$. If this condition is met for all such pairs, then the substitution satisfies the coincidence condition. This method has several serious drawbacks, however.

First, it is inefficient. For each iterate of the $\varphi(i)$ and $\varphi(j)$, one must check strings of symbols for equality in the same position, and then check the prefixes for equality up to a reordering of their letters. If the coincidence condition is then confirmed, then all is well, but if not, that check must be made again for the next letter in the given iterate, and so on until that iterate has been completely checked. The computational process of checking for coincidence once is not insignificant for a single iterate. Having to repeat this check many times, especially as the size of the words to be examined grows rapidly under iteration, quickly becomes infeasible from a computational standpoint, especially when coincidence is computed for substitutions defined on alphabets of order greater than 2. Second, there is also the danger that this process may not terminate. If coincidence is not achieved, then the algorithm will continue *ad infinitum*. There seems to be no way to guarantee that coincidence will or will not be achieved in a finite number of steps determined by some quality of the substitution.

That said, for the substitutions investigated in this research, coincidence is achieved fairly rapidly, even for higher-order alphabets ($n \leq 5$ in all tested cases). Therefore a "brute force" approach obtained directly from Definition 5.1.1 is sufficient for the purposes of this research. More advanced algorithms approach the issue of computing coincidence from entirely different angles, and are able to avoid the above pitfalls (see [2] for more information). These algorithms require significant additional theory and context to justify and explain, and we choose to not discuss them further in this paper.

Given a method of testing for coincidence, a possible approach is then to generate substitutions and test them for coincidence. While attempting to verify the Coincidence Conjecture by an exhaustive search of all Pisot substitutions is guaranteed to prove insufficient if the Coincidence Conjecture is indeed true, such a search would add to the evidence supporting the truth of the Coincidence Conjecture. Moreover, such a search could yield a counterexample, however unlikely that case might be, which would also definitively disprove the Coincidence Conjecture.

To that end, we investigated algorithms that generate substitutions. While it is relatively easy to create an algorithm that generates all substitutions on x letters with images up to length y, it is more difficult to create an algorithm that generates all *Pisot* substitutions, that does not merely generate a substitution and then check it for Pisot type. To help develop such an algorithm, we first define the notion of a *reordering* of a substitution.

Definition 5.2.1. Let φ be a substitution on alphabet $\mathcal{A}$. A *reordering of* φ, $\tilde{\varphi}$, is a substitution such that $l(\varphi(a_i)) = l(\tilde{\varphi}(a_i))$ for all $a_i \in \mathcal{A}$ (where l is the abelianization map of Definition 3.1.4).

With the above notion, we submit the following conjecture, and provide an example to demonstrate and motivate the claim.

Conjecture 5.2.2. *Let φ be a substitution of Pisot type that satisfies the coincidence condition. Then every reordering of φ also satisfies the coincidence condition.*

Example 5.2.3. Let φ be the Fibonacci substitution, $\varphi(1) = 12$, $\varphi(2) = 1$. Then $\tilde{\varphi}$ defined by $\tilde{\varphi}(1) = 21$, $\tilde{\varphi}(2) = 1$ is a reordering of φ. $\tilde{\varphi}$ satisfies the coincidence condition for $k = 3, n = 3$.

Note, however, that the above proposition relies on the initial assumption that φ is Pisot, as shown by the example below.

Example 5.2.4. The Morse substitution $\varphi(1) = 12$, $\varphi(2) = 21$ is not Pisot (see Example 3.2.9. The Morse substitution also does not satisfy the coincidence condition; where a 1 appears in $\varphi^n(1)$, a 2 appears in $\varphi^n(2)$, and vice versa. The reordering $\tilde{\varphi}(1) = 12$, $\tilde{\varphi}(2) = 12$ does satisfy the coincidence condition for $k = 1, n = 1$.

Given a substitution of Pisot type, every reordering will also be of Pisot type (since Pisot type relies only on the incidence matrix, which ignores the

order of letters in words), so the Coincidence Conjecture implies the above conjecture. In the absence of a proof of the Coincidence Conjecture, however, Conjecture 5.2.2 could have significant value, both from a computational and theoretical standpoint.

Computationally, the conjecture would reduce the number of Pisot substitutions that would have to be generated, and allow them to be generated and considered in a somewhat canonical form (*e.g.*, fix an order on the alphabet and order the letters in images of φ in a non-decreasing manner). Similarly, from a theoretical angle, researchers could assume without loss of generality that Pisot substitutions are of a convenient ordering. Unfortunately, we were unable to obtain a proof of the above conjecture.

5.2.2   Multiple Coincidence Parameters

Recall the coincidence condition, Definition 5.1.1: a substitution φ on the alphabet $\mathcal{A} = \{1, 2, ..., d\}$ is said to satisfy the *coincidence condition* if for every $i, j \in \mathcal{A}$, there exist integers k, n such that $\varphi^n(i)$ and $\varphi^n(j)$ have the same kth letter, and the prefixes of length $k - 1$ of $\varphi^n(i)$ and $\varphi^n(j)$ have the same image under the abelianization map. The coincidence condition is satisfied only if such k, n exist for each $i, j \in \mathcal{A}$; it does not make any other demands on the substitution. It could be the case that coincidence is satisfied for many such k, n parameters, and even that for a given n, there might be several k that satisfy the coincidence condition. We first define the notions of a *pairwise coincidence condition* and *coincidence parameters*.

Definition 5.2.5. Given a substitution φ, we say that the letters $i, j \in \mathcal{A}$ satisfy the *pairwise coincidence condition* if there exist k, n such that $[\varphi^n(i)]_k = [\varphi^n(j)]_k$ and $(\varphi^n(i))_{[1,k-1]}$ and $(\varphi^n(j))_{[1,k-1]}$ have the same image under the abelianization map. For any k, n that satisfy the above conditions for i, j, we say that (k, n) is a *coincidence parameter pair* for i, j, denoted $(k, n)_{i,j}$.

We can rewrite the coincidence condition in terms of the pairwise coincidence condition by stating that if the pairwise coincidence condition is satisfied for all $i, j \in \mathcal{A}$, the substitution satisfies the coincidence condition. Next, we turn our attention to the occurance of coincidence as a substitution is iterated.

Proposition 5.2.6. *Let φ be a substitution that satisfies the pairwise coincidence condition for parameter pair $(k, n)_{i,j}$ associated with $i, j \in \mathcal{A}$. Then there exists k' so that $(k', n+1)_{i,j}$ is also a coincidence parameter for φ.*

Proof. First, if $(\varphi^n(i))_1 = (\varphi^n(j))_1$, then $\varphi((\varphi^n(i))_1) = \varphi((\varphi^n(j))_1)$, and so $(\varphi^{n+1}(i))_1 = (\varphi^{n+1}(j))_1$. Then $(1, n+1)_{i,j}$ is a coincidence parameter pair for φ.

Otherwise, by assumption we know $(\varphi^n(i))_{[1,k-1]}$ and $(\varphi^n(j))_{[1,k-1]}$ are the same up to a reordering of the letters. Let $W = w_1 w_2 ... w_r$ be a reordering of $(\varphi^n(i))_{[1,k-1]}$. Then $\varphi(W) = \varphi(w_1)\varphi(w_2)...\varphi(w_r)$ is a reordering of $\varphi((\varphi^n(i))_{[1,k-1]})$, and also $\varphi((\varphi^n(j))_{[1,k-1]})$. Rewriting, we have $(\varphi^{n+1}(i))_{[1,|W|]}$ is the same as $(\varphi^{n+1}(j))_{[1,|W|]}$ up to a reordering of the letters.

We also know that $(\varphi^n(i))_k = (\varphi^n(j))_k$. Then $\varphi((\varphi^n(i))_k) = \varphi((\varphi^n(j))_k)$, and in particular, $(\varphi((\varphi^n(i))_k))_1 = (\varphi((\varphi^n(j))_k))_1$. Since the prefix of $(\varphi^n(i))_k$ is $(\varphi^n(i))_{[1,k-1]}$, the prefix of $(\varphi((\varphi^n(i))_k))_1$ must be $(\varphi^{n+1}(i))_{[1,|W|]}$, where W is determined as above, and similarly for j. So this equality happens at position $k' = |W| + 1$, and $[\varphi^{n+1}(i)]_{k'} = [\varphi^{n+1}(j)]_{k'}$.

Therefore we have $(\varphi^{n+1}(i))_{[1,k'-1]}$ and $(\varphi^{n+1}(j))_{[1,k'-1]}$ the same up to the reordering of the letters, and $[\varphi^{n+1}(i)]_{k'} = [\varphi^{n+1}(j)]_{k'}$. Therefore $(k', n+1)_{i,j}$ is a coincidence parameter for φ. $\qquad\square$

Corollary 5.2.7. *Let φ be a substitution that satisfies the pairwise coincidence condition for parameter $(k, n)_{i,j}$. Then for each $n' > n$, there exists k' so that $(k', n')_{i,j}$ is a coincidence parameter of φ.*

Proof. Follows directly from induction on the previous theorem, when we let $(k', n+1)$ be our new coincidence parameter. $\qquad\square$

We know that once a substitution satisfies the pairwise coincidence condition for letters i, j, all further iterates of that substitution also satisfy the pairwise coincidence condition within that iterate. So we may turn our attention to investigating the different sorts of coincidence parameters $(k, n)_{i,j}$ that occur when we fix n. In particular, we chose to investigate the minimal k and the maximal k that occur in a given iterate n. We use a MATLAB program to plot $\min k$ and $\max k$ against n for a given substitution φ. If the pairwise coincidence condition is not satisfied within a given iterate n, we let $k = 0$. Some results are below.

The results for the Fibonacci substitution are seen in Figure 1. We see that $\min k$ remains constant as n increases, which is no surprise, as a quick
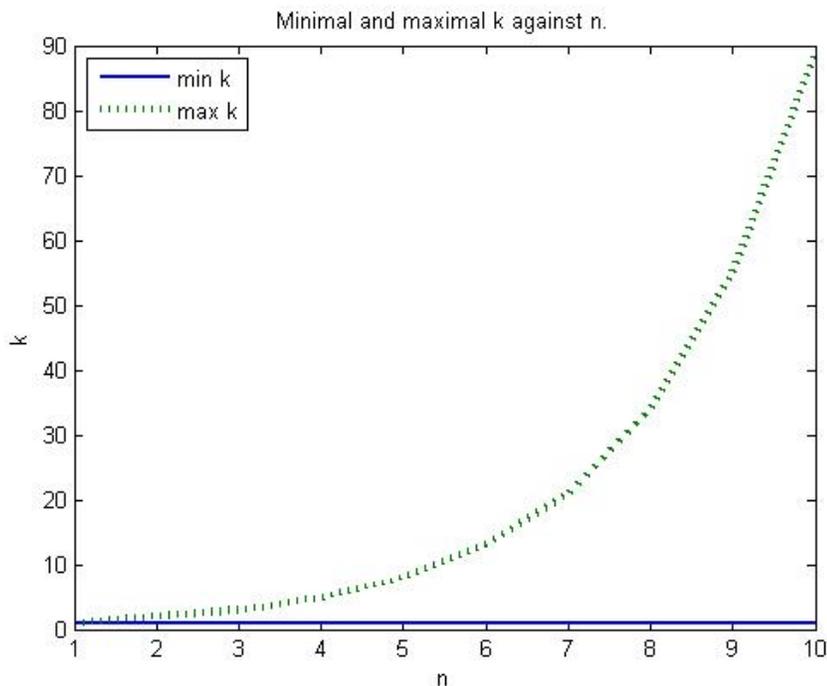
Figure 5.1: $\min k$ and $\max k$ versus n for the substitution $\varphi(1) = 12$, $\varphi(2) = 1$.

look reveals that both $\varphi^n(1)$ and $\varphi^n(2)$ will always begin with 1. The $\max k$ increases rapidly, however.

We observe the same sort of behavior for the substitution $\varphi(1) = 112$, $\varphi(2) = 21$ in Figure 2, with the exception that $\max k$ grows more rapidly. This can be attributed to the added letters in the images of the substitution, which contribute to a faster growth rate of the lengths of words as the substitution is iterated.

Since the maximal value of k grows rapidly as n increases, we then chose to investigate that growth rate relative to the growth rate of the lengths of the relevant words. That is, given distinct $i, j \in \mathcal{A}$, investigate the ratio of $\max k$ to $|\varphi^n(i)|$ and $|\varphi^n(j)|$ as n increases. We did exactly this via MATLAB. Figure 3 shows the results for the Fibonacci substitution.

We see the ratio of $\max k$ to $|\varphi^n(2)|$ stays at 1 as n increases. We also see that the ratio of $\max k$ to $|\varphi^n(1)|$ stabilizes to a constant value as n increases.
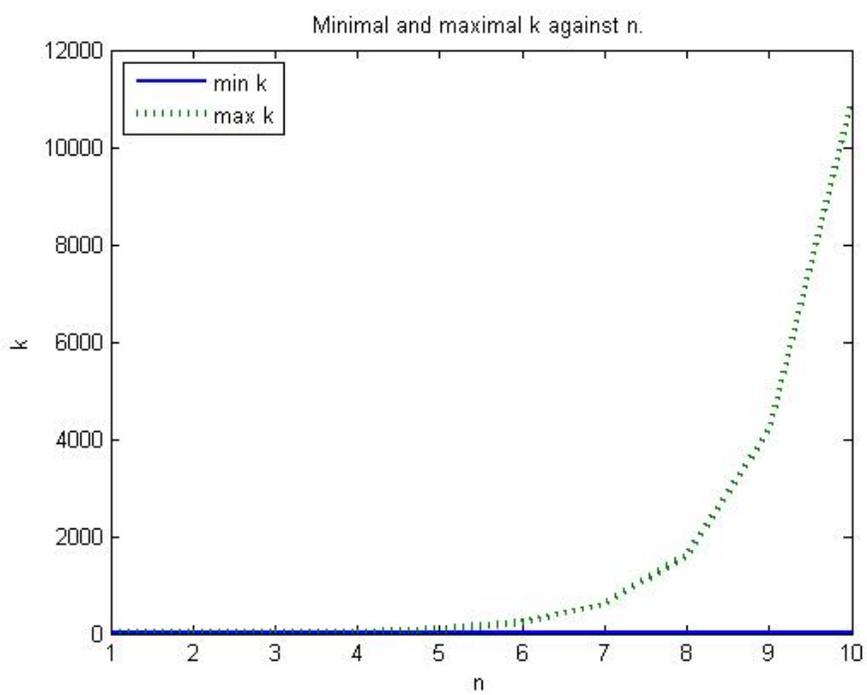
Figure 5.2: $\min k$ and $\max k$ versus n for the substitution $\varphi(1) = 122$, $\varphi(2) = 21$.
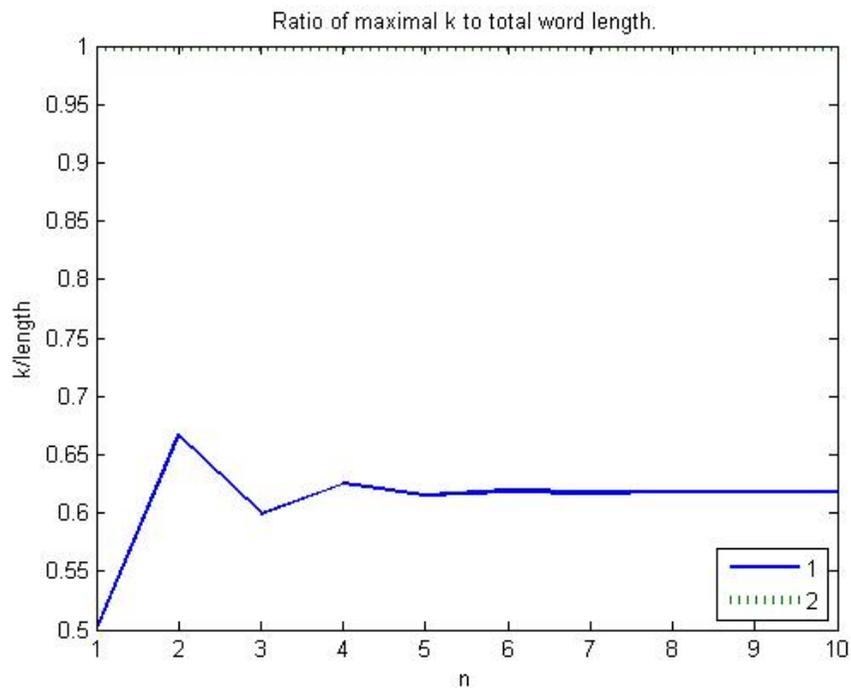
Figure 5.3: $\max k \,/\, |\varphi^n(1)|$ and $|\varphi^n(2)|$ versus n for the substitution $\varphi(1) = 12$, $\varphi(2) = 1$.
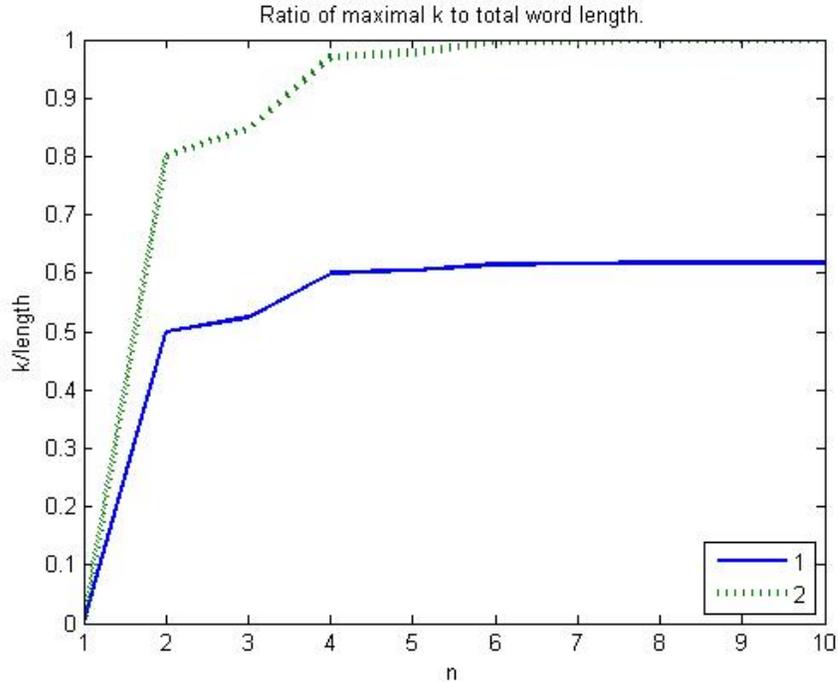
Figure 5.4: $\max k$ / $|\varphi^n(1)|$ and $|\varphi^n(2)|$ versus n for the substitution $\varphi(1) = 112$, $\varphi(2) = 21$.

We observe the same behavior for the substitution $\varphi(1) = 112$, $\varphi(2) = 21$, shown in Figure 4, with the exception that it takes some time for both ratios to stabilize to constant values. In all cases, however, the constant values are actually achieved.

This trend also seems to be present in substitutions on alphabets with more than two letters, for each distinct $i, j \in \mathcal{A}$. Starting with the substitution $\varphi(1) = 123$, $\varphi(2) = 23$, $\varphi(3) = 31$, Figure 5 shows the ratio of $\max k_{i,j}$ to $|\varphi^n(i)|$ and $|\varphi^n(j)|$ as n increases, where $\max k_{i,j}$ denotes the maximal value of k so that, for a fixed n, $(k, n)_{i,j}$ is a coincidence parameter for the letters i, j. The legend of Figure 5 indicates i, j.

With other results showing the same trend, we state it formally as a conjecture.

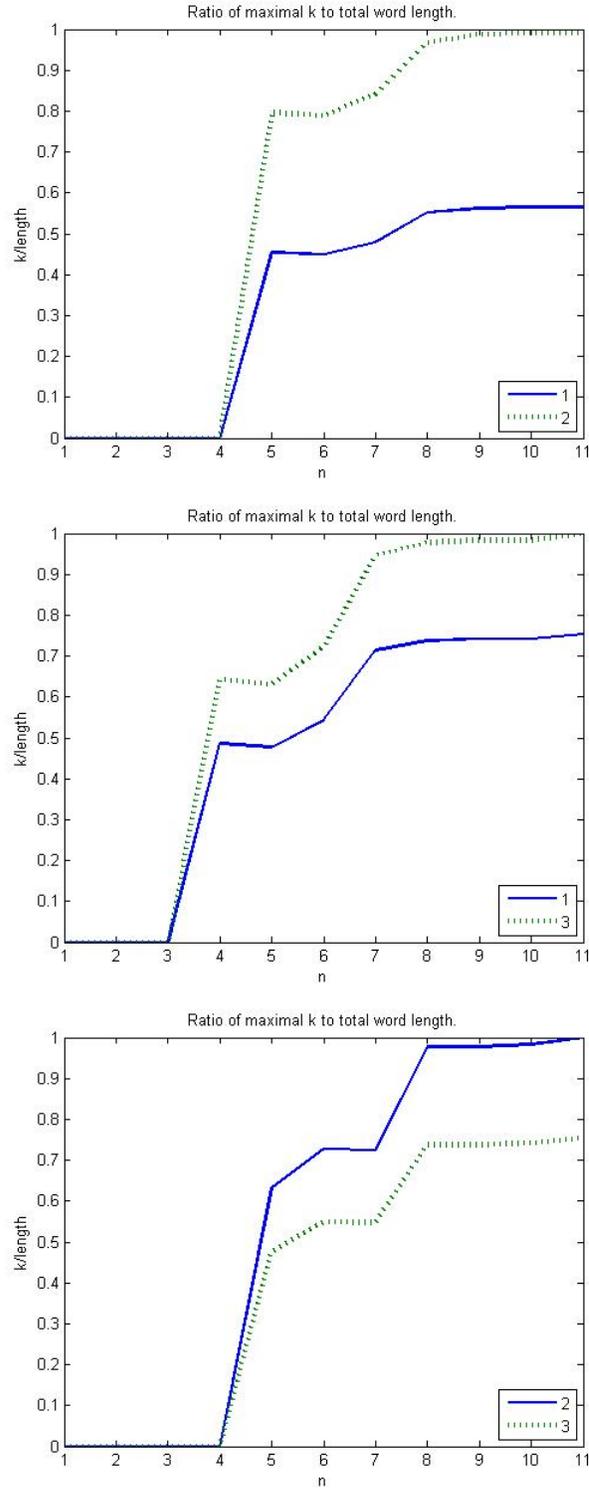**Conjecture 5.2.8.** *Given a Pisot substitution φ on alphabet $\mathcal{A}$, for any*

Figure 5.5: $\max k \,/\, |\varphi^n(i)|$ and $|\varphi^n(j)|$ versus n for the substitution $\varphi(1) = 123$, $\varphi(2) = 23$, $\varphi(3) = 31$.

$i, j \in \mathcal{A}$, $\dfrac{\max k_{i,j}}{|\varphi^n(i)|}$ and $\dfrac{\max k_{i,j}}{|\varphi^n(j)|}$ converge to a constant nonzero value as n increases, with one of those constant values being 1. Moreover, these constant values are actually achieved for some sufficiently high iterate n.

If this trend is true in general for Pisot substitutions, the Coincidence Conjecture follows immediately, as the substitution satisfies the pairwise coincidence condition for each i, j pair in $\mathcal{A}$. Therefore, the substitution satisfies the general coincidence condition (though the above conjecture does state more than that). It also means that we can consider the images of letters under the substitution in *coincident* and *non-coincident* parts. We define these notions below.

Definition 5.2.9. Let u and v be words with lengths $|u| = r$, $|v| = s$. We say $u_{[1,k]}$ is *coincidenct* with $v_{[1,k]}$ if k is maximal so that $u_k = v_k$ and $l(u_{[1,k]}) = l(v_{[1,k]})$ (where $k = 0$ if prior conditions are not met). If $k = r$ (or $k = s$), we say u is *completely coincident with v* (or v is *completely coincident with u*). Otherwise, we say $u_{[k+1,r]}$ and $v_{[k+1,s]}$ are *non-coincident*.

If we assume $\varphi^n(i) < \varphi^n(j)$ as n increases, this trend indicates that $\varphi^n(i)$ is eventually completely coincident, that is, $\dfrac{\max k_{i,j}}{|\varphi^n(i)|}$ stabilizes to 1 as n increases; at the same time, $\dfrac{\max k_{i,j}}{|\varphi^n(i)|}$ stabilizes to some constant as n increases, so the ratio of the lengths of the coincident and non-coincident parts stabilizes.

If true, this conjecture would have several implications. It would offer researchers a new approach to verifying coincidence; instead of checking any position in an iterate of the substitution for a pair of letters, it would be sufficient to check the last position of the word of smaller length for a sufficiently high iterate. Though this modification does not offer much by itself (since iterating a substitution still requires much computational work), it might be coupled with other modifications to produce new, more efficient algorithms for verifying coincidence. It also offers researchers a new way of thinking about the iterative behavior of substitutions with regards to the coincidence condition, which may be useful to prove additional statements, or lead to other interesting results.

Unfortunately, this phenomenon is currently unproven and unjustified. While the lack of a proof is unsurprising (since the truth of this trend would

imply the truth of the Coincidence Conjecture), what is more interesting is that nothing we found in the literature on Pisot substitutions mentioned anything similar to this observed trend. Why substitutions would be behaving this way is still unexplained, and would be an interesting subject of future study.

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