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ON THE EXISTENCE AND UNIQUENESS OF A LIMIT CYCLE FOR A LIENARD SYSTEM WITH A DISCONTINUITY LINE

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On the existence and uniqueness of limit cycles in Liénard differential equations allowing discontinuities

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Abstract

In this paper we study the non-existence and the uniqueness of limit cycles for the Liénard differential equation of the form $x'' - f(x)x' + g(x) = 0$ where the functions f and g satisfy $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$ but can be discontinuous at $x = 0$.

In particular, our results allow us to prove the non-existence of limit cycles under suitable assumptions, and also prove the existence and uniqueness of a limit cycle in a class of discontinuous Liénard systems which are relevant in engineering applications.

Mathematics Subject Classification: 58F21, 34C05, 58F14

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Piecewise smooth dynamical systems serve as models for a great variety of engineering devices and they deserve considerable attention, see for instance the recent book [1] and references therein. For instance, in modern nonlinear control techniques the lack of smoothness is sometimes enforced either by the consideration of hybrid systems or by the artificial introduction of discontinuities, see [23].

Even for low-dimensional continuous models the analysis of non-smooth systems is an intricate problem. Perhaps one of the most striking examples is the seemingly simplest case of continuous piecewise linear differential systems with only two regions separated by a hyperplane which contains the unique equilibrium point. Surprisingly enough, in dimension three or higher the stability of such an equilibrium has not yet been explicitly characterized, see [3].

When the vector field defining the differential system is discontinuous, the necessity to enlarge the mathematical concept of solution arises from the inclusion of non-standard solutions such as sliding solutions or impact solutions. Another obstacle to overcome in the analysis of discontinuous differential systems is the absence of canonical forms that can cope with a sufficiently broad class of systems, in contrast to what can be done for continuous non-smooth systems, see [2]. The above drawbacks frequently compel researchers to analyse the possible different models using a case-by-case approach.

One of the main problems in the qualitative theory of planar differential equations is to identify the existence of the limit cycle and its number. The restriction of this problem to polynomial differential equations is the well-known 16th Hilbert's problem [11]. Dealing with planar differential systems, a celebrated and rather general canonical form is the Liénard equation. Since Hilbert's problem turns out to be a strongly difficult one, Smale [20] has particularized it to Liénard differential systems in his list of problems for the present century.

For just continuous or even smooth Liénard systems there are many results on the non-existence, existence and uniqueness of limit cycles, see for instance [6, 8, 15, 24, 25]. Going beyond the smooth case, the first natural step is to allow non-smoothness while keeping the continuity, as has been done in some recent works [9, 12, 13]. In a further step, other authors have considered a line of discontinuity in the vector field defining the planar system, see [10,27].

In [10] a complete study of the number and stability of limit cycles was done for planar piecewise linear systems with both linear dynamics of focus type, and having a symmetry. Under such rather strong assumptions, those authors were able to solve the problem by writing the corresponding Liénard form and computing appropriate return maps on the discontinuity line. In [27] the authors studied a generalization of the Hopf bifurcation for systems with one parameter and having a line of discontinuity passing through the origin, which is the only point where both vector fields vanish for all values of the parameter. It was assumed that for a parameter critical value the equilibrium point at the origin becomes non-hyperbolic, giving rise to the bifurcation. The analysis was done in a neighbourhood of the origin assuming dynamics of focus type for the origin on both sides of the discontinuity line. Both works [10, 27] show, as suggested previously, how the advance in the analysis of non-smooth systems must proceed by considering specific situations with some restrictive hypotheses, which cover only partially the huge set of all possible cases.

In this paper we provide a new contribution to the study of the existence of limit cycles for Liénard differential systems which are allowed to have a line with some kind of discontinuity. In particular, we consider for $x \in [a, b]$, with $-\infty < a < 0 < b < \infty$, the Liénard differential equation

$$x'' - f(x)x' + g(x) = 0, \tag{1}$$

where a jump discontinuity is allowed for both functions f and g at $x = 0$. As will be seen, this case gives rise to discontinuous planar Liénard differential systems without sliding solutions but their analysis, far from being trivial, is important as a natural first step in the study of other discontinuous systems.

The hypotheses to be fulfilled by the corresponding vector field are rather general and no special symmetry will be assumed. Since our motivation comes from the analysis of piecewise linear systems, we focus our attention on such a case but the main results are valid for general non-smooth systems.

The rest of the paper is organized as follows. The statement of the assumptions and the main theoretical results along with an illustrative example of application appear in section 2. The specific case of discontinuous piecewise linear systems is presented in section 3, where a theorem of the existence of limit cycles is also included. In section 4 a discussion on the obtained results and how to improve them in the future is given. Some preliminary results about the local behaviour near the origin and its relation to possible periodic orbits are given in section 5. The proofs of the main theorems appear in sections 6 and 7.

2. Statement of the main results

Consider the Liénard differential equation (1) for $x \in [a, b]$, where $-\infty < a < 0 < b < \infty$, and the functions f and g are given by

$$f(x) = \begin{cases} f_1(x) & \text{if } x < 0, \\ f_2(x) & \text{if } x > 0, \end{cases} \qquad g(x) = \begin{cases} g_1(x) & \text{if } x < 0, \\ g_2(x) & \text{if } x > 0. \end{cases} \tag{2}$$

It is assumed that f_1 and g_1 are continuously differentiable in $[a, 0]$ and f_2, g_2 continuously differentiable in $[0, b]$. Note that the functions f and g are not defined at $x = 0$, so that if we eventually define $f(0)$ and $g(0)$, they are allowed to have a jump discontinuity at the origin.

By using the classical Liénard plane we can obtain the equivalent differential system:

$$\begin{aligned} x' &= F(x) - y, \\ y' &= g(x), \end{aligned} \qquad \text{where } F(x) = \int_0^x f(s)\,\mathrm{d}s, \tag{3}$$

and it is understood that $F(0) = 0$, while $g(0)$ is not defined for now.

This system has the associated vector field

$$\boldsymbol{X(x)} = \begin{cases} \boldsymbol{X}_1(\boldsymbol{x}) & \text{if } x \leqslant 0, \\ \boldsymbol{X}_2(\boldsymbol{x}) & \text{if } x \geqslant 0, \end{cases} \qquad \text{where } \boldsymbol{X}_i(\boldsymbol{x}) = \begin{pmatrix} F(x) - y \\ g_i(x) \end{pmatrix}, \tag{4}$$

with $\boldsymbol{x} = (x, y)^{\mathrm{T}}$. The ambiguity in the definition of $\boldsymbol{X(x)}$ on $x = 0$ is clarified below.

Since the system can be discontinuous we must adopt some criteria in order to define solutions starting at or passing through the allowed discontinuity line $x = 0$. Typically this is done by using the so-called Filippov approach, see for instance [16]. However, here only the vertical component of the vector field (4) can be discontinuous at the y-axis, while its horizontal component turns out to be continuous. In fact, we have $x' = -y$ on $x = 0$. Thus, if we consider, for instance, orbits starting at points with $x < 0$, then these orbits are well defined whenever they do not touch the y-axis but can arrive at this straight line (obviously only at points $(0, y)$ with $y \leqslant 0$) by extending $g(x)$ as if $g(0)$ were equal to $g_1(0)$. Now starting from the point $(0, y)$ with $y < 0$ we assume that $g(0) = g_2(0)$ and we continue the orbit inside $x > 0$ using system (3).

From the above paragraph and using the standard terminology of planar Filippov systems [16], the *crossing set* of the discontinuity line of system (3) includes the negative y-axis. Similar arguments for $x > 0$ imply that the crossing set is the y-axis without the origin and so no *sliding segments* appear. In [16] the origin is then called a *singular isolated sliding point*.

In short, except for orbits arriving at the origin and assuming that the system is actually discontinuous, it is natural to allow concatenation of solutions in an obvious way so that the system has no sliding (Filippov) solutions. The only possible singular point may be the origin, where each vector field can either vanish or have a tangency with the y-axis. If at least one vector field vanishes at the origin we say that it is a *boundary equilibrium point*. If both vector fields are not zero at the origin we still can have a *pseudo-equilibrium point* when both vector fields are anti-collinear (i.e. $g_1(0)g_2(0) < 0$). Then it behaves like an equilibrium point that may be reached in finite time. Its stability and local phase portrait will be determined by studying its nearby orbits, see figure 1.

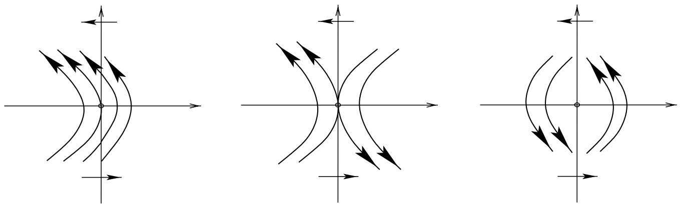

Figure 1. The three main cases for the local phase plane at the origin when it is not a boundary equilibrium point: regular point, pseudo-saddle and pseudo-focus.

Proposition 1. *For system (3) the following statements hold.*

(a) *If $g_1(0)g_2(0) > 0$ then the origin can be thought of as a regular point.*
(b) *If $g_1(0)g_2(0) = 0$ then the origin is a boundary equilibrium point.*
(c) *If $g_1(0)g_2(0) < 0$ then the origin is a pseudo-equilibrium point, being of saddle type if $g_1(0) > 0$ and $g_2(0) < 0$ and of focus type if $g_1(0) < 0$ and $g_2(0) > 0$.*

Proposition 1 is proved in section 5.

In the absence of any symmetry and similarly to what is done in [27], the existence of only one singular point at the origin (a pseudo-equilibrium point or a proper equilibrium point) is assumed. This assumption is not so strong if one recalls that all the results of this paper are stated for the restriction of a planar vector field to the band $a < x < b$ with $a < 0 < b$. Furthermore, in contrast to the approach followed in [27], the analysis to be done has a non-local character and the continuity at the origin is not required at all. Then, from the point of view of practical engineering problems, the case of a singular point of focus type is the most interesting case, because then systems can exhibit oscillatory behaviour, and frequently it is desirable to generate or avoid self-sustained oscillations.

Thus, we are mainly interested in giving conditions for the existence of periodic orbits. In order to ensure that there are no more singular points different from the origin and to exclude the saddle case of proposition 1, the following hypothesis is assumed.

(H1) The function g satisfies $xg(x) > 0$ for $x \neq 0$.

We will also require for the sake of simplicity that the divergence of the vector field does not change its sign on each side of the discontinuity line, i.e.

(H2) the function f satisfies $xf(x) > 0$ for $x \neq 0$.

Under this hypothesis we have a positive divergence for $x > 0$ and a negative divergence for $x < 0$. Then in order to have some periodic orbit surrounding the origin, there must be some balance between the x-positive and the x-negative parts of the interior of the bounded region limited by the periodic orbit. This idea is precisely stated in lemma 6, but in the same spirit of comparing the x-positive and the x-negative half-planes and following [4], it will be useful to introduce some auxiliary functions as follows.

Under hypothesis H2 and recalling the definition of F in (3), we define a variable $p = p(x) = F(x)$. As $p'(x) = f(x)$, then $p(x) \geqslant 0$ for all x and $\operatorname{sgn}(p'(x)) = \operatorname{sgn}(x)$ for $x \neq 0$. We deduce that the function $p(x)$ has inverse functions both for $x \leqslant 0$ and for $x \geqslant 0$, namely, the non-positive decreasing function

$$x_1 : [0, F(a)] \to [a, 0], \qquad \text{such that } F(x_1(p)) = p, \tag{5}$$

and the non-negative increasing function

$$x_2 : [0, F(b)] \to [0, b], \qquad \text{such that } F(x_2(p)) = p. \tag{6}$$

Hence, for $x \neq 0$ we have that both systems (3) and (4) are equivalent to the two differential equations

$$\frac{\mathrm{d}y(x_i(p))}{\mathrm{d}p} = \frac{g(x_i(p))}{F(x_i(p)) - y}\frac{1}{f(x_i(p))} = \frac{1}{p-y}\frac{g(x_i(p))}{f(x_i(p))}, \tag{7}$$

where $i = 1, 2$, according to $x < 0$ or $x > 0$, respectively, and these new differential equations are both meaningful only for $p > 0$. Now by considering the functions

$$h_i(p) = \frac{g\,(x_i(p))}{f\,(x_i(p))}; \tag{8}$$

equation (7) can be written in a more compact form:

$$\frac{\mathrm{d}y\,(x_i(p))}{\mathrm{d}p} = \frac{h_i(p)}{p-y}. \tag{9}$$

Note that $h_i(p) > 0$ for $p > 0$ and $i = 1, 2$ and that the effect of considering equation (9) instead of the original system (3) or (4) can be thought of as the plane (x, y) having been folded into the half-plane (p, y) with $p > 0$.

When $h_1(p) = h_2(p)$ for p sufficiently small and the origin is a topological focus it is not difficult to show that we have indeed a centre, see for instance theorem 11.3 in [14]. We add a third hypothesis precluding such a possibility. It is written in a dual way to facilitate the checking of its validity in the applications.

(H3) Assume that there exist the two limits

$$\lim_{x\to 0^-}\frac{g(x)}{f(x)} = \lim_{p\to 0^+} h_1(p) = l_1, \qquad \lim_{x\to 0^+}\frac{g(x)}{f(x)} = \lim_{p\to 0^+} h_2(p) = l_2$$

satisfying

$$0 \leqslant l_2 \leqslant l_1 < \infty,$$

and if $l_2 = l_1$ then $h_2(p) < h_1(p)$ for $p > 0$ and sufficiently small (when $l_2 < l_1$ this last requirement is always fulfilled).

It is worth mentioning that this hypothesis implies that the origin is topologically an unstable focus when $l_2 > 0$, see lemma 10 below. The next result states a necessary condition for the existence of periodic orbits under the above hypotheses, also giving an estimate of their minimal size.

Theorem 2. *Let f and g be the functions defined in (2) such that f_i and g_i are of class $\mathcal{C}^1$ in $[a, 0]$ and $[0, b]$ for $i = 1, 2$, respectively, where $-\infty < a < 0 < b < \infty$. Let F and h_i be the functions defined in (3) and (8) and assume that hypotheses H1–H3 are fulfilled. If system (3) has a periodic orbit contained in the band $a < x < b$, then the system*

$$F(x_1) = F(x_2), \qquad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}, \tag{10}$$

has at least one solution $(x_1, x_2) = (s_1, s_2)$ with $a < s_1 < 0 < s_2 < b$, or equivalently there exists at least one solution $\hat{p} \in (0, F(a)) \cap (0, F(b))$ for the equation $h_1(p) = h_2(p)$.

When such a periodic orbit does exist, it surrounds the origin and cuts the two verticals $x = s_1$ and $x = s_2$, (s_1, s_2) being the smallest solution of (10).

Theorem 2 is proved in section 5. To illustrate how it can be applied, let us consider a discontinuous piecewise quadratic polynomial case, namely,

$$f(x) = \begin{cases} f_1(x) = -2 - 8x - 24x^2, & \text{if } x < 0, \\ f_2(x) = 1 - 2x + 3x^2, & \text{if } x > 0 \end{cases} \tag{11}$$

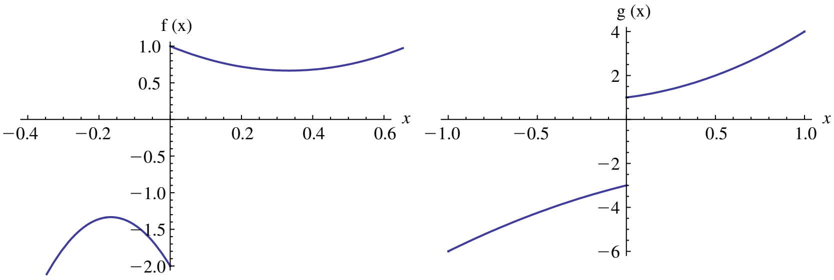

Figure 2. Graphs of functions f and g of the example given by (11) and (12). Hypotheses H1 and H2 are satisfied. Hypothesis H3 is fulfilled with $0 < l_2 = 1 < l_1 = 3/2$.

and

$$g(x) = \begin{cases} g_1(x) = -3 + 2x - x^2, & \text{if } x < 0, \\ g_2(x) = 1 + x + 2x^2, & \text{if } x > 0, \end{cases} \tag{12}$$

see figure 2. According to (3), we obtain

$$F(x) = \begin{cases} -2x(1 + 2x + 4x^2), & \text{if } x < 0, \\ x(1 - x + x^2), & \text{if } x > 0, \end{cases} \tag{13}$$

and then equation $F(x_1) = F(x_2)$ in (10) is satisfied only if $x_2 = -2x_1$. Substituting this relation in the second equation of (10), it is easy to conclude that the only solution is $x_1 = -1/5$, $x_2 = 2/5$. Then from theorem 2 the system cannot have limit cycles totally contained in the band $-1/5 < x < 2/5$, but it can exhibit limit cycles cutting the vertical lines $x = -1/5$ and $x = 2/5$. In fact, by numerical simulation, one limit cycle is observed, as shown in figure 3, but for the moment we have no rigorous information about its stability and uniqueness.

Now we give a result on the uniqueness of limit cycles for discontinuous Liénard equations satisfying hypotheses H1–H3 which complements theorem 2.

Theorem 3. *Under the same conditions of theorem 2, assume that system (10) has exactly one solution* $(x_1, x_2) = (s_1, s_2)$ *with* $a < s_1 < 0 < s_2 < b$*, or equivalently there exists exactly one solution* $\hat{p} \in (0, F(a)) \cap (0, F(b))$ *for the equation* $h_1(p) = h_2(p)$*. The following statement holds.*

If the positive function

$$\alpha(x) = \frac{g(x)}{f(x)F(x)} \tag{14}$$

is increasing for $x \in (a, 0)$*, or equivalently the positive function*

$$\frac{h_1(p)}{p} \tag{15}$$

is decreasing for $p \in (0, F(a))$*, then system (3) has at most one periodic orbit contained in the band* $a < x < b$ *and, if it exists, it has a negative characteristic exponent.*

Theorem 3 is proved in section 5. Returning to the previous example, and using the fact that the function defined in (14), namely,

$$\alpha(x) = \frac{-3 + 2x - x^2}{2x(2 + 8x + 24x^2)(1 + 2x + 4x^2)},$$

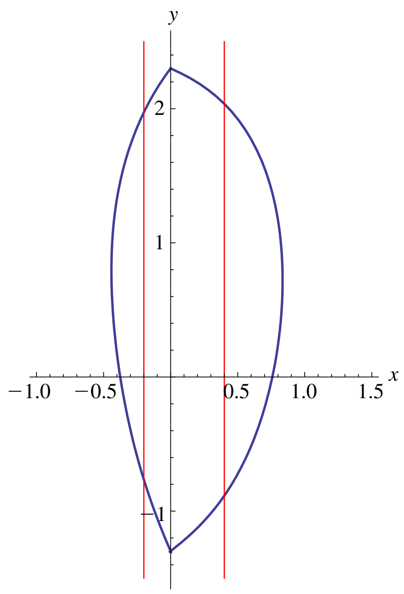

Figure 3. The limit cycle of the illustrative example cuts the two vertical lines $x = -1/5$ and $x = 2/5$. From theorem 3 this limit cycle is unique and stable.

turns out to be increasing for $x < 0$, we conclude from theorem 3 that the observed limit cycle is unique and stable.

Note that even though our main motivation is the case of discontinuous systems, the above results can also be applied to continuous differential systems. Particular applications of theorems 2 and 3 to specific families of Liénard equations are presented in the following sections.

3. Application to discontinuous piecewise linear systems

We devote this section to the application of the above results to discontinuous piecewise linear differential systems. This class is increasingly used in engineering and applied sciences to model a large variety of technological devices and physical or biological systems [5, 22, 26]. The study of these systems is also useful in order to be able to tackle other piecewise smooth systems not necessarily composed by linear ones, in the same spirit as done in [19].

The analysis of these kinds of systems seems to be simple as one can easily integrate the system in each linear region. However, the matching of the different parts of a given solution is a difficult task since the knowledge of the different flight times in every region is only implicit, see for instance [9, 10]. Thus, the existence of limit cycles in piecewise linear systems is a non-elementary problem which sometimes forces researchers to extend perturbation methods from smooth systems to cope with these non-smooth ones, as done in [18, 21] for example.

Here our attention is restricted to the case of piecewise linear systems with only two linear regions. Similar differential systems had been considered before in [9] but under the assumption of continuity for the corresponding vector field. It should be remarked that in practice models with only two linear regions satisfying our hypotheses are hardly found, because then we have a region not bounded in the x-variable with a positive divergence. However, such a structure frequently appears when focusing attention on an x-bounded part of the phase plane where only two zones are involved.

Theorem 4. *Consider the Liénard piecewise linear differential system:*

$$\begin{cases} \dot{x} = t_1 x - y, \\ \dot{y} = d_1 x + a_1, \end{cases} \quad \text{if } x < 0, \qquad \begin{cases} \dot{x} = t_2 x - y, \\ \dot{y} = d_2 x + a_2, \end{cases} \quad \text{if } x \geqslant 0, \tag{16}$$

where it is assumed that

$$t_1 < 0, \qquad d_1 > 0, \qquad a_1 < 0, \qquad t_2 > 0, \qquad d_2 > 0, \qquad a_2 > 0.$$

Then the following statements hold.

(a) If $a_2/t_2 < a_1/t_1$ then a necessary condition for the existence of periodic orbits is $d_2/t_2^2 > d_1/t_1^2$. If the system has periodic orbits, then it has a unique periodic orbit which is a stable limit cycle.

(b) If $a_1/t_1 < a_2/t_2$ then a necessary condition for the existence of periodic orbits is $d_1/t_1^2 > d_2/t_2^2$. If the system has periodic orbits, then it has a unique periodic orbit which is an unstable limit cycle.

(c) If $a_2/t_2 = a_1/t_1$ then either the system has no periodic orbits when $d_1/t_1^2 \neq d_2/t_2^2$ or it has a centre at the origin when $d_1/t_1^2 = d_2/t_2^2$.

Theorem 4 is proved in section 7. Observe that statement (c) of theorem 4 when $0 < a_2/t_2 = a_1/t_1$ and $d_1/t_1^2 = d_2/t_2^2$ says that the origin is a centre even when the dynamics of the linear differential system in each half-plane could be of node type. This situation occurs when

$$\frac{d_i}{t_i^2} \leqslant \frac{1}{4}$$

for $i = 1, 2$. When both dynamics are of focus type, and under the assumptions of statements (a) and (b) of theorem 4, the necessary condition for the existence of limit cycles is also sufficient, as stated in our last main result.

Theorem 5. *Under the assumptions of theorem 4, and if*

$$\frac{d_i}{t_i^2} > \frac{1}{4}$$

for $i = 1, 2$, then the following statements hold.

(a) If $a_2/t_2 < a_1/t_1$ then the system has periodic orbits if and only if $d_2/t_2^2 > d_1/t_1^2$, and in such a case it has a unique periodic orbit which is a stable limit cycle.

(b) If $a_1/t_1 < a_2/t_2$ then the system has periodic orbits if and only if $d_1/t_1^2 > d_2/t_2^2$, and in such a case it has a unique periodic orbit which is an unstable limit cycle.

Theorem 5 is proved in section 7. Note that this result completely characterizes the number and stability of limit cycles in the focus–focus case.

4. Discussion and future work

In previous sections some new results have been stated regarding possible limit cycles of Liénard systems having a line of discontinuity and no sliding solutions. Under appropriate hypotheses on the vector field on both sides of the discontinuity line, a necessary condition for the existence of limit cycles is given, and when some additional conditions are fulfilled a uniqueness result has been stated. The approach that follows, which is inspired by the work in [4], relies on a comparison method of the resulting semi-orbits after folding the plane along the discontinuity line.

Obviously some existence results ensuring that limit cycles do exist are required to complete the analysis. For instance, in the piecewise linear case the corresponding existence theorem is given in theorem 4. A natural question then is how to proceed in a general case to achieve such existence theorems. To answer this question in the continuous case it is usual to look for positive invariant annular regions; we strongly feel that with the adequate requirements to validate some extension of the Poincaré–Bendixson theorem, the analogous approach is the right way. In this sense special attention should be paid to possible sliding solutions.

Thus, future work on this subject must fill the actual gap on limit cycle existence results for general discontinuous systems. Keeping the approach followed in this paper, and with reference to the (p, y)-plane and looking for invariant annular regions, one possibility is to obtain estimates for the arriving points on the negative y-axis of semi-orbits starting from the same point on the positive y-axis. Another possible approach is to attempt the extension to the discontinuous case of known results on the existence of limit cycles for continuous planar vector fields. Perhaps one good candidate is a result of Filippov, see [24], which has a similar flavour to that stated here, but folding the plane by using the function g instead of f.

In order to be precise, let us comment on the difficulties to be overcome by studying a concrete example. In [27] the generalized Hopf bifurcation of a discontinuous mathematical model of a mechanical brake system is considered, namely, the planar system

$$\begin{aligned} x' &= y, \\ y' &= b^{\pm}(\lambda)y - a^{\pm}x - x^3, \qquad \pm x > 0, \end{aligned} \tag{17}$$

where $b^{\pm}(\lambda) = b_0^{\pm} + b_1^{\pm}\lambda$ is a function of the bifurcation parameter λ. The system is continuous at the origin, and other parameter values are $a^+ = 1/10, a^- = 1/5, b_0^+ = 1/20, b_0^- = -\sqrt{2}/20$ and $b_1^{\pm} = \pm 1$ (after correcting some evident typographical mistakes in [27]). For such values, after a local analysis the origin turns out to be a non-hyperbolic focus for $\lambda = 0$ and a generalized Hopf bifurcation is then predicted. According to the results in [27], we can ensure that one stable limit cycle appears for $\lambda > 0$ and sufficiently small.

Let us write the corresponding Liénard equation, namely,

$$x'' - b^{\pm}(\lambda)x' + a^{\pm}x + x^3 = 0,$$

so that, regarding (1), $f(x) = b^{\pm}(\lambda)$ and $g(x) = a^{\pm}x + x^3$ for $\pm x > 0$. In what follows it will be shown that theorem 2 suggests the existence of the aforementioned generalized Hopf bifurcation after straightforward computations.

Obviously the function g is non-smooth but continuous, so that the discontinuity of system (17) can be removed by writing its Liénard form (3), namely,

$$\begin{aligned} x' &= b^{\pm}(\lambda)x - y, \\ y' &= a^{\pm}x + x^3, \end{aligned} \qquad \pm x > 0, \tag{18}$$

which is clearly continuous. This remark emphasizes that some discontinuous systems are no longer discontinuous after some variable changes.

Regarding system (18), since hypothesis H1 is always satisfied but hypothesis H2 is only fulfilled for $\lambda > -1/20$, from now on we restrict our attention to this parameter range. Regarding hypothesis H3, we see that $l_1 = l_2 = 0$ and so we need to check the condition $h_2(p) < h_1(p)$ for p sufficiently small. From (5), (6) and (18), we see that

$$x_1(p) = \frac{p}{b^-(\lambda)}, \qquad x_2(p) = \frac{p}{b^+(\lambda)}$$

so that

$$h_1(p) = \frac{a^- p}{b^-(\lambda)^2} + \frac{p^3}{b^-(\lambda)^4}, \qquad h_2(p) = \frac{a^+ p}{b^+(\lambda)^2} + \frac{p^3}{b^+(\lambda)^4}.$$

Then hypothesis H3 is satisfied if $h_2'(0) < h_1'(0)$, which amounts to the condition

$$\frac{a^+}{b^+(\lambda)^2} < \frac{a^-}{b^-(\lambda)^2}. \tag{19}$$

After some simple calculations we get the equivalent condition

$$\lambda\left(\lambda + \frac{2-\sqrt{2}}{10}\right) > 0,$$

which in the parameter range under study leads to $\lambda > 0$. Assuming this condition, in order to apply theorem 2 we solve the equation $h_1(p) = h_2(p)$ for $p > 0$, arriving at the solution

$$\hat{p}^2 = \frac{\dfrac{a^-}{b^-(\lambda)^2} - \dfrac{a^+}{b^+(\lambda)^2}}{\dfrac{1}{b^+(\lambda)^4} - \dfrac{1}{b^-(\lambda)^4}}, \tag{20}$$

which, assuming inequality (19), corresponds to one real positive value if and only if $|b^+(\lambda)| < |b^-(\lambda)|$, and this happens for $\lambda > -(1+\sqrt{20})/40$ and in particular for $\lambda > 0$. The above value of $\hat{p}$ vanishes for $\lambda = 0$ and it is an increasing function of λ.

In order to apply theorem 2 when $\lambda < 0$, let us make in (18) the change in variables $(x, y, t) \to (-x, y, -t)$ to obtain its specular image with time reversal, namely,

$$\begin{aligned} x' &= -b^{\mp}(\lambda)x - y, \\ y' &= a^{\mp}x + x^3, \end{aligned} \qquad \pm x > 0.$$

Here all the hypotheses H1–H3 are fulfilled for $-1/20 < \lambda < 0$, but now expression (20) turns out negative and according to theorem 2 no limit cycles are possible.

Thus, theorem 2 precludes the existence of limit cycles for $\lambda < 0$ but allows it when $\lambda > 0$. All these facts suggest the bifurcation predicted in [27] and also that the size of the limit cycle that bifurcates grows with λ.

Unfortunately, in this case the required condition on the function $\alpha(x)$ defined in theorem 3 is not fulfilled and so we cannot deduce the stability and uniqueness of the limit cycle that bifurcates. This fact is not so deceptive, since system (18) is continuous with a boundary equilibrium point and we already know that this is one of the most intricate situations in non-smooth systems, see [3]. On the other hand, in this continuous case it is not difficult to show the existence of the limit cycle when $\lambda > 0$ using the aforementioned theorem of Filippov, see [24].

5. On the origin and the periodic orbits

In this section we prove proposition 1 and we give some preliminary results necessary for the proof of theorem 2.

Proof of proposition 1. The vector fields at the origin are $\boldsymbol{X}_1(0,0) = (0, g_1(0))$ and $\boldsymbol{X}_2(0,0) = (0, g_2(0))$, see (4). Then from (1) we have $x''(0) = -g_i(0)$ for $i = 1, 2$. Therefore, if $g_1(0)$ and $g_2(0)$ are both positive or both negative then $\boldsymbol{X}_1(0,0)$ and $\boldsymbol{X}_2(0,0)$ are collinear and the orbits of both vector fields in a neighbourhood of the origin have the same convexity. Consequently we can define the vector field at the origin in such a way that the orbit through the origin has a quadratic tangency with the y-axis. This completes the proof of statement (a).

Statement (b) follows directly from the definitions.

If $g_1(0)g_2(0) < 0$ then the vector fields at the origin are anti-collinear and so the origin is a pseudo-equilibrium point. Assume $g_1(0) > 0$ and $g_2(0) < 0$. Then the vector field $\boldsymbol{X}_1$

has a *visible quadratic tangency*, that is, the orbit of $\boldsymbol{x}' = \boldsymbol{X}_1(\boldsymbol{x})$ through the origin is locally contained in $x \leqslant 0$ for backward and forward times. Similarly, the vector field $\boldsymbol{X}_2$ also has a visible quadratic tangency in $x \geqslant 0$, see figure 1. Hence, the origin is a topological saddle.

When $g_1(0) < 0$ and $g_2(0) > 0$ the vector fields $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ have *invisible quadratic tangencies*. That is, the unique point of the orbit of $\boldsymbol{x}' = \boldsymbol{X}_1(\boldsymbol{x})$ through the origin locally contained in $x \leqslant 0$ for backward and forward times is the origin itself, and similarly for $\boldsymbol{X}_2$; see figure 1. Now the origin is a topological focus. This ends the proof of statement (c). □

Now we extend a necessary condition for the existence of periodic orbits fulfilled by smooth vector fields to the case of our discontinuous differential systems. The following result, which is not valid when there exists the possibility of sliding solutions, is included with its proof for the sake of completeness, because we have not previously found it explicitly.

Lemma 6. *Consider the functions f and g defined as in (2). If system (3) has a periodic orbit Γ and the interior of the bounded region limited by Γ includes the origin and it is denoted by Δ, then Γ crosses the y-axis at two points different from the origin, and the function f satisfies the condition*

$$\iint_{\Delta} f(x)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Proof. Since $x' = -y$ on $x = 0$ and the origin is in Δ, it follows that Γ intersects the y-axis at two points $M = (0, y_M)$ and $N = (0, y_N)$ with $y_M < 0 < y_N$.

We define Δ_1, Γ_1, and Δ_2 , Γ_2 to be part of Δ and Γ contained in $x < 0$ and $x > 0$, respectively. We denote by Λ the oriented segment on the y-axis from point M to point N while the same segment with the opposite orientation is denoted by $-\Lambda$, see figure 4. Then by applying Green's theorem we have

$$\begin{aligned}
\iint_{\Delta} f(x)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\Delta_1} f(x)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Delta_2} f(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Gamma_1} [F(x) - y]\,\mathrm{d}y - g(x)\,\mathrm{d}x + \int_{\Lambda} [F(x) - y]\,\mathrm{d}y - g(x)\,\mathrm{d}x \\
&\quad + \int_{\Gamma_2} [F(x) - y]\,\mathrm{d}y - g(x)\,\mathrm{d}x + \int_{-\Lambda} [F(x) - y]\,\mathrm{d}y - g(x)\,\mathrm{d}x \\
&= 0 + \int_{y_M}^{y_N} (-y)\,\mathrm{d}y + 0 + \int_{y_N}^{y_M} (-y)\,\mathrm{d}y = 0,
\end{aligned}$$

and the conclusion follows. □

6. Proof of theorems 2 and 3

First we prove theorem 2.

Proof of theorem 2. We start by noting that if system (3) has singular points they must be on the y-axis because $xg(x) > 0$ if $x \neq 0$. Also we have $g_1(0) \leqslant 0$ and $g_2(0) \geqslant 0$. Since $x' = -y$ when $x = 0$ the unique possible singular point is the origin, and from proposition 1 it is a boundary equilibrium point or a pseudo-focus because $g_1(0)g_2(0) \leqslant 0$.

Assume that system (3) has a periodic orbit Γ contained in the band $a < x < b$. Obviously, the orbit Γ cannot be totally contained in the region $a < x < 0$ because then, as a consequence of the Poincaré–Bendixson theorem for phase portraits of smooth systems in the plane (see for instance [7]), the interior of the bounded region limited by Γ should contain a singular point.

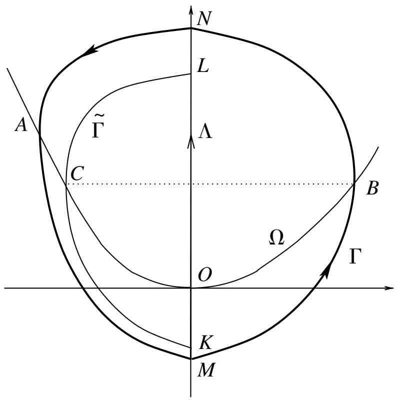

Figure 4. Some notable points associated with a periodic orbit (thick line). The points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, where $y_A = F(x_A)$ and $y_B = F(x_B)$, are the leftmost and rightmost points of the periodic orbit, respectively. For $x < 0$, the orbit $\tilde{\Gamma}$ passing through $C = (x_C, y_C)$ is sketched, where $y_B = y_C = F(x_C)$. The line Ω is the graph of the curve $y = F(x)$.

However, we know that the only possible singular point is the origin. Analogously, the orbit Γ cannot be totally contained in the region $0 < x < b$. Thus, Γ being a closed curve and since $x' = -y$ on the y-axis, we conclude that it crosses the y-axis from the left to the right at a point with $y < 0$ and from the right to the left at a point with $y > 0$, surrounding the origin in the counterclockwise sense.

Next other geometrical properties of the periodic orbit Γ will be established. Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be the points on Γ for which the variable x assumes its minimum and maximum values, then $x_A < 0 < x_B$. Since for $x \neq 0$ we have

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{F(x) - y}{g(x)}, \tag{21}$$

and this derivative vanishes for $x = x_A$ and $x = x_B$, one obtains $y_A = F(x_A)$ and $y_B = F(x_B)$. Moreover this derivative only vanishes at points A and B. Indeed when $\mathrm{d}x/\mathrm{d}y = 0$ the second derivative is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = \left. \frac{\left(\dfrac{\mathrm{d}F}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}y} - 1\right) g(x) - [F(x) - y]\dfrac{\mathrm{d}g}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}y}}{g(x)^2} \right|_{\frac{\mathrm{d}x}{\mathrm{d}y} = 0} = -\frac{1}{g(x)},$$

which has a definite sign, in fact, the opposite sign to x. Then derivative (21) vanishes only once for $x > 0$ and only once for $x < 0$, and so points A and B are the unique points where the orbit Γ intersects the curve defined by the equation $y = F(x)$ denoted by Ω.

It follows that Γ intersects any straight line L defined by $x = q$ with $x_A < q < x_B$ in exactly two points (q, y_α) and (q, y_β) with $y_\alpha < F(q) < y_\beta$. In particular, for $q = 0$ such points are denoted by $M = (0, y_M)$ and $N = (0, y_N)$ with $y_M < 0 < y_N$. Moreover the path Γ can be described as the graph of $y = y_l(x)$ on the lower arc AMB and by the graph of $y = y_u(x)$ on the upper arc ANB. Clearly $y_l(x) < F(x) < y_u(x)$; that is, Γ is below the curve Ω on the lower arc AMB, while it is over the curve Ω on the upper arc ANB, see figures 4 and 5.

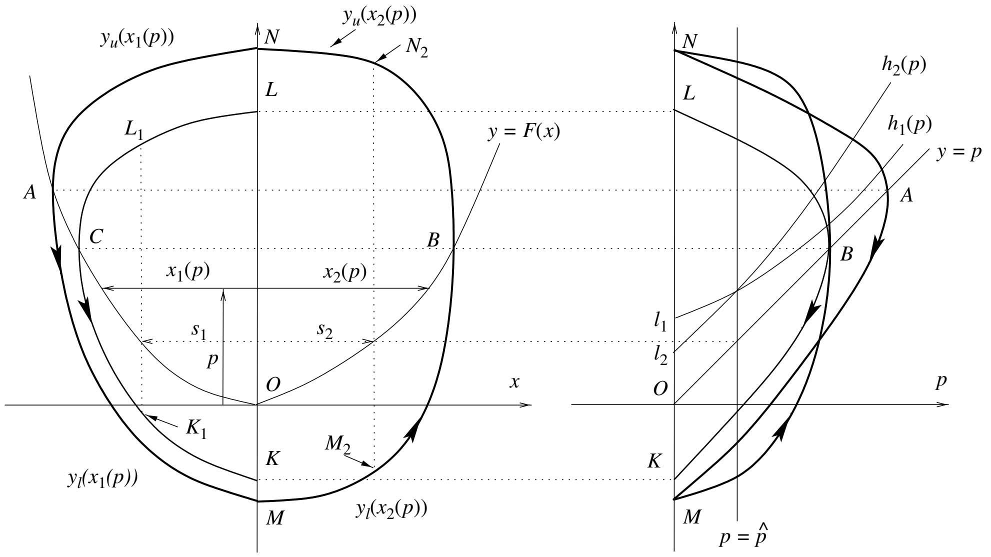

Figure 5. On the left, figure 4 is repeated and all the notable points are labelled. On the right, and by using the coordinate p instead of x, the complete plane is folded keeping the y-coordinate, so that the two parts of the periodic orbit can be compared. Thus, in the new coordinates (p, y) both semi-orbits $\Gamma_1 \equiv NAM$ and $\Gamma_2 \equiv MBN$ are in the half-plane $p > 0$, and they must enclose the same area with the y-axis.

Differential equations (9) can be continuously extended to $x = 0$ by putting $h_i(0) = l_i$, so that they define the orbits of the following two differential systems, both defined for $p \geqslant 0$:

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}\tau} &= p - y,\\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= h_i(p),\end{aligned} \tag{22}$$

for $i = 1, 2$. The arc NAM of the periodic orbit Γ can be parametrized as

$$\Gamma_1(p) = \begin{cases} y_l(x_1(p)) & \text{if } y_l(x_1(p)) \leqslant y_A, \\ y_u(x_1(p)) & \text{if } y_A \leqslant y_u(x_1(p)), \end{cases}$$

while the arc MBN of Γ can be parametrized as

$$\Gamma_2(p) = \begin{cases} y_l(x_2(p)) & \text{if } y_l(x_2(p)) \leqslant y_B, \\ y_u(x_2(p)) & \text{if } y_B \leqslant y_u(x_2(p)), \end{cases}$$

where

$$y_l(x_1(0)) = y_l(x_2(0)) = y_M < 0 \qquad \text{and} \qquad y_u(x_1(0)) = y_u(x_2(0)) = y_N > 0. \tag{23}$$

Before proceeding further we now state some results from the theory of differential inequalities, providing their proof for the sake of completeness.

Lemma 7. *Assume that the graphs of two continuous functions $y_i : [c, d] \to \mathbb{R}$ are solution curves of some given Lipschitz differential systems*

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}\tau} &= p - y,\\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= \phi_i(p),\end{aligned}$$

for $i = 1, 2$, *respectively. Assume also that the inequalities*

$$p - y_1(p) > 0 \quad \text{and} \quad p - y_2(p) > 0, \qquad \text{for all } p \in (c, d),$$

and

$$0 < \phi_1(p) < \phi_2(p), \qquad \text{for all } p \in (c, d),$$

are satisfied. The following statements hold.

(a) If $y_1(c) \leqslant y_2(c)$ *then* $y_1(p) < y_2(p)$ *for all* $p \in (c, d]$.
(b) If $y_1(d) \geqslant y_2(d)$ *then* $y_1(p) > y_2(p)$ *for all* $p \in [c, d)$.

Proof. For all $p \in (c, d)$ such that $y_1(p) \leqslant y_2(p)$ we have

$$\frac{\mathrm{d}y_1}{\mathrm{d}p} = \frac{\phi_1(p)}{p - y_1(p)} \leqslant \frac{\phi_1(p)}{p - y_2(p)} < \frac{\phi_2(p)}{p - y_2(p)} = \frac{\mathrm{d}y_2}{\mathrm{d}p},$$

so that the function $y_2 - y_1$ is strictly increasing in (c, d) and the conclusion of statement (a) follows easily.

To show statement (b) suppose in contrast that there exists $\bar{p} \in [c, d)$ such that $y_1(\bar{p}) \leqslant y_2(\bar{p})$. Then by statement (a) we conclude that $y_1(p) < y_2(p)$ for all $p \in (\bar{p}, d]$, and in particular that $y_1(d) < y_2(d)$, which is a contradiction. □

Remark 8. An analogous result for lemma 7 is also true by reversing all the inequalities in the statements when $p - y_1(p) < 0$ and $p - y_2(p) < 0$ (i.e. when the orbits are in the region $y > p$) while $0 < \phi_1(p) < \phi_2(p)$ still holds for all $p \in (c, d)$.

From the hypotheses we get that $0 \leqslant h_2(0) \leqslant h_1(0)$ and $h_2(p) < h_1(p)$ for $0 < p \ll 1$. Then from (9) and (23) and using statement (a) of lemma 7 in the interval $[0, \bar{p}]$ with $\bar{p}$ sufficiently small, we obtain

$$y_l(x_2(p)) < y_l(x_1(p)) \qquad \text{for } 0 < p \ll 1.$$

On the other hand, by using remark 8 we have analogously

$$y_u(x_1(p)) < y_u(x_2(p)) \qquad \text{for } 0 < p \ll 1. \tag{24}$$

Next we will show that both paths $\Gamma_1(p)$ and $\Gamma_2(p)$ cross themselves at least at one point. Let us build a system that 'unfolds' the two systems (22) in the complete plane, namely,

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = -p - y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = -h_1(-p), \end{cases} \quad \text{if } p < 0, \qquad \begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = p - y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = h_2(p). \end{cases} \quad \text{if } p > 0, \tag{25}$$

System (25) must have a counterclockwise periodic orbit $\hat{\Gamma}$ constituted by a path $\hat{\Gamma}_1$, which is the symmetrical one with respect to the y-axis of the path $\Gamma_1(p)$ and the path $\Gamma_2(p)$. System (25) is the Liénard system

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = |p| - y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = h(p),$$

where

$$h(p) = \begin{cases} h_2(p) & \text{if } p > 0, \\ -h_1(-p) & \text{if } p < 0. \end{cases}$$

Then by applying lemma 6 we have

$$\iint_{\Delta} \operatorname{sgn}(p)\,\mathrm{d}p\,\mathrm{d}y = 0 = -S_1 + S_2, \tag{26}$$

where Δ is the interior of the region limited by $\hat{\Gamma}$ and S_1 and S_2 are the areas of Δ on the left- and right-hand sides of the line $p = 0$, respectively. If the path $\Gamma_1(p)$ does not cut the path $\Gamma_2(p)$, then $S_1 \neq S_2$ and (26) cannot be fulfilled. So the path $\Gamma_1(p)$ must cut the path $\Gamma_2(p)$ to enclose the same area with the y-axis, see the right part of figure 5.

Assume now that $h_2(p) < h_1(p)$ for $0 < p \leqslant \min\{y_A, y_B\}$. Since $y_l(x_2(p))$ and $y_l(x_1(p))$ are solutions of the equations

$$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{h_2(p)}{p-y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}p} = \frac{h_1(p)}{p-y},$$

respectively, with $y_l(x_2(0)) = y_l(x_1(0))$, then by statement (a) of lemma 7 we must have $y_l(x_2(p)) < y_l(x_1(p))$ for $0 < p \leqslant \min\{y_A, y_B\}$. Similarly, by using remark 8 we must have $y_u(x_1(p)) < y_u(x_2(p))$ for $0 < p \leqslant \min\{y_A, y_B\}$, and then the paths $\Gamma_1(p)$ and $\Gamma_2(p)$ do not cross themselves, which is a contradiction. Hence, there must exist $\hat{p}$ such that $h_1(\hat{p}) = h_2(\hat{p})$, i.e. the system

$$\hat{p} = F(x_1) = F(x_2), \qquad \frac{f(x_1)}{g(x_1)} = \frac{f(x_2)}{g(x_2)},$$

must have a solution $(x_1, x_2) = (s_1, s_2)$ with $x_A < s_1 < 0 < s_2 < x_B$, and theorem 2 is proved. □

Proof of theorem 3. We start by assuming again the existence of a periodic orbit Γ contained in the band $a < x < b$ with all the geometric properties already established in the proof of theorem 2. Furthermore, we assume that there is a unique value $\hat{p} < \min\{y_A, y_B\}$ such that $h_2(p) < h_1(p)$ for $0 < p < \hat{p}$, and $h_1(p) < h_2(p)$ for $p > \hat{p}$.

We claim first that $y_A > y_B$, as shown in figures 4 and 5. Now we start the proof of the claim. By using statement (a) of lemma 7 in the interval $[0, \hat{p}]$ it follows that

$$y_l(x_2(p)) < y_l(x_1(p)) \qquad \text{for } 0 < p \leqslant \hat{p},$$

and analogously, by using remark 8 for the upper part, we have

$$y_u(x_1(p)) < y_u(x_2(p)) \qquad \text{for } 0 < p \leqslant \hat{p}.$$

From the above inequalities we see that when the paths start to separate from the y-axis the two arcs of path $\Gamma_2(p)$ are farther from the p-axis than the two arcs of path $\Gamma_1(p)$, see figure 5. We already know from the proof of theorem 2 that both paths intersect and now from the relative position of their beginning arcs at the y-axis we can ensure that their crossing points appear in even-number multiplicities. In fact, due to the uniqueness of solutions of system (10), we conclude now that there exists a unique value $\delta_1 > \hat{p}$ such that

$$y_l(x_2(p)) < y_l(x_1(p)) \qquad \text{for } 0 < p < \delta_1,$$

$$y_l(x_2(p)) > y_l(x_1(p)) \qquad \text{for } \delta_1 < p < \min\{y_A, y_B\}.$$

This lower crossing point at $p = \delta_1$ for $y_l(x_1(p))$ and $y_l(x_2(p))$ must be unique because resorting to lemma 7(a) in the interval $[\delta_1, \min\{y_A, y_B\}]$ we have $y_l(x_1(p)) < y_l(x_2(p))$ in such an interval. Similarly, by using remark 8 there is a unique value $\delta_2 > \hat{p}$ such that for the upper parts

$$y_u(x_1(p)) < y_u(x_2(p)) \qquad \text{for } 0 < p < \delta_2,$$

$$y_u(x_1(p)) > y_u(x_2(p)) \qquad \text{for } \delta_2 < p < \min\{y_A, y_B\}.$$

Therefore, as these two crossing points are only possible when $y_A > y_B$, if system (3) has a periodic orbit then the condition $y_A > y_B$ holds and our first claim is proved.

We now claim that the characteristic exponent of a periodic orbit of system (3) is negative; that is, the periodic orbit is a stable limit cycle. Hence, the system has at most one periodic orbit because we cannot have two consecutive stable periodic orbits. This should complete the proof of theorem 3. Now we prove this second claim.

Let $C = (x_C, y_C)$ be the point on the curve Ω for which $x_A < x_C < 0$ and $y_C = F(x_C) = F(x_B) = y_B > y_l(x_C)$, and let $\tilde{\Gamma}$ be the orbit of (3) passing through the point C. Then the orbit $\tilde{\Gamma}$ meets the y-axis in the points K and L (see figure 5), where $y_M < y_K < 0 < y_L < y_N$. The orbit $\tilde{\Gamma}$ is given by the graph of $y = \tilde{y}_l(x)$ on the arc CK and by the graph of $y = \tilde{y}_u(x)$ on the arc LC. Since $y_M = y_l(0) < y_K = \tilde{y}_l(0)$, lemma 7(a) in the interval $[0, \hat{p}]$ implies

$$y_l(x_2(p)) < \tilde{y}_l(x_1(p)) \qquad \text{for } 0 \leqslant p \leqslant \hat{p}. \tag{27}$$

The previous inequality can be extended to ensure that

$$y_l(x_2(p)) < \tilde{y}_l(x_1(p)) \qquad \text{for } \hat{p} \leqslant p \leqslant y_B, \tag{28}$$

by using statement (b) of lemma 7 in the interval $[\hat{p}, y_B]$ because we know that $y_l(x_2(y_B)) = y_B = y_C = \tilde{y}_l(x_1(y_B))$ and that $h_1(p) < h_2(p)$ for $p > \hat{p}$.

By using remark 8 in an analogous way, we can show that

$$\tilde{y}_u(x_1(p)) < y_u(x_2(p)) \qquad \text{for } 0 < p < y_B. \tag{29}$$

Next we compute the characteristic exponent ρ of the periodic orbit Γ, i.e.

$$\rho = \int_\Gamma f(x(t))\,\mathrm{d}t,$$

where the line integral is described in the sense of the flow, that is counterclockwise.

The periodic orbit $\Gamma = \{(x(t), y(t))\}$ intersects the line $x = s_2$ at the points M_2 and N_2, and the orbit $\tilde{\Gamma} = \{(\tilde{x}(t), \tilde{y}(t))\}$ intersects the line $x = s_1$ at the points K_1 and L_1, see figure 5. We first compute the integral

$$I = \int_{MBN} f(x(t))\,\mathrm{d}t + \int_{LCK} f(\tilde{x}(t))\,\mathrm{d}t$$

along the arc MBN of the periodic orbit Γ and along the arc LCK of the path $\tilde{\Gamma}$.

To this end we compute the following integral

$$\begin{aligned} I_1 &= \int_{\Gamma:MM_2} f(x(t))\,\mathrm{d}t + \int_{\tilde{\Gamma}:K_1K} f(\tilde{x}(t))\,\mathrm{d}t = \int_0^{s_2} \frac{f(x)}{F(x) - y_l(x)}\,\mathrm{d}x + \int_{s_1}^0 \frac{f(x)}{F(x) - \tilde{y}_l(x)}\,\mathrm{d}x \\ &= \int_0^{\hat{p}} \frac{\mathrm{d}p}{p - y_l(x_2(p))} - \int_0^{\hat{p}} \frac{\mathrm{d}p}{p - \tilde{y}_l(x_1(p))} = \int_0^{\hat{p}} \frac{[y_l(x_2(p)) - \tilde{y}_l(x_1(p))]\,\mathrm{d}p}{[p - y_l(x_2(p))][p - \tilde{y}_l(x_1(p))]}, \end{aligned}$$

and from (27) we conclude that $I_1 < 0$. Now we consider

$$\begin{aligned} I_2 &= \int_{\Gamma:M_2B} f(x(t))\,\mathrm{d}t + \int_{\tilde{\Gamma}:CK_1} f(\tilde{x}(t))\,\mathrm{d}t = \int_{s_2}^{x_B} \frac{f(x)}{F(x) - y_l(x)}\,\mathrm{d}x + \int_{x_C}^{s_1} \frac{f(x)}{F(x) - \tilde{y}_l(x)}\,\mathrm{d}x \\ &= \int_{\hat{p}}^{y_B} \frac{\mathrm{d}p}{p - y_l(x_2(p))} - \int_{\hat{p}}^{y_B} \frac{\mathrm{d}p}{p - \tilde{y}_l(x_1(p))} = \lim_{\eta \to y_B} \int_{\hat{p}}^{\eta} \frac{[y_l(x_2(p)) - \tilde{y}_l(x_1(p))]\,\mathrm{d}p}{[p - y_l(x_2(p))][p - \tilde{y}_l(x_1(p))]}, \end{aligned}$$

and from (28) we conclude that $I_2 < 0$. We have

$$\begin{aligned} I_3 &= \int_{\Gamma:BN_2} f(x(t))\,\mathrm{d}t + \int_{\tilde{\Gamma}:L_1C} f(\tilde{x}(t))\,\mathrm{d}t = \int_{x_B}^{s_2} \frac{f(x)}{F(x) - y_u(x)}\,\mathrm{d}x + \int_{s_1}^{x_C} \frac{f(x)}{F(x) - \tilde{y}_u(x)}\,\mathrm{d}x \\ &= \int_{\hat{p}}^{y_B} \frac{\mathrm{d}p}{y_u(x_2(p)) - p} - \int_{\hat{p}}^{y_B} \frac{\mathrm{d}p}{\tilde{y}_u(x_1(p)) - p} = \lim_{\eta \to y_B} \int_{\hat{p}}^{\eta} \frac{[\tilde{y}_u(x_1(p)) - y_u(x_2(p))]\,\mathrm{d}p}{[y_u(x_2(p)) - p][\tilde{y}_u(x_1(p)) - p]}, \end{aligned}$$

and from (29) we conclude that $I_3 < 0$. We compute

$$I_4 = \int_{\Gamma:N_2N} f(x(t))\,\mathrm{d}t + \int_{\tilde{\Gamma}:LL_1} f(\tilde{x}(t))\,\mathrm{d}t = \int_{s_2}^{0} \frac{f(x)}{F(x) - y_u(x)}\,\mathrm{d}x + \int_{0}^{s_1} \frac{f(x)}{F(x) - \tilde{y}_u(x)}\,\mathrm{d}x$$
$$= \int_0^{\hat{p}} \frac{\mathrm{d}p}{y_u(x_2(p)) - p} - \int_0^{\hat{p}} \frac{\mathrm{d}p}{\tilde{y}_u(x_1(p)) - p} = \int_0^{\hat{p}} \frac{\left[\tilde{y}_u(x_1(p)) - y_u(x_2(p))\right]\mathrm{d}p}{\left[y_u(x_2(p)) - p\right]\left[\tilde{y}_u(x_1(p)) - p\right]},$$

and from (29) we conclude that $I_4 < 0$. Hence $I = I_1 + I_2 + I_3 + I_4 < 0$ and

$$\rho = I + \int_{\Gamma:NAM} f(x(t))\,\mathrm{d}t - \int_{\tilde{\Gamma}:LCK} f(\tilde{x}(t))\,\mathrm{d}t.$$

We define

$$J_1 = \int_{\Gamma:NA} f(x(t))\,\mathrm{d}t - \int_{\tilde{\Gamma}:LC} f(\tilde{x}(t))\,\mathrm{d}t,\;\; J_2 = \int_{\Gamma:AM} f(x(t))\,\mathrm{d}t - \int_{\tilde{\Gamma}:CK} f(\tilde{x}(t))\,\mathrm{d}t,$$

so that $\rho = I + J_1 + J_2$. Now we will show that $J_1 < 0$ and $J_2 < 0$.

We compute the integral

$$J_1 = \int_0^{y_A} \frac{\mathrm{d}p}{p - y_u(x_1(p))} - \int_0^{y_B} \frac{\mathrm{d}p}{p - \tilde{y}_u(x_1(p))} = \int_0^{y_A} \frac{\mathrm{d}p}{p - y_u(x_1(p))} - \int_0^{y_A} \frac{\mathrm{d}p}{p - \hat{y}_u(p)}$$
$$= \int_0^{y_A} \frac{y_u(x_1(p)) - \hat{y}_u(p)}{\left[p - y_u(x_1(p))\right]\left[p - \hat{y}_u(p)\right]}\,\mathrm{d}p,$$

where the function $\hat{y}_u(p)$ is given by

$$\hat{y}_u(p) = \mu \tilde{y}_u\left(x_1\left(\mu^{-1}p\right)\right) \qquad \text{and} \qquad \mu = \frac{y_A}{y_B} > 1.$$

Clearly the function $\hat{y}_u(p)$ is a solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{\hat{h}_1(p)}{p - y},$$

where $\hat{h}_1(p) = \mu h_1(\mu^{-1}p)$.

The function α defined in (14), which can be written for $x < 0$ as

$$\alpha(x_1(p)) = \frac{g(x_1(p))}{f(x_1(p))p} = \frac{h_1(p)}{p},$$

is an increasing function of x_1, and so a decreasing function of p. Then $h_1(\mu^{-1}p) > \mu^{-1}h_1(p)$, so $h_1(p) < \mu h_1(\mu^{-1}p) = \hat{h}_1(p)$ for $p \geqslant 0$. We recall that for $y = y_u(x_1(p))$ we knew that

$$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{h_1(p)}{p - y}.$$

Now from the equality

$$\hat{y}_u(y_A) = \frac{y_A}{y_B}\tilde{y}_u\left(x_1\left(\frac{y_B}{y_A}y_A\right)\right) = y_A = y_u(x_1(y_A))$$

and using statement (b) corresponding to remark 8 for h_1 and $\hat{h}_1$ in interval $[0, y_A]$ we get $y_u(x_1(p)) < \hat{y}_u(p)$ for $0 < p < y_A$ and consequently $J_1 < 0$.

Similarly we can show that $J_2 < 0$, and the proof is complete. □

7. Proof of theorems 4 and 5

First we prove theorem 4.

Proof of theorem 4. We first check the hypotheses H1–H3 in order to see that both theorems 2 and 3 can be applied.

Hypotheses H1 and H2 are immediate.

We will use the functions h_i in checking hypothesis H3, and noting that $x_i(p) = p/t_i$ for $i = 1, 2$, we have

$$h_i(p) = \frac{d_i}{t_i^2} p + \frac{a_i}{t_i},$$

for $i = 1, 2$. Now hypothesis H3 is fulfilled whenever $l_2 = a_2/t_2 < l_1 = a_1/t_1$ or if $a_2/t_2 = a_1/t_1$ when $d_2/t_2^2 < d_1/t_1^2$. The equation $h_1(p) = h_2(p)$ becomes equivalent to

$$\left(\frac{d_1}{t_1^2} - \frac{d_2}{t_2^2}\right) p = \frac{a_2}{t_2} - \frac{a_1}{t_1}, \tag{30}$$

which has a unique positive solution only if

$$\left(\frac{d_1}{t_1^2} - \frac{d_2}{t_2^2}\right)\left(\frac{a_2}{t_2} - \frac{a_1}{t_1}\right) > 0.$$

Now statement (a) of theorem 4 is a direct consequence of theorems 2 and 3.

Statement (b) can be shown by using statement (a) applied to the system

$$\begin{cases} \dot{x} = (-t_2)x - y, \\ \dot{y} = d_2 x + (-a_2), \end{cases} \quad \text{if } x \leqslant 0, \qquad \begin{cases} \dot{x} = (-t_1)x - y, \\ \dot{y} = d_1 x + (-a_1), \end{cases} \quad \text{if } x > 0,$$

which corresponds to systems (16) after the change in variable $(x, y, \tau) \to (-x, y, -\tau)$.

Regarding statement (c), it is obvious that equation (30) has no solutions different from zero when $d_1/t_1^2 - d_2/t_2^2 \neq 0$ and the first assertion then comes from theorem 2. In the remaining case we have $h_1(p) = h_2(p)$ for all p and the conclusion on having a centre comes from the application of theorem 11.3 in [14] to system (25). This completes the proof of theorem 4. □

Starting from system (25) corresponding to system (16), when both dynamics are of focus type (stable for $x < 0$, unstable for $x > 0$) it is possible to globally define a Poincaré return map by introducing a transversal section to the flow. For that we select the negative y-axis and define $P : (0, \infty) \to (0, \infty)$ which maps the coordinate $y > 0$ of the point $(0, -y)$ into the vertical coordinate $P(y)$ of the point $(0, -P(y))$, where both points are the initial and the final points, respectively, of one orbit that gives a complete counterclockwise turn around the origin. For more details on the definition of this Poincaré map see proof of lemma 9. The explicit computation of this map P should solve the problem of determining the exact number of periodic orbits; however, this is not possible in general. The following result will be needed later.

Lemma 9. *Under the assumptions of theorem 5 the derivative of the Poincaré map P satisfies*

$$\lim_{y\to\infty} \frac{\mathrm{d}P}{\mathrm{d}y} = \mathrm{e}^{\pi(\kappa_2 - \kappa_1)},$$

where for $i = 1, 2$,

$$\kappa_i = \frac{1}{\sqrt{\dfrac{4d_i}{t_i^2} - 1}}.$$

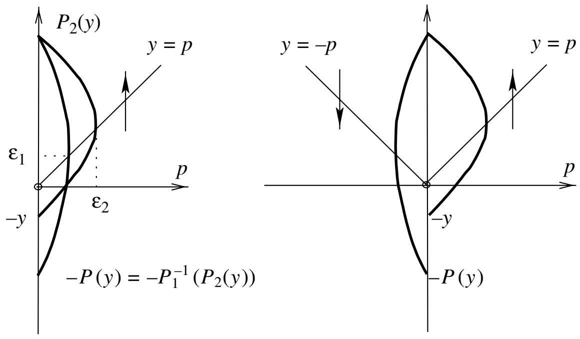

Figure 6. Building the Poincaré map near the origin by using the half-return maps P_1 and P_2.

Proof. To work in a more compact way we choose the folded plane, that is systems (22), namely,

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = p - y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{d_i}{t_i^2} p + \frac{a_i}{t_i},$$

for $p \geqslant 0$, and we define

$$\omega_i = \sqrt{\frac{d_i}{t_i^2} - \frac{1}{4}}, \qquad p_i^e = -\frac{a_i t_i}{d_i},$$

for $i = 1, 2$. Integrating both linear systems taking as initial point $(0, -y)$, we have

$$\begin{pmatrix} p_i(\tau) - p_i^e \\ y_i(\tau) - y_i^e \end{pmatrix} = \exp\left(\frac{\tau}{2}\right) C_i(\tau) \begin{pmatrix} 0 - p_i^e \\ -y - y_i^e \end{pmatrix},$$

where $y_i^e = p_i^e$, and

$$C_i(\tau) = \begin{pmatrix} \cos(\omega_i\tau) + \dfrac{\sin(\omega_i\tau)}{2\omega_i} & -\dfrac{\sin(\omega_i\tau)}{\omega_i} \\ \dfrac{d_i \sin(\omega_i\tau)}{t_i^2\omega_i} & \cos(\omega_i\tau) - \dfrac{\sin(\omega_i\tau)}{2\omega_i} \end{pmatrix}.$$

After one half-turn around the origin following these solutions $(p_i(\tau), y_i(\tau))$, we will arrive at the positive part of the y-axis for certain values τ_i such that $p_i(\tau_i) = 0$ with $0 < \omega_i\tau_i < \pi$, see figure 6. The corresponding values of $y_i(\tau_i)$ allow us to define the half-return maps

$P_i : (0, \infty) \to (0, \infty)$ with $P_i(y) = y_i(\tau_i)$ and $p_i(\tau_i) = 0$ with $0 < \omega_i\tau_i < \pi$,

for $i = 1, 2$. Now the return map $P(y)$ of system (25) corresponding to system (16) can be recovered by taking $P(y) = P_1^{-1}(P_2(y))$.

As shown in [9] for the continuous case, the study of such half-return Poincaré maps is not possible explicitly and must be done in a parametric way. Thus the notation $\theta_i = \omega_i\tau_i$ and $\kappa_i = 1/(2\omega_i)$ for $i = 1, 2$, the map P_i is determined by the equation

$$\mathrm{e}^{\kappa_i\theta_i} \begin{pmatrix} \cos\theta_i + \kappa_i \sin\theta_i & -2\kappa_i \sin\theta_i \\ \dfrac{1+\kappa_i^2}{2\kappa_i} \sin\theta_i & \cos\theta_i - \kappa_i \sin\theta_i \end{pmatrix} \begin{pmatrix} -p_i^e \\ -y - p_i^e \end{pmatrix} = \begin{pmatrix} -p_i^e \\ P_i(y) - p_i^e, \end{pmatrix}$$

and it is parametrically described for each value of $\theta_i \in (0, \pi)$ as follows,

$$y = -p_i^e \frac{e^{-\kappa_i \theta_i} - \cos\theta_i + \kappa_i \sin\theta_i}{2\kappa_i \sin\theta_i},$$

$$P_i(y) = -p_i^e \frac{e^{\kappa_i \theta_i} - \cos\theta_i - \kappa_i \sin\theta_i}{2\kappa_i \sin\theta_i},$$

for $i = 1, 2$. Hence a straightforward computation now shows that for the derivatives we also have the parametric representation

$$\frac{dP_i}{dy}(\theta_i) = \frac{1 - e^{\kappa_i\theta_i}(\cos\theta_i - \kappa_i \sin\theta_i)}{1 - e^{-\kappa_i\theta_i}(\cos\theta_i + \kappa_i \sin\theta_i)} = e^{2\kappa_i\theta_i} \frac{y}{P_i(y)},$$

so that

$$\lim_{y\to\infty} \frac{dP_i(y)}{dy} = \lim_{\theta_i \to \pi^-} \frac{1 - e^{\kappa_i\theta_i}(\cos\theta_i - \kappa_i \sin\theta_i)}{1 - e^{-\kappa_i\theta_i}(\cos\theta_i + \kappa_i \sin\theta_i)} = \frac{1 + e^{\kappa_i\pi}}{1 + e^{-\kappa_i\pi}} = e^{\kappa_i\pi}.$$

We can conclude by the chain rule and the inverse function theorem that

$$\lim_{y\to\infty} \frac{dP(y)}{dy} = \lim_{y\to\infty} \frac{dP_1^{-1}(P_2(y))}{dy} = \frac{1}{e^{\kappa_1\pi}} e^{\kappa_2\pi} = e^{\pi(\kappa_2 - \kappa_1)},$$

and the lemma follows. □

This last result can also be obtained by resorting to the techniques followed in [17].

We finish by giving the proof of theorem 5. For that we first show another technical result.

Lemma 10. *Hypothesis H3 implies that the origin is an unstable topological focus if $l_2 > 0$.*

Proof. We will show that when $y > 0$ is sufficiently small the Poincaré map introduced in this section satisfies $P(y) > y$. Taking a point $(\varepsilon_i, \varepsilon_i)$ on the line $y = p$ sufficiently near the origin, we know that for the orbit passing through this point

$$\frac{dp_i}{dy} = 0, \qquad \frac{d^2 p_i}{dy^2} = -\frac{1}{h_i(\varepsilon_i)},$$

and then the corresponding orbits can be approximated by

$$p_i(y) = \varepsilon_i - \frac{1}{2h_i(\varepsilon_i)}(y - \varepsilon_i)^2 + \mathcal{O}(y - \varepsilon_i)^3,$$

which cuts the y-axis at the points

$$y_i^{\pm} \approx \pm\sqrt{2h_i(\varepsilon_i)\varepsilon_i} + \varepsilon_i \approx \pm\sqrt{2l_i\varepsilon_i} + \varepsilon_i.$$

Note that for ε_i sufficiently small we have that $y_i^- < 0$.

We choose ε_1 and ε_2 for the system with $i = 1, 2$ in such a way that the two quadratic approximations for the orbits coincide in the positive y-axis, namely,

$$\varepsilon_1 + \sqrt{2h_1(\varepsilon_1)\varepsilon_1} = \varepsilon_2 + \sqrt{2h_2(\varepsilon_2)\varepsilon_2}. \tag{31}$$

From remark 8, and considering only the part of the orbits contained in the region $y > p$, we can assure that $\varepsilon_1 < \varepsilon_2$, see figure 6. Now taking $y = -y_2^-$ we can make the approximation $P(y) \approx -y_1^-$ so that

$$P(y) - y \approx -(\varepsilon_1 - \sqrt{2h_1(\varepsilon_1)\varepsilon_1}) + \varepsilon_2 - \sqrt{2h_2(\varepsilon_2)\varepsilon_2} = 2(\varepsilon_2 - \varepsilon_1) > 0,$$

where we have taken into account the equality (31). This implies that the origin is an unstable topological focus, see figure 6. □

Proof of theorem 5. Reasoning as in the proof of theorem 4 we need only show statement (a).

From lemma 10 we know that the origin is unstable and in particular that for the Poincaré map P in this section we have $P(y) > y$ for $y > 0$ and sufficiently small.

The assumptions assure that $d_2/t_2^2 > d_1/t_1^2$, and using that the function $1/\sqrt{4x-1}$ is decreasing for $x > 1/4$ we see that $\kappa_2 - \kappa_1 < 0$. Therefore, from lemma 9 we have

$$L = \lim_{y\to\infty} \frac{\mathrm{d}P}{\mathrm{d}y} = \mathrm{e}^{\pi(\kappa_2-\kappa_1)} < 1.$$

We will now claim that there exists $y^* > 0$ with $P(y^*) < y^*$ so that from the intermediate value theorem we deduce the existence of a periodic orbit. Then the conclusion of the theorem follows from theorem 3.

Effectively we can assure that there exists a certain value $\bar{y}$ such that for $y \geqslant \bar{y}$ we have

$$\frac{\mathrm{d}P}{\mathrm{d}y} < \frac{1+L}{2} = \bar{L} < 1.$$

If $P(\bar{y}) < \bar{y}$ we are done. Otherwise taking $y^* > \bar{y}$ and invoking the mean value theorem we have

$$P(y^*) - P(\bar{y}) < \bar{L}(y^* - \bar{y}),$$

which implies that

$$P(y^*) - y^* < P(\bar{y}) - \bar{L}\bar{y} - (1-\bar{L})y^*,$$

which is clearly negative if y^* is big enough and the claim is true. □

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