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Abstract

Let G be a graph whose vertices are labeled $1,\dots,n$, and π be a permutation on $[n] := \{1,2,\dots,n\}$. A pebble p_i that is initially placed at the vertex i has destination $\pi(i)$ for each $i \in [n]$. At each step, we choose a matching and swap the two pebbles on each of the edges. Let $rt(G,\pi)$, the routing number for π, be the minimum number of steps necessary for the pebbles to reach their destinations.

Li, Lu and Yang proved that $rt(C_n,\pi) \leqslant n-1$ for every permutation π on the n-cycle C_n and conjectured that for $n \geqslant 5$, if $rt(C_n,\pi) = n-1$, then $\pi = 23\cdots n1$ or its inverse. By a computer search, they showed that the conjecture holds for $n < 8$. We prove in this paper that the conjecture holds for all even $n \geqslant 6$.

Keywords: Routing number, permutation, sorting algorithm, Cayley graphs

1 Introduction

Routing problems occur in many areas of computer science. Sorting a list involves routing each element to the proper location. Communication across a network involves routing

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messages through appropriate intermediaries. Message passing between multiprocessors requires the routing of signals to correct processors.

In each case, one would like the routing to be done as quickly as possible. Let us consider the routing model introduced by Alon, Chung, and Graham [2] in 1994. Let $G = (V, E)$ be a graph whose vertices are labeled as $1, \ldots, n$. For a permutation π on $[n]$, a pebble p_i, which has destination $\pi(i)$, is placed at i for each $i \in [n]$. For example, let $\pi = 342165 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 6 & 5 \end{pmatrix}$, then the destination of $(p_1, p_2, p_3, p_4, p_5, p_6)$ are $(3, 4, 2, 1, 6, 5)$.

We wish to move pebbles to their destinations. To do so, at each round we select a matching of G and swap the pebbles at the endpoints of each edge, and repeat rounds until all pebbles are in place.

Let $rt(G, \pi)$ denote the minimum number of rounds necessary to route π on G. Then, the **routing number** of G is defined as:

$$rt(G) = \max_{\pi} \{rt(G, \pi)\}.$$

Note that when each matching in the routing process consists of one edge, the routing number corresponds to the diameter of Cayley graph generated by (cyclically) adjacent transpositions. The diameter problem of Cayley graphs has a rich research literature, and we refer the reader to the book [3] for a comprehensive survey.

Very few results are known for the exact values of the routing numbers of graphs.

Theorem 1 (Alon, Chung, and Graham [2]). *The following are true:*

1. $rt(P_n) = n$, $rt(K_n) = 2$ *and* $rt(K_{n,n}) = 4$;
2. $rt(G) \geqslant diam(G)$ *and* $rt(G) \geqslant \frac{2}{|C|} \min\{|A|, |B|\}$, *where* $diam(G)$ *is the diameter of* G *and* C *is a set that cuts* G *into parts* A *and* B;
3. $rt(G) \leqslant rt(H)$ *and* $rt(T_n) < 3n$, *where* H *is a spanning subgraph of* G *and* T_n *is a tree on* n *vertices;*
4. $rt(G_1 \times G_2) \leqslant 2rt(G_1) + rt(G_2)$, *and* $n \leqslant rt(Q_n) \leqslant 2n - 1$.

Zhang [6] improved the above bound on trees, showing that $rt(T_n) \leqslant \lfloor \frac{3n}{2} \rfloor + O(\log n)$.

Li, Lu, and Yang [5] showed that $n + 1 \leqslant rt(Q_n) \leqslant 2n - 2$, improving both the previous upper and lower bounds on hypercubes. Among other results, they showed that $rt(C_n) = n - 1$, and made the following conjecture.

Conjecture 2 (Li, Lu, Yang [5]). For $n \geqslant 5$, if $rt(C_n, \pi) = n - 1$, then π is the rotation $23 \cdots n1$ or its inverse $n12 \cdots (n-1)$.

The conjecture does not hold for $n = 4$; the permutation that transposes two non-adjacent vertices and fixes the other two serves as a counterexample. They verified the conjecture for $n < 8$ through a computer search. The conjecture is kind of counter-intuitive: the permutations on the cycle requiring most time to route are the ones that each pebble is very close to (actually at distance one from) its destination.

In this article, we give a proof of the conjecture when n is even.

Theorem 3. *For even $n \geqslant 6$, if $rt(C_n, \pi) = n - 1$, then π is the rotation $23 \cdots n1$ or its inverse.*

Some new tools are introduced in the proof, beyond the ideas from [1] by Albert, Li, Strang, and the last author. These tools, introduced in Section 2, enable us to describe precisely the swapping process of each pebble, and determine the pebbles that need $n-1$ steps to route, see Section 3 for detail. In Section 4, we show that the only permutations that need $n-1$ steps to route must be the two permutations in the theorem.

2 Important notion and tools

Let $G = C_n$ with even $n \geqslant 6$ and label the vertices of C_n as $1, 2, \ldots, n$ in the clockwise order. Let the clockwise direction be the positive direction and counter clockwise be the negative direction. In the rest of the paper, when list pebbles or sets of pebbles in a row, we always think them to be in the clockwise order on the cycle.

2.1 The Odd-Even Routing Algorithm

An *odd-even sort* or *odd-even transposition sort* is a classic sorting algorithm (see [4]) used to sort a list of numbers on parallel processors. To describe this algorithm, we may place the n numbers to be sorted on the vertices of the path $P_n = v_1 v_2 \ldots v_n$. An edge $e = v_i v_{i+1}$ is odd if and only if i is odd. At each odd step (respectively, even step) of the routing process, we select a matching consisting of odd edges (respectively, even edges) whose two numbers have the wrong order and swap the numbers on the endpoints.

We apply a similar algorithm on even cycles, whose edges can be partitioned into two perfect matchings. We shall call edges in one perfect matching to be even and the others to be odd. Thus, once the parity of one edge is specified, the parity of all the edges is determined. During the odd steps (respectively, even steps) we choose a matching consisting of odd edges (respectively, even edges) whose two pebbles are comparable (defined in subsequent subsections). If an edge e is chosen to be an odd edge, we would call this algorithm the *odd-even routing algorithm with odd edge e*.

2.2 Spins and Disbursements

There are exactly two paths for pebble p_i to reach its destination, by traveling either in the positive or negative direction. Let $d^+(i,j)$ denote the distance from the vertex i to the vertex j along the positive direction. Then if $i < j$, then $d^+(i,j) = j - i$ if $i < j$, and $d^+(i,j) = j - i + n$ if $i > j$. For simplicity, for pebbles p_i and p_j, we define $d^+(p_i, p_j) = d^+(i,j)$, and if p_i and p_j are on the endpoints of an edge, we sometimes call the edge $p_i p_j$.

Consider a routing process of a permutation π on C_n with pebble set $P = \{p_1, \dots, p_n\}$. For each pebble p_i, let $s(p_i)$, the *spin* of p_i, represent the displacement for p_i to reach its destination from its current position. So, $s(p_i) \in \{d^+(i, \pi(i)), d^+(i, \pi(i)) - n\}$. Note that the spin of a pebble changes with its movement.

A sequence $B = (s(p_1), s(p_2), \dots, s(p_n))$ is called a *valid disbursement* of π if the spins can be realized by a routing process on π. Not all combinations of spins give valid disbursements. The following lemma gives a necessary and sufficient condition for a set of spins to be a valid disbursement.

Lemma 4. *Let $B = (s(p_1), s(p_2), \dots, s(p_n))$ be an assignment of the spins to the pebbles. It is a valid disbursement if and only if $\sum_{i=1}^{n} s(p_i) = 0$.*

Proof. To see the necessity, we observe that when two pebbles are swapped, one moves forward one step and one moves backward one step, so the sum of spins remains invariant. Since B is a valid disbursement, the final spins are all zeroes, so the sum is also zero.

For sufficiency, we can move the pebbles one by one along their assigned directions. □

From this lemma, a valid disbursement of a non-identity permutation π must contain both positive and negative spins. Let $s(p_i) > 0$ and $s(p_j) < 0$ in a valid disbursement B of π. By *flipping the spins of* p_i *and* p_j, we change the spins of p_i and p_j to $s(p_i) - n$ and $s(p_j) + n$, respectively. Clearly, after one flip, we obtain a new valid disbursement.

A valid disbursement $(s(p_1), \dots, s(p_n))$ is minimized if $\sum_{p \in P} |s(p)|$ is minimized. The following simple fact is very important.

Lemma 5. *If a valid disbursement is minimized, then $s(p_i) - s(p_j) \leqslant n$ for all $i, j \in [n]$.*

Proof. For otherwise, one can make the sum smaller by flipping the spins of p_i and p_j. □

2.3 An order relation

Once a valid disbursement $B = (s(p_1), s(p_2), \dots, s(p_n))$ of π is given, some restrictions are placed on the routing processes realizing B. For example, if $s(p_i) - s(p_j) > d^+(p_i, p_j)$, then p_i and p_j must swap at some round in the routing processes. In other words, each valid disbursement is associated with an order relation on the pebbles.

Definition 6. Let $B = (s(p_1), s(p_2), \dots, s(p_n))$ be a valid disbursement. We call $p_i \succ p_j$ if $s(p_i) - s(p_j) > d^+(p_i, p_j)$

Note that the order relation is transitive. To see that, let $p_i \succ p_j$ and $p_j \succ p_k$. Then $s(p_i) - s(p_j) > d^+(p_i, p_j)$ and $s(p_j) - s(p_k) > d^+(p_j, p_k)$. It follows that $s(p_i) - s(p_k) > d^+(p_i, p_j) + d^+(p_j, p_k) \geqslant d^+(p_i, p_k)$, which implies $p_i \succ p_k$.

As two pebbles have different destinations, $s(p_i) - s(p_j) \neq d^+(p_i, p_j)$, so if $p_i \succ p_j$ is not true, then $s(p_i) - s(p_j) < d^+(p_i, p_j)$. When $p_i \succ p_j$, we say that p_i and p_j are *comparable*, or more precisely, p_i *is bigger than* p_j and p_j *is smaller than* p_i. If p_i is neither bigger nor smaller than p_j, we call them *incomparable*. If each pebble in set P_1 is bigger than every pebble in P_2, we also write $P_1 \succ P_2$.

The following lemma provides a convenient way to determine order relations.

Lemma 7. *Let x, y, z be three pebbles in the clockwise order sitting on the cycle. If $x \succ z$, then $x \succ y$ or $y \succ z$. Furthermore, if $x \succ z$, then y is not smaller than z and not bigger than x.*

Proof. For otherwise, $s(x) - s(y) < d^+(x, y)$ and $s(y) - s(z) < d^+(y, z)$. It follows that $s(x) - s(z) < d^+(x, y) + d^+(y, z) = d^+(x, z)$. Then x is not bigger than z, a contradiction.

For the furthermore part, if $x \succ z$ and $z \succ y$, then $s(x) - s(z) \geqslant d^+(x, z)$ and $s(z) - s(y) \geqslant d^+(z, y)$, and it follows that $s(x) - s(y) \geqslant d^+(x, z) + d^+(z, y) > n$, a contradiction. Likewise, if $x \succ z$ and $y \succ x$, then $s(x) - s(z) \geqslant d^+(x, z)$ and $s(y) - s(x) \geqslant d^+(y, x)$, and it follows that $s(y) - s(z) \geqslant d^+(x, z) + d^+(y, x) > n$, a contradiction. □

The following lemma says that we only need to swap comparable pebbles to route the permutation.

Lemma 8. *Let B be a minimized disbursement of π. If a pebble p is incomparable with all other pebbles, then $s(p) = 0$, i.e., the pebble p is at its destination vertex.*

Proof. Suppose that $s(p) \neq 0$. By symmetry, let $s(p) > 0$.

Let $\pi = \Pi_i \pi_i$ be a cycle decomposition of π, where $\pi_i = (i_1, \cdots, i_{r_i})$. Then the pebble placed at i_k, which we call pebble i_k to save symbols, has destination i_{k+1} for all $k \leqslant r_i$, with $i_{r_i+1} = i_1$. Let P_i be the set of pebbles on π_i. We say that π_i is the orbit of the pebbles in P_i. We may assume that $p = i_1$.

We claim that for each j, $\sum_{q \in P_j} s(q) = an$ for some integer a. To see this, we note that $s(j_k) \in \{d^+(j_k, j_{k+1}), d^+(j_k, j_{k+1}) - n\}$. Thus, if the spins are all positive, the sum equals bn for some positive integer b. However, each switch of a spin from positive to negative would cause a change of $-n$ in the sum. So the sum of spins remains a multiple of n.

We further claim that all pebbles in P_i have positive spins, which also implies that $\sum_{q \in P_i} s(q) = bn$ for some positive integer b. Note that $s(i_1) = s(p) > 0$. Let j be the smallest integer so that $s(i_j) < 0$. Then $s(i_j) = d^+(i_j, i_{j+1}) - n \leqslant -d^+(i_1, i_j)$. So $s(i_1) > 0 \geqslant s(i_j) + d^+(i_1, i_j)$. It follows that $s(i_1) - s(i_j) > d^+(i_1, i_j)$, that is, $p = i_1 \succ i_j$, a contradiction.

As the sum of all spins is zero, there exists some orbit π_j with spin sum cn for some integer $c < 0$. In particular, there exists a pebble $q \in P_j$ such that q passes $p = i_1$ in the negative direction to arrive its destination. So $s(q) + d^+(p, q) < 0 < s(p)$ and it follows that $p \succ q$, a contradiction. □

Note that the order of pebbles is always associated with the current disbursement, which may not be the same as the initial disbursement. The following lemma says that whether or not two pebbles swap is determined by the initial disbursement. So we will not keep tracking of the spins, but just see whether necessary swaps are performed.

Lemma 9. *If p_i and p_j are incomparable, then in the sorting process, they will always be incomparable. If $p_i \succ p_j$, then p_i and p_j are incomparable after the swap of p_i and p_j.*

Proof. If p_i and p_j are incomparable, then in the sorting process, $(s(p_i)-s(p_j))-d^+(p_i,p_j)$ does not change: if a pebble swaps with both p_i and p_j, then it must be bigger than or smaller than both p_i and p_j, thus the distance from p_i to p_j does not change and $s(p_i)-s(p_j)$ does not change; If a pebble swaps only with p_i, then $s(p_i)$ increases by one and $d^+(p_i,p_j)$ increases by one; If a pebble swaps only with p_j, then $s(p_j)$ increases by one and $d^+(p_i,p_j)$ decreases by one.

Now let $p_i \succ p_j$. We only need to show that they become incomparable right after the swap of p_i and p_j, by what we just proved. Since $p_i \succ p_j$, $n \geqslant s(p_i)-s(p_j) \geqslant d^+(p_i,p_j)+1 \geqslant 2$. Let $s'(p_i)$ and $s'(p_j)$ respectively be the new spins of p_i and p_j right after the swap of p_i and p_j. Note that p_i and p_j are adjacent to each other only if the pebbles on the segment from p_i to p_j along the positive direction have swapped with p_i (for those smaller than p_i) or p_j (for those bigger than p_j). Then $s'(p_i)-s'(p_j) = s(p_i)-s(p_j)-d^+(p_i,p_j)-2$. Therefore, $s'(p_j)-s'(p_i) < 0$, and p_j cannot be bigger than p_i. Also, $s'(p_i)-s'(p_j) \leqslant n-2 < n-1$, thus after the swap, p_i cannot be bigger than p_j. □

When B is minimized and two pebbles p_i and p_j satisfy $s(p_i)-s(p_j)=n$, we still get a minimized disbursement after flipping the spins of p_i and p_j. The following lemma tells us how the order relation changes when we do such a flip.

Lemma 10. *Let B be a minimized disbursement of π, and $s(p_i)-s(p_j)=n$. Let B' be the disbursement after flipping the spins of p_i and p_j. Let $k \notin \{i,j\}$. Then*

1. *$p_j \succ p_i$ under B', and the order relation remains unchanged for pebbles other than p_i and p_j;*
2. *$p_k \succ p_i$ under B' if p_i and p_k are incomparable under B; and p_i and p_k are incomparable under B' if $p_i \succ p_k$ under B;*
3. *$p_j \succ p_k$ under B' if p_j and p_k are incomparable under B; and p_j and p_k are incomparable under B' if $p_k \succ p_j$ under B.*

Proof. Let $s'(p_i)=s(p_i)-n$ and $s'(p_j)=s(p_j)+n$.

(1) Clearly $p_j \succ p_i$ under B', since $s'(p_j)-s'(p_i) = 2n-(s(p_i)-s(p_j)) = n > d^+(p_j,p_i)$. For pebbles not in $\{p_i,p_j\}$, the spins and distance do not change from B to B', so their order relation does not change as well.

(2) For $k \notin \{i,j\}$, we know that

$$s(p_k)-s'(p_i)-d^+(p_k,p_i) = s(p_k)-(s(p_i)-n)-(n-d^+(p_i,p_k)) = s(p_k)-s(p_i)+d^+(p_i,p_k).$$

Therefore, $s(p_k)-s'(p_i) > d^+(p_k,p_i)$ if and only if $s(p_i) < s(p_k)+d^+(p_i,p_k)$. Note that p_k cannot be bigger than p_i under B, for otherwise, $s(p_k)-s(p_j) > s(p_i)-s(p_j) = n$. It follows that $p_k \succ p_i$ under B' if and only if p_k and p_i are incomparable under B.

By flipping the spins of p_i and p_j in B', we get B. So we have the other part as well.

(3) Since $s'(p_j)-s'(p_i)=n$ under B', and one gets B after flipping the spins of p_i and p_j in B', these two statements follow from (2). □

2.4 The window of a pebble

Let B be a minimized disbursement of π with associated order $\succ$. For an arbitrary pebble p_0, let

$$U = \{p \in P : p \succ p_0\} \text{ and } W = \{p \in P : p_0 \succ p\}.$$

By Lemma 8, the routing process ends when no pebble has a bigger or smaller pebble. So we have the following equation, which is heavily used to determine the spins of the pebbles in our later proofs.

$$s(p_0) = |W| - |U|. \tag{1}$$

By Lemma 7, there are no $u \in U, w \in W$ such that u, w, p_0 or p_0, u, w on C_n. So if $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_t\}$, then we may assume that the pebbles in $U \cup W$ and p_0 are ordered as $u_r, \ldots, u_1, p_0, w_1, \ldots, w_t$ on C_n. We denote the set of pebbles incomparable to p_0 between p_0 and w_t (between u_r and p_0 resp.) by X (Y resp.).

A *segment* is a sequence of consecutive pebbles. An U-segment is a segment whose pebbles are all in U, and likewise, we have W-segments, X-segments and Y-segments. So we can group the pebbles between u_r and w_t along the positive direction as

$$win(p_0) = (U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l),$$

where X_1, Y_1 may be empty, and $win(p_0)$ is called the *initial window* of p_0. So in the window $win(p_0)$ of the pebble p_0, from the leftmost (the U-segment) to p_0, the segments are alternatively U- and Y-segments, and from the rightmost (the W-segment) to p_0, the segments are alternatively W- and X-segments. We shall use this notation without further notice.

We denote the set of all other pebbles as Z. So sometimes we write π as

$$\pi = (Z, U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l).$$

By transitivity, we have $u_i \succ w_j$ since $u_i \succ p_0 \succ w_j$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant t$, and in particular, $u_r \succ w_t$, hence $n \geqslant s(u_r) - s(w_t) > d^+(u_r, w_t)$. By Lemma 7, if $i \geqslant j$, then $u \succ y$ for all $u \in U_i,\ y \in Y_j$; If $k \geqslant l$, then $w \prec x$ for all $w \in W_k,\ x \in X_l$.

2.5 Two important lemmas

A nice property of the odd-even routing algorithm is the following

Lemma 11. *Let p be a pebble and Q be a segment of pebbles. If $p \succ Q$ or $Q \succ p$, then once p starts to swap with a pebble in Q in an odd-even routing algorithm, p will not stop swapping until p swaps with all pebbles in Q (in the following $|Q| - 1$ or more steps).*

Proof. By symmetry, we let $p \succ Q$. An *enlargement of* Q is a segment obtained from Q by mixing some pebbles that are smaller than p. So $p \succ Q'$ if Q' is an enlargement of Q.

We first claim that if $x, y \succ Q$, then after swapping with some pebbles in Q, x and y can never be on an edge unless they have swapped with all pebbles in Q. Suppose that the first time such an edge occurs at step s. That is, after step $s-1$, we have q, x, y, q', where $x, y \succ \{q, q'\}$. Now, at step $s-1$, xq and yq' are among the chosen edges. So, after step $s-1$, we should have q, x, q', y, a contradiction.

Note that for a pebble q between p and Q along the positive direction, by Lemma 7, either $p \succ q$ or $q \succ Q$; in the former case, q either becomes a pebble in an enlargement of Q or p, q swap before p meets a pebble in Q. So we may assume that all pebbles from p to Q along the positive direction are bigger than Q. Now by the claim, p never meets a pebble incomparable with p before it finishes swapping with Q. So p moves continuously. $\square$

Lemma 12 (Rotation Lemma). *Let q be an integer with $-\frac{n}{2} < q \leqslant \frac{n}{2}$, and π be the permutation that satisfies $\pi(a) = a+q \pmod n$ for each $a \in [n]$. Then, $rt(C_n, \pi) = n-|q|$.*

Proof. By symmetry we only consider the case when $q > 0$. For each pebble p, the spin of p is either q or $q-n$. Since the sum of spins is zero, there must be exactly $n-q$ pebbles with the positive spin and q pebbles with the negative spin. So, $n-q \leqslant rt(C_n, \pi)$.

Now, we show that $rt(C_n, \pi) \leqslant n-q$. Let the pebbles be $p_1, p_2, \ldots, p_n$ on C_n. We assign spins to the pebbles so that $s(p_{2i-1}) = q-n$ for $1 \leqslant i \leqslant q$ and each of the remaining $n-q$ pebbles has the spin q. We use an odd-even routing algorithm with odd edge p_1p_2. As $q \leqslant n/2$, no two pebbles with spin $q-n$ are adjacent. In the routing process, p_1 will be paired with $p_2, p_4, \ldots, p_{2q}, p_{2q+1}, \ldots, p_n$ in the first $n-q$ steps, thus reach its destination, and similar things occur for all other pebbles with negative spins. When all pebbles with negative spins reach their destinations, there is no comparable pebbles, so each pebble will be in place. Thus, π is routed in $n-q$ steps. $\square$

3 Extremal Windows

Now we count the steps needed for an arbitrary pebble, say p_0, to swap with all comparable pebbles. Let the initial window of p_0 be

$$win(p_0) = (U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l).$$

As we only consider how p_0 swaps with other pebbles, we ignore the swaps between pebbles inside each of the segments, and regard them to be incomprable for now.

By Lemma 11, when we apply the odd-even routing algorithm, p_0 will meet a segment $S \in \{U_1, \ldots, U_k, W_1, \ldots, W_l\}$ and swap with all the pebbles in S in the following $|S|$ steps. Assume that p_0 meets the segments in the order $S_1, S_2, \ldots, S_{k+l}$, where $S_i \in \{U_1, \ldots, U_k, W_1, \ldots, W_l\}$.

For $i = 1, 2, \ldots, k+l-1$, let ω_i be the waiting time between S_{i-1} and S_i, that is, the number of steps that p_0 waits between swapping with the last pebble of S_{i-1} and swapping with the first pebble of S_i. Let α be the largest index such that $\omega_\alpha \neq 0$.

By symmetry, we assume that $S_\alpha = W_t$. Because of the parity, a swap of p_0 and W (or p_0 and U) cannot be followed by a swap of p_0 and U (or p_0 and W). Therefore, as $\omega_{\alpha+1} = \cdots = \omega_{k+l} = 0$, $\cup_{i \geqslant \alpha} S_i = \cup_{i=t}^{l} W_i$ and p_0 will swap with them continuously until it reaches its destination. Let w be the pebble in W_t next to X_t. Since we ignore the swaps between pebbles in W, w only moves in one direction (counter-clockwise). Then the steps for p_0 to be in place are the steps for p_0 and w to meet plus $|\cup_{i=t}^{l} W_i|$.

To meet p_0, w has to swap with pebbles in $\cup_{j=1}^{t} X_j$ and $\cup_{i=1}^{k} U_i$. We also note that, w is always paired with a comparable pebble starting from the first or the second step, depending on the parity of the first edge with which w is incident. Thus the total number of steps for p_0 to be in place is:

$$\sum_{j=1}^{t} |X_j| + \sum_{j=t}^{l} |W_j| + \sum_{i=1}^{k} |U_i| + \delta, \tag{2}$$

where $\delta = 0$ if $w \in W_t$ is paired with an X-pebble in the first step, and $\delta = 1$ otherwise.

By symmetry, if $S_\alpha = U_t$, then the number of steps for p_0 to be in place is

$$\sum_{j=1}^{t} |Y_j| + \sum_{j=t}^{k} |U_j| + \sum_{i=1}^{l} |W_i| + \delta, \text{ where } \delta \in \{0, 1\}. \tag{3}$$

Therefore, every permutation that takes $n-1$ steps to route must contain a pebble p_0 such that (when $S_\alpha = W_t$)

$$\sum_{j=1}^{k} |Y_j| + \sum_{j=t+1}^{l} |X_j| + \sum_{j=1}^{t-1} |W_j| + |Z| = \delta, \text{ where } \delta \in \{0, 1\}. \tag{4}$$

or (when $S_\alpha = U_t$)

$$\sum_{j=1}^{l} |X_j| + \sum_{j=t+1}^{k} |Y_j| + \sum_{j=1}^{t-1} |U_j| + |Z| = \delta, \text{ where } \delta \in \{0, 1\}. \tag{5}$$

Now we are ready to determine the extreme windows that need $n-1$ steps to route.

Lemma 13. *Every permutation that takes $n-1$ steps to route must contain a pebble p_0 whose window is one of the following*

(i) $|win(p_0)| = n$ *and* $win(p_0) = (p_0, X, W)$ *(or* $win(p_0) = (U, Y, p_0)$*).*

(ii) $|win(p_0)| = n-1$ *and* $win(p_0) = (U, p_0, X, W)$ *(or* $win(p_0) = (U, Y, p_0, W)$*).*

(iii) $|win(p_0)| = n$*, and* $win(p_0) = (p_0, X_1, W_1, X_2, W_2)$ *and* $\min(|W_1|, |X_2|) = 1$ *(or* $win(p_0) = (U_2, Y_2, U_1, Y_1, p_0)$ *and* $\min(|Y_2|, |U_1|) = 1$*).*

In other words, if the window of a pebble is not one of the above ones, then in $n-2$ steps, the pebble will be in place.

Proof. By symmetry, we may assume that (4) holds. As $\delta = 0$ or 1, all the terms in the left-hand side of (4) are zeros or ones.

We first claim that when $|win(p_0)| = n$, U_k or W_l must be empty. For otherwise, let $u_p \in U_k$ and $w_q \in W_l$ be the furthest U-pebble and W-pebble to p_0, respectively. As $|win(p_0)| = n$, no pebble is bigger than u_p and no pebble is smaller than w_q by Lemma 7, so $s(u_p) \geqslant 1 + |Y| + |W|$ and $s(w_q) \leqslant -(1 + |X| + |U|)$, and it follows that $s(u_p) - s(w_q) \geqslant n + 1$, a contradiction.

Case 1. $\delta = 0$ or $\delta = |Z| = 1$. Then $\sum_{j=1}^{k} |Y_j| = \sum_{j=t+1}^{l} |X_j| = \sum_{j=1}^{t-1} |W_j| = 0$. It follows that $Y = \emptyset$, $t = 1 = l$. So $win(p_0) = (U, p_0, X, W)$. By the above claim, when $|win(p_0)| = n$, U or W must be empty, so we have (i) or (ii) in the lemma, where X (or Y) could be empty.

Case 2. $\delta = 1$ and $|Z| = 0$. Then $|win(p_0)| = n$, and one of the following holds:

- $\sum_{j=1}^{k} |Y_j| = 1$, and $\sum_{j=t+1}^{l} |X_j| = \sum_{j=1}^{t-1} |W_j| = 0$. Then $|Y| = 1, t = l = 1$. So there are at most two U-sets, U_1 and U_2, and when there are two, $Y_1 = \emptyset$ and $|Y_2| = 1$. Because of the above claim, we have $win(p_0) = (U_2, y, U_1, p_0)$.
- $\sum_{j=t+1}^{l} |X_j| = 1$ and $\sum_{j=1}^{k} |Y_j| = \sum_{j=1}^{t-1} |W_j| = 0$. Then $Y = \emptyset$ and $t = |X_2| = 1$. Because of the above claim, $win(p_0) = (p_0, X_1, W_1, x_2, W_2)$.
- $\sum_{j=1}^{t-1} |W_j| = 1$ and $\sum_{j=1}^{k} |Y_j| = \sum_{j=t+1}^{l} |X_j| = 0$. Then $Y = \emptyset$, $t = 2$ and $|W_1| = 1$, and $X_i = \emptyset$ for $i \geqslant 3$. Because of the above claim, $win(p_0) = (p_0, X_1, w_1, X_2, W_2)$.

So we have the desired extremal windows in the lemma. □

4 Proof of Theorem 3

In this section, we show how to deal with the extremal situations in Lemma 13.

For each of the extreme windows, we will decompose it into blocks of the following kinds. Let $q_1, q_2, \ldots, q_s$ be a segment of pebbles. It is called a *block with head* q_1 if $q_1 \succ q_i$ for $i \geqslant 2$ and the other pebbles are incomparable; it is called a *block with tail* q_s if $q_i \succ q_s$ for $i < s$ and the other pebbles are incomparable; it is called an *isolated block* if none of the pebbles is comparable.

We start with a minimized disbursement B of π, the permutation that cannot be routed in $n-2$ steps. By Lemma 13, π should contain a pebble with one of the extreme windows. We shall determine π explicitly and alter the disbursement and/or the odd-even routing algorithm to show that it can be routed in $n-2$ steps.

4.1 Extremal window type 1: $win(p_0) = (p_0, X, W)$ and $|win(p_0)| = n$

Lemma 14. *If permutation π needs $n-1$ steps to route and some pebble p_0 in π has $win(p_0) = (p_0, X, W)$ and $|win(p_0)| = n$, then π is $23\ldots n1$ or its inverse.*

Proof. Let $X = x_1x_2 \dots x_a$ and $W = w_1w_2 \dots w_b$. Consider the spins of p_0 and w_b. Note that $spin(p_0) = |W|$ and $s(w_b) \leqslant -(1 + |X|)$ since no pebble is smaller than w_b (by Lemma 7) and $\{p_0\} \cup X \succ w_b$. So $s(p_0) - s(w_b) \geqslant n$ and it follows that $s(w_b) = -1 - |X|$ and w_b is only comparable with $X \cup \{p_0\}$. Now repeat the argument for $w_{b-1}, \dots, w_1$ successively, we have $s(w_i) = -1 - |X|$ for $1 \leqslant i \leqslant b$.

Consider the spin of x_1. It is clear that $s(x_1) \geqslant |W|$ since $x_1 \succ W$ and no pebble is bigger than x_1 (by Lemma 7). Then $s(x_1) - s(w_b) \geqslant n$. It follows that $s(x_1) = |W|$ and $x_1 \succ W$ is the only order relation involving x_1. Inductively we have $s(x_i) = |W|$ for all $x_i \in X$ and $\{p_0\} \cup X \succ W$ is the only order relation in the permutation.

So along the positive direction every pebble is $|W|$ steps away from its destination. So π is a rotation. By Lemma 12, π must be $23 \cdots n1$ or its inverse. □

4.2 Extremal window type 2: $win(p_0) = (U, p_0, X, W)$ and $|win(p_0)| = n-1$

Lemma 15. *If a permutation π contains a pebble p_0 such that $win(p_0) = (U, p_0, X, W)$ and $\pi = (\{z\}, U, p_0, X, W)$, where $U, W \neq \emptyset$, then U and W are isolated blocks, and X can be partitioned into $X_1, \dots, X_r$ such that X_i is either an isolated block or a block with tail x_i. Furthermore, $s(z) = c \leqslant 0$, and if $c < 0$, then X_r is an isolated block with $-c$ pebbles, and the order relations are*

$$U \cup \{p_0\} \cup X \succ W, \qquad X_r \succ \{z\}, \qquad X_i - x_i \succ x_i \text{ for some } 1 \leqslant i \leqslant r.$$

Proof. Let $U = u_1u_2 \dots u_p, X = x_1x_2 \dots x_a$ and $W = w_1w_2 \dots w_b$. Consider the spins of u_1 and w_b. As no pebble is bigger than u_1 and $u_1 \succ \{p_0\} \cup W$, $s(u_1) \geqslant 1 + |W| = 1 + b$. Similarly, no pebble is smaller than w_b and $U \cup \{p_0\} \cup X \succ w_b$, so $s(w_b) \leqslant -(1+|U|+|X|) = -(1 + a + p)$. So $s(u_1) - s(w_b) \geqslant 1 + b + 1 + a + p = n$, and the equality must hold. So $s(u_1) = 1 + b, s(w_b) = -(1 + a + p)$ and the only order relation involving u_1 and w_b are $u_1 \succ \{p_0\} \cup W$ and $U \cup \{p_0\} \cup X \succ w_b$. Inductively we can consider u_2 and w_{b-1} and all pebbles in U and W and conclude that $U \cup \{p_0\} \cup X \succ W$ is the only order relation involving U and W.

Now consider the spins of pebbles in X. As $s(w_b) = -(1+a+k)$ and $s(x) - s(w_b) \leqslant n$ for each $x \in X$, we have $s(x) \leqslant b + 1$. Note that z cannot be bigger than any pebble in X, for otherwise $z \succ W$ and contradict to what we just concluded. But z may be smaller than some pebbles in X, thus $s(z) \leqslant 0$.

Consider x_1. By Lemma 7, no pebble is bigger than x_1. As $x_1 \succ W$, $s(x_1) \geqslant |W| = b$. So $s(x_1) \in \{b, b + 1\}$. Let $s(x_1) = b + 1$. Then the order relations involving x_1 are $x_1 \succ W \cup \{x_i\}$ for some $2 \leqslant i \leqslant a$ or $x_1 \succ W \cup \{z\}$; if $x_1 \succ z$, then $x_j \succ z$ for $1 \leqslant j \leqslant a$ by Lemma 7 and we inductively conclude $s(x_j) = b + 1$, thus X is an isolated block and $X \succ z$; if $x_1 \succ x_i$ for some $2 \leqslant i \leqslant a$, then $x_j \succ x_i$ for $1 \leqslant j < i$ by Lemma 7, and no other pebble in X is smaller than x_i, for otherwise it would be smaller than x_1 which contradicts what we just concluded. So $x_1x_2 \dots x_i$ is a block with tail x_i. Now we similarly consider x_{i+1} and get a block partition of X. Now let $s(x_1) = b$. Then $x_1 \succ W$ is the only order relation involving x_1, and we will inductively consider x_2 and get a block partition of X. □

Now we are ready to show that such permutations can be routed in $n-2$ steps.

Lemma 16. *If a permutation π contain a pebble p_0 such that $win(p_0) = (U, p_0, X, W)$ and $\pi = (\{z\}, U, p_0, X, W)$, where $U, X, W \neq \emptyset$, then π can be routed in at most $n-2$ steps.*

Proof. First we assume that $X \neq \emptyset$. Let $\pi = zu_1 \dots u_k p_0 x_1 \dots x_a w_1 \dots w_b$, with $u_i \in U, x_i \in X$ and $w_i \in W$. We use an odd-even routing algorithm so that $x_a w_1$ is an odd edge. The order relations are shown in Lemma 15.

By Lemma 11, x_a swaps with w_1 in the first step, thus swaps with W in the following $|W|-1$ steps, so w_b meets (i.e., is paired with a pebble in) $U \cup \{p_0\} \cup X$ after $|W|-1$ steps, then w_b would swap with $U \cup \{p_0\} \cup X$ in the following $|U \cup \{p_0\} \cup X|$ steps, so it takes $|W| - 1 + |U \cup \{p_0\} \cup X| = n-2$ steps for w_b to be in place. As a pebble in $U \cup \{p_0\} \cup X$ has to pass $W - w_b$ to meet w_b, all pebbles in W would be in place after $n-2$ steps.

For z, its window $win(z) = (X_r, W, z)$, so $|win(z)| = n - |U| - 1 - |X - X_r| \leqslant n-2$, thus, z will be in place after at most $n-2$ steps, by Lemma 13. For a $x_j \in X$ that is in a block with tail x_i, its window $win(x_j) = (x_j, \{x_{j+1}, \dots, x_{i-1}\}, x_i, \{x_{i+1}, \dots, x_a\}, W)$, so $|win(x_j)| \leqslant n - 2 - |U| \leqslant n-3$, thus x_j will be in place after at most $n-2$ steps, by Lemma 13.

So after $n-2$ steps, there are no comparable pebbles, as each order relation involves a pebble in $W \cup \{z\}$ or a pebble in a block with a head in X. By Lemma 8, π is routed in at most $n-2$ steps. ☐

Lemma 17. *If a permutation π contain a pebble p_0 such that $win(p_0) = (U, p_0, W)$ and $\pi = (\{z\}, U, p_0, W)$, where $U, W \neq \emptyset$, then π can be routed in at most $n-2$ steps.*

Proof. Let $\pi = zu_1 \dots u_k p_0 w_1 \dots w_b$. By Lemma 15, $s(u) = 1 + b$ and $s(w) = -1 - k$ for $u \in U, w \in W$ and the only order relation is $U \cup \{p_0\} \succ W$.

First we consider the case when $k = 1$ or $b = 1$. By symmetry, let $k = 1$. As $n \geqslant 6$, $b \geqslant 4$. Flip the spins of u and w_2. By Lemma 8, the order relations under the new disbursement are $p_0 \succ W - w_2, w_2 \succ (W - w_2) \cup \{z, u\}$. We use an odd-even routing algorithm with odd edge $p_0 w_1$. Then w_2 is paired with a smaller pebble in each step, thus will be in place after $n-2$ steps; similarly, p_0 is paired with a smaller pebble in each of the first $n-3$ steps, thus will be in place after $n-3$ steps. Therefore, after $n-2$ steps, there are no comparable pebbles since each order relation involves p_0 or w_2. By Lemma 8, π is routed in $n-2$ steps.

Now, let $k, b \geqslant 2$. We first flip the spins of u_1 and w_b. By Lemma 10, the order relations under the new disbursement are

$$U - u_1 \succ \{p_0\} \cup (W - w_b), \quad w_b \succ (W - w_b) \cup \{z, u_1\}, \quad (U - u_1) \cup \{z, w_b\} \succ u_1.$$

The window for p_0 is $win(p_0) = (U - u_1, p_0, W - w_b)$ and the window for z is $win(z) = (w_b, z, u_1)$, so $|win(p_0)| = n-3$ and $|win(z)| = 3 \leqslant n-3$ (as $n \geqslant 6$), so by Lemma 13, p_0 and z will be in place after $n-2$ steps. The window for u_1 is $win(u_1) = (U - u_1, W -$

$w_b + p_0, \{w_b, z\}, u_1)$. As $b \geqslant 2$, it is not one of the extreme windows in Lemma 13, so u_1 will be in place in at most $n-2$ steps.

Now we show that all pebbles in $W - w_b$ are in place after $n-2$ steps, which by Lemma 8 implies that all pebbles are in place since each order relation involves a pebble in $(W - w_b) \cup \{p_0, z, u_1\}$. Note that for $1 \leqslant i \leqslant b-1$, $win(w_i) = (w_b, \{z, u_1\}, U - u_1 + p_0, \{w_1, \ldots, w_{i-1}\}, w_i)$. Since $k \geqslant 2$, $win(u_i)$ is not one of the extreme windows in Lemma 13; thus, w_i will be in place after $n-2$ steps. □

4.3 Extremal window type 2a: $win(p_0) = (p_0, X, W)$ and $|win(p_0)| = n-1$

This is the case of $win(p_0) = (U, p_0, X, W)$ with $U = \emptyset$. In this case, W is not necessarily an isolated block. The following lemma tells the possible structures in π.

Lemma 18. *If a permutation π has a pebble p_0 with $win(p_0) = (p_0, X, W)$ and $\pi = (\{z\}, p_0, X, W)$, then one of the following must be true (note that X could be empty)*

1. *if $s(z) = c > 0$, then X is an isolated block and W can be partitioned into isolated blocks and blocks with heads, say $W_1, W_2, \ldots, W_r$, so that W_1 is isolated with c pebbles and $z \succ W_1$, and the only other order relation is $\{p_0\} \cup X \succ W$.*

2. *if $s(z) = c < 0$, then W is isolated and X can be partitioned into isolated blocks and blocks with tails, say $X_1, \ldots, X_r$, so that X_r is isolated with $|c|$ pebbles and $X_r \succ z$, and the only other order relation is $\{p_0\} \cup X \succ W$.*

3. *if $s(z) = 0$, then either X can be partitioned into isolated blocks and blocks with tails and W is an isolated block, or W can be partitioned into isolated blocks and blocks with heads and X is an isolated block, and the only other order relation is $\{p_0\} \cup X \succ W$.*

The proof of this lemma is very similar to Lemma 15, and for completeness, we include a proof below.

Proof. Let $\pi = zp_0x_1x_2\ldots x_a w_1 w_2 \ldots w_b$, where $x_i \in X$ and $w_i \in W$. Clearly, $s(p_0) = |W| = b$. By Lemma 7, $X \succ W$. As z is incomparable with p_0, no pebble in W is bigger than z.

Consider w_b. Since $\{p_0\} \cup X \succ w_b$, and no pebble is smaller than w_b by Lemma 7, $s(w_b) \leqslant -(|X|+1) = -(a+1)$. On the other hand, $s(p_0) - s(w_b) \leqslant n$, for otherwise the flip of spins of p_0 and w_b gives a smaller disbursement, so $s(w_b) \geqslant -(a+2)$. Therefore $s(w_b) \in \{-(a+1), -(a+2)\}$. Clearly, if $s(w_b) = -(a+1)$, then $\{p_0\} \cup X \succ w_b$ is the only order relation involving w_b, thus w_b is in an isolated block. We shall move to consider w_{b-1}.

Now let $s(w_b) = -(a+2)$. Then exactly one pebble in $\{z\} \cup (W - w_b)$ is bigger than w_b. If $z \succ w_b$, then by Lemma 7, $z \succ w_j$ for each $w_j \in W - w_b$; thus, inductively we conclude that the only order relation involving W is $\{z, p_0\} \cup X \succ W$, and W is an isolated block.

If $w_i \succ w_b$ for some $i < b$, then for $i < j < b$, w_j is not comparable with w_b and $w_i \succ w_j$ by Lemma 7. Note that no pebble w_l with $l < i$ could be bigger than w_i, as otherwise it would be bigger than w_b which is impossible. Now inductively we conclude that $w_{b-1}, \ldots, w_{i+1}$ all have spin $-(a+2)$ and are only smaller than $\{p_0, w_i\} \cup X$. That is, $\{w_i, w_{i+1}, \ldots, w_b\}$ is a block with head w_i.

We now repeat the above argument for w_{b-1} (if $s(w_b) = -(a+1)$) or w_{i-1} (if w_b is in a block with head w_i), and eventually W can be partitioned into isolated blocks and/or blocks with heads. In particular, if $z \succ w_r$, then w_r must be in an isolated block, and by Lemma 7, $z \succ \{w_1, w_2, \ldots, w_r\}$, that is, $w_1, w_2, \ldots, w_r$ are in an isolated block.

Note that z is not bigger than a pebble in X. For otherwise, let $z \succ x$ for some $x \in X$. Then $z \succ W$. Thus, $s(z) \geqslant 1 + b$ and $s(w_b) = -(a+2)$, which implies that $s(z) - s(w_b) \geqslant n + 1$. Now a flip of the spins of z and w_b gives a smaller disbursement, a contradiction.

We claim that if $s(w) = -(a+2)$ for some $w \in W$, then the only order relation involving X is $X \succ W$ (thus X is an isolated block). Consider x_1. Since no pebble is bigger than x_1, thus $s(x_1) \geqslant |W| = b$; Since $s(x_1) - s(w) \leqslant n$, $s(x_1) \leqslant b$. So $s(x_1) = b$ and the only order relation involving x_1 is $x_1 \succ W$. Now we consider $x_2, x_3, \ldots, x_a$ successively and similarly to get the conclusion. This means also that if $s(z) = c \geqslant 0$, then the isolated block $w_1 w_2 \ldots w_i$ has c pebbles.

Similar to the analysis of the structure in W, when the order relation involving W is $\{p_0\} \cup X \succ W$, we get the partition and structure in X. □

Lemma 19. *If a permutation π contains a pebble p_0 with $win(p_0) = (p_0, X, W)$ and $\pi = (\{z\}, p_0, X, W)$, then π can be routed in at most $n-2$ steps or is $2 \ldots n1$ or its inverse.*

Proof. We only consider the case $s(z) = c \geqslant 0$. By Lemma 18, we assume that $X = x_1 x_2 \ldots x_a$ is an isolated block, and $W = w_1 \ldots w_b$ has the block decomposition $W_1, \ldots, W_k$ so that W_0 is an isolated block with c pebbles and $z \succ W_1$, and for $i > 0$, W_i is either an isolated block or a block with a head (say $w_{i'}$).

If $c = b$, then the spins of $\{z, p_0\} \cup X$ are all b and the spins of X are all $-(a+2)$, and π is a rotation. By Lemma 12, if π needs $n-1$ steps to route, π must be one of the two extremal permutations. So we assume $c < b$.

When $c = s(z) = 0$, we may assume that W is not a block with head w_1 (that is, $w_1 \succ W - w_1$). Suppose otherwise. We flip the spins of p_0 and w_b, by Lemma 10, $w_b \succ \{z, p_0\} \cup (W - w_b)$, and $X \cup \{z, w_b\} \succ p_0$. Now we use the odd-even sorting algorithm so that $w_b z$ is an odd edge. By Lemma 11, w_b will be in place in $n-2$ steps, w_1 swaps from the second step and takes $n-4$ steps to be in place, and p_0, z will be in place in 3 steps. So in at most $n-2$ steps all pebbles will be in place.

Now consider the rest of the cases. We use the odd-even routing algorithm so that $x_a w_1$ ($p_0 w_1$ if $X = \emptyset$) is an odd edge. By Lemma 11, x_i with $1 \leqslant i \leqslant a$ meets W after $a - i$ steps and swaps with W in the following $|W|$ steps, so it will be in place after $a - i + |W| = a + b - i = n - 2 - i \leqslant n - 2$ steps; p_0 could be regarded as x_0, so takes at most $n-2$ steps; z meets W_1 after $a+1$ steps and takes c swaps, so will be in place in at most $a + 1 + c < a + b + 1 \leqslant n - 2$ steps. The head w in block W_i swaps with W_i at the first step, or the second step (if it is not adjacent to x and is not incident with an odd edge in the first step), or after $a+2$ steps (if $w = w_1$), and in the former two cases it will be in place after at most $1 + (|W_i| - 1) + a + 1 = a + |W_i| + 1 \leqslant a + b = n - 2$ steps. Now consider the last case. If $z \succ w_1$, then w_1 is in an isolated block in W, thus will be in place after $a+2 = n-b \leqslant n-2$ steps (as $b > c \geqslant 1$). If w_1 is not smaller than z, then $s(z) = 0$, and w_1 will be in place after at most $(a+1)+1+|W_i|-1 < a+1+|W| = n-1$ steps. So after $n-2$ steps, the above pebbles are in place, thus, all pebbles will be in place by Lemma 8, since each order relation involves one of the pebbles. □

4.4 Extremal window type 3: $win(p_0) = \pi = (p_0, X_1, W_1, x, W_2)$ or $win(p_0) = \pi = (p_0, X_1, w, X_2, W_2)$

Lemma 20. *If a permutation π contains a pebble p_0 with $win(p_0) = \pi = (p_0, X_1, W_1, x, W_2)$ or $win(p_0) = \pi = (p_0, X_1, w, X_2, W_2)$, then X_1 and W_2 are isolated blocks, and*

- *if $win(p_0) = (p_0, X_1, W_1, x, W_2)$, then W_1 can be partitioned into isolated blocks and blocks with heads. Furthermore, $c := s(x) - |W_2| \geqslant 0$, and if $c > 0$, then the block W_0 in W_1 next to X_1 is an isolated block with c elements and are all smaller than x; and the only other order relation between segments are $\{p_0\} \cup X_1 \succ W_1 \cup W_2, x \succ W_2 \cup W_0$.*

- *if $win(p_0) = (p_0, X_1, w, X_2, W_2)$, then X_2 can be partitioned into isolated blocks and blocks with tails. Furthermore, $c := s(w) + (1 + |X_1|) \leqslant 0$, and if $c < 0$, then the block X_0 in X_2 next to W_2 is an isolated block with $-c$ elements and are all bigger than w; and the order relation between segments are $\{p_0\} \cup X_1 \succ \{w\} \cup W_2$ and $X_0 \succ w$.*

Proof. We only prove the case when $win(p_0) = (p_0, X_1, W_1, x, W_2)$, and the other case is very similar. Let $W_1 = w_1 w_2 \dots w_a$ and $W_2 = w_{a+1} w_2 \dots w_b$. Consider the spins of p_0 and w_b. Since $s(p_0) = b$ and by Lemma 7, no pebble is smaller than w_b, thus $s(w_b) \leqslant -(2 + |X|) = -(n - b)$; furthermore, as $s(p_0) - s(w_b) \leqslant n$, we have $s(w_b) \geqslant -(n - b)$, so $s(w_b) = -(n - b)$, and the order relation involving w_b is $\{p_0\} \cup X \succ w_b$. Now we can repeat the argument for $w_{b-1}, w_{b-2}, \dots, w_{a+1}$ successively and get $s(w_i) = -(n - b)$ for all $w_i \in W_2$. Similarly, by comparing the spins of pebbles in X_1 to that of w_b, we have $s(x_i) = b$ for all $x_i \in X_1$. So X_1 and W_2 are isolated blocks. Since $s(p_0) = s(x)$ for each $x \in X_1$, let $X_1' = X_1 \cup \{p_0\}$.

Now we consider the spins of pebbles in W_1.

We note that no pebble in W_1 is bigger than x, for otherwise p_0 would be bigger than x as $p_0 \succ W_1$. Consider w_a. The pebbles in X_1' are bigger than w_a and by Lemma 7, no pebble is smaller than w_a, so $s(w_a) \leqslant -|X_1'| = b - n + 1$. On the other hand, $s(p_0) - s(w_a) \leqslant n$ and $s(p_0) = b$ implies $s(w_a) \geqslant b - n$. That is, $s(w_a) \in \{b-n, b-n+1\}$, and at most one pebble other than those in X_1' is bigger than w_a.

If $s(w_a) = b - n + 1$, then $\{p_0\} \cup X_1 \succ w_a$ is the only order relation involving w_a, and we turn to consider w_{a-1}. If $s(w_a) = b - n$, then w_a is smaller than x or some pebble $w_i \in W_1$; in the former case, all pebbles in W_1 are smaller than x and inductively one can show that they are incomparable and thus W_1 is an isolated block; in the latter case,

$w_i \succ w_j$ for $i+1 \leqslant j \leqslant a$ and w_i is the only such pebble other than those in X_1', so $w_i w_{i+1} \dots w_a$ is a block with head w_i. Inductively one can have a partition of W_1 into blocks, as desired.

We observe that if $x \succ w_i \in W_1$ then w_i is in an isolated block and $x \succ \{w_1, w_2, \dots, w_i\}$ by Lemma 7. Let $c := s(x) - |W_2|$. Then $W_0 := (w_1, \cdots, w_c)$ is an isolated block, $x \succ W_2 \cup W_0$ is the only order relation involving x, as desired. □

Lemma 21. *If a permutation π contains a pebble p_0 such that $win(p_0) = \pi = (p_0, X_1, W_1, x, W_2)$ or $win(p_0) = \pi = (p_0, X_1, w, X_2, W_2)$, then π can be routed in $n-2$ steps.*

Proof. Again we only consider the case $win(p_0) = (p_0, X_1, W_1, x, W_2)$, as the other one is very similar. Let $win(p_0) = (p_0 x_1 x_2 \dots x_k w_1 \dots w_a x w_{a+1} \dots w_b)$ with $x_i \in X_1$ and $w_i \in W_1 \cup W_2$. By Lemma 20, X_1, W_2 are isolated blocks, and $W_1 = \cup_{i=0}^r W_1^i$, where $x \succ W_1^0$, and W_1^i for $i > 0$ is an isolated block or a block with head w_1^i. We flip the spins of p_0 and w_b, and then use an odd-even routing algorithm with odd edge $x_k w_1$ to route the permutation.

Now by Lemma 10 and Lemma 20, the order relations under the new disbursement are

$$X_1 \cup \{w_b\} \succ W_1 \cup (W_2 - w_b) \cup \{p_0\}, \quad x \succ (W_2 - w_b) \cup W_1^0 \cup \{p_0\}, \quad w_1^i \succ W_1^i - w_1^i \text{ for some } i.$$

We list the windows for the pebbles in $X_1 \cup \{w_b, x, w_1^i : 1 \leqslant i \leqslant r\}$:

$$\begin{aligned} win(x_i) &= (x_i, X_1 - \{x_j : 1 \leqslant j \leqslant i\}, W_1, x, W_2 - w_b, p_0), \\ win(w_1^i) &= (w_b, p_0, X_1, \cup_{j=0}^{i-1} W_1^j, w_1^i, W_1^i - w_1^i), \\ win(x) &= (x, W_2 - w_b, w_b, p_0, X_1, W_1^0), \\ win(w_b) &= (w_b, p_0, X_1, W_1, x, W_2 - w_b). \end{aligned}$$

Then $|win(x_i)| = n-(i-1)$, $|win(w_1^i)| \leqslant n-1-|W_2| \leqslant n-2$, $|win(x)| = n-|W_1 - W_1^0|$ and $|win(w_b)| = n$, and none of the windows is among the extreme windows in Lemma 13. Therefore, in at most $n-2$ steps, the pebbles in $X_1 \cup \{w_b, x, w_1^i : 1 \leqslant i \leqslant r\}$ will be in place. However, each order relation on π involves one of the pebbles, so by Lemma 8, after $n-2$ steps, there are no comparable pebbles, that is, π is routed. □

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