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Poles and Zeros of Generalized Carathéodory Class Functions

Yael Gilboa
College of William and Mary

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Yael Gilboa

Accepted for ______________________
(Honors)

____
Vladimir Bolotnikov, Committee Chair

____
Ilya Spiktovsky

____
Leiba Rodman

____
Julie Agnew

Williamsburg, VA
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YAEL GILBOA

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Contents

1. Introduction

The goal of this thesis is to establish certain representations for generalized Carathéodory functions in terms of related classical Carathéodory functions.

Let $\mathcal{C}$ denote the Carathéodory class of functions f that are analytic and that have a nonnegative real part in the open unit disk $\mathbb{D} = \{z: \ |z| < 1\}$. In his papers [10] and [11], Constantin Carathéodory characterized such functions in terms of their Taylor coefficients around the origin. Carathéodory class of functions play an important role in the theory of electrical networks. Since the 1920s, there has been extensive research done on related problems. Once Cauer [12] first established the connection between positive functions and electrical circuits, an interest in specifically Carathéodory class functions was prodded on by electrical networks theory. In the 1960s, possible uses for Carathéodory class functions were found in absolute stability theory [17]. An important property of the Carathéodory class is that the kernel

$$C_f(z,\zeta) = \frac{f(z) + \overline{f(\zeta)}}{1 - z\bar{\zeta}}$$

is positive on $\mathbb{D}$.

During this time, people started to become interested in a wider class of functions such that the kernel $C_f(z,\zeta)$ has a finite number of negative squares. This means that the Hermitian matrix

$$\left[\frac{f(z_i) + \overline{f(z_j)}}{1 - z_i\bar{z}_j}\right]_{i,j=1}^{n} \tag{1.1}$$

has at most κ negative eigenvalues (counted with multiplicities) for any choice of an integer n and of n points $z_1, \dots, z_n \in \mathbb{D}$ at which f is analytic, and it has exactly κ negative eigenvalues for at least one such choice. The positivity of matrices (1.1) is an extremely strong condition that implies, in particular, the analyticity of f even if this analyticity is not an a'priori assumption. However, the uniform bound for the number of negative eigenvalues makes the generalized Carathéodory class $\mathcal{C}_\kappa$ quite special and closely connected to the classical Carathéodory class $\mathcal{C}$.

The class $\mathcal{C}_\kappa$ was introduced in the 1960s [4] as the class of functions f that are meromorphic in $\mathbb{D}$ and such that $C_f(z,\zeta)$ has κ negative squares. Extensions of previous results were found to be relevant to generalized Carathéodory class functions and have applications to electrical networks as well.

In the contrast to the classical Carathéodory functions, which do not have poles in $\mathbb{D}$ and cannot have zeros in $\mathbb{D}$, the functions from the class $\mathcal{C}_\kappa$ may have up to κ zeros and up to κ poles in $\mathbb{D}$.

In this thesis, we get a representation formula for $\mathcal{C}_\kappa$ functions in terms of their poles and certain related $\mathcal{C}$ functions. Namely, we will show that every function $f \in \mathcal{C}_\kappa$ can be written in the form

$$f(z) = \frac{p(z)\Phi(z) + q(z)}{\prod_{i=1}^{r}(z - z_i)^{n_i}(1 - z\overline{z}_i)^{n_i} \cdot \prod_{i=r+1}^{\ell}(z - z_i)^{2n_i}}. \tag{1.2}$$

where Φ belongs to $\mathcal{C}$, p and q are polynomials of degree at most 2κ, and z_i are the poles or the generalized boundary poles of nonpositive type of f.

This thesis contains six sections, including the Introduction, and is organized as follows: Section 2 introduces the classical Schur class $\mathcal{S}$ of complex-valued analytic functions mapping the open unit disk $\mathbb{D} = \{z : \ |z| \leq 1\}$ into the closed unit disk $\overline{\mathbb{D}}$. Section 3 introduces the generalized Schur class $\mathcal{S}_\kappa$. Section 4 introduces the classical and generalized Carathéodory classes, mentioned above. In Section 5, we derive representation (1.2) as well as an example for the representation of a generalized Carathéodory class function in terms of its poles, zeros and a Carathéodory class function, which inspires our potential future research, briefly outlined in Section 6.

2. Schur Class Functions

The classical Schur class $\mathcal{S}$ consisting of complex-valued analytic functions mapping the unit disk $\mathbb{D} = \{z : \ |z| \le 1\}$ into the closed unit disk $\overline{\mathbb{D}}$ has been a source of much study and inspiration for over a century now, beginning with the seminal work of Schur (for the original paper of Schur and a survey of some of its impact and applications in signal processing, see [18]).

Definition 2.1. A complex-valued function s is called a Schur function if it is analytic on $\mathbb{D}$ and satisfies $|s(z)| \le 1$ for every $z \in \mathbb{D}$.

Schur-class functions enjoy many remarkable properties. To present some of them we first recall some needed definitions.

Definition 2.2. A matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian if $A = A^*$. A Hermitian matrix is called positive semidefinite if all its eigenvalues are nonnegative, and it is called positive definite if all its eigenvalues are positive.

Alternatively, we may define positive semidefinite and positive definite matrices as those for which the inner product $\langle Ax, x\rangle$ is real and, respectively, nonnegative or positive for every nonzero vector $x \in \mathbb{C}^n$.

On the set of Hermitian $n \times n$ matrices we introduce Loewner's partial order: we will say that $A \ge B$ if the matrix $A - B$ is positive semidefinite. We will say that $A > B$ if the matrix $A - B$ is positive definite.

Definition 2.3. A matrix $T \in \mathbb{C}^{m\times n}$ is called *contractive* if $TT^* \le I_m$.

Lemma 2.4. *Let $T \in \mathbb{C}^{m\times n}$. The following is equivalent:*
(1) *T is contractive;*
(2) *T^* is contractive;*
(3) *$\|Tx\| \le \|x\|$ for every $x \in \mathbb{C}^n$;*
(4) *$\|T^*y\| \le \|y\|$ for every $y \in \mathbb{C}^m$.*

Proof: The matrix $\begin{bmatrix} I_m & T \\ T^* & I_n \end{bmatrix}$ can be factored in two different ways as follows:

$$\begin{bmatrix} I_m & T \\ T^* & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ T^* & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & I_n - T^*T \end{bmatrix} \begin{bmatrix} I_m & T \\ 0 & I_n \end{bmatrix} \tag{2.1}$$

$$= \begin{bmatrix} I_m & T \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m - TT^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ T^* & I_n \end{bmatrix}. \tag{2.2}$$

Assume that $T^*T \le I$; then $I - T^*T \ge 0$, so the middle matrix in (2.1) is positive semidefinite. Then $\begin{bmatrix} I_m & T \\ T^* & I_n \end{bmatrix}$ is positive semidefinite. Since $\begin{bmatrix} I_m & T \\ T^* & I_n \end{bmatrix}$ is invertible, it follows from (2.2) that the middle matrix is positive semidefinite. Then $I_m - TT^* \ge 0$ or $TT^* \le I_m$. By reversing the arguments, we get the equivalence of (1) and (2).

Assume that $||Tx|| \leq \|x\|$ for every x. Then $||Tx||^2 \leq ||x||^2$, which can be written as

$$0 \leq ||x||^2 - ||Tx||^2 = \langle x, x\rangle - \langle Tx, Tx\rangle = x^*x - x^*T^*Tx = x^*(I - T^*T)x.$$

So $x^*(I - T^*T)x \geq 0$ for every x, which means that $(I - T^*T)$ is positive semidefinite and $T^*T \leq I$. We have proved (3) $\Rightarrow$ (2). By reversing the arguments, we get (2) $\Rightarrow$ (3).

Assume that $||T^*y|| \leq \|y\|$ for every y. Then $||T^*y||^2 \leq ||y||^2$, which can be written as

$$0 \leq ||y||^2 - ||T^*y||^2 = \langle y, y\rangle - \langle T^*y, T^*y\rangle = y^*y - y^*TT^*y = y^*(I - TT^*)y.$$

So $y^*(I - TT^*)y \geq 0$ for every y, which means that $(I - TT^*)$ is positive semidefinite and $TT^* \leq I$. We have proved (4) $\Rightarrow$ (1). By reversing the arguments, we get (1) $\Rightarrow$ (4). □

To present a remarkable property of Schur-class functions, let us recall the Hardy space H^2 consisting of all functions f with square-summable Taylor coefficients:

$$H^2 = \left\{ f(z) = \sum_{k=0}^{\infty} f_k z^k : \sum_{k=0}^{\infty} |f_k|^2 < \infty \right\}.$$

This space can be endowed with inner product

$$\langle f, g\rangle_{H^2} = \sum_{k=0}^{\infty} f_k \overline{g}_k \quad \text{for} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k, \; g(z) = \sum_{k=0}^{\infty} g_k z^k. \tag{2.3}$$

Then for every $f \in H^2$ we can define the norm

$$\|f\|_{H^2} = \langle f, f\rangle_{H^2}^{\frac{1}{2}} = \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{\frac{1}{2}}.$$

Lemma 2.5. *The function* $k_a(z) = \dfrac{1}{1 - z\overline{a}}$ *belongs to* H^2 *for every fixed* $a \in \mathbb{D}$ *and* $\|k_a\|_{H^2} = \dfrac{1}{\sqrt{1 - |a|^2}}$.

Proof: The function $k_a(z)$ can be represented by the Taylor expansion:

$$k_a(z) = \sum_{n=0}^{\infty} (z\overline{a})^n = \sum_{n=0}^{\infty} \overline{a}^n z^n \tag{2.4}$$

Using the definition of the norm in H^2 and the Taylor coefficients of $\overline{a}^n$, one can see

$$\|k_a\|_{H^2}^2 = \langle k_a, k_a\rangle_{H^2} = \sum_{n=0}^{\infty} |a|^{2n} = \frac{1}{1 - |a|^2}.$$

So, $\|k_a\|_{H^2} = \dfrac{1}{\sqrt{1 - |a|^2}}$. □

The functions k_a have the following reproducing property: the inner product of k_a with any function $f \in H^2$ recovers the value of f at a.

Lemma 2.6. *For every $f \in H^2$ and any $a \in \mathbb{D}$, we have*

$$\langle f, k_a \rangle_{H^2} = f(a). \tag{2.5}$$

Proof: Using definition (2.3) of the inner product of H^2 and the Taylor expansion for $k_a(z)$ (2.4), we have

$$\langle f, k_a \rangle_{H^2} = \sum_{n=0}^{\infty} f_n a^n = f(a). \quad \square$$

Let us define:

$$k_{a,n}(z) = \frac{1}{n!}\frac{\partial^n}{\partial \overline{a}} k_a(z) = \frac{z^n}{(1 - z\overline{a})^{n+1}}. \tag{2.6}$$

Lemma 2.7. *For every $f \in H^2$ and any $a \in \mathbb{D}$, we have $k_{a,n} \in H^2$ as a rational function with no poles in D and*

$$\langle f, k_{a,n} \rangle_{H^2} = \frac{f^{(n)}(a)}{n!}. \tag{2.7}$$

Proof: The Taylor series for $f(z) = \dfrac{1}{(1 - z\overline{a})^{n+1}}$ is

$$f(z) = \sum_{k=0}^{\infty} \binom{n+k}{k} \overline{a}^k z^k.$$

Thus, the Taylor series for $k_{a,n}(z) = z^n f(z)$ is

$$k_{a,n}(z) = \sum_{k=0}^{\infty} \binom{n+k}{k} \overline{a}^k z^{n+k}. \tag{2.8}$$

Using the definition (2.3) of inner product of H^2, the Taylor expansion (2.8) for $k_{a,n}(z)$, and (2.6) we have

$$\langle f, k_{a,n} \rangle_{H^2} = \sum_{k=0}^{\infty} f_{n+k} \binom{n+k}{k} a^k = \frac{f^{(n)}(a)}{n!}. \quad \square$$

It was shown by Pierre Fatou [14] that for any Schur-class function s, the radial boundary limit

$$s(e^{it}) = \lim_{r \to 1^-} s(re^{it})$$

exists for almost all $t \in [0, 2\pi)$. It is clear that this boundary limit does not exceed one in modulus whenever it exists. Fatou also showed that if f belongs to H^2, then for almost all $t \in [0, 2\pi)$, the radial boundary limit

$$f(e^{it}) = \lim_{r \to 1^-} f(re^{it})$$

exists and moreover,

$$\|f\|_{H^2}^2 := \sum_{k=0}^{\infty} |f_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt. \tag{2.9}$$

Every Schur-class function $s \in \mathcal{S}$ defines the linear map $M_s : \ f \to s \cdot f$. Assuming that f belongs to H^2 let us compare the norms $\|f\|$ and $\|sf\|$ using the formula (2.9) and the fact that $|s(e^{it})| \leq 1$ for almost all $t \in [0, 2\pi)$:

$$\begin{aligned}
\|sf\|_{H^2}^2 &= \frac{1}{2\pi} \int_0^{2\pi} |s(e^{it}) f(e^{it})|^2 dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} |s(e^{it})|^2 \cdot |f(e^{it})|^2 dt \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt = \|f\|_{H^2}^2.
\end{aligned}$$

The latter calculation shows that $M_s f$ belongs to H^2 whenever f does and moreover, that M_s does not increase the norm. Thus, M_s is a contraction by Lemma 2.4. Therefore the adjoint map M_s^* is also contractive by Lemma 2.4.

Lemma 2.8. *Let $s(z) = \sum s_n z^n \in \mathcal{S}$ and $f(z) = \sum f_n z^n \in H^2$. Then*

$$M_s^* f = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \overline{s_k} f_{n+k} \right) z^n. \tag{2.10}$$

Proof: The function $H = M_s^* f$ belongs to H^2. Let us expand H in Taylor series

$$H(z) = \sum_{n=0}^{\infty} \frac{H^{(n)}(0)}{n!} z^n$$

and show that

$$\frac{H^{(n)}(0)}{n!} = \sum_{k=0}^{\infty} \overline{s_k} f_{n+k}.$$

Indeed, by (2.7),

$$\begin{aligned}
\frac{H^{(n)}(0)}{n!} &= \langle H, k_{0,n}\rangle = \langle H, z^n\rangle \\
&= \langle M_s^* f, z^n\rangle = \langle f, M_s z^n\rangle = \langle f, s z^n\rangle \\
&= \langle \sum_{k=0}^{\infty} f_k z^k, \sum_{k=0}^{\infty} s_{k+n} z^k\rangle \\
&= \sum_{k=0}^{\infty} f_{k+n}\overline{s}_k. \quad \square
\end{aligned}$$

The formula (2.10) takes a particularly simple form in the case where f equals the elementary kernel k_a.

Corollary 2.9. *Let $s \in \mathcal{S}$ and let $k_a(z) = \dfrac{1}{1 - z\overline{a}}$. Then*

$$M_s^* k_a = \overline{s(a)} k_a. \tag{2.11}$$

Proof: Let us apply (2.10) to $f = k_a$. So $f_n = \overline{a}^n$ and for $H = M_s^* k_a$, we have

$$\frac{H^{(n)}(0)}{n!} = \sum_{k=0}^{\infty} \overline{s_k} f_{k+n} = \sum_{k=0}^{\infty} \overline{a^{k+n} s_k} = \overline{a^n} \cdot \sum_{k=0}^{\infty} \overline{a^k s_k} = \overline{a^n}\overline{s(a)}.$$

Therefore,

$$M_s^* k_a = \sum_{n=0}^{\infty} \frac{H^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \overline{a}^n \overline{s(a)} z^n = \overline{s(a)} \cdot \sum_{n=0}^{\infty} \overline{a}^n z^n = \overline{s(a)} k_a. \quad \square$$

Remark 2.10. *Equality (2.11) means that elementary kernels k_a are eigenvectors of the operator M_s^* while the complex-conjugate values $\overline{s(a)}$ are the corresponding eigenvalues.*

Let us consider the function

$$g(z) = c_1 k_{a_1}(z) + \ldots + c_n k_{a_n}(z) = \sum_{j=1}^{n} \frac{c_j}{1 - z\overline{a_j}} \tag{2.12}$$

where $a_1, \ldots, a_n$ are any n distinct points in $\mathbb{D}$ and $c_1, \ldots, c_n$ are arbitrary fixed complex numbers. This function belongs to H^2 since H^2 is a vector space.

Since the operator M_s^* is linear, we have from (2.11),

$$M_s^* g = M_s^* \sum_{i=1}^{n} c_i k_{a_i} = \sum_{i=1}^{n} c_i M_s^* k_{a_i} = \sum_{i=1}^{n} c_i \overline{s(a_i)} k_{a_i}. \tag{2.13}$$

We now use the reproducing property (2.5) to get

$$\begin{aligned}
\|g\|_{H^2}^2 &= \|\sum_{i=1}^n c_i k_{a_i}\|_{H^2}^2 \\
&= \left\langle \sum_{j=1}^n c_j k_{a_j}, \sum_{i=1}^n c_i k_{a_i} \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j k_{a_j}(a_i) \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j \frac{1}{1 - a_i \overline{a}_j}. \qquad (2.14)
\end{aligned}$$

Quite similarly we get from (2.13)

$$\begin{aligned}
\|M_s^* g\|_{H^2}^2 &= \langle M_s^* g, M_s^* g \rangle_{H^2} \\
&= \langle \sum_{i=1}^n c_i \overline{s(a_i)} k_{a_i}, \sum_{j=1}^n c_j \overline{s(a_j)} k_{a_j} \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j s(a_i) \overline{s(a_j)} k_{a_j}(a_i) \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j \frac{s(a_i)\overline{s(a_j)}}{1 - a_i \overline{a}_j}. \qquad (2.15)
\end{aligned}$$

If s is a Schur-class function, the map M_s is a contraction. Therefore, $\|M_s^* g\| \leq \|g\|$, for every $g \in H^2$. In particular, for the function g of the form (2.12) we have

$$\|M_s^* g\|^2 \leq \|g\|^2,$$

which on account of (2.14) and (2.15) can be written as

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j \frac{s(a_i)\overline{s(a_j)}}{1 - a_i \overline{a}_j} \leq \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j \frac{1}{1 - a_i \overline{a}_j},$$

which is the same as

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c}_j \frac{1 - s(a_i)\overline{s(a_j)}}{1 - a_i \overline{a}_j} \geq 0. \qquad (2.16)$$

Let us consider the matrix

$$P^s(a_1, \ldots, a_n) = \left[\frac{1 - s(a_i)\overline{s(a_j)}}{1 - a_i \overline{a}_j} \right]_{i,j=1}^n. \qquad (2.17)$$

Then condition (2.16) means that

$$\begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} P^s(a_1,\dots,a_n) \begin{bmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_n \end{bmatrix} \geq 0$$

for every choice of $c_1,\dots,c_n$. In other words, $x^* P^s(a_1,\dots,a_n)x \geq 0$ for every vector-column from $\mathbb{C}^n$, which, in turn, means that the matrix $P^s(a_1,\dots,a_n)$ is positive semidefinite. We thus proved that if s is a Schur-class function, then for every choice of a positive integer n and for any n distinct points in $\mathbb{D}$, the matrix $P^s(a_1,\dots,a_n)$ of the form (2.17) is positive semidefinite.

The converse statement is of special interest. As Alan Hindmarsh showed in [19], if s is a function defined on $\mathbb{D}$ and such that the matrices $P^s(a_1,a_2,a_3)$ are positive semidefinite for any triple of distinct points a_1,a_2,a_3 in $\mathbb{D}$, then s is a Schur-class function. The boundedness property $|s(z)| \leq 1$ is immediate even from positivity of 1×1 matrices. A remarkable part of Hindmarsh's result is that positivity of all 3×3 matrices implies that s is analytic and therefore, the matrices P_n are positive semidefinite for all integers $n \geq 1$.

Definition 2.11. Let Ω be a domain in $\mathbb{C}$. A function $K(z,\zeta)$ is called *a positive kernel on* $\Omega\times\Omega$ if it is defined for all $z,\zeta\in\Omega$ and

$$\sum_{i,j=1}^{r} c_i\bar{c}_j K(z_i,z_j) \geq 0$$

for every choice of a positive integer r and of points $z_1,\dots,z_n\in\Omega$ or equivalently, if the matrix $[K(z_i,z_j)]_{i,j=1}^{r}$ is positive semidefinite for any such choice.

Our previous result just means that if s is a Schur-class function, then the kernel

$$K_s(z) = \frac{1-s(z)\overline{s(\zeta)}}{1-z\bar{\zeta}} \tag{2.18}$$

is positive on $\mathbb{D}\times\mathbb{D}$. We summarize the above discussion in the following theorem.

Theorem 2.12. *Let s be a complex-valued function analytic on $\mathbb{D}$. Then the following are equivalent:*

(1) *s belongs to $\mathcal{S}$, i.e., $|s(z)|\leq 1$ for every $z\in\mathbb{D}$.*
(2) *The kernel $K_s(z,\zeta)$ defined in (2.18) is positive on $\mathbb{D}\times\mathbb{D}$.*
(3) *The matrix $P_s(a_1,\dots,a_n)$ is positive semidefinite for any choice of $a_1,\dots,a_n\in\mathbb{D}$.*

In what follows, we will refer to the matrix $P_s(a_1,\dots,a_n)$ as to the *Pick matrix* of the function s based on points $a_1,\dots,a_n\in\mathbb{D}$.

We now present simple examples of Schur-class functions. The function

$$b_a(z) = \frac{z-a}{1-z\bar{a}} \quad (a \in \mathbb{D}), \tag{2.19}$$

is called a *Blaschke factor*. This is a linear fractional function with its only zero at $z = a$ and its only pole at $z = 1/\overline{a}$. Let us show that b_a belongs to $\mathcal{S}$. Indeed,

$$\begin{aligned} 1 - \left|\frac{z-a}{1-z\bar{a}}\right|^2 &= \frac{|1-z\bar{a}|^2 - |z-a|^2}{|1-z\bar{a}|^2} \\ &= \frac{(1-z\bar{a})(1-\bar{z}a) - (z-a)(\bar{z}-\bar{a})}{|1-z\bar{a}|^2} \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|1-z\bar{a}|^2}, \end{aligned}$$

which implies that $|b_a(z)| < 1$ if $z \in \mathbb{D}$ and that $|b_a(z)| = 1$ if $|z| = 1$. The product of finitely many Blaschke factors

$$f(z) = \prod_{j=1}^{n} b_{a_j}(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-z\bar{a}_j} \tag{2.20}$$

where a_i's are not necessarily distinct, is called a *finite Blaschke product*. This function is rational of degree n and has zeros at a_i and poles at $1/\overline{a}_i$. Since the modulus of the product of two complex numbers is equal to the product of moduli, it follows that $|f(z)| < 1$ if $z \in \mathbb{D}$ and that $|f(z)| = 1$ if $|z| = 1$ and in particular, that any finite Blaschke product belongs to the Schur class. The set of all Blaschke products of degree k will be denoted by $\mathcal{B}_k$. By $\mathcal{B}_0$ we denote the set of all constant unimodular functions.

The class $\mathcal{B}_k$ admits the following characterization in terms of Pick matrices (see e.g., [6] for a proof).

Theorem 2.13. *A function f belongs in $\mathcal{B}_k$ if and only if for every $n \geq 1$ and any choice of $a_1, \ldots, a_n \in \mathbb{D}$, the Pick matrix $P_s(a_1, \ldots, a_n)$ is positive semidefinite and* $\operatorname{rank} P_s(a_1, \ldots, a_n) = \min\{k, n\}$.

3. Generalized Schur Class Functions

For a fixed integer $\kappa \geq 0$, we define the *generalized Schur class* $\mathcal{S}_\kappa$ as the set of all meromorphic functions of the form

$$f(z) = \frac{s(z)}{b(z)}, \tag{3.1}$$

where $s \in \mathcal{S}$ and $b \in \mathcal{B}_\kappa$ do not have common zeros (in particular, $\mathcal{S}_0 = \mathcal{S}$). Originally, the classes $\mathcal{S}_\kappa$ appeared implicitly in approximation theory [2] and in interpolation theory [20]. Later V. Adamyan, D. Arov and M.G. Krein [1] discovered deep connections of $\mathcal{S}_\kappa$ with spectral theory of Hankel operators. The thorough study of $\mathcal{S}_\kappa$ classes is presented in [15, 16], while more recent advances can be found in the monograph [3].

Formula (3.1) is called the Krein-Langer representation of a generalized Schur function f; the entries s and b are determined by f uniquely up to a unimodular constant. Since every finite Blaschke product is a rational function analytic on a neighborhood of $\mathbb{D}$, the function f of the form (3.1) admits radial boundary limits whenever the denominator $s \in \mathcal{S}$ does, that is, almost everywhere on $\mathbb{T}$. Since $|b(e^{it})| = 1$ for every $t \in [0, 2\pi)$, it follows that

$$|f(e^{it})| = |s(e^{it})| \leq 1.$$

Therefore, via radial boundary limits, the $\mathcal{S}_\kappa$ functions can be identified with the functions defined almost everywhere on $\mathbb{T}$, which are bounded by one in modulus (that is, with the functions from the unit ball of $L^\infty(\mathbb{T})$) and which admit the meromorphic continuation inside the unit disk with total pole multiplicity equal to κ. On the other hand, the $\mathcal{S}_\kappa$ functions f can be characterized as meromorphic functions on $\mathbb{D}$ for which all the associated Pick matrices

$$P^s(a_1, \ldots, a_n) = \left[\frac{1 - s(a_i)\overline{s(a_j)}}{1 - a_i\overline{a}_j}\right]_{i,j=1}^n \tag{3.2}$$

have not more than κ negative eigenvalues (counted with multiplicities) and some of these matrices have exactly κ negative eigenvalues. Of course, the points $a_1, \ldots, a_n$ are chosen in the domain of definition of f. It was shown in [9] that for an $f \in \mathcal{S}_\kappa$, there always exist $\kappa \times \kappa$ Pick matrices that have κ negative eigenvalues.

Definition 3.1. Let Ω be a domain in $\mathbb{C}$. The kernel $K(z, \zeta)$ is said to have κ negative squares – in notation, $\mathrm{sq}\_K(z, \zeta) = \kappa$ if

(1) It is defined for all $z, \zeta \in \Omega$;
(2) It is Hermitian in the sense that $K(z, \zeta) = \overline{K(\zeta, z)}$ for all $z, \zeta \in \Omega$;
(3) For every choice of a positive integer r and of points $z_1, \ldots, z_n \in \Omega$, the Hermitian matrix $[K(z_i, z_j)]_{i,j=1}^r$ has at most κ negative eigenvalues and it has exactly κ negative eigenvalues for at least one such choice.

This definition and the preceding discussion imply that $\mathcal{S}_\kappa$ can be characterized as the class of meromorphic functions f such that the associated kernel

$$K_f(z,\zeta) := \frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} \tag{3.3}$$

has κ negative squares on $\rho(f)$, the domain of analyticity of f: $\mathrm{sq}_-(K_f) = \kappa$.

We conclude this section with a result concerning Hermitian kernels.

Theorem 3.2. *Let $K(z,\zeta)$ be a Hermitian kernel on $\Omega \times \Omega$ and let $A(z)$ be a function such that $A(z) \neq 0$ for all $z \in \Omega$. Then the kernel $\widetilde{K}(z,\zeta) = A(z)K(z,\zeta)\overline{A(\zeta)}$ has the same number of negative squares as $K(z,\zeta)$:*

$$\mathrm{sq}_-\widetilde{K}(z,\zeta) = \mathrm{sq}_-A(z)K(z,\zeta)\overline{A(\zeta)} = \mathrm{sq}_-K(z,\zeta).$$

Proof: Choose $z_1, \ldots, z_n \in \Omega$ and consider the matrices $P = [K(z_i,z_j)]_{i,j=1}^n$ and $\tilde{P} = \left[\tilde{K}(z_i,z_j)\right]_{i,j=1}^n = \left[A(z_i)K(z_i,z_j)\overline{A(z_j)}\right]_{i,j=1}^n$.

Then it is easily seen that $\tilde{P} = CPC^*$, where

$$C = \begin{bmatrix} A(z_1) & & 0 \\ & \ddots & \\ 0 & & A(z_n) \end{bmatrix}.$$

Since $A(z) \neq 0$, then C is invertible. If the kernel K has m negative squares, then the matrix P has at most m negative eigenvalues, and for some choice of points, it has exactly m. Since $\tilde{P}$ has the same inertia as P(meaning they have the same numbers of positive, negative and zero eigenvalues, including multiplicities), that means that $\tilde{P}$ has at most m negative eigenvalues, and for some choice of points, it has exactly m. So, the kernel $\tilde{K}$ has exactly m negative squares. □

4. Classical and Generalized Carathéodory Class Functions

Define the Carathéodory class $\mathcal{C}$ to be a class of functions f that are analytic and have a nonnegative real part in $\mathbb{D}$. It will be useful for us to be able to switch between the classes $\mathcal{C}$ and $\mathcal{S}$ with relative ease. For this reason, the following theorems have been formulated, which will be applied to functions in $\mathcal{C}$ and $\mathcal{S}$ throughout what follows in this thesis:

Theorem 4.1. *A function f belongs to $\mathcal{C}$ if and only if it can be represented as $f(z) = \frac{1+s(z)}{1-s(z)}$ for some s in $\mathcal{S}$. Clearly, this s equals $s(z) = \frac{f(z)-1}{f(z)+1}$.*

Proof: For $s = \frac{f-1}{f+1}$, we have

$$\begin{aligned}
1 - s(z)s(\zeta) &= 1 - \frac{(f(z)-1)(\overline{f(\zeta)}-1)}{(f(z)+1)(\overline{f(\zeta)}+1)} \\
&= \frac{(f(z)+1)(\overline{f(\zeta)}+1) - (f(z)-1)(\overline{f(\zeta)}-1)}{(f(z)+1)(\overline{f(\zeta)}+1)} \\
&= \frac{f(z)\overline{f(\zeta)} + \overline{f(\zeta)} + f(z) + 1 - f(z)\overline{f(\zeta)} + f(z) + \overline{f(\zeta)} - 1}{(f(z)+1)(\overline{f(\zeta)}+1)} \\
&= \frac{2f(z) + 2\overline{f(\zeta)}}{(f(z)+1)(\overline{f(\zeta)}+1)}.
\end{aligned} \tag{4.1}$$

In particular, for $z = \zeta$, we have

$$1 - |s(z)|^2 = \frac{2f(z) + 2\overline{f(z)}}{|f(z)+1|^2} \geq 0.$$

The inequality is true because f belongs to the Carathéodory class. Since $\mathrm{Re} f(z) \geq 0$, it follows that $f(z)+1 \neq 0$ for all z in $\mathbb{D}$. Then $s(z) = \frac{f(z)-1}{f(z)+1}$ is analytic on $\mathbb{D}$. Further, since $|s(z)| \leq 1$, then s is indeed in the Schur class. $\square$

Note also that if $f \in \mathcal{C}$, then the kernel

$$C_f(z,\zeta) = \frac{f(z) + \overline{f(\zeta)}}{1 - z\bar{\zeta}}$$

is positive on $\mathbb{D}$. Indeed, it follows from (4.1) that

$$K_s(z,\zeta) = \frac{1 - s(z)\overline{s(\zeta)}}{1 - z\bar{\zeta}} = \frac{\sqrt{2}}{f(z)+1} C_f(z,\zeta) \frac{\sqrt{2}}{\overline{f(\zeta)}+1} \tag{4.2}$$

where s is defined as in Theorem 4.1. Since $A(z) = \dfrac{\sqrt{2}}{f(z)+1} \neq 0$, the kernels K_s and C_f have the same number of negative squares. Since $f \in \mathcal{C}$, we have $s \in \mathcal{S}$. Then K_s is positive on $\mathbb{D}$, and so C_f is positive on $\mathbb{D}$.

Define $\mathcal{C}_\kappa$ to be a class of functions f that are meromorphic in $\mathbb{D}$ and such that $C_f(z,\zeta)$ has κ negative squares. We need the following well-known result.

Lemma 4.2. *For any n distinct points $z_1, \ldots, z_n \in \mathbb{D}$, the matrix*

$$F = \left[\frac{1}{1 - z_i\overline{z_j}}\right]_{i,j=1}^n$$

is positive definite.

Proof: The functions $k_{z_j}(z) = \dfrac{1}{1 - z\overline{z_j}}$, $j = 1, \ldots, n$ are linearly independent functions in H^2. Therefore, the Gram matrix consisting of all their inner products

$$\left[\langle k_{z_j}, k_{z_i}\rangle\right]_{i,j=1}^n = \begin{bmatrix} \langle k_{z_1}, k_{z_1}\rangle & \cdots & \langle k_{z_1}, k_{z_n}\rangle \\ \vdots & \ddots & \vdots \\ \langle k_{z_n}, k_{z_1}\rangle & \cdots & \langle k_{z_n}, k_{z_n}\rangle \end{bmatrix}$$

is positive definite. Since

$$\left\langle \frac{1}{1 - z\overline{z_j}}, \frac{1}{1 - z\overline{z_i}} \right\rangle_{H^2} = \frac{1}{1 - z_i\overline{z_j}}$$

by Lemma 2.6, the Gram matrix equals F.

The next theorem provides a connection between functions in the two classes $\mathcal{S}_\kappa$ and $\mathcal{C}_\kappa$.

Theorem 4.3. *A function f belongs to $\mathcal{C}_\kappa$ if and only if it can be represented as $f = \dfrac{1+s}{1-s}$ for some $s \in S_\kappa$.*

Proof: Given $f \in \mathcal{C}_\kappa$, define $s = \dfrac{f-1}{f+1}$. First let us show that this s is well-defined, meaning that $f \not\equiv -1$. Indeed, if $f(z) \equiv -1$, then the kernel C_f has infinitely many negative squares. To show this, take $z_1, \ldots, z_n \in \mathbb{D}$.

$$P = [C_f(z_i, z_j)]_{i,j=1}^n = \left[-\frac{2}{1 - z_i\overline{z_j}}\right]_{i,j=1}^n = -2\left[\frac{1}{1 - z_i\overline{z_j}}\right]_{i,j=1}^n$$

has n negative eigenvalues. If $f(z) = -1$ for all z, then P would have infinitely many negative eigenvalues as you increase the size of the matrix.

If f and s are related as above, then the kernels K_s and C_f are related as in (4.2). Thus, $\text{sq}\_K_s = \kappa$ by Theorem 3.2. Therefore, $s \in \mathcal{S}_\kappa$. The reverse argument is also true by Theorem 3.2. □

Corollary 4.4. *Every function $f \in \mathcal{C}_\kappa$ admits a representation*

$$f = \frac{b+g}{b-g} \tag{4.3}$$

where $g \in \mathcal{S}$ and $b \in \mathcal{B}_\kappa$ have no common zeros.

Proof: Combining (4.3) and the Krein-Langer representation of a generalized Schur function from (3.1), we define $f = \dfrac{1+s_k}{1-s_k}$ and have the equality

$$f = \frac{1+s}{1-s} = \frac{1+\frac{g}{b}}{1-\frac{g}{b}} = \frac{b+g}{b-g},$$

where $s \in \mathcal{S}$. □

5. Poles and Zeros of Generalized Carathéodory Functions

We now take the advantage of the representation formula (4.3) to study poles and zeros of generalized Carathéodory functions. We first note that for a finite Blaschke product $b \in \mathcal{B}_\kappa$ and a Schur-class function $g \in \mathcal{S}$, the function $h = g - b$ may have at most κ zeros inside $\mathbb{D}$ counted with multiplicities. Let $z_1, \ldots, z_r$ be all such zeros and let $n_1, \ldots, n_r$ be their respective multiplicities. Then

$$g^{(j)}(z_i) = b^{(j)}(z_i) \quad \text{for} \quad j = 0, \ldots, n_i - 1;\ i = 1, \ldots, r. \tag{5.1}$$

It was shown in [7] that if $n_1 + \ldots + n_r$ is less than κ, then the function $h = g - b$ must have certain zeros at the boundary of $\mathbb{D}$. More precisely, there exist points $z_{r+1}, \ldots, z_\ell \in \mathbb{T}$ with respective multiplicities $n_{r+1}, \ldots, n_\ell$ such that the nontangential boundary limits $h_{ij} = \lim_{z \to z_i} h^{(j)}(z)$ exist for $j = 0, \ldots, 2n_i - 1$ and are such that $h_{ij} = 0$ for $j = 0, \ldots, 2n_i - 2$ and

$$(-1)^{n_i} z_i^{2n_i - 1} \overline{b(z_i)} h_{i,2n_i-1} \geq 0 \quad \text{for} \quad i = r+1, \ldots, \ell.$$

Since b is analytic on $\mathbb{T}$, the latter is equivalent to the existence of the nontangential boundary limits $\lim_{z \to z_i} g^{(j)}(z)$ and the relations

$$\lim_{z \to z_i} g^{(j)}(z) = b^{(j)}(z_i) \quad \text{for} \quad j = 0, \ldots, 2n_i - 2;\ i = r+1, \ldots, \ell \tag{5.2}$$

and

$$(-1)^{n_i} z_i^{2n_i - 1} \overline{b(z_i)} (\lim_{z \to z_i} g^{(2n_i - 1)}(z) - b^{(2n_i-1)}(z_i)) \geq 0 \quad \text{for} \quad i = r+1, \ldots, \ell. \tag{5.3}$$

An interesting thing to note here is that $n_1 + \ldots + n_r + \ldots + n_\ell = \kappa$.

Given a function f with a representation (4.3), let us say that z_i is a *generalized pole* of f of nonpositive type of multiplicity $2n_i$ if conditions (5.2) and (5.3) are satisfied. It is clear that if conditions (5.1) are satisfied at $z_1, \ldots, z_r \in \mathbb{D}$, then z_i is a usual pole of f of order n_i.

Given $b \in \mathcal{B}_\kappa$ and $g \in \mathcal{S}$, let $z_1, \ldots, z_r \in \mathbb{D}$ and $z_{r+1}, \ldots, z_\ell \in \mathbb{T}$ be the zeros of the function $g - b$ described as above. Let us introduce the numbers

$$c_{ij} := \frac{b^{(j)}(z_i)}{j!} \quad \text{for} \quad \begin{array}{ll} j = 0, \ldots, n_i - 1, & \text{if} \quad i \in \{1, \ldots, r\}, \\ j = 0, \ldots, 2n_i - 1, & \text{if} \quad i \in \{r+1, \ldots, \ell\} \end{array} \tag{5.4}$$

and let us consider the following interpolation problem:

IP: *find all $g \in \mathcal{S}$ such that*

$$g^{(j)}(z_i) = j! \cdot c_{ij} \quad \textit{for} \quad j = 0, \ldots, n_i - 1;\ i = 1, \ldots, r, \tag{5.5}$$

$$g^{(j)}(z_i) := \lim_{z \to z_i} g^{(j)}(z) = j! \cdot c_{ij} \tag{5.6}$$

for $j = 0, \ldots, 2n_i - 2$; $i = r+1, \ldots, \ell$, and

$$(-1)^{n_i - 1} z_i^{2n_i - 1} \overline{c}_{i,0} \left(g^{(2n_i-1)}(z_i) - (2n_i - 1)! \cdot c_{i,2n_i-1} \right) \leq 0 \tag{5.7}$$

for $i = r+1, \dots, \ell$. This problem is well known (see e.g., [5, 8]) and its solution is recalled below.

Let us associate with the given tuples $\mathbf{z} = (z_1, \dots, z_\ell)$ and $\mathbf{n} = (n_1, \dots, n_\ell)$ the matrices

$$T = \begin{bmatrix} J_{n_1}(z_1) & & \\ & \ddots & \\ & & J_{n_k}(z_k) \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_{n_1} \\ \vdots \\ E_{n_k} \end{bmatrix}, \tag{5.8}$$

where we denote by $J_n(z)$ and E_n respectively the $n \times n$ Jordan block with $z \in \mathbb{C}$ on the main diagonal and the vector of the length n where the first coordinate equals one and other coordinates equal zero:

$$J_n(z) = \begin{bmatrix} z & 0 & \dots & 0 \\ 1 & z & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & z \end{bmatrix}, \quad E_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.9}$$

The numbers c_{ij} from (5.5)–(5.7) are arranged in the column

$$C = \begin{bmatrix} C_{1,n_1} \\ \vdots \\ C_{k,n_k} \end{bmatrix}, \quad \text{where} \quad C_{i,n_i} = \begin{bmatrix} c_{i,0} \\ \vdots \\ c_{i,n_i-1} \end{bmatrix}. \tag{5.10}$$

Observe that $C_{\mathbf{n}}$ contains all the numbers c_{ij} but $c_{i,n_i}, \dots, c_{i,2n_i-1}$ for all $i = r+1, \dots, \ell$. These extra numbers are used to define the structured matrix P_{ii} as follows:

$$P_{ii} = \begin{bmatrix} c_{i,1} & c_{i,2} & \cdots & c_{i,n_i} \\ c_{i,2} & c_{i,3} & \cdots & c_{i,n_i+1} \\ \vdots & \vdots & & \vdots \\ c_{i,n_i} & c_{i,n_i+1} & \dots & c_{i,2c_i-1} \end{bmatrix} \mathbf{\Psi}_{n_i}(z_i) \begin{bmatrix} \overline{c}_{i,0} & \dots & \overline{c}_{i,n_i-1} \\ & \ddots & \vdots \\ 0 & & \overline{c}_{i,0} \end{bmatrix} \tag{5.11}$$

($i = r+1, \dots, \ell$) where the first factor is a Hankel matrix, the third factor is an upper triangular Toeplitz matrix and where $\mathbf{\Psi}_{n_i}(z_1) = [\Psi_{j\ell}]_{j,\ell=0}^{n_1-1}$ is the upper triangular matrix

$$\mathbf{\Psi}_{n_i}(z_i) = \begin{bmatrix} z_i & -z_i^2 & z_i^3 & \cdots & (-1)^{n_i-1}\binom{n_i-1}{0} z_i^{n_i} \\ 0 & -z_i^3 & 2z_i^4 & \cdots & (-1)^{n_i-1}\binom{n_i-1}{1} z_i^{n_1+1} \\ \vdots & & z_i^5 & \cdots & (-1)^{n_i-1}\binom{n_i-1}{2} z_i^{n_i+2} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & (-1)^{n_i-1}\binom{n_i-1}{n_i-1} z_i^{2n_i-1} \end{bmatrix}, \tag{5.12}$$

with entries

$$\Psi_{j\ell} = (-1)^\ell \binom{\ell}{j} z_i^{\ell+j+1}, \quad 0 \le j \le \ell \le n_i - 1. \tag{5.13}$$

The matrix P_{ii} is not Hermitian in general. However, if the numbers c_{ij} are the Taylor coefficients of a finite Blaschke product at a boundary point, then the matrix P_{ii} is not only Hermitian, but also positive semidefinite. This was proved in [8]. Moreover, whenever P_{ii} is Hermitian and $c_{i,0}$ is unimodular, the matrix P_{ii} satisfies the following Stein identity:

$$P_{ii} - J_{n_i}(z_i)P_{ii}J_{n_i}(z_i)^* = E_{n_i}E_{n_i}^* - C_{n_i}C_{n_i}^* \tag{5.14}$$

where $J_{n_i}(z_i)$, E_{n_i} and C_{n_i} are given in (5.9), (5.10). Furthermore, if $i \neq j$ or if $i = j \in \{1, \dots, r\}$, we define the $n_i \times n_j$ matrix P_{ij} as the unique matrix satisfying the Stein identity

$$P_{ij} - J_{n_i}(z_i)P_{ij}J_{n_j}(z_j)^* = E_{n_i}E_{n_j}^* - C_{n_i}C_{n_j}^*. \tag{5.15}$$

The latter equality indeed has a unique solution since the only eigenvalues of $J_{n_i}(z_i)$ and $J_{n_j}(z_j)$ are z_i and z_j respectively and $z_i\overline{z}_j \neq 1$. In this case, the entries of P_{ij} can be found recursively (see e.g., [5]):

$$\begin{aligned}
[P_{ij}]_{\ell,r} &= \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!} \frac{z_i^{r-s}\overline{z}_j^{\ell-s}}{(1-z_i\overline{z}_j)^{\ell+r-s+1}} \\
&- \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!} \frac{z_i^{\beta-s}\overline{z}_j^{\alpha-s}c_{i,\ell-\alpha}\overline{c}_{j,r-\beta}}{(1-z_i\overline{z}_j)^{\alpha+\beta-s+1}}.
\end{aligned} \tag{5.16}$$

It now follows that the block matrix P given by

$$P = [P_{ij}]_{i,j=1}^{\ell} \tag{5.17}$$

with the blocks defined in (5.11) and (5.17) is Hermitian and satisfies the Stein identity

$$P - TT^* = EE^* - CC^*. \tag{5.18}$$

where T, E and C are given in (5.9), (5.10). Moreover, since the numbers c_{ij} are the Taylor coefficients of a finite Blaschke product of degree κ, the matrix P is positive definite. With all these ingredients in hands, we introduce the 2×2 matrix function

$$\begin{aligned}
\Theta(z) &= \begin{bmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (z-\mu)\begin{bmatrix} E^* \\ C^* \end{bmatrix}(I - zT^*)^{-1}P^{-1}(\mu I - T)^{-1}\begin{bmatrix} E & -C \end{bmatrix},
\end{aligned} \tag{5.19}$$

where μ is an arbitrary point in $\mathbb{T}\backslash\{x_1, \dots, x_\ell\}$. The following theorem (see [5], [8] for the proof) describes all functions $g \in \mathcal{S}$ satisfying conditions (5.5)–(5.7).

Theorem 5.1. *A function g belongs to the Schur class $\mathcal{S}$ and satisfies conditions* (5.5)–(5.7) *if and only if it is of the form*

$$g = \frac{\theta_{11}\mathcal{E} + \theta_{12}}{\theta_{21}\mathcal{E} + \theta_{22}} \tag{5.20}$$

for some $\mathcal{E} \in \mathcal{S}$. Moreover, if $\mathcal{E} \in \mathcal{B}_m$, then $g \in \mathcal{B}_{m+\kappa}$.

The functions θ_{ij} are rational and of degree at most κ. Let us introduce the signature matrix $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. A straightforward calculation based solely on the identity (5.18) shows that for every z, ζ at which Θ is analytic,

$$J - \Theta(z) J \Theta(\zeta)^* = (1 - z\bar{\zeta}) \begin{bmatrix} E^* \\ C^* \end{bmatrix} (I - zT^*)^{-1} P^{-1} (I - \bar{\zeta} T)^{-1} \begin{bmatrix} E & C \end{bmatrix}.$$

In particular,

$$\Theta(z) J \Theta(\zeta)^* \leq J \quad \text{if} \quad |z| < 1$$

and

$$\Theta(z) J \Theta(\zeta)^* = J \quad \text{if} \quad |z| = 1.$$

Another calculation based on the identity (5.18) gives

$$\det \Theta(z) = \prod_{i=1}^{\ell} \left(\frac{(z - z_i)(1 - \mu \overline{z}_i)}{(1 - z\overline{z}_i)(\mu - z_i)} \right)^{n_i}. \tag{5.21}$$

Indeed, since $\det(I + AB) = \det(I + BA)$, we have

$$\begin{aligned}
\det \Theta(z) &= \det \left(I + (z - \mu) \begin{bmatrix} E^* \\ C^* \end{bmatrix} (I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix} \right) \\
&= \det \left(I + (z - \mu)(I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix} \begin{bmatrix} E^* \\ C^* \end{bmatrix} \right) \\
&= \det \left((I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \right) \\
&\quad \cdot \det \left((\mu I - T) P (I - zT^*) + (z - \mu)(EE^* - CC^*) \right) \\
&= \det \left((I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \right) \\
&\quad \cdot \det \left((\mu I - T) P (I - zT^*) + (z - \mu)(P - TPT^*) \right) \\
&= \det \left((I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \right) \\
&\quad \cdot \det \left(zP - TP - \mu z P T^* + \mu T P T^* \right) \\
&= \det \left((I - zT^*)^{-1} P^{-1} (\mu I - T)^{-1} \right) \cdot \det \left((zI - T) P (I - \mu T^*) \right) \\
&= \det \left((I - zT^*)^{-1} (\mu I - T)^{-1} (zI - T)(I - \mu T^*) \right).
\end{aligned}$$

Due to the special structure (5.8), (5.9) of T,

$$\det(zI - T) = \prod_{i=1}^{\ell} \det(zI_{n_i} - J_{n_i}(z_i)) = \prod_{i=1}^{\ell} (z - z_i)^{n_i}$$

and quite similarly,

$$\det(I - zT^*) = \prod_{i=1}^{\ell} (1 - z\overline{z}_i)^{n_i}.$$

The three latter formulas imply (5.21). Observe that since $z_{r+1}, \ldots, z_\ell$ fall on the unit circle, we have

$$\frac{z - z_i}{1 - z\overline{z}_i} = \frac{z_i(z\overline{z}_i - 1)}{1 - z\overline{z}_i} = -z_i$$

for $i = r+1, \ldots, \ell$. Thus,

$$\det \Theta(z) = \gamma \cdot \prod_{i=1}^{r} \left(\frac{z - z_i}{1 - z\overline{z}_i} \right)^{n_i}. \tag{5.22}$$

where

$$\gamma = \prod_{i=1}^{\ell} \left(\frac{1 - \mu\overline{z}_i}{\mu - z_i} \right)^{n_i} \cdot \prod_{i=r+1}^{\ell} (-z_i)^{n_i} \tag{5.23}$$

is a unimodular number, since

$$\prod_{i=1}^{\ell} \left| \frac{1 - \mu\overline{z}_i}{\mu - z_i} \right|^{n_i} = \prod_{i=1}^{\ell} \left| \mu \cdot \frac{\overline{\mu} - \overline{z}_i}{\mu - z_i} \right|^{n_i} = \prod_{i=1}^{\ell} |\mu|^{n_i} \left| \cdot \frac{\overline{\mu} - \overline{z}_i}{\mu - z_i} \right|^{n_i} = 1.$$

Theorem 5.2. *Let f be a function in $\mathcal{C}_\kappa$ with poles $z_1, \ldots, z_r \in \mathbb{D}$ of respective orders $n_1, \ldots, n_r$ and with generalized boundary poles $z_{r+1}, \ldots, z_\ell$ of nonpositive type of respective orders $2n_{r+1}, \ldots, 2n_\ell$. Then there exist polynomials p and q of degree at most 2κ and a Carathéodory-class function $\Phi \in \mathcal{C}$ such that*

$$f(z) = \frac{p(z)\Phi(z) + q(z)}{\prod_{i=1}^{r}(z - z_i)^{n_i}(1 - z\overline{z}_i)^{n_i} \cdot \prod_{i=r+1}^{\ell}(z - z_i)^{2n_i}}. \tag{5.24}$$

Proof: If $f \in \mathcal{S}$ and $b \in \mathcal{B}_\kappa$ are the functions from the representation (4.3), they are solutions to the problem **IP** by construction. Therefore, g is of the form (5.20) for some $\mathcal{E} \in \mathcal{S}$ and b admits a similar representation

$$b = \frac{\theta_{11}\widetilde{b} + \theta_{12}}{\theta_{21}\widetilde{b} + \theta_{22}} \tag{5.25}$$

with $\widetilde{b} \in \mathcal{S}$. By Theorem 5.1, $\widetilde{b} \in \mathcal{B}_0$ that is $\widetilde{b}$ is a unimodular constant. Evaluating (5.25) at $z = \mu$ gives $b(\mu) = \widetilde{b}$. Since we may assume without loss of generality that $b(\mu) = 1$, we get

$$b = \frac{\theta_{11} + \theta_{12}}{\theta_{21} + \theta_{22}}. \tag{5.26}$$

Using (5.20), (5.26) and Corollary 4.4 , we have

$$\begin{aligned}\frac{b+g}{b-g} &= \frac{\dfrac{\theta_{11}+\theta_{12}}{\theta_{21}+\theta_{22}} + \dfrac{\theta_{11}\mathcal{E}+\theta_{12}}{\theta_{21}\mathcal{E}+\theta_{22}}}{\dfrac{\theta_{11}+\theta_{12}}{\theta_{21}+\theta_{22}} - \dfrac{\theta_{11}\mathcal{E}+\theta_{12}}{\theta_{21}\mathcal{E}+\theta_{22}}} \\ &= \frac{(\theta_{11}\mathcal{E}+\theta_{12})(\theta_{21}+\theta_{22}) + (\theta_{11}+\theta_{12})(\theta_{21}\mathcal{E}+\theta_{22})}{(\theta_{11}\mathcal{E}+\theta_{12})(\theta_{21}+\theta_{22}) - (\theta_{11}+\theta_{12})(\theta_{21}\mathcal{E}+\theta_{22})} \\ &= \frac{2\mathcal{E}\theta_{11}\theta_{21} + 2\theta_{12}\theta_{22} + (\mathcal{E}+1)(\theta_{11}\theta_{22}+\theta_{12}\theta_{21})}{(\theta_{11}\theta_{22}-\theta_{12}\theta_{21})(1-\mathcal{E})}.\end{aligned} \tag{5.27}$$

Define a function $\Phi \in \mathcal{C}$ using Theorem 4.1 to be $\Phi = \dfrac{1+\mathcal{E}}{1-\mathcal{E}}$. Thus, $\mathcal{E} = \dfrac{\Phi-1}{\Phi+1}$. So, $1-\mathcal{E} = \dfrac{2}{\Phi+1}$ and $1+\mathcal{E} = \dfrac{2\Phi}{\Phi+1}$. Substituting the latter equalities into (5.27), we have

$$\begin{aligned}\frac{b+g}{b-g} &= \frac{2\mathcal{E}\theta_{11}\theta_{21} + 2\theta_{12}\theta_{22} + (\mathcal{E}+1)(\theta_{11}\theta_{22}+\theta_{12}\theta_{21})}{\det\Theta\cdot(1-\mathcal{E})} \\ &= \frac{\dfrac{\Phi-1}{\Phi+1}\theta_{11}\theta_{21} + \theta_{12}\theta_{22} + \dfrac{\Phi}{\Phi+1}(\theta_{11}\theta_{22}+\theta_{12}\theta_{21})}{\det\Theta\cdot\dfrac{1}{\Phi+1}} \\ &= \frac{(\Phi-1)\theta_{11}\theta_{21} + (\Phi+1)\theta_{12}\theta_{22} + \Phi(\theta_{11}\theta_{22}+\theta_{12}\theta_{21})}{\det\Theta} \\ &= \frac{(\Phi-1)\theta_{11}\theta_{21} + (\Phi+1)\theta_{12}\theta_{22} + (\Phi)(\theta_{11}\theta_{22}+\theta_{12}\theta_{21})}{\theta_{11}\theta_{22}-\theta_{12}\theta_{21}} \\ &= \frac{\Phi(\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+\theta_{11}\theta_{22}+\theta_{12}\theta_{21}) - \theta_{11}\theta_{21} + \theta_{12}\theta_{22}}{\det\Theta} \\ &= \frac{(\theta_{11}+\theta_{12})(\theta_{21}+\theta_{22})\Phi + \theta_{12}\theta_{22} - \theta_{11}\theta_{21}}{\det\Theta}.\end{aligned} \tag{5.28}$$

Let us look at the formula (5.19). It is clear due to the block-diagonal structure (5.8) of T that

$$(I - zT^*)^{-1} = \begin{bmatrix} (I - zJ_{n_1}(z_1)^*)^{-1} & & 0 \\ & \ddots & \\ 0 & & (I - zJ_{n_\ell}(z_\ell)^*)^{-1} \end{bmatrix}$$

and since

$$(I - zJ_{n_1}(z_1)^*)^{-1} = \begin{bmatrix} \frac{1}{1-z\bar{z}_i} & 0 & \cdots & 0 \\ \frac{1}{(1-z\bar{z}_i)^2} & \frac{1}{1-z\bar{z}_i} & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ \frac{1}{(1-z\bar{z}_i)^{n_i}} & & \frac{1}{(1-z\bar{z}_i)^2} & \frac{1}{1-z\bar{z}_i} \end{bmatrix}$$

(see (5.9)), it follows that the function

$$\prod_{i=1}^{\ell}(1-z\bar{z}_i)^{n_i}\cdot(I-zT^*)^{-1}$$

is a matrix polynomial. Now it follows from (5.19) that the functions

$$\widetilde{\theta}_{ij}(z)=\prod_{i=1}^{\ell}(1-z\bar{z}_i)^{n_i}\cdot\theta_{ij}(z). \qquad i,j=1,2,$$

are polynomials. Let us set

$$p=\overline{\gamma}(\widetilde{\theta}_{11}+\widetilde{\theta}_{12})(\widetilde{\theta}_{21}+\widetilde{\theta}_{22}) \quad \text{and} \quad q=\overline{\gamma}(\widetilde{\theta}_{12}\widetilde{\theta}_{22}-\widetilde{\theta}_{11}\widetilde{\theta}_{21}).$$

where γ is the unimodular number defined in (5.23). If we multiply the denominator on the right hand side of (5.28) by the function

$$\overline{\gamma}\prod_{i=1}^{\ell}(1-z\bar{z}_i)^{2n_i},$$

we get the expression equal the denominator in (5.24). Multiplying the numerator on the right hand side of (5.28) by the same function gives $p\Phi+q$ by the very definition of p and q. This completes the proof of the theorem. □

Theorem 5.2 elaborated the poles of a generalized Carathéodory function. The zeros of f can be elaborated in much the same way. All zeros of such an f are the points $\zeta_1,\dots,\zeta_\alpha\in\mathbb{D}$ at which the functions $g\in\mathcal{S}$ and $-b\in\mathcal{B}_\kappa$ coincide up to certain order:

$$g^{(j)}(\zeta_i)=-b^{(j)}(\zeta_i) \quad \text{for} \quad j=0,\dots,m_i-1;\ i=1,\dots,\alpha. \tag{5.29}$$

Due to a result from [7], if $m_1+\ldots+m_\alpha$ is less than κ, then there exist points $\zeta_{\alpha+1},\dots,\zeta_\beta\in\mathbb{T}$ with such that the nontangential boundary limits $\lim\limits_{z\to\zeta_i} g^{(j)}(z)$ exist and satisfy

$$\lim_{z\to\zeta_i} g^{(j)}(z)=-b^{(j)}(\zeta_i) \quad \text{for} \quad j=0,\dots,2m_i-2;\ i=\alpha+1,\dots,\beta \tag{5.30}$$

and

$$(-1)^{m_i}z_i^{2m_i-1}\overline{b(\zeta_i)}(\lim_{z\to\zeta_i} g^{(2m_i-1)}(z)+b^{(2m_i-1)}(\zeta_i))\le 0 \quad \text{for} \quad i=\alpha+1,\dots,\beta. \tag{5.31}$$

Once again, $m_1+\ldots+m_\beta=\kappa$. We may think of (5.29)–(5.31) as of an interpolation problem similar to **IP** but with different data $d_{ij}=-b^{(j)}(\zeta_i)/j!$. We may follow the above strategy to get a representation of f emphasizing its zeros. However, there are several indications that these two descriptions can be combined. We plan to pursue our investigations in this direction. Here we present a simple case where f is of the form

$$f(z)=\frac{g(z)+z}{z-g(z)} \tag{5.32}$$

and has one pole and one zero in $\mathbb{D}$. Thus, we assume that

$$g(z_1)=z_1 \quad \text{and} \quad g(z_2)=-z_2. \tag{5.33}$$

Since g is a Schur-class function,

$$\left[\frac{1-g(z_i)\overline{g(z_j)}}{1-z_i\bar{z}_j}\right]_{i,j=1}^{2} = \begin{bmatrix} 1 & \dfrac{1+z_1\bar{z}_2}{1-z_1\bar{z}_2} \\ \dfrac{1+\bar{z}_1 z_2}{1-\bar{z}_1 z_2} & 1 \end{bmatrix} \geq 0.$$

Therefore, the determinant of this matrix is nonnegative, i.e.,

$$\left|\frac{1+z_1\bar{z}_2}{1-z_1\bar{z}_2}\right|^2 \leq 1 \quad \Leftrightarrow \quad |1+z_1\bar{z}_2| \leq |1-z_1\bar{z}_2|$$

which, in turn, is equivalent to

$$\operatorname{Re}(z_1\bar{z}_2) \leq 0. \tag{5.34}$$

Actually, it follows that the points z_1 and z_2 in $\mathbb{D}$ are, respectively, a pole and a zero of some function $f \in \mathcal{C}_1$ of the form (5.32) if and only if condition (5.34) is satisfied. Although we can use Theorem 5.1 to get a linear fractional representation of g, we will proceed differently. Due to the first condition in (5.33), the function

$$s(z) = \frac{g(z)-z_1}{1-g(z)\bar{z}_1} \cdot \frac{1-z\bar{z}_1}{z-z_1} \tag{5.35}$$

belong to $\mathcal{S}$ by the Schwarz-Pick lemma. We can write the latter equality equivalently as

$$g(z) = \frac{\dfrac{z-z_1}{1-z\bar{z}_1}s(z)+z_1}{1+\dfrac{z-z_1}{1-z\bar{z}_1}\bar{z}_1 s(z)}. \tag{5.36}$$

Combining the second condition in (5.33) with (5.35) gives

$$s(z_2) = A := -\frac{z_1+z_2}{1+z_2\bar{z}_1} \cdot \frac{1-z_2\bar{z}_1}{z_2-z_1}. \tag{5.37}$$

By virtue of (5.36),

$$s(z) = \frac{\dfrac{z-z_2}{1-z\bar{z}_2}\mathcal{E}(z)+A}{1+\dfrac{z-z_2}{1-z\bar{z}_2}\bar{A}\mathcal{E}(z)}, \tag{5.38}$$

where A is given in (5.37). We now substitute (5.38) into (5.36):

$$g(z) = \frac{\dfrac{z-z_1}{1-z\bar{z}_1} \cdot \dfrac{\dfrac{z-z_2}{1-z\bar{z}_2}\mathcal{E}(z)+A}{1+\dfrac{z-z_2}{1-z\bar{z}_2}\bar{A}\mathcal{E}(z)} + z_1}{1+\dfrac{z-z_1}{1-z\bar{z}_1}\bar{z}_1 \cdot \dfrac{\dfrac{z-z_2}{1-z\bar{z}_2}\mathcal{E}(z)+A}{1+\dfrac{z-z_2}{1-z\bar{z}_2}\bar{A}\mathcal{E}(z)}} = \frac{N(z)}{D(z)}. \tag{5.39}$$

Long but straightforward calculations show that

$$z-g(z)=-\frac{2(z-z_1)(1-|z_1|^2)}{D(z)(1-z\bar{z}_2)}\cdot\left[\frac{(z-z_2)\bar{z}_2\mathcal{E}(z)}{(\bar{z}_2-\bar{z}_1)(1+z_1\bar{z}_2)}-\frac{z_2(1-z\bar{z}_2)}{(z_2-z_1)(1+z_2\bar{z}_1)}\right]$$

and

$$z+g(z)=-\frac{2(z-z_2)(1-|z_1|^2)}{D(z)(1-z\bar{z}_1)}\cdot\left[\frac{(z\bar{z}_1+z_1\bar{z}_2)\mathcal{E}(z)}{(\bar{z}_2-\bar{z}_1)(1+z_1\bar{z}_2)}+\frac{z_1+zz_2\bar{z}_1}{(z_2-z_1)(1+z_2\bar{z}_1)}\right].$$

We summarize these calculations for $z-g(z)$. The result for $z+g(z)$ is derived similarly.

$$\begin{aligned} z-g(z)&=z-\frac{\frac{z-z_1}{1-z\bar{z}_1}s(z)+z_1}{1+\frac{z-z_1}{1-z\bar{z}_1}\bar{z}_1s(z)}\\ &=\frac{z+zz_1\frac{z-z_1}{1-z\bar{z}_1}s(z)-\frac{z-z_1}{1-z\bar{z}_1}s(z)-z_1}{1+\frac{z-z_1}{1-z\bar{z}_1}\bar{z}_1s(z)} \end{aligned}\tag{5.40}$$

Given that the denominator is the same for both $z+g(z)$ and $z-g(z)$, we analyze only the numerator and find it simplifies to:

$$\begin{aligned} (z-z_1)(1-s(z))&=(z-z_1)(1+\frac{z-z_2}{1-z\bar{z}_2}\bar{A}\mathcal{E}(z)-\frac{z-z_2}{1-z\bar{z}_2}\mathcal{E}(z)-A)\\ &=(z-z_1)(\frac{z-z_2}{1-z\bar{z}_2}(\bar{A}-1)\mathcal{E}(z)+1-A). \end{aligned}\tag{5.41}$$

Recall A from (5.37) and see that

$$1-A=1+\frac{z_1+z_2}{1+z_2\bar{z}_1}\cdot\frac{1-z_2\bar{z}_1}{z_2-z_1}=\frac{2z_2(1-|z_1|^2)}{(1+\bar{z}_1z_2)(z_2-z_1)}.$$

Substituting this and

$$\bar{A}-1=-\frac{2\bar{z}_2(1-|z_1|^2)}{(1+z_1\bar{z}_2)(\bar{z}_2-\bar{z}_1)}$$

into (5.41), adding back the denominator and simplifying, we have

$$z-g(z)=-\frac{2(z-z_1)(1-|z_1|^2)}{D(z)(1-z\bar{z}_2)}\cdot\left[\frac{(z-z_2)\bar{z}_2\mathcal{E}(z)}{(\bar{z}_2-\bar{z}_1)(1+z_1\bar{z}_2)}-\frac{z_2(1-z\bar{z}_2)}{(z_2-z_1)(1+z_2\bar{z}_1)}\right].$$

Using the two equalities for $z+g(z)$ and $z-g(z)$ above, we substitute into (5.32) and get:

$$f(z)=\frac{(z-z_2)(1-z\bar{z}_1)}{(z-z_1)(1-z\bar{z}_2)}\cdot\Phi(z),\tag{5.42}$$

where

$$\Phi(z) = \frac{\dfrac{(z\bar{z}_1 + z_1\bar{z}_2)\mathcal{E}(z)}{(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)} + \dfrac{z_1 + zz_2\bar{z}_1}{(z_2 - z_1)(1 + z_2\bar{z}_1)}}{\dfrac{(z - z_2)\bar{z}_2\mathcal{E}(z)}{(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)} - \dfrac{z_2(1 - z\bar{z}_2)}{(z_2 - z_1)(1 + z_2\bar{z}_1)}}.$$

Let us show that $\Phi(z)$ has a nonnegative real part, and thus is in the $\mathcal{C}$ class, whenever condition (5.34) is satisfied.

Multiplying both numerator and denominator by $(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)$, we have

$$\Phi(z) = \frac{(z\bar{z}_1 + z_1\bar{z}_2)\mathcal{E}(z) + \dfrac{(z_1 + zz_2\bar{z}_1)(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)}{(z_2 - z_1)(1 + z_2\bar{z}_1)}}{(z - z_2)\bar{z}_2\mathcal{E}(z) - \dfrac{z_2(1 - z\bar{z}_2)(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)}{(z_2 - z_1)(1 + z_2\bar{z}_1)}}.$$

Introducing the unimodular number $\Gamma = \dfrac{(\bar{z}_2 - \bar{z}_1)(1 + z_1\bar{z}_2)}{(z_2 - z_1)(1 + z_2\bar{z}_1)}$ allows us to write the last formula as

$$\Phi(z) = \frac{(z\bar{z}_1 + z_1\bar{z}_2)\mathcal{E}(z) + (z_1 + zz_2\bar{z}_1)\Gamma}{(z - z_2)\bar{z}_2\mathcal{E}(z) - z_2(1 - z\bar{z}_2)\Gamma}.$$

Let us now demonstrate that $\Phi(z) + \overline{\Phi(z)} \geq 0$.

$$\begin{aligned} \Phi(z) + \overline{\Phi(z)} = {} & |\mathcal{E}(z)|^2(z_1\bar{z}_2 + \bar{z}_1z_2)(|z|^2 - |z_2|^2) \\ & + \mathcal{E}(z)\overline{\Gamma}\bar{z}_2(z_1\bar{z}_2 + \bar{z}_1z_2)(|z|^2 - 1) \\ & + \overline{\mathcal{E}(z)}\Gamma z_2(z_1\bar{z}_2 + \bar{z}_1z_2)(|z|^2 - 1) \\ & + |\Gamma|^2(z_1\bar{z}_2 + \bar{z}_1z_2)(|z|^2|z_2|^2 - 1). \end{aligned} \tag{5.43}$$

Keeping in mind that Γ is unimodular, this simplifies to:

$$\begin{aligned} \Phi(z) + \overline{\Phi(z)} &= (z_1\bar{z}_2 + \bar{z}_1z_2)[|\mathcal{E}(z)|^2(|z|^2 - |z_2|^2) \\ &\quad + \mathcal{E}(z)\overline{\Gamma}\bar{z}_2(|z|^2 - 1) + \overline{\mathcal{E}(z)}\Gamma z_2(|z|^2 - 1) \\ &\quad + (|z|^2|z_2|^2 - 1)] \\ &= -(z_1\bar{z}_2 + \bar{z}_1z_2)[|\mathcal{E}(z)|^2(|z_2|^2 - |z|^2) \\ &\quad + \mathcal{E}(z)\overline{\Gamma}\bar{z}_2(1 - |z|^2) + \overline{\mathcal{E}(z)}\Gamma z_2(1 - |z|^2) \\ &\quad + (1 - |z|^2|z_2|^2)] \end{aligned} \tag{5.44}$$

Factoring out $1 - |z|^2$, completing the square and factoring again, we have

$$\Phi(z) + \overline{\Phi(z)} = -(z_1\bar{z}_2 + \bar{z}_1z_2)[(1 - |z|^2)(|\mathcal{E}(z) + \Gamma z_2|^2) + (1 - |z_2|^2)(1 - |\mathcal{E}(z)|^2)] \tag{5.45}$$

Given that each of these pieces is nonnegative, except for $(z_1\bar{z}_2 + \bar{z}_1z_2)$, which is necessarily nonpositive by (5.34), then (5.45) is nonnegative. □

The result of this example is the direction for future research.

6. Future Research and Acknowledgments

The objective of this paper was to find something similar to the Krein-Langer representation for $f \in C_\kappa$. Our result was a representation of generalized Carathéodory class functions in terms of Carathéodory class functions, as shown in (5.28).

However, there is still work left to be done in reducing the equation to a form that can account for both the zeros and poles of the function simultaneously.

The formula (5.45) shows that this is possible for the example given the constraint of (5.34). We hope that this result can be obtained for n poles and n zeros, including boundary poles and boundary zeros.

"If I have seen further it is by standing on the shoulders of giants." Sir Isaac Newton

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