A New Lower Bound on the Minimum Density of Vertex Identifying Codes for the Infinite Hexagonal Grid

Ariel J. Cukierman

College of William and Mary

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A New Lower Bound on the Minimum Density of a Vertex Identifying Code for the Infinite Hexagonal Grid

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science in Mathematics from The College of William and Mary

by

Ariel Jozef Cukierman

Accepted for Honors

Gexin Yu, Director
Ilya Spitkovsky
Christopher Carone

Williamsburg, VA
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A New Lower Bound on the Minimum Density of a Vertex Identifying Code for the Infinite Hexagonal Grid

Ari Cukierman
Abstract

For a graph, $G$, and a vertex $v \in V(G)$, let $N[v]$ be the set of vertices adjacent to and including $v$. A set $D \subseteq V(G)$ is a vertex identifying code if for any two distinct vertices $v_1, v_2 \in V(G)$, the vertex sets $N[v_1] \cap D$ and $N[v_2] \cap D$ are distinct and non-empty. We consider the minimum density of a vertex identifying code for the infinite hexagonal grid. In 2000, Cohen et al. constructed two codes with a density of $\frac{3}{7} \approx 0.428571$, and this remains the best known upper bound. Until now, the best known lower bound was $\frac{12}{29} \approx 0.413793$ and was proved by Cranston and Yu in 2009. We present three new codes with a density of $\frac{3}{7}$, and we improve the lower bound to $\frac{5}{12} \approx 0.416667$. 
Acknowledgment

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Chapter 1

Introduction

The study of vertex identifying codes is motivated by the desire to detect failures efficiently in a multi-processor network. Such a network can be modeled as a simple undirected graph, $G$, where $V(G)$ represents the set of processors and $E(G)$ represents the set of connections among processors. Suppose we place detectors on a subset of these processors. These detectors monitor all processors within a neighborhood of radius $r$ and send a signal to a central controller when a failure occurs. We assume that no two failures occur simultaneously. A signal from a detector, $d$, indicates that a processor in the $r$-neighborhood of $d$ has failed but provides no further information.

Now, any given processor, $p$, might be in the $r$-neighborhood of several detectors, $d_1$, $d_2$, $d_3$... Then, when $p$ fails, the central controller receives signals from $d_1$, $d_2$, $d_3$... Let us call \{$d_1, d_2, d_3,...$\} the trace of $p$ in $G$. If each processor has a unique and non-empty trace, then the central controller can determine which processor failed simply by noting the detectors from which signals were received. In this case, we call the subset of processors on which detectors were placed an identifying code.

Vertex identifying codes were first introduced in 1998 by Karpovsky, Chakrabarty and Levitin [5]. The processors of the preceding paragraph become the vertices of a graph, and the processors on which detectors have been placed become the vertex
subset called a vertex identifying code. In the example above, we considered detectors which monitor a neighborhood of radius $r$. In this paper, we concern ourselves with the case in which $r = 1$.

Let $N_i(v)$ be the set of vertices at distance-$i$ from a vertex, $v$, and let $N[v] = N_1(v) \cup \{v\}$.

**Definition 1.1.** For a graph, $G$, a set $D \subseteq V(G)$ is a **vertex identifying code** if

(i) for all $v \in V(G)$, we have $N[v] \cap D \neq \emptyset$

(ii) for all $v_1, v_2 \in V(G)$ with $v_1 \neq v_2$, we have $N[v_1] \cap D \neq N[v_2] \cap D$

From Definition 1.1, we see that some graphs do not admit vertex identifying codes. In particular, if $N[v_1] = N[v_2]$ for some distinct $v_1, v_2 \in V(G)$ then $G$ does not admit a vertex identifying code because $N[v_1] \cap D = N[v_2] \cap D$ for any $D \subseteq V(G)$. On the other hand, if $N[v_1] \neq N[v_2]$ for all distinct $v_1, v_2 \in V(G)$ then $G$ admits a vertex identifying code because $V(G)$ is such a code.

Of particular interest are vertex identifying codes of minimal cardinality. When dealing with infinite graphs, we consider instead the **density** of a vertex identifying code, i.e., the ratio of the number of vertices in the code to the total number of vertices. Let $G$ be an infinite graph, and let $D \subseteq V(G)$ be a vertex identifying code for $G$. Then, for some $v \in V(G)$, the set of vertices in $D$ within distance-$k$ of $v$ is given by $\bigcup_{i=0}^{k} N_i(v) \cap D$. Let $\sigma(D, G)$ be the density of $D$ in $G$. Then,

$$\sigma(D, G) = \limsup_{k \to \infty} \frac{\left| \bigcup_{i=0}^{k} N_i(v) \cap D \right|}{\left| \bigcup_{i=0}^{k} N_i(v) \right|} \quad (1.1)$$

Let $\sigma_0(G)$ be the minimum density of a vertex identifying code for $G$; that is,

$$\sigma_0(G) = \min_D \{ \sigma(D, G) \} \quad (1.2)$$
Karpovsky et al. [5] considered the minimum density of vertex identifying codes for the infinite triangular \((G_T)\), square \((G_S)\) and hexagonal \((G_H)\) grids. They showed \(\sigma_0(G_T) = 1/4\). In 1999, Cohen et al. [2] proved \(\sigma_0(G_S) \leq 7/20\), and, in 2005, Ben-Haim and Litsyn [1] completed the proof by showing \(\sigma_0(G_S) \geq 7/20\).

We concern ourselves in this paper with \(\sigma_0(G_H)\). In 1998, Karpovsky et al. [5] showed \(\sigma_0(G_H) \geq 2/5 = 0.4\). In 2000, Cohen et al. [3] improved this result to \(\sigma_0(G_H) \geq 16/39 \approx 0.410256\) and constructed two codes with a density of \(3/7 \approx 0.428571\) (included in Chapter 2) implying \(\sigma_0(G_H) \leq 3/7\). In 2009, Cranston and Yu [4] proved \(\sigma_0(G_H) \geq 12/29 \approx 0.413793\). For other results on identifying codes for the hexagonal grid, see [6, 7].

In this paper, we present three new codes with a density of \(3/7\) and prove \(\sigma_0(G_H) \geq 5/12 \approx 0.416667\). In conclusion, it is now known that \(5/12 \leq \sigma_0(G_H) \leq 3/7\).

Suppose \(\beta\) is an upper bound on \(\sigma_0(G_H)\). To prove this, we need only show the existence of a code, \(D\), with \(\sigma(D, G_H) \leq \beta\). When constructing such codes, we usually look for tiling patterns. Since the pattern repeats ad infinitum, the density of one tile is the density of the whole graph. Examples are included in Chapter 2.

**Theorem 1.2.** The minimum density of a vertex identifying code for the infinite hexagonal grid is greater than or equal to \(5/12\).

To prove Theorem 1.2, we employ the discharging method. Let \(D\) be an arbitrary vertex identifying code for \(G_H\). We assign 1 “charge” to each vertex in \(D\) which we then redistribute so that every vertex in \(G_H\) retains at least \(5/12\) charge. The charge is redistributed in accordance with a set of “Discharging Rules”. Since \(D\) was chosen arbitraril, we then conclude that \(5/12\) is a lower bound on \(\sigma_0(G_H)\).

Our results regarding the upper bound are presented in Chapter 2. The rest of the paper is devoted to the proof of Theorem 1.2. As the proof is rather lengthy, we include a sketch in Chapter 3. In Chapter 4, we introduce several properties of vertex codes.
identifying codes for $G_H$ which we will reference throughout the paper. Chapter 5 is devoted to terminology and notations; the vast majority of relevant notions are defined here. In Chapter 6, we state and prove several lemmas concerning the structure of vertex identifying codes for $G_H$. The main result of this paper, Theorem 1.2, is proved in Chapter 7.

For the rest of the paper, if not explicitly stated, $D$ is to be interpreted as a vertex identifying code for the infinite hexagonal grid.

![Figure 1.1: By our convention, $u \in D$ and $v \notin D$, while $w$ is unknown.](image)

We introduce the following convention which we will use throughout the paper. Let $G$ be a graph, and suppose $D \subseteq V(G)$ is a vertex identifying code for $G$. We use a solid vertex to denote that a vertex is in $D$, and we use a hollow vertex to denote that a vertex is not in $D$. The status of all other vertices is undetermined. In Figure 1.1, for instance, $u \in D$ and $v \notin D$, while the status of $w$ is undetermined.
Chapter 2

Upper Bound

The proof of an upper bound on the minimum density of a vertex identifying code is essentially a proof by example. If we can find a code with a density of $\beta$, then we have proven that $\beta$ is a possible density. Now, it might be the case that $\beta$ is not only a possible density but also the minimum density. However, our example code does not suffice to prove this. The most we can say is that the minimum density is less than or equal to $\beta$. In other words, $\beta$ is an upper bound on the minimum density.

When dealing with finite graphs, it is a relatively straightforward exercise to find an example of a vertex identifying code. How does one construct an identifying code for an infinite graph? Mostly we look for repeating patterns. Since the behavior of the code is regular, this allows us to determine the limit of the density as the number of vertices approaches infinity. The codes presented in the following pages are all rectangular tiling patterns. Since the tiles extend ad infinitum, the density of any one tile is the density of the whole graph. Then Equation 1.1 (p. 8) becomes

$$\sigma(D, G_H) = \frac{\text{number of vertices in } D \text{ per tile}}{\text{number of vertices per tile}}$$

Equation (2.1)

All of the constructions shown in this chapter are rectangular tiling patterns with a density of $3/7$, and this remains the best known upper bound.
In 2000, Cohen et al. [3] constructed the two codes shown in Figure 2.1. These are both brick tiling patterns; that is, the corners of any four tiles do not meet at a single point. We introduce the following convention to describe the dimensions of a tile. We look for disjoint paths of equal length running from one side of the tile to the other such that at most three vertices lie on a given 6-cycle and such that their union contains all the vertices in the tile (and no vertices from outside of the tile). The number of vertices in each of these paths is one dimension, and the number of paths is the other dimension. A quick analysis will show that according to this convention, the tiles shown in Figure 2.1a are $7 \times 2$ and those shown in Figure 2.1b are $4 \times 7$. Notice also that these dimensions give the total number of vertices in each tile.

In 2010, we constructed the codes shown in Figures 2.2-2.4. All of these are the products of computer searches. The program which yielded the code shown in Figure 2.2 was written by Jeff Soosiah using a “brute force” method. This is also a brick pattern but of different dimensions than those presented by Cohen et al. The program
which yielded the codes shown in Figures 2.3 and 2.4 was written by Chase Albert using integer linear programming techniques introduced to us by David Phillips. This approach allows us to perform much faster searches and can be formulated as follows. We think of each vertex, $v$, as a bit satisfying

$$
v = \begin{cases} 
1, & v \in D \\
0, & v \notin D 
\end{cases}
$$

Then the definition of a vertex identifying code (Definition 1.1) can be reformulated as follows:

**Condition 1.** For all $v \in V(G_H)$, we have $\sum_{u \in N[v]} u \geq 1$.

**Condition 2.** For all $v_1, v_2 \in V(G_H)$, we have $\sum_{u \in N[v_1] \Delta N[v_2]} u \geq 1$.

Condition 1 guarantees the first part of Definition 1.1, namely that $N[v] \cap D \neq \emptyset$ for all $v \in V(G_H)$. Condition 2 guarantees the second part of Definition 1.1, namely that $N[v_1] \cap D \neq N[v_2] \cap D$ for all distinct $v_1, v_2 \in V(G_H)$.

Of course, we cannot run searches on infinite graphs. Instead, we consider the finite
Figure 2.3: Regular $14 \times 4$ tiling pattern with density $3/7$

graph composed only of a single tile but with one modification: we tie the borders together. That is, the border vertices become connected to the border vertices on the opposite side. Then the graph “looks like” an infinite rectangular tiling of the infinite hexagonal grid. Obviously, if we are trying for a brick tiling pattern, then we must be careful about how we tie the borders together.
Figure 2.4: Regular $14 \times 6$ tiling pattern with density $3/7$
Chapter 3

Sketch of the Proof

As mentioned in the introduction, our proof of Theorem 1.2 makes use of the discharging method. We assign 1 charge to each vertex in $D$ and then redistribute this charge so that each vertex in $G_H$ retains at least $5/12$. To design the proper discharging rules, we start with the following (Rule 1 in Chapter 6):

- If a vertex, $v$, is not in $D$ and has $k$ neighbors in $D$, then $v$ receives $\frac{5}{12k}$ from each of these neighbors.

We can easily verify that Rule 1 suffices to allow each vertex in $G_H \setminus D$ to retain $5/12$ charge (Claim 7.1). As a result, the remaining discharging rules are concerned exclusively with vertices in $D$. Now, any vertex, $v$, in $D$ with a neighbor in $G_H \setminus D$ loses charge by Rule 1. We show in Chapter 7 that only one type of vertex loses too much by Rule 1; we call such a vertex a poor $1$-cluster (Definition 5.1). Consequently, we must find charge to send to poor $1$-clusters from nearby vertices. We find that it is helpful to consider a cluster (Definition 4.1) as a single entity. Thus we first need to determine the surplus charge each cluster may have after Rule 1.

We observe that some 1-clusters may have surplus charge and that their surplus differs according to the neighbors they may have; for this reason we define non-poor 1-clusters (Definition 5.12) and one-third vertices (Definition 5.13). In Lemmas 6.2-
6.4, we determine how many poor 1-clusters can lie in the neighborhood of a non-poor 1-cluster, and then in Rules 2, 3d and 3e, we design the appropriate discharging rules to distribute the surplus charge. In Claim 7.4, we show that non-poor 1-clusters ultimately retain a charge of at least 5/12.

For 3\(^+\)-clusters, the situation is more complicated. We first see a difference of surplus charge according to the distribution of vertices at distance-2 from a given 3\(^+\)-cluster; for this reason we define open/closed \(k\)-clusters (Definition 5.3), crowded \(k\)-clusters (Definition 5.4) and the \(P\)-function (Definition 5.5). These definitions allow us to distinguish among 3\(^+\)-clusters with varying amounts of surplus charge. We will see in Chapter 7 that for very large \(k\), a \(k\)-cluster can always afford to send charge to all nearby poor 1-clusters. Consequently, we are mostly concerned with \(k\)-clusters with \(3 \leq k \leq 6\). In Lemmas 6.6-6.16, we determine the number of poor 1-clusters that can lie in the neighborhood of a given \(k\)-cluster. Discharging Rules 3a-3c are designed in accordance with these lemmas to send charge from 3\(^+\)-clusters to poor 1-clusters lying in a distance-2 or distance-3 neighborhood.

Now, some poor 1-clusters do not lie in a neighborhood that receives charge by Rule 3. We call these very poor 1-clusters (Definition 5.14), and we distinguish between two orientations: symmetric and asymmetric (Definition 5.15). In Lemmas 6.17, 6.20 and 6.24 we scan the neighborhood of a very poor 1-cluster for clusters with charge available for redistribution after Rule 3. Crucially, we find in Lemma 6.20 that if there is no other way to squeeze charge for a given very poor 1-cluster from a single nearby cluster, there must be type-1 paired 3-clusters or type-2 paired 3-clusters (Definition 5.9) in the neighborhood. These are structures which tend to form in the extended neighborhood of an asymmetric very poor 1-cluster and which always have extra charge after Rule 3. In order to reserve this extra charge for very poor 1-clusters, several discharging rules make exceptions for type-1 and type-2 paired 3-clusters. That this creates no new deficiency of charge is proved in Chapter 7. We
prove some properties of type-1 and type-2 paired 3-clusters in Lemmas 6.25 and 6.26. Discharging Rules 4-7 are designed in accordance with the above-mentioned lemmas to send charge to very poor 1-clusters.

On an additional note, the structure of type-1 and type-2 paired 3-clusters is very specific, and this forces us to introduce some very specific notions (for example, Definitions 4.5 and 4.6). This is done so that our analysis can penetrate to the properties of individual vertices. As a result, the proofing process is somewhat tedious though more or less straightforward.

3.1 Discharging Examples

As the discharging process employed in the proof of Theorem 1.2 is rather involved, it may be helpful to provide a simple example of a proof involving the discharging method. We will begin by considering a finite graph with a known identifying code. We will use the discharging method to prove the exact density of this code. Then we will consider the infinite hexagonal grid with an arbitrary identifying code. We will use the discharging method to prove a weak lower bound on the minimum density of this code. These proofs are merely instructive, and we fully acknowledge that one can arrive at the same conclusions by much simpler techniques.

![Figure 3.1: D = \{a, c\}]

Let \( G \) be the graph shown in Figure 3.1, and let \( D = \{a, c\} \) be the vertex identifying code in question. One can easily verify that \( D \) does indeed satisfy Definition 1.1. One can also easily see that the density of \( D \) in \( G \) is \( 2/3 \). In Proposition 3.1, however, we will prove this using the discharging method.
Proposition 3.1. *The density of* \( D \) in \( G \) *is* \( \frac{2}{3} \).

\[ \text{Proposition 3.1.} \]

\[ \text{Proof.} \] We employ the discharging method. We assign 1 charge to each vertex in \( D \), and we redistribute this charge according to the following rule:

- Each vertex in \( D \) sends \( \frac{1}{3} \) charge to each neighbor not in \( D \).

Let \( f(v) \) be the charge of a vertex, \( v \), after all discharging has been completed. Now, \( a, c \in D \), so both begin with 1 charge but must send \( \frac{1}{3} \) to \( b \). Then \( f(a) = f(c) = \frac{2}{3} \). On the other hand, \( b \) begins with no charge but receives \( \frac{1}{3} \) from both \( a \) and \( c \) yielding \( f(b) = \frac{2}{3} \). Then \( f(v) = \frac{2}{3} \) for all \( v \in V(G) \); therefore, the density of \( D \) in \( G \) is \( \frac{2}{3} \). \( \square \)

Compared with the following proof, the above proof of Proposition 3.1 is relatively simple for two reasons. The first is that \( G \) is finite. The second is that we know which vertices are in the code. In the proof of Proposition 3.2, we consider an infinite graph and we must design our discharging rules so that they are valid for any vertex identifying code.

Proposition 3.2. *The minimum density of a vertex identifying code for the infinite hexagonal grid is greater than or equal to* \( \frac{2}{5} \).

\[ \text{Proposition 3.2.} \]

\[ \text{Proof.} \] We employ the discharging method. We assign 1 charge to each vertex in \( D \), and we redistribute this charge according to the following rule:

- If a vertex, \( v \), is not in \( D \) and has \( k \) neighbors in \( D \), then \( v \) receives \( \frac{2}{5k} \) from each of these neighbors.
Now we verify that the above discharging rule suffices to allow each vertex in $G_H$ to retain $2/5$ charge. As in the proof of Proposition 3.1, let $f(v)$ denote the final charge of a vertex, $v$.

First, consider $v \notin D$. Suppose $v$ has $k$ neighbors in $D$. Then, by our discharging rule, $v$ receives $2/k$ from each of these neighbors. That is, $f(v) = k \cdot \frac{2}{5k} = \frac{2}{5}$.

Now consider $v \in D$. There are four cases: $v$ has 3 neighbors in $D$, 2 neighbors in $D$, 1 neighbor in $D$ or no neighbors in $D$.

If $v$ has 3 neighbors in $D$, then $v$ is not required to send any charge. Therefore, $f(v) = 1$. Therefore, $f(v) \geq \frac{2}{5}$.

Suppose $v$ has 2 neighbors in $D$. Then, since $G_H$ is 3-regular, $v$ has only one neighbor not in $D$. Then $v$ must send at most $\frac{2}{5}$. Therefore, $f(v) \geq 1 - \frac{2}{5} = \frac{3}{5}$. Therefore, $f(v) \geq \frac{2}{5}$.

Now suppose $v$ has 1 neighbor in $D$. Then $v$ has 2 neighbors not in $D$. Let these vertices be $a_1$ and $a_2$. Suppose $a_1$ and $a_2$ each have only one neighbor in $D$. Then this neighbor must be $v$. But then $N[a_1] \cap D = \{v\}$ and $N[a_2] \cap D = \{v\}$ which contradicts the definition of a vertex identifying code (Definition 1.1). Therefore, at least one of $a_1$ and $a_2$ has more than one neighbor in $D$ (see Proposition 4.6). Then $v$ sends at most $\frac{2}{5}$ to one of $a_1$ and $a_2$ and at most $\frac{2}{10}$ to the other. Therefore, $f(v) \geq 1 - \left(\frac{2}{5} + \frac{2}{10}\right) = \frac{2}{5}$.

Finally, suppose $v$ has no neighbors in $D$. Then $v$ has 3 neighbors not in $D$. Notice that $N[v] \cap D = \{v\}$. Therefore, each of the neighbors of $v$ must have more than one neighbor in $D$. Otherwise, $N[a] \cap D = \{v\}$ for some neighbor, $a$. But then $N[v] \cap D = N[a] \cap D$ which contradicts Definition 1.1 (see Proposition 4.5). So each neighbor of $v$ has at least 2 neighbors in $D$. Then $v$ sends at most $\frac{2}{10}$ to each of these neighbors. Therefore, $f(v) \geq 1 - 3 \cdot \frac{2}{10} = \frac{2}{5}$.

So we have shown that any vertex not in $D$ retains $2/5$ charge. And we have shown that any vertex in $D$ retains at least $2/5$ charge. But every vertex in $G_H$
belongs to one of these two classes. Therefore, our discharging rule suffices to allow each vertex in $G_H$ to retain at least $\frac{2}{5}$ charge. Therefore, $\sigma_0(G_H) \geq \frac{2}{5}$. 

There are two points to be made about the above proof. The first deals with aesthetics, and the second deals with structural properties of the hexagonal grid.

In the proof of Proposition 3.2, when discussing vertices in $D$ with one neighbor or no neighbors in $D$, we were forced to engage in a certain amount of reasoning regarding the types of vertices that might appear in the extended neighborhood of $v$. Since this kind of reasoning does not depend on the discharging rules, we find that it is preferable to include results regarding the structure of identifying codes as separate propositions, lemmas, etc. This allows us to abbreviate the discharging process. In fact, had we proved Proposition 3.2 after Chapter 4, we would have been able to shorten the proof by simply referring to Propositions 4.5 and 4.6. It is for these aesthetic reasons that we prove several structural results in Chapters 4 and 6 before beginning the proof of Theorem 1.2 in Chapter 7.

The second point to be made is about the depth of analysis required to prove Proposition 3.2. Notice that the only structural property of the infinite hexagonal grid that was used in the proof was the fact that $G_H$ is 3-regular. In other words, the proof of Proposition 3.2 is actually valid for any 3-regular graph. Notice also that the proof only considers neighborhoods of radius at most 2. In order to improve the result, we must achieve a more robust analysis of the hexagonal grid. We must discover more precise properties of $G_H$, and we must extend our analysis to much larger neighborhoods.
Chapter 4

General Structural Properties

Definition 4.1. A component of the subgraph induced by $D$ is called a cluster. A cluster containing $k$ vertices is called a $k$-cluster; a cluster containing $k$ or more vertices is called a $k^+$-cluster. Let $D_k$ be the set of all vertices in $k$-clusters; and let $K_k$ be the set of all $k$-clusters. Let $d_C(v)$ be the degree of a vertex, $v$, in a $3^+$-cluster, $C$; and let $\Delta(C) = \max\{d_C(v) : v \in C\}$.

Proposition 4.2. There exist no 2-clusters.

Proof. Suppose by contradiction that there exists a 2-cluster, $C$, and let $V(C) = \{v, w\}$ as in Figure 4.1. Then, $N[v] \cap D = \{v, w\}$ and $N[w] \cap D = \{v, w\}$. Now, if $N[v] \cap D = N[w] \cap D$, then $v = w$ (Definition 1.1), which is a contradiction.

Figure 4.1: Proposition 4.2
Corollary 4.3. If a vertex, \( v \), is not in a \( 3^+ \)-cluster, then either \( v \) is not in \( D \) or \( v \) is a 1-cluster.

Proposition 4.4. If a vertex not in \( D \) has 2 adjacent vertices not in \( D \), then the remaining adjacent vertex is in a \( 3^+ \)-cluster.

![Figure 4.2: Proposition 4.4](image)

**Proof.** Consider a vertex, \( v \), such that that \( N_1(v) = \{a, b, c\} \) and let \( a, b, v \not\in D \) as in Figure 4.2. Suppose by contradiction that \( c \not\in D_{3^+} \). Then, \( c \not\in D \) or \( c \in D_1 \) (Corollary 4.3). If \( c \not\in D \), then \( N[v] \cap D = \emptyset \), which is a contradiction (Definition 1.1). If \( c \in D_1 \), then \( N[v] \cap D = N[c] \cap D = \{c\} \); therefore, \( c = v \) (Definition 1.1), which is a contradiction. \( \square \)

Proposition 4.5. Each of the vertices adjacent to a 1-cluster, \( v \), has at least one adjacent vertex in \( D \setminus \{v\} \).

![Figure 4.3: Proposition 4.5](image)

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Proof. Let \( v \in D_1 \), and let \( u \) be an adjacent vertex as in Figure 4.3. Suppose by contradiction that \( u \) has no adjacent vertices in \( D \setminus \{v\} \). Then, \( v \in D_{3^+} \) (Proposition 4.4), which is a contradiction.

Proposition 4.6. Each leaf of a \( 3^+ \)-cluster, \( C \), has at least one distance-2 vertex in \( D \setminus C \).

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.4.png}
\caption{Proposition 4.6}
\end{figure} \]

Proof. Let \( v \) be a leaf of a \( 3^+ \)-cluster, \( C \). Then, exactly 2 of the vertices adjacent to \( v \) are not in \( D \); let \( u \) and \( w \) be these vertices as in Figure 4.4. Suppose by contradiction that \( v \) has no distance-2 vertices in \( D \setminus C \). Then, \( N[u] \cap D = N[w] \cap D = \{v\} \); therefore, \( u = w \) (Definition 1.1), which is a contradiction. \( \square \)
Chapter 5

Terminology and Notations

Definition 5.1. A 1-cluster with exactly 3 distance-2 vertices in $D$ is called a poor 1-cluster. Let $D^p_1$ be the set of all poor 1-clusters.

Corollary 5.2. Each of the neighbors of a poor 1-cluster, $v$, has exactly one neighbor in $D \setminus \{v\}$.

Proof. This follows from Proposition 4.5 and Definition 5.1.

Definition 5.3. For $k \geq 3$, let $C$ be a $k$-cluster with $\Delta(C) = 2$. If none of the non-leaf vertices of $C$ has a distance-2 vertex in $D \setminus C$, then $C$ is an open $k$-cluster. If at least one of the non-leaf vertices of $C$ has a distance-2 vertex, $v$, in $D \setminus C$, then $C$ is a closed $k$-cluster and $v$ closes $C$. Let $D^o_k$ be the set of all vertices in open $k$-clusters and $D^c_k$ the set of all vertices in closed $k$-clusters; let $K^o_k$ be the set of all open $k$-clusters and $K^c_k$ the set of all closed $k$-clusters.

Definition 5.4. If an open $k$-cluster, $C$, has exactly 2 distance-2 vertices in $D$, both of which are poor 1-clusters, then $C$ is uncrowded. Otherwise, $C$ is crowded.
**Definition 5.5.** For a given cluster, $C$, let $P(C) = \sum_{v \in C} |N_2(v) \cap D \setminus C|$.

![3-Cluster Diagram](image)

**Figure 5.1: 3-Cluster**

**Definition 5.6.** Let $C$ be the 3-cluster shown in Figure 5.1. Vertices $a$ and $b$ are in the **head positions** of $C$; $c$ and $e$ are in the **shoulder positions**; $f$ and $g$ are in the **arm positions**; $h$ and $m$ are in the **hand positions**; $i$ and $k$ are in the **foot positions**; $j$ is in the **tail position**; and $n$ and $q$ are in the **fin positions**. If $q$ is not in $D$, then $b, d, e, g, k$ and $m$ are on the **finless side** of $C$. If $d$ is in $D$, then $b, e, g, k, m$ and $q$ are on the **closed side** of $C$.

![4-Clusters Diagram](image)

**Figure 5.2: 4-Clusters**

**Definition 5.7.** Let $C$ be a 4-cluster with $\Delta(C) = 2$. If the leaves of $C$ do not lie on the same 6-cycle, then $C$ is a **linear 4-cluster**. Otherwise, $C$ is a **curved 4-cluster**. Let $C_1$ be the linear 4-cluster shown in Figure 5.2a. Vertices $a$ and $b$ are
in the one-turn positions of $C_1$. Let $C_2$ be the curved 4-cluster shown in Figure 5.2b. Vertices $c$ and $d$ are in the backwards positions of $C_2$.

**Definition 5.8.** A vertex, $v$, is distance-$k$ from a cluster, $C$, if $k$ is the minimum distance from $v$ to any of the vertices of $C$. If $k \leq \ell$, then $v$ is within distance-$\ell$ of $C$. If a vertex, $v$, is within distance-3 of a cluster, $C$, then $v$ is nearby $C$.

![Figure 5.3: Paired 3-Clusters](image)

**Definition 5.9.** Let $C_1$ be the 3-cluster described by $a$, $j$ and $c$ in Figure 5.3a, and let $C_2$ be the 3-cluster described by $s$, $k$ and $r$. Then, $C_1$ and $C_2$ are paired 3-clusters. Let $C_3$ be the 3-cluster described by $b$, $d$ and $e$ in Figure 5.3b, and let $C_4$ be the 3-cluster described by $f$, $g$ and $h$. Then, $C_3$ and $C_4$ are type-1 paired, and $C_3$ is type-1 paired on top. Let $C_5$ be the 3-cluster described by $i$, $m$ and $n$ in Figure 5.3c, and let $C_6$ be the 3-cluster described by $p$, $q$ and $t$. Then, $C_5$ and $C_6$ are type-2 paired.

**Corollary 5.10.** If a 3-cluster, $C$, is type-1 paired, then $C$ is not type-2 paired, and vice versa.
Definition 5.11. A poor 1-cluster, $v$, is **stealable** if $v$ is distance-3 from a $4^+$-cluster and distance-2 from a $3^+$-cluster, $C$, such that if $C$ is an open 3-cluster, then

(i) $v$ is not in a shoulder position;

(ii) if $v$ is in an arm position, then $C$ is neither type-1 nor type-2 paired.

Definition 5.12. A 1-cluster that is not poor is called a **non-poor 1-cluster**. Let $D_{np}^1$ be the set of all non-poor 1-clusters. If 3 non-poor 1-clusters, $u$, $v$ and $w$, are adjacent to the same one-third vertex, then $u$, $v$ and $w$ are referred to as a **group of non-poor 1-clusters**. A vertex, $v$, is **distance-$k$ from a group of non-poor 1-clusters**, $H$, if $v$ is distance-$k$ from any of the 1-clusters in $H$.

Definition 5.13. If a vertex, $v$, is not in $D$ and has 3 neighbors in $D$, then $v$ is called a **one-third vertex**.

Definition 5.14. If a poor 1-cluster, $v$, is neither distance-2 from a $3^+$-cluster or a non-poor 1-cluster nor distance-3 from a closed 3-cluster or $4^+$-cluster, then $v$ is called a **very poor 1-cluster**. Let $D_{vp}^1$ be the set of all very poor 1-clusters.

![Diagram](image.png)

(a) Symmetric  
(b) Asymmetric

Figure 5.4: Orientations of Very Poor 1-Clusters

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Definition 5.15. If a very poor 1-cluster, \( v \), has 3 distance-2 vertices in \( D \) which are all distance-4 from each other, then \( v \) is in a symmetric orientation (see Figure 5.4a). A very poor 1-cluster which is not in a symmetric orientation is in an asymmetric orientation (see Figure 5.4b). The vertex \( u \) in Figure 5.4b is in the \textbf{u-position} of \( v \), the vertex \( w \) is in the \textbf{w-position} of \( v \), and the vertex \( x \) is in the \textbf{x-position} of \( v \).
Chapter 6

Structural Lemmas

In this chapter, we state several lemmas concerning the structure of a vertex identifying code for the infinite hexagonal grid. The primary purpose of these lemmas is to abridge the proof of Theorem 1.2. As such, the proofs presented in this chapter may be skipped on a first reading.

6.1 Non-Poor 1-Clusters

Proposition 6.1. If a poor 1-cluster, \( v \), is distance-2 from exactly one of the 1-clusters in a group of non-poor 1-clusters, then \( v \) is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or 4\(^+\)-cluster.

Proof. Let \( H \) be the group of non-poor 1-clusters described by \( b, d \) and \( e \) in Figure 6.1. We choose \( i \in D_1^p \). Then, \( i \) is distance-2 from \( e \) and not distance-2 from the other 1-clusters in \( H \). By symmetry, this is the general case. Since \( i \in D_1^p \), we have \( j, s \notin D \) and, by Corollary 5.2, \( h \notin D \). Therefore, \( q \in D_{3^+} \) (Proposition 4.4) and \( g \in D \) (Proposition 4.5). Let \( C \) be the 3\(^+\)-cluster at \( q \). If \( r \in D \), then \( r \in C \); therefore, \( i \) is distance-2 from a 3\(^+\)-cluster. If \( r \notin D \), then \( p \in C \). If \( n \in D \), then \( g, n, p, q \in C \) and \( C \) is a 4\(^+\)-cluster; therefore, \( i \) is distance-3 from a 4\(^+\)-cluster. If
n \not\in D$, then $v \in D$ and $p, q, v \in C$. Now, $g$ closes $C$. Therefore, $i$ is distance-3 from a closed $3^+$-cluster.

\[ \square \]

**Figure 6.1:** Proposition 6.1 and Lemma 6.2

**Lemma 6.2.** Consider a group of non-poor 1-clusters, $H$. There exist at most 2 poor 1-clusters which are distance-2 from $H$ and neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or $4^+$-cluster.

**Proof.** Let $H$ be the group of non-poor 1-clusters described by $b, d$ and $e$ in Figure 6.1. Now, if a poor 1-cluster, $w$, is distance-2 from exactly one of $b, d$ and $e$, then $w$ is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or $4^+$-cluster (Proposition 6.1). Thus, we need only consider poor 1-clusters which are distance-2 from 2 of the 1-clusters in $H$. There are 3 possibilities: $a, c$ and $h$. Suppose by contradiction that each of $a, c$ and $h$ is a poor 1-cluster that is not distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or $4^+$-cluster. Since $h \in D_1^p$, we have $p \in D$ or $r \in D$ but not both (Corollary 5.2). By symmetry, we choose $r \in D$. Now, by hypothesis, $h$ is not distance-2 from any $3^+$-cluster; therefore, $r \in D_1$ (Corollary 4.3). So we have $s \not\in D$. Since $h \in D_1^p$ and $e \in D$, we also have $i \not\in D$ (Corollary 5.2). Therefore, $j \in D_3^+$ (Proposition 4.4). Also, since $i, s \not\in D$ and $r \in D_1$, we have $t \in D$ (Proposition 4.5). Now, $e \in D$ and, by hypothesis, $c \in D_1^p$;
therefore, \( f \not\in D \) (Corollary 5.2). Let \( C \) be the \( 3^+ \)-cluster at \( j \). Since \( f, i \not\in D \), we have \( k \in C \). Now, if \( u \in D \), then \( j, k, t, u \in C \); therefore, \( c \) and \( h \) are distance-3 from a \( 4^+ \)-cluster, which is a contradiction. If \( u \not\in D \), then \( m \in C \) and \( t \) closes \( C \); therefore, \( c \) and \( h \) are distance-3 from a closed \( 3^+ \)-cluster, which is a contradiction.

**Lemma 6.3.** Let \( v \) be a one-third vertex, and let \( v \) have exactly 2 adjacent 1-clusters, \( c \) and \( d \). Each of \( c \) and \( d \) has at most one distance-2 poor 1-cluster that is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster.

**Proof.** Let \( v \) be the one-third vertex shown in Figure 6.2, and let \( c \) and \( d \) be 1-clusters. Then, by hypothesis, \( a \in D \setminus D_1 \); therefore, \( a \in D_{3^+} \) (Corollary 4.3). Suppose by contradiction that one of \( c \) and \( d \) has at least 2 distance-2 poor 1-clusters that are not distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster. By symmetry, we consider \( d \). Now, there are 4 candidates for distance-2 poor 1-clusters: \( b, e, h \) and \( i \). However, \( b \) is distance-2 from the \( 3^+ \)-cluster at \( a \), so we need not consider \( b \).

If \( h \in D_1^p \), then \( n \not\in D \) and \( i \not\in D \) (Corollary 5.2). So we have \( e \in D_1^p \); therefore, \( f, j \not\in D \) and \( k \in D \) (Proposition 4.5). Since \( i, j \not\in D \), we have \( q \in D_{3^+} \) (Proposition
4.4). Let $C$ be the $3^+$-cluster at $q$. If $p \in D$, then $p \in C$ and $h$ is distance-2 from a $3^+$-cluster, which is a contradiction. If $p \not\in D$, then $r \in C$; therefore, either $k \in C$ and $e$ is distance-2 from a $4^+$-cluster, or $k$ closes $C$ and both $e$ and $h$ are distance-3 from a closed $3^+$-cluster, which is a contradiction.

If $h \not\in D^p_1$, then we have $e, i \in D^p_1$. Therefore, $f, j, q \not\in D$ and, by Corollary 5.2, $h \not\in D$. Then $n \in D_{3^+}$ (Proposition 4.4) and $g \in D$ (Proposition 4.5). Let $C$ be the $3^+$-cluster at $n$. If $p \in D$, then $p \in C$ and $i$ is distance-2 from a $3^+$-cluster, which is contradiction. If $p \not\in D$, then $m \in C$. Then, either $g \in C$ and $i$ is distance-3 from a $4^+$-cluster, or $g$ closes $C$ and $i$ is distance-3 from a closed $3^+$-cluster, which is a contradiction. \hfill \square

**Lemma 6.4.** If a one-third vertex has exactly one adjacent 1-cluster, $d$, then $d$ has at most 2 distance-2 poor 1-clusters.

![Figure 6.3: Lemma 6.4](image)

**Proof.** Let $v$ be the one-third vertex shown in Figure 6.3, and let $d$ be a 1-cluster. Then, by hypothesis, $a, c \in D \setminus D_1$; therefore, $a, c \in D_{3^+}$ (Corollary 4.3). Now, $d$ has 6 distance-2 vertices: $a, b, c, e, f$ and $g$. However, $a, c \in D_{3^+}$. If $b \in D^p_1$, then $e \not\in D$ (Corollary 5.2); and vice versa. If $f \in D^p_1$, then $g \not\in D$ (Corollary 5.2); and vice versa. Therefore, at most 2 of $b, e, f$ and $g$ are poor 1-clusters. \hfill \square
6.2 $3^+$-Clusters

**Lemma 6.5.** A 3-cluster has at most one finless side.

*Proof.* Let $C$ be the 3-cluster shown in Figure 5.1 (p. 26). Suppose by contradiction that $C$ has 2 finless sides; then, $n, p \notin D$ (Definition 5.6). If $j \notin D$, then $p \in D_{3^+}$ (Proposition 4.4). But none of the vertices adjacent to $p$ is in $D$; therefore, $p \in D_1$, which is a contradiction. If $j \in D$ and $p \in D$, then $j, p \in D_2$, which is a contradiction (Proposition 4.2). If $j \in D$ and $p \notin D$, then $j \in D_{3^+}$ (Proposition 4.4). But none of the vertices adjacent to $j$ is in $D$; therefore, $j \in D_1$, which is a contradiction. \hfill $\square$

**Lemma 6.6.** Let $C_1$ be a closed 3-cluster with $P(C_1) = 3$.

(i) $C_1$ has at most 8 nearby poor 1-clusters. If $C_1$ has 8 such clusters, at least one of the poor 1-clusters at distance-3, $v$, is distance-2 from another $3^+$-cluster, $C_2$, such that

(a) if $C_2$ is an open 3-cluster, then $v$ is not in a shoulder position;

(b) if $C_2$ is an open 3-cluster and $v$ is in an arm position, then $C_2$ is not type-1 paired; if $C_2$ is type-2 paired, then $C_1$ is type-2 paired with $C_2$.

(ii) If neither the shoulder positions nor the tail position are in $D$, then $C_1$ has at most 5 nearby poor 1-clusters.

*Proof.* Let $C_1$ be the 3-cluster shown in Figure 6.4. If $C_1 \in K_3^5$, then the non-leaf vertex of $C_1$ has at least one distance-2 vertex in $D \setminus C$ (Definition 5.3); by symmetry, we choose $f \in D$. Now, $P(C_1) = 3$ and each leaf of $C$ has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6); therefore, $e \notin D$ and

$$|\{d, j, p, q\} \cap D| = |\{g, k, r, q\} \cap D| = 1$$
There are 11 candidates for nearby poor 1-clusters: $a, c/d, f, h, i/j, k/m, n, p/t, q/v, r/x$ and $s$.

First we consider the cases for which $q \not\in D$. To begin, we show that there are at most 9 nearby poor 1-clusters. Now, $v \in D_{3^+}$ (Proposition 4.4); therefore, there are at most 10 nearby poor 1-clusters. If $p \in D$, then $n \not\in D_1^p$ and there are at most 9 nearby poor 1-clusters. If $p \not\in D$ and $t \not\in D$, then there are at most 9 nearby poor 1-clusters. If $r \in D$, then $s \not\in D_1^p$ and there are at most 9 nearby poor 1-clusters. If $r \not\in D$ and $x \not\in D$, then there are at most 9 nearby poor 1-clusters. So we consider $p, r \not\in D$ and $t, x \in D$. Let $C_v$ be the $3^+$-cluster at $v$. At least one of $u$ and $w$ is in $C_v$; therefore, at least one of $t$ and $x$ is not a 1-cluster. Therefore, there are at most 9 nearby poor 1-clusters.

Now we consider the cases for which at least one of $d$ and $g$ is in $D$. If $g \in D$, then $f, h \not\in D_1^p$. Therefore, there are at most 7 nearby poor 1-clusters. So now we consider $d \in D$ and $g \not\in D$. Either $k \in D$ or $r \in D$ but not both. If $k \in D$, then $s \not\in D_1^p$ and there are at most 8 nearby poor 1-clusters. Now, if $k \not\in D_1^p$, then there are at most 7 nearby poor 1-clusters. If $k \in D_1^p$, then $s \not\in D$. Since $r \not\in D$, we have $x \in D_{3^+}$ (Proposition 4.4). If $t \not\in D_1^p$, then $C_1$ has at most 7 nearby poor 1-clusters. If $t \in D_1^p$, then $t$ is distance-2 from $C_v$. If $C_v \in K_3^3$, then $v, w, x \in C_v$ and $t$ is in an arm position. If $C_v$ is paired, then it is type-2 paired with $C_1$. Therefore, the lemma holds with $k \in D$. If $r \in D$, then $s \not\in D_1^p$ and there are at most 8 nearby poor 1-clusters. If $r \not\in D_1^p$, then $C_1$ has at most 7 nearby poor 1-clusters. If $r \in D_1^p$, then $s \not\in D$. Since $k \not\in D$, we have $m \in D_{3^+}$ (Proposition 4.4); therefore, $C_1$ has at most 7 nearby poor 1-clusters and the lemma holds.

Now we consider the cases for which neither shoulder position is in $D$. Since $d \not\in D$, we have $a, c \in D_{3^+}$. Therefore, $C_1$ has at most 7 nearby poor 1-clusters. Either $j \in D$ or $p \in D$; in both cases, $n \not\in D_1^p$. Therefore, $C_1$ has at most 6 nearby poor 1-clusters. Either $k \in D$ or $r \in D$; in both cases, $s \not\in D_1^p$. Therefore, $C_1$ has at most 6 nearby poor 1-clusters. Either $k \in D$ or $r \in D$; in both cases, $s \not\in D_1^p$. Therefore, $C_1$ has at most 6 nearby poor 1-clusters.
most 5 nearby poor 1-clusters and the lemma holds.

Now we consider the case for which \( q \in D \). If \( q \in D \), then \( d, g, j, k, p, r \notin D \). Then, \( a, c \in D_{3+} \) (Proposition 4.4). Therefore, \( C_1 \) has at most 9 nearby poor 1-clusters. If \( q \notin D_{1} \), then \( C_1 \) has at most 8 nearby poor 1-clusters. If \( q \in D_{1} \), then either \( u \in D \) or \( w \in D \) (Proposition 4.5). If \( u \in D \), then \( t \notin D_{1} \) and there are at most 8 nearby poor 1-clusters; if \( w \in D \), then \( x \notin D_{1} \) and there are at most 8 nearby poor 1-clusters. Therefore, \( C_1 \) has at most 8 nearby poor 1-clusters. Now, if \( i \notin D_{1} \), then \( C_1 \) has at most 7 nearby poor 1-clusters. If \( i \in D_{1} \), then \( i \) is distance-2 from the \( 3^+ \)-cluster at \( c, C_c \); since \( a \in D \), we have \( C_c \notin K_{3} \).

\[ \square \]

**Lemma 6.7.** Let \( C \) be a closed 3-cluster with \( P(C) = 4 \). If \( C \) is adjacent to a one-third vertex, then \( C \) has at most 8 nearby poor 1-clusters. Furthermore, if an arm position or a foot position of \( C \) is a poor 1-cluster, then \( C \) has at most 7 nearby poor 1-clusters. If 2 arm or foot positions are poor 1-clusters, then \( C \) has at most 6 nearby poor 1-clusters.

**Proof.** Let \( C \) be the closed 3-cluster shown in Figure 6.4. By symmetry, we choose \( f \in D \). There are 5 possible one-third vertices adjacent to \( C \), and there are 11
candidates for nearby poor 1-clusters: a/e, c/d, f, h, i/j, k/m, n, p/t, q/v, r/x and s.

Suppose e ∈ D. Then, a, e, f ∉ D₁; therefore, C has at most 9 nearby poor 1-clusters. Since P(C) = 4, each leaf has exactly one distance-2 vertex in D \ C. If d, g ∈ D, then c, d, h ∉ D₁; therefore, C has at most 7 nearby poor 1-clusters. If q ∈ D, then either q ∈ D₁ or q ∉ D₁. If q ∉ D₁, then C has at most 8 nearby poor 1-clusters. If q ∈ D₁, then p, r ∉ D (Corollary 5.2). If t ∉ D₁ or x ∉ D₁, then C has at most 8 nearby poor 1-clusters. So assume t, x ∈ D₁. Since q ∈ D₁, we have u ∈ D or w ∈ D (Proposition 4.5); therefore, at least one of t and x is not a poor 1-cluster. Therefore, C has at most 8 nearby poor 1-clusters. If q ∉ D, then v ∈ D₃+ (Proposition 4.4); therefore, C has at most 8 nearby poor 1-clusters. If j ∈ D or p ∈ D, then n ∉ D₁; and if k ∈ D or r ∈ D, then s ∉ D₁. Therefore, if one foot or arm position is a poor 1-cluster, then C has at most 7 nearby poor 1-clusters; and if 2 foot or arm positions are poor 1-clusters, then C has at most 6 nearby poor 1-clusters.

Suppose d, j ∈ D. Then c, d, i, j, n ∉ D₁. Therefore, C has at most 8 nearby poor 1-clusters. If k ∈ D or r ∈ D, then s ∉ D₁; in this case, C has at most 7 nearby poor 1-clusters.

The argument is nearly identical to the one above for the cases in which g, k ∈ D, p, q ∈ D and q, r ∈ D.

Lemma 6.8. Let C₁ be a linear open 4-cluster with P(C₁) = 2.

(i) C₁ has at most 8 nearby poor 1-clusters. Furthermore, if C₁ has 8 such clusters, then at least 2 of the distance-3 poor 1-clusters are stealable; if C₁ has 7 such clusters, then at least one is stealable.

(ii) If one one-turn position is not in D, then C₁ has at most 6 nearby poor 1-clusters. If C₁ has exactly 6 such clusters, then at least one of the distance-3
poor 1-clusters is stealable.

(iii) If neither one-turn position is in $D$, then $C_1$ has at most 4 nearby poor 1-clusters.

Proof. Let $C_1$ be the linear open 4-cluster shown in Figure 6.5. Both leaves of $C_1$ must have at least one distance-2 vertex in $D$ (Proposition 4.6), and, by hypothesis, $P(C_1) = 2$; therefore, $f, g, r, s \notin D$ and

$$|\{e, k, q\} \cap D| = |\{h, m, t\} \cap D|$$

By Proposition 4.4, we have $b, w \in D_{3^+}$. Let $C_b$ and $C_w$ be the $3^+$-clusters at $b$ and $w$, respectively. There are 10 candidates for nearby poor 1-clusters: $a, c/h, d/e, i, j/k, m/n, p, q/v, t/u$ and $x$. Now, at least one of $a$ and $c$ is adjacent to or in $C_b$; therefore, at least one of $a$ and $c$ is not a poor 1-cluster. A similar argument may be made for $v$ and $x$. Therefore, there are at most 8 nearby poor 1-clusters.

If both one-turn positions are in $D$, then we have $e, t \in D$. If $C_1$ has 8 nearby poor 1-clusters, then at most one of $a$ and $c$ is not a poor 1-cluster. If $a \notin D_{1^+}^p$, then $c$ is distance-2 from $C_b$. If $C_b \in K_3$, then $c$ is either in a foot position or arm position. If $c$ is in an arm position, then $a \in C_b$ and $C_b$ is not paired. If $c \notin D_{1^+}^p$, then a similar argument may be made for $a$. And a symmetric argument may be made for $v$ and $x$ and $C_w$. Therefore, at least 2 of the poor 1-clusters at distance-3 are stealable. If $C_1$ has exactly 7 nearby poor 1-clusters, then at least one of $a, c, v$ and $x$ is a poor 1-cluster. Then, the above argument suffices; therefore, at least one of the distance-3 poor 1-clusters is stealable.

If exactly one one-turn position is in $D$, then $e \notin D$ or $t \notin D$. By symmetry we choose $e \notin D$. Then, $d \in D_{3^+}$ (Proposition 4.4). Therefore, there are at most 7 nearby poor 1-clusters. By hypothesis, at least one of $k$ and $q$ is in $D$. In both cases, $p \notin D_{1^+}^p$. Therefore, there are at most 6 nearby poor 1-clusters. Now, $a \in D_{3^+}$
Then $c$ is distance-2 from $C_b$ and $a \in C_b$. If $C_b \in K_3$, then $c$ is in an arm position and $C_b$ is not paired. Therefore, $c$ is stealable.

If neither one-turn position is in $D$, then $e, t \not\in D$. Therefore, $a, d, u, x \in D_3^p$ (Proposition 4.4). Therefore, $C_1$ has at most 6 nearby poor 1-clusters. By hypothesis, at least one of $k$ and $q$ and at least one of $h$ and $m$ is in $D$. If $k \in D$ or $q \in D$, then $p \not\in D_1^p$; and if $h \in D$ or $m \in D$, then $i \not\in D_1^p$. Therefore, $C_1$ has at most 4 nearby poor 1-clusters.

\[ \square \]

**Lemma 6.9.** Let $C$ be a linear 4-cluster with $P(C) = 3$.

(i) If $C$ is adjacent to no one-third vertices, then $C$ has at most 9 nearby poor 1-clusters.

(ii) If $C$ is adjacent a one-third vertex, then $C$ has at most 6 nearby poor 1-clusters.

**Proof.** Let $C$ be the linear 4-cluster shown in Figure 6.5. First, suppose $C$ is adjacent to no one-third vertices. Now, either $g \in D$ or $g \not\in D$. If $g \in D$, then one of the leaves of $C$ and one of the middle vertices has a distance-2 vertex in $D \setminus C$. Now, each leaf has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6) and, by hypothesis,
\[ P(C) = 3; \text{ therefore, } f, h, m, r, s, t \not\in D \text{ and } |\{e, k, q\} \cap D| = 1. \text{ Then, } u, w, x \in D_{3+} \] (Proposition 4.4) and there are at most 9 nearby poor 1-clusters: \( a, c, d/e, g, i, j/k, n, p \) and \( q/v \). Now, suppose \( g \not\in D \). If \( r \in D \), then this case can be reduced, by symmetry, to the above case. So assume \( r \not\in D \). Then, \( b, w \in D_{3+} \) (Proposition 4.4).

There are 10 candidates for nearby poor 1-clusters: \( a/f, c/h, d/e, i, j/k, m/n, p, q/v, s/x \) and \( t/u \). Suppose by contradiction that there exist 10 nearby poor 1-clusters. Then, \( i, p \in D^p \). Therefore, \( h, m \not\in D \) and \( n, v \in D \) (Proposition 4.5). And we must have \( n, v \in D^p \); otherwise, \( C \) has fewer than 10 nearby poor 1-clusters. Since \( g, h, m \not\in D \), we have \( t \in D \) (Proposition 4.6). And, as above, we must have \( t \in D^p \); therefore, \( x \in D \) (Proposition 4.5). And, again, we must have \( x \in D^p \). Therefore, \( v, x \in D^p \). But \( w \in D_{3+} \) and \( r \not\in D \); therefore, at least one of \( v \) and \( x \) is not a poor 1-cluster, which is a contradiction.

Now, suppose \( C \) is adjacent to a one-third vertex, \( v_{\bar{3}} \). Since \( P(C) = 3 \) and each leaf must have at least one distance-2 vertex in \( D \setminus C \) (Proposition 4.6), \( v_{\bar{3}} \) must be adjacent to \( e \) and \( k \) or \( m \) and \( t \). By symmetry, we choose \( e, k \in D \). Then, \( |\{h, m, t\} \cap D| = 1 \) and \( f, g, q, r, s \not\in D \); therefore, \( b, w \in D_{3+} \) (Proposition 4.4) and there are at most 7 nearby poor 1-clusters: \( a, c/h, m/n, p, t/u, v \) and \( x \). Suppose by contradiction that \( C \) has 7 nearby poor 1-clusters. Then, \( v, x \in D^p \). But \( w \in D_{3+} \) and \( r \not\in D \); therefore, at least one of \( v \) and \( x \) is not a poor 1-cluster, which is a contradiction.

\[ \Box \]

**Lemma 6.10.** Let \( C_1 \) be a curved open 4-cluster with \( P(C_1) = 2 \).

(i) \( C_1 \) has at most 8 nearby poor 1-clusters. Furthermore, if \( C_1 \) has 8 such clusters, then at least 2 of the distance-3 poor 1-clusters are stealable; if \( C_1 \) has 7 such clusters, then at least one is stealable.

(ii) If one backwards position is not in \( D \), then \( C_1 \) has at most 6 nearby poor 1-clusters. If \( C_1 \) has exactly 6 such 1-clusters, then at least one of the distance-3
poor 1-clusters is stealable.

(iii) If neither backwards position is in $D$, then $C_1$ has at most 2 nearby poor 1-clusters.

Proof. Let $C_1$ be the curved open 4-cluster shown in Figure 6.6. Both leaves of $C_1$ must have at least one distance-2 vertex in $D$ (Proposition 4.6), and, by hypothesis, $P(C_1) = 2$; therefore, $f,j,p,u \notin D$ and

$$|\{d,e,i\} \cap D| = |\{n,s,t\} \cap D| = 1$$

By symmetry, there are only 6 cases to consider: $e,t \in D$; $e,s \in D$; $e,n \in D$; $d,s \in D$; $d,n \in D$; and $i,n \in D$. Note that $k,q \in D_{3^+}$ in every case (Proposition 4.4). First, we consider the cases with backwards positions.

$e,t \in D$: There are 9 candidates for nearby poor 1-clusters: $a,c,e,g,h,r,s,v$ and $w$. We could have chosen $m$ instead of $h$, but the proof would be symmetric so we consider only $h$ as a candidate. Now, at most one of $h$ and $r$ is a poor 1-cluster; therefore, $C_1$ has at most 8 nearby poor 1-clusters. When $C_1$ has exactly 8 such 1-clusters, all of the candidates other than $h$ and $r$ are poor 1-clusters. Therefore, $g,v \in D_1^p$ and $q$ and $k$ are in the same $4^+$-cluster, $C_2$. Then we have $g$ and $v$ at distance-2 from $C_2$, where $C_2$ is not an open 3-cluster. When $C_1$ has exactly 7 nearby poor 1-clusters, at most one of $g$ and $v$ is no longer a poor 1-cluster. Therefore, at least one of $g$ and $v$ is distance-2 from a $3^+$-cluster. If $g \in D_1^p$, then $k \in D_{3^+} \setminus D_2^3$ and $g$ is distance-2 from $k$; a symmetric argument can be made for $v$ and $q$. Therefore, at least one of $g$ and $v$ is distance-2 from a $3^+$-cluster, $C_2$, where $C_2$ is not an open 3-cluster.

$e,s \in D$: Since $t \notin D$, we have $k,q,v,x \in D_{3^+}$. There are 6 candidates for nearby poor 1-clusters: $a,c,e,g,s$ and $h/m$. However, $h$ and $s$ cannot both be poor 1-clusters; and $m$ and $c$ cannot both be poor 1-clusters. Therefore, there are at most
5 nearby poor 1-clusters.

\( e, n \in D \): Again, \( k, q, v, x \in D_{3^+} \). There are 7 candidates for nearby poor 1-clusters: \( a, c, e, g, h, n \) and \( w \). However, at most one of \( n \) and \( w \) is a poor 1-cluster. Therefore, there are at most 6 nearby poor 1-clusters. If \( C_1 \) has exactly 6 such 1-clusters, then \( g \in D_1^p \). Then, \( g \) is distance-2 from the \( 3^+ \)-cluster at \( k \), \( C_k \), and \( C_k \) is not an open 3-cluster.

\( d, s \in D \): Since \( e, t \not\in D \), we have \( b, g, k, q, v, x \in D_{3^+} \). There are 3 candidates for nearby poor 1-clusters: \( d, h \) and \( s \). We could have chosen \( m \) instead of \( h \) but the proof would be symmetric. It cannot be the case that both \( h \) and \( s \) are poor 1-clusters. Therefore, there are at most 2 nearby poor 1-clusters.

\( d, n \in D \): Again, \( b, g, k, q, v, x \in D_{3^+} \). There are 4 candidates for nearby poor 1-clusters: \( d, h, n \) and \( w \). However, at most one of \( d \) and \( h \) is a poor 1-cluster; likewise for \( n \) and \( w \). Therefore, there are at most 2 nearby poor 1-clusters.

\( i, n \in D \): Once again, \( b, g, k, q, v, x \in D_{3^+} \). There are 4 candidates for nearby poor 1-clusters: \( a, i, n \) and \( w \). However, at most one of \( a \) and \( i \) is a poor 1-cluster; likewise for \( n \) and \( w \). Therefore, there are at most 2 nearby poor 1-clusters.
Lemma 6.11. Let $C$ be a curved 4-cluster with $P(C) = 3$.

(i) If $C$ is adjacent to no one-third vertices, then $C$ has at most 11 nearby poor 1-clusters. Furthermore, if $C$ has $k$ backwards positions not in $D$, then $C$ has at most $11 - k$ nearby poor 1-clusters.

(ii) If $C$ is adjacent to a one-third vertex, then $C$ has at most 6 nearby poor 1-clusters.

Proof. Let $C$ be the curved 4-cluster shown in Figure 6.6. First, suppose $C$ is adjacent to no one-third vertices and both backwards positions are in $D$; that is, $e, t \in D$. Then, either $j \in D$ or $j \not\in D$. Now, $P(C) = 3$; therefore, if $j \in D$ then $d, f, i, n, p, s, u \not\in D$. Therefore, $C$ has at most 11 nearby poor 1-clusters: $a, b, c, e, g, h/m, j, q, r, t$ and $v$. Now, consider the case in which $j \not\in D$. If $p \in D$, then this case can be reduced by symmetry to the previous case. So we assume $p \not\in D$. Then, $k, q \in D_{3+}$ (Proposition 4.4), and $C$ has at most 10 nearby poor 1-clusters: $a/d, c, e, g, h/i, m/n, r, s/w, t/x$ and $u/v$.

Now, suppose $C$ is adjacent to no one-third vertices and one backwards position is not in $D$. By symmetry, we choose $e \not\in D$. Again, either $j \in D$ or $j \not\in D$. First, assume $j \in D$. Since $P(C) = 3$ and each leaf has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6), we must have $f \not\in D$; then, $b, g \in D_{3+}$ (Proposition 4.4), and $C$ has at most 10 nearby poor 1-clusters: $a/d, c, h/i, j, m/n, p/q, r, s/w, t/x$ and $u/v$. Now, assume $j \not\in D$. If $p \in D$, then this case can be reduced by symmetry to the previous case. So we assume $p \not\in D$; then, $k, q \in D_{3+}$ (Proposition 4.4), and $C$ has at most 10 nearby poor 1-clusters: $a/d, b, c, f/g, h/i, m/n, r, s/w, t/x$ and $u/v$.

Now, suppose $C$ is adjacent to no one-third vertices and both backwards positions are not in $D$; that is, $e, t \not\in D$. First, assume $j \in D$. Since $P(C) = 3$ and each leaf has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6), we must have $f, p, u \not\in D$; then, $b, g, v, x \in D_{3+}$ (Proposition 4.4), and $C$ has at most 8 nearby poor 1-clusters:
a/d, c, h/i, j, m/n, q, r and s/w. Now, assume \( j \notin D \). If \( p \in D \), then this case can be reduced by symmetry to the previous case. So we assume \( p \notin D \). Then, \( k, q \in D_{3+} \) (Proposition 4.4). There are 10 candidates for nearby poor 1-clusters: \( a/d, b, c, f/g, h/i, m/n, r, s/w, u/v \) and \( x \). Each leaf of \( C \) has at least one distance-2 vertex in \( D \setminus C \) (Proposition 4.6). Therefore, \( d \in D \) or \( i \in D \); in both cases, \( c \notin D_1^p \). Therefore, \( C \) has at most 9 nearby poor 1-clusters.

Finally, suppose \( C \) is adjacent to a one-third vertex, \( v_{\frac{1}{3}} \). Since \( P(C) = 3 \) and each leaf must have at least one distance-2 vertex in \( D \setminus C \) (Proposition 4.6), \( v_{\frac{1}{3}} \) must be adjacent to a leaf of \( C \). By symmetry, we choose \( d, e \in D \). Then, for the same reasons, we must have \( f, i, j, p, u \notin D \). Then, \( k, q \in D_{3+} \) (Proposition 4.4), and \( C \) has at most 6 nearby poor 1-clusters: \( g, h/m, n/r, s/w, t/x \) and \( v \).

**Lemma 6.12.** An open 5-cluster, \( C \), has at most 9 nearby poor 1-clusters. If \( C \) has exactly 9 such 1-clusters, then at least one of the distance-3 poor 1-clusters is stealable.

**Proof.** By symmetry, there are only 3 cases to consider.

Let \( C_1 \) be the open 5-cluster shown in Figure 6.7a. By Proposition 4.4, we have \( c, u, v \in D_{3+} \). Then the candidates for nearby poor 1-clusters are \( a, e, f/g, h/i, j/k, \)

![Figure 6.7: Lemma 6.12](image-url)
There are 10 candidates in total. However, \(c \in D_{3^+}\). Let \(C_c\) be the \(3^+\)-cluster at \(c\). Now, \(b \in C_c\) or \(d \in C_c\) or both. If both, then \(a, e \notin D_1^p\) and there are at most 8 nearby poor 1-clusters. So consider the case in which only one of \(b\) and \(d\) is in \(D\). By symmetry, we choose \(d \in D\); therefore, \(e \notin D_1^p\) and there are at most 9 nearby poor 1-clusters. However, \(a\) is distance-2 from \(C_c\). Now, if \(e \notin D\), then \(h, i \in D_{3^+}\) (Proposition 4.4) and there are at most 8 nearby poor 1-clusters. Therefore, if \(C_c \in K_3^o\) and \(C_1\) has 9 nearby poor 1-clusters, then \(c, d, e \in C_c\). Then, \(a\) is not in a shoulder position and \(C_c\) is not paired.

Let \(C_2\) be the open 5-cluster shown in Figure 6.7b. By Proposition 4.4, we have \(r, t, u \in D_{3^+}\). There are 9 candidates for nearby poor 1-clusters: \(a/b, c/d, e/f, g/h, i/j, k/m, n, p/q\) and \(s\). However, \(s\) is distance-2 from the \(3^+\)-cluster at \(t\); let \(C_t\) be this \(3^+\)-cluster. If \(C_t \in K_3^o\), then \(u \in C_t\); furthermore, \(s\) is in an arm position but \(C_t\) is not paired.

Let \(C_3\) be the open 5-cluster shown in Figure 6.7c. By Proposition 4.4, we have \(g, j, n, q \in D_{3^+}\). There are 7 candidates for nearby poor 1-clusters: \(a/b, c, d, e/f, h/i, k/m\) and \(p\).

\[\square\]

**Lemma 6.13.** If a 4-cluster, \(C\), has one degree-3 vertex and \(P(C) = 3\), then \(C\) has at most 8 nearby poor 1-clusters.

![Figure 6.8: Lemma 6.13](image-url)
Proof. Let $C$ be the 4-cluster shown in Figure 6.8. Then $C$ has one degree-3 vertex. Each of the 3 leaves of $C$ has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6) and, by hypothesis, $P(C) = 3$; therefore

$$|\{e, f, j, p\} \cap D| = |\{f, g, k, q\} \cap D| = |\{p, q, t, v\} \cap D| = 1$$

By symmetry, we must consider only 2 cases: $f \in D$ and $g \in D$. There are 12 candidates for nearby poor 1-clusters: $a/e, b/f, c/g, d, i/j, k/m, n/p, q/r, s/t, u$ and $v/w$.

Now, if $f \in D$, then $e, g, j, k, p, q \notin D$. Therefore, $n, r \in D_{3+}$ (Proposition 4.4). Thus, $C$ has at most 10 nearby poor 1-clusters. Then, either $t \in D$ or $v \in D$; in both cases, $u \notin D_1^p$. Therefore, $C$ has at most 9 nearby poor 1-clusters. By symmetry, we choose $v \in D$ and $t \notin D$. If $v \notin D_1^p$, then the lemma holds; so we assume $v \in D_1^p$. Then, $u \notin D$. But we also have $t \notin D$; therefore, $s \in D_{3+}$ (Proposition 4.4). Therefore, $C$ has at most 8 nearby poor 1-clusters.

If $g \in D$, then $f, k, q \notin D$. Therefore, $b, r \in D_{3+}$ and $h \notin D_1^p$. Therefore, $C$ has at most 9 nearby poor 1-clusters. If $g \notin D_1^p$, then the lemma holds; so we assume $g \in D_1^p$. Then, $h \notin D$. But we also have $k \notin D$; therefore, $m \in D_{3+}$ (Proposition 4.4). Therefore, $C$ has at most 8 nearby poor 1-clusters.

Lemma 6.14. If a 5-cluster, $C$, has one degree-3 vertex, then $C$ has at most 12 nearby poor 1-clusters.

Proof. Let $C$ be the 5-cluster shown in Figure 6.9. Then $C$ has one degree-3 vertex. There are 13 candidates for nearby poor 1-clusters: $a/h, b/c, d, e/f, g, i/j, k/m, n/p, q/r, s/t, u/v, w$ and $x/y$.

Now, if $c \in D$, then $d \notin D_1^p$. Therefore, $C$ has at most 12 nearby poor 1-clusters.
Now we consider the case for which $c \notin D$. If $b \notin D_1^p$, then $C$ has at most 12 nearby poor 1-clusters and the lemma holds. So we assume $b \in D_1^p$. Then we have $a \notin D_1^p$. If $h \notin D_1^p$, then $C$ has at most 12 nearby poor 1-clusters and the lemma holds. So we assume $h \in D_1^p$. Then $g \notin D_1^p$; therefore, $C$ has at most 12 nearby poor 1-clusters.

\[\square\]

**Lemma 6.15.** An open 6-cluster, $C$, with $\Delta(C) = 2$ has at most 10 nearby poor 1-clusters.

*Proof.* By symmetry there are only 4 cases to consider. Let $C_1$ be the open 6-cluster shown in Figure 6.10a. Then, $b, c, t, u \in D_3^+$ (Proposition 4.4). There are 10 candidates for nearby poor 1-clusters: $a, d/g, e/f, h, i/j, k/m, n, p/s, q/r$ and $v$. Therefore, $C_1$ has at most 10 nearby poor 1-clusters. Let $C_2$ be the open 6-cluster shown in Figure 6.10b. Then, $b, c, f, q \in D_3^+$. Therefore, $C_2$ has at most 9 nearby poor 1-clusters: $a, d/e, g/h, i, j/k, m/n, p, r/s$ and $t/u$. Let $C_3$ be the open 6-cluster shown in Figure 6.10c. Then, $b, e, p, s \in D_3^+$. Therefore, $C_3$ has at most 8 nearby poor 1-clusters: $a, c/d, f/g, h/i, j/k, m/n, q/r$ and $t$. Let $C_4$ be the open 6-cluster shown in Figure 6.10d. Then, $p, s, u, v \in D_3^+$. Therefore, $C_4$ has at most 9 nearby poor 1-clusters: $a/e, b/f, c/d, g, h/i, j/k, m/n, q/r$ and $t$. \[\square\]
Lemma 6.16 was proved by Cranston and Yu [4, p. 14] in 2009. We state it here without proof.

**Lemma 6.16.** For \( k \geq 3 \), a \( k \)-cluster has at most \( k + 8 \) nearby clusters.

### 6.3 Very Poor 1-Clusters

**Lemma 6.17.** Let \( v \) be a very poor 1-cluster in a symmetric orientation, and let \( a, b \) and \( c \) be the vertices in \( D \) at distance-2 from \( v \). There exist 3 open 3-clusters, \( C_1, C_2 \) and \( C_3 \), such that \( v \) is in a head position of all 3 and exactly one of \( a, b \) and \( c \) is in a shoulder position of each of \( C_1, C_2 \) and \( C_3 \). Furthermore, if each of \( C_1, C_2 \) and \( C_3 \) is uncrowded, then each of \( a, b \) and \( c \) is distance-3 from a closed 3-cluster or \( 4^+ \)-cluster.
Proof. Let \( v, a, b \) and \( c \) be as shown in Figure 6.11a. Now, \( v \in D_1^{op} \) and \( a, b \) and \( c \) are distance-2 from \( v \); therefore, \( a, b, c \in D_1^p \). Therefore, \( g, n, p \notin D \) and \( d, m, q \in D \) (Proposition 4.5). Then \( h, k, s \in D_{3^+} \) (Proposition 4.4). Let \( C_h \) be the \( 3^+ \)-cluster at \( h \). Now, \( C_h \) is distance-3 from \( v \); therefore, \( C_h \in K_3^a \). Since \( b, d \in D \), we must have \( e, h, i \in C_h \). Symmetric arguments may be made to show \( f, j, k \in D_{3^+} \) and \( r, s, t \in D_{3^+} \). Therefore, \( v \) is in a head position of 3 open 3-clusters and exactly one of \( a, b \) and \( c \) is in a shoulder position of each of these open 3-clusters.

Now suppose each of the open 3-clusters at distance-3 from \( v \) is uncrowded as in Figure 6.11b. All of the vertices have been relabeled except \( v, a, b \) and \( c \). Now, the graph is rotationally symmetric about \( v \), so we need only consider one of \( a, b \) and \( c \). We choose \( c \). Now, \( d \) is in a shoulder position of an open 3-cluster, \( C \), which is distance-3 from \( v \). By hypothesis, \( C \) is uncrowded; thus, \( d \in D_1^p \). Therefore, \( f \notin D \) and \( e \in D \) (Proposition 4.5). Then we have \( i \in D_{3^+} \). Let \( C_i \) be the \( 3^+ \)-cluster at \( i \). Since 2 of the 3 neighbors of \( i \) are not in \( D \), we also have \( h \in C_i \). If \( g \in D \), then \( e, g, h, i \in C_i \). If \( g \notin D \), then \( h, i, j \in C_i \) and \( e \) closes \( C_i \). In both cases, \( C_i \) is a closed 3-cluster or \( 4^+ \)-cluster at distance-3 from \( c \).
Corollaries 6.18 and 6.19 follow directly from the proof of Lemma 6.17.

**Corollary 6.18.** None of the open 3-clusters of which a very poor 1-cluster in a symmetric orientation is in a head position is type-1 paired on top.

**Corollary 6.19.** Each of the vertices in $D$ at distance-2 from a very poor 1-cluster in a symmetric orientation is distance-2 from no other very poor 1-cluster.

**Lemma 6.20.** Let $v$ be a very poor 1-cluster, and let $u$, $w$ and $x$ be in the $u$-position, $w$-position and $x$-position, respectively, of $v$. There exists an open 3-cluster, $C_0$, such that $v$ and $x$ are in the head positions and $w$ is in a shoulder position of $C_0$; and one of the following holds:

(i) $C_0$ is crowded.

(ii) There exists a closed 3-cluster or $4^+$-cluster at distance-3 from $w$.

(iii) There exists an open 3-cluster, $C$, such that the tail position of $C$ is in $D$ and $u$, $v$ or $w$ is in the hand position on the finless side of $C$.

(iv) There exists a leaf, $\ell$, of a $4^+$-cluster, $C$, at distance-2 from $u$ such that $u$ is the only vertex in $D \setminus C$ at distance-2 from $\ell$. If $C$ is a linear 4-cluster, then $C$ has at most one one-turn position. If $C$ is a curved 4-cluster, then $C$ has at most one backwards position in $D$.

(v) There exists a leaf, $\ell$, of a closed 3-cluster, $C$, at distance-2 from $u$ such that $u$ is the only vertex in $D \setminus C$ at distance-2 from $\ell$ and $u$ is in a foot or arm position. Either $C$ is type-2 paired and $u$ is in the arm position on the closed side of $C$, or $C$ has at most 6 nearby poor 1-clusters.

(vi) There exists an open 3-cluster, $C$, such that $v$ or $w$ is in a hand position of $C$ and the hand and arm positions on the opposite side of $C$ are both in $D$. 

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(vii) There exists an open 3-cluster, \( C \), such that \( u \) is in a foot position and \( C \) is type-1 paired on top.

**Proof.** Let \( u, v, w \) and \( x \) be as shown in Figure 6.12. Now, by hypothesis, \( v \in D_1^p \); therefore, \( u, w, x \in D_1^p \). Then, we have \( f \not\in D \); therefore, \( n \in D_3^+ \). Since \( x \in D_1^p \), we have \( g \in D \) (Proposition 4.5). Let \( C_0 \) be the 3+-cluster at \( n \). Now, \( v \) is distance-3 from \( C \), so \( C \in \mathcal{K}_3^+ \). The only possibility is to have \( m, n, p \in C \). Therefore, \( v \) and \( x \) are in the head positions of an open 3-cluster, and \( w \) is in a shoulder position.

![Figure 6.12: Lemma 6.20](image)

If \( C_0 \) is crowded, then (i) is satisfied and the lemma holds. So assume \( C_0 \) is uncrowded. Then, \( j, k, s \not\in D \), and since \( u \in D_1^p \) we have \( e \not\in D \) (Corollary 5.2). If \( i \not\in D \), then \( h, r \in D_3^+ \). Let \( C_r \) be the 3+-cluster at \( r \). If \( h \in C_r \), then \( w \) is distance-3 from a 4+-cluster. If \( h \not\in C_r \), then \( h \) closes \( C_r \) and \( w \) is distance-3 from a closed 3+-cluster. In both cases, (ii) is satisfied. So assume \( w \) is not distance-3 from a closed 3-cluster or 4+-cluster. Then, we have \( i \in D \). Now, if \( r \not\in D \), then \( i, q \in D_3^+ \). Let \( C_i \) be the 3+-cluster at \( i \). If \( q \in C_i \), then \( w \) is distance-3 from a 4+-cluster; and if \( q \not\in C_i \), then \( q \) closes \( C_i \) and \( w \) is distance-3 from a closed 3+-cluster. But we assumed that (ii) is not satisfied; therefore, \( r \in D \).

If \( c \not\in D \), then \( a, b, d \in D_3^+ \). Let \( C_b \) be the 3+-cluster at \( b \). If \( b \) is a leaf of \( C_b \), then either \( u \in C_b \) and \( C_b \in \mathcal{K}_4^+ \), or \( u \) closes \( C_b \) and \( C_b \in \mathcal{K}_3^c \), or \( a \in C_b \) and \( C_b \in \mathcal{K}_4^+ \), or

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a closes $C_b$ and $C_b \in K_{3+}$. But $v$ is very poor and distance-3 from $b$, so we must have $C_b \in K_{3_0}$. This is only possible if $b$ is the middle vertex of $C_b$. Let $C_a$ be the $3^+$-cluster at $a$, and let $C_d$ be the $3^+$-cluster at $d$. Now, $C_b \in K_{3}^o$, so $a$ is a leaf of $C_a$ and either $d \in C_a$ or $d$ closes $C_a$. In the first case, $v$ is distance-3 from a $4^+$-cluster. But, by hypothesis, $v$ is not distance-3 from a $4^+$-cluster. Therefore, $d$ closes $C_a$. But then $d$ is a leaf of $C_d$ and $x$ closes $C_d$; therefore, $v$ is distance-3 from a closed $3^+$-cluster. But, by hypothesis, $v$ is not distance-3 from a closed $3^+$-cluster. Therefore, $c \in D$.

Figure 6.13: Lemma 6.20, where (i) and (ii) are not satisfied.

Figure 6.13 shows the surrounding vertices of $v$ when neither (i) nor (ii) is satisfied. All the vertices except $u,v,w$ and $x$ have been relabeled. Now, $u \in D_{1}^p$; therefore, exactly one of $g$ and $m$ is in $D$ (Corollary 5.2).

First, we consider the case for which $m \in D$ and $g \notin D$. If $p \in D$, then $m,p \in D_{3+}$. Let $C_{m,p}$ be the $3^+$-cluster at $m$ and $p$. If $k,m,p \in C_{m,p}$, then $u$ closes $C_{m,p}$. But $p$ is distance-3 from $w$ and, by assumption, $w$ is not distance-3 from a closed $3^+$-cluster.

If $m,n,p \in C_{m,p}$, then $s$ closes $C_{m,p}$. But, again, by assumption, $w$ is not distance-3 from a closed $3^+$-cluster. Therefore, $p \notin D$. Then, by Proposition 4.4, we have $s \in D_{3+}$. Let $C_s$ be the $3^+$-cluster at $s$. Since 2 of the 3 neighbors of $s$ are not in $D$, we have $r \in C_s$. Now, $C_s$ is distance-3 from $w$; therefore, by assumption, $C_s \in K_{3}^o$. 

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Therefore, either \( q, r, s \in C_s \) or \( r, s, t \in C_s \). In both cases we have \( n, y \not\in D \). Then, by Proposition 4.4, we have \( m \in D_{3^+} \). Let \( C_m \) be the \( 3^+ \)-cluster at \( m \). Since 2 of the 3 neighbors of \( m \) are not in \( D \), we must have \( k \in C_m \).

![Figure 6.14: Lemma 6.20, where (i) and (ii) are not satisfied and \( m \in D \).](image-url)

Figure 6.14 shows the surrounding vertices of \( v \) when neither (i) nor (ii) is satisfied and \( m \in D \). All of the vertices except \( g, m, u, v, w \) and \( x \) have been relabeled.

Now, if \( z \in D \), then \( z \in D_{3^+} \). Let \( C_z \) be the \( 3^+ \)-cluster at \( z \). By assumption, (ii) is not satisfied, so \( C_z \in \mathcal{K}^2_3 \). Then \( w \) is in the hand position of an open 3-cluster, \( C_z \), such that the tail position is in \( D \) and \( w \) is on the finless side. Therefore, (iii) is satisfied.

Now assume (iii) is not satisfied. Then \( z \not\in D \) and \( y \in D_{3^+} \). Let \( C_y \) be the \( 3^+ \)-cluster at \( y \). Since \( w \) is distance-3 from \( C_y \), we have \( C_y \in \mathcal{K}^2_3 \). Let \( C_m \) be the \( 3^+ \)-cluster at \( m \). Then, \( m \) is a leaf of \( C_m \) and \( u \) is the only vertex in \( D \setminus C_m \) at distance-2 from \( m \). If \( C_m \) is a linear 4-cluster, then either \( q, r \in C_m \) or \( l, k \in C_m \); in both cases, the one-turn position at distance-2 from \( m \) is not in \( D \). If \( C_m \) is a curved 4-cluster, then either \( r, t \in C_m \) or \( l, n \in C_m \); in both cases, the backwards position at distance-2 from \( m \) is not in \( D \). Therefore, if \( C_m \in \mathcal{K}_{4^+} \), then (iv) is satisfied.

Now assume (iv) is not satisfied. Then, \( C_m \in \mathcal{K}_3 \). Either \( l \in C_m \) or \( r \in C_m \);
in both cases, \( u \) is in a foot or arm position. First, we consider the case in which \( r \in C_m \). Then, \( l, q, t \notin D \). If \( n \notin D \), then \( d, o \in D_{3^+} \) (Proposition 4.4). Since 2 of the 3 neighbors of \( o \) are not in \( D \), we also have \( f \in D_{3^+} \). Let \( C_{f,o} \) be the \( 3^+ \)-cluster at \( f \) and \( o \). If \( e \in D \), then \( d, e, f, o \in C_{f,o} \) and \( C_{f,o} \in K_{4^+} \). If \( e \notin D \), then \( h \in C_{f,o} \) and \( d \) closes \( C_{f,o} \). But, by hypothesis, \( v \) is not within distance-3 of a closed \( 3^+ \)-cluster. Therefore, \( n \in D \) and \( C_m \in K_3^o \). Then, \( C_m \) is type-2 paired with \( C_y \) and \( u \) is in the arm position on the closed side of \( C_m \). Then, \( (v) \) is satisfied. Therefore, with \( r \in D \), the lemma holds.

Now assume \( (v) \) is not satisfied. So we have \( l \in C_m \). Recall \( C_m \in K_3^o \); therefore, \( k, n, r \notin D \). Then, \( f, o \in D_{3^+} \) (Proposition 4.4). Again, let \( C_{f,o} \) be the \( 3^+ \)-cluster at \( f \) and \( o \). Now, \( C_{f,o} \) is distance-3 from \( v \), so we must have \( C_{f,o} \in K_3^o \). If \( h \in C_{f,o} \), then the tail position of \( C_{f,o} \) is in \( D \) and \( u \) is in the hand position on the finless side. But, by assumption, \( (iii) \) is not satisfied. Therefore, \( e \in C_{f,o} \) and \( h \notin D \). If \( C_{f,o} \in K_3^o \), then \( C_{f,o} \) has at most 6 nearby poor 1-clusters: \( a/b, c, i/j, p/q, s/t \) and \( u \). But, by assumption, \( (v) \) is not satisfied. So we have \( C_{f,o} \in K_3^o \). Then, \( q, t \notin D \); therefore, \( s \in D_{3^+} \). Let \( C_s \) be the \( 3^+ \)-cluster at \( s \). If \( C_s \) occupies the arm position of \( C_y \), then \( w \) is in the hand position of an open \( 3 \)-cluster satisfying \( (vi) \).

Now assume \( (vi) \) is not satisfied. Then, then arm position of \( C_y \) is not in \( D \). Therefore, \( u \) is in the foot position of an open \( 3 \)-cluster which is type-1 paired on top. Therefore, \( (vii) \) is satisfied, and the lemma holds.

Now we return to Figure 6.13 and consider the case for which \( g \in D \) and \( m \notin D \).

If \( h \in D \), then \( g, h \in D_{3^+} \) and either \( a \in D_{3^+} \) or \( f \in D_{3^+} \). In both cases, \( v \) is distance-3 from a closed \( 3^+ \)-cluster. Therefore, \( h \notin D \). Then, by Proposition 4.4, we have \( i \in D_{3^+} \). Let \( C_i \) be the \( 3^+ \)-cluster at \( i \). Since 2 of the 3 neighbors of \( i \) are not in \( D \), we have \( c \in C_i \) and \( i \) is a leaf of \( C_i \). Therefore, \( j \in D \) (Proposition 4.6). Now, \( C_i \) is distance-3 from \( v \); therefore, \( C_i \in K_{3^+}^o \). Either \( b \in C_i \) or \( d \in C_i \). In both cases, we
have $a, e \not\in D$. Then, $g \in D_{3^+}$ (Proposition 4.4). Let $C_g$ be the $3^+$-cluster at $g$. Since 2 of the 3 neighbors of $g$ are not in $D$, we have $f \in C_g$. If $p \not\in D$, then $n, s \in D_{3^+}$ (Proposition 4.4). Let $C_s$ be the $3^+$-cluster at $s$. Since 2 of the 3 neighbors of $s$ are not in $D$, we have $r \in C_s$. If $q \in D$, then $n, q, r, s \in C_s$ and $C_s \in \mathcal{K}_{4^+}$. If $q \not\in D$, then $r, s, t \in C_s$ and $n$ closes $C_s$. But $s$ is distance-3 from $w$ and we assumed (ii) is not satisfied. Therefore, $p \in D$.

![Figure 6.15: Lemma 6.20, where (i) and (ii) are not satisfied and $g \in D$.](image)

Figure 6.15 shows the surrounding vertices of $v$ when neither (i) nor (ii) is satisfied and $g \in D$. All of the vertices except $g, m, u, v, w$ and $x$ have been relabeled.

If $j \in D$, then $j \in D_{3^+}$. Let $C_j$ be the $3^+$-cluster at $j$. Since, $C_j$ is distance-3 from $v$, we must have $C_j \in \mathcal{K}_3^+$. Then, the tail position of $C_j$ is in $D$ and $v$ is in the hand position on the finless side. Therefore, (iii) is satisfied.

Now assume (iii) is not satisfied. Then we have $i \in D_3^+$ and $j \not\in D$. Let $C_i$ be the open 3-cluster at $i$. Now, $u$ is distance-2 from the $3^+$-cluster at $g$; let $C_g$ be this $3^+$-cluster. Then, $g$ is a leaf of $C_g$ and $u$ is the only distance-2 vertex of $g$ in $D \setminus C_g$. If $C_g$ is a linear 4-cluster, then either $n, p \in C_g$ or $e, f \in C_g$; in both cases, the one-turn position at distance-2 from $g$ is not in $D$. If $C_g$ is a curved 4-cluster then either $f, h \in C_g$ or $p, s \in C_g$; in both cases, the backwards position at distance-2 from $g$ is not in $D$. Therefore, if $C_g \in \mathcal{K}_{4^+}$, then (iv) is satisfied.
Now assume (iv) is not satisfied. Then, \( C_g \in \mathcal{K}_3 \). Either \( f \in C_g \) or \( p \in C_g \); in both cases, \( u \) is in a foot or arm position. First we consider the case in which \( f \in C_g \). Then, \( e, h, p \notin D \). If \( s \notin D \), then \( r, y \in D_{3^+} \) (Proposition 4.4). Let \( C_y \) be the \( 3^+ \)-cluster at \( y \). If \( t \in D \), then \( r, t \in C_y \) and \( C_y \in \mathcal{K}_{4^+} \). If \( t \notin D \), then \( z \in C_y \) and \( r \) closes \( C_y \). In both cases, \( w \) is distance-3 from a closed 3-cluster or \( 4^+ \)-cluster. But we assumed that (ii) is not satisfied. Therefore, \( s \in D \). Then, \( u \) is in an arm position on the closed side of \( C_g \), and \( C_g \) is type-2 paired with \( C_i \). Then, (v) is satisfied. Therefore, with \( f \in D \), the lemma holds.

Now assume (v) is not satisfied. Then \( f \notin D \). Therefore, \( p \in C_g \) and \( f, n, s \notin D \). Then, \( y \in D_{3^+} \) (Proposition 4.4). Let \( C_y \) be the 3+-cluster at \( y \). Now, \( u \) is distance-3 from \( C_y \) and we assumed (ii) is not satisfied. Therefore, \( C_y \in \mathcal{K}'_3 \). If \( z \in C_y \), then the tail position of \( C_y \) is in \( D \) and \( u \) is in the hand position on the finless side of \( C_y \). But we assumed that (iii) is not satisfied. Therefore, \( t \in C_y \). Now, if \( C_g \) is a closed 3-cluster, then there are at most 6 nearby poor 1-clusters: \( b/h, a/e, d, k/l, q \) and \( u \). But we assumed (v) is not satisfied. Therefore, \( C_g \in \mathcal{K}'_3 \). Then, \( h \notin D \). By Proposition 4.4, we have \( b \in D_{3^+} \). If \( c \in D \), then \( v \) is in the hand position of \( C_i \) and the hand and arm positions on the other side of \( C_i \) are both in \( D \). Therefore, (vi) is satisfied.

Now assume (vi) is not satisfied. Then we have \( c \notin D \). Therefore, \( u \) is in the foot position of an open 3-cluster which is type-1 paired on top. Therefore, (vii) is satisfied and the lemma holds.

Corollary 6.21 follows directly from the proof of Lemma 6.20.

**Corollary 6.21.** Let \( v \) be a very poor 1-cluster in an asymmetric orientation, and let \( w \) be in the \( w \)-position of \( v \). The open 3-cluster of which \( v \) is in a head position and \( w \) is in a shoulder position is not type-1 paired on top.
Corollary 6.22. Let \( u \) and \( w \) be vertices in the \( u \)-position and \( w \)-position, respectively, of a very poor 1-cluster, \( v \). Neither \( u \) nor \( w \) is distance-2 from any very poor 1-cluster other than \( v \).

Proof. Let \( x \) be in the \( x \)-position of \( v \). Now, \( v \in D_{1}^{vp} \); therefore, \( u, w, x \in D_{1}^{p} \). Additionally, \( v \) is in a head position of an open 3-cluster, \( C_0 \), and \( w \) is in a shoulder position (Lemma 6.20). Let \( u, v, w, x \) and \( C_0 \) be as shown in Figure 6.16.

Suppose by contradiction that \( u \) is distance-2 from a very poor 1-cluster other than \( v \). There are 2 possibilities: \( c \in D_{1}^{vp} \) or \( h \in D_{1}^{vp} \). If \( c \in D_{1}^{vp} \), then \( d \notin D \) and \( a \in D \) (Proposition 4.5). Then, \( e \in D_{3}^{+} \) (Proposition 4.4). Let \( C_e \) be the \( 3^+ \)-cluster at \( e \). Either \( a \in C_e \) or \( a \) closes \( C_e \); in both cases \( c \) is within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster. Therefore, \( c \not\in D_{1}^{vp} \). If \( h \in D_{1}^{vp} \), then \( k \not\in D \) and \( j \in D \) (Proposition 4.5). Then, \( m \in D_{3}^{+} \) (Proposition 4.4). Let \( C_m \) be the \( 3^+ \)-cluster at \( m \). Either \( j \in C_m \) or \( j \) closes \( C_m \); in both cases \( h \) is within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster. Therefore, \( h \not\in D_{1}^{vp} \).

Now, suppose by contradiction that \( w \) is distance-2 from a very poor 1-cluster other than \( v \). The only possibility is \( u \). If \( u \in D_{1}^{vp} \), then \( c \in D_{1}^{p} \) or \( h \in D_{1}^{p} \). But we already saw that \( c \in D_{1}^{p} \) or \( h \in D_{1}^{p} \) implies \( e \in D_{3}^{+} \cup D_{4}^{+} \) or \( m \in D_{3}^{+} \cup D_{4}^{+} \). Since \( u \) is distance-3 from both \( e \) and \( m \), we have \( u \not\in D_{1}^{vp} \). \( \square \)
Corollary 6.23. Consider a very poor 1-cluster, $v$, in an asymmetric orientation, and let $x$ be in the $x$-position of $v$. If $x$ is a very poor 1-cluster, then $x$ is in an asymmetric orientation and $v$ is in the $x$-position of $x$.

Proof. Let $v$ and $x$ be as shown in Figure 6.16. If $x \in D_1^{vp}$, then $g \in D_1^p$ or $i \in D_1^p$. If $g \in D_1^p$, then $f \notin D$ and $b \in D$ (Proposition 4.5). Therefore, $e \in D_{3^+}$ (Proposition 4.4). Let $C_e$ be the $3^+$-cluster at $e$. Either $b \in C_e$ or $b$ closes $C_e$; in both cases, $x$ is distance-3 from a closed 3-cluster or $4^+$-cluster. But, by hypothesis, $v \in D_1^{vp}$. Therefore, $i \in D_1^p$. Therefore, $x$ is in an asymmetric orientation and $v$ is in the $x$-position of $x$. \hfill \Box

Lemma 6.24. If a very poor 1-cluster is in a head position of an open 3-cluster, $C$, then both shoulder positions of $C$ are in $D$.

Proof. Let $C$ be the open 3-cluster shown in Figure 6.17, and let $v$ be a very poor 1-cluster. Then, $v$ is in a head position of $C$. Since $v \in D_1$, we have $b \in D$ (Proposition 4.5). Suppose by contradiction that $c \notin D$. Then, $a \in D_{3^+}$ (Proposition 4.4). Then $v \in D_1^{vp}$ and $v$ is distance-2 from a $3^+$-cluster, which is a contradiction. Therefore, $c \in D$. \hfill \Box
6.4 Paired 3-Clusters

Lemma 6.25. If a poor 1-cluster, \( v \), is in a shoulder or arm position of an open 3-cluster that is type-1 paired on top, then \( v \) is nearby another 3\(^+\)-cluster, \( C_2 \), such that if \( C_2 \) is an open 3-cluster then \( v \) is distance-2 from \( C_2 \) but not in an arm position and \( C_2 \) is not type-1 paired on top.

![Figure 6.18: Lemma 6.25](image)

Proof. Let \( C_1 \) be the open 3-cluster described by \( j, k \) and \( m \) in Figure 6.18. Then \( C_1 \) is type-1 paired on top. Now, \( e \in D_{3^+} \) (Proposition 4.4). Let \( C_e \) be the 3\(^+\)-cluster at \( e \). Since 2 of the 3 neighbors of \( e \) are not in \( D \), we have \( d \in C_e \). If \( C_1 \) has a poor 1-cluster in a shoulder or arm position, then either \( h \in D_{1^p} \) or \( i \in D_{1^p} \).

First suppose \( h \in D_{1^p} \). Then \( c \in D \) (Proposition 4.5). Therefore, \( c \in C_e \) and \( h \) is distance-2 from \( C_e \). If \( C_e \in \mathcal{K}_{3^+} \), then \( h \) is in a foot position and \( C_e \) is not paired.

Now suppose \( i \in D_{1^p} \). Then \( h \not\in D \) (Corollary 5.2). Therefore, \( c, g \in D_{3^+} \) (Proposition 4.4). Let \( C_g \) be the 3\(^+\)-cluster at \( g \). If \( f \in D \), then \( i \) is distance-2 from \( C_g \). If \( f \not\in D \), then either \( a, b, c, g \in C_g \) or \( c \) closes \( C_g \); in both cases, \( i \) is distance-3 from a closed 3-cluster or 4\(^+\)-cluster. Thus, if \( C_g \in \mathcal{K}_{3^+} \), then \( a, f, g \in C_g \). Then, \( c \) and \( i \) are in the shoulder positions of \( C_g \) and, hence, \( C_g \) is not type-1 paired on top. \( \square \)
Lemma 6.26. Let $C_1$ be a closed 3-cluster that is type-2 paired with the open 3-cluster, $C_2$. If $C_1$ has 7 nearby poor 1-clusters, then the arm position, $n$, of $C_2$ is in $D$ and the hand position, $k$, on the same side is not in $D$. Furthermore, if $n$ is a poor 1-cluster, then $n$ is nearby a third $3^+$-cluster, $C_3$, such that if $C_3$ is an open 3-cluster then $n$ is distance-2 from $C_3$ but not in an arm position and $C_3$ is not type-1 paired on top.

Figure 6.19: Lemma 6.26

Proof. Let $C_1$ be the type-2 paired closed 3-cluster shown in Figure 6.19, and let $C_2$ be the type-2 paired open 3-cluster. Suppose by contrapositive that $n \notin D$ or $k \in D$. First, we deal with case in which $n \notin D$. Then $C_1$ has 7 candidates for nearby poor 1-clusters: $a/d$, $b/c$, $e$, $f$, $g/h$, $i$ and $j/k$. It suffices to eliminate one of these candidates. If $k \notin D$, then $j \in D_{3^+}$ (Proposition 4.4) and the lemma holds. If $k \notin D^p_1$, then the lemma holds; so assume $k \in D^p_1$. Then, $j \notin D$. If $h \notin D$, then $g \in D_{3^+}$ (Proposition 4.4) and the lemma holds. If $h \notin D^p_1$, then the lemma holds; so assume $h \in D^p_1$. Then, $c \notin D$ (Corollary 5.2). If $b \notin D^p_1$, then the lemma holds; so assume $b \in D^p_1$. Then, $d \in D$ (Proposition 4.5). But then $e$ is adjacent to a one-third vertex; therefore, $e \notin D^p_1$. Therefore, $C_1$ does not have 7 nearby poor 1-clusters. Now, we deal with the case in which $k \in D$. Then, $C_1$ has 7 candidates
for nearby poor 1-clusters: $a/d$, $b/c$, $e$, $f$, $g/h$, $i$ and $k$. It suffices to eliminate one of these candidates. If $k \notin D_1^p$, then the lemma holds; so assume $k \in D_1^p$. Then, $j \notin D$. If $h \notin D$, then $g \in D_3^+$ (Proposition 4.4) and the lemma holds; so assume $h \in D$. If $h \notin D_1^p$, then the lemma holds; so assume $h \in D_1$. Then, $c \notin D$ (Corollary 5.2). If $b \notin D_1^p$, then the lemma holds; so assume $b \in D_1^p$. Then, $d \in D$ (Proposition 4.5). But then $e$ is adjacent to a one-third vertex; therefore, $e \notin D_1^p$. Therefore, $C_1$ does not have 7 nearby poor 1-clusters.

If $n \in D_1^p$, then $m \notin D$ and $r \notin D$ (Corollary 5.2). Then, $q, t \in D_3^+$ (Proposition 4.4). Let $C_q$ be the $3^+$-cluster at $q$. If $C_q \in \mathcal{K}_3^+$, then $p, q, s \in C_q$. However, $n$ and $t$ are in shoulder positions; therefore, $C_q$ is not type-1 paired on top. If $C_q \notin \mathcal{K}_3^+$, then $n$ closes $C_q$ or $t$ closes $C_q$ or $t \in C_q$; in each case, $n$ is nearby a closed 3-cluster or $4^+$-cluster. \hfill \Box
Chapter 7

Proof of Theorem 1.2

We employ the discharging method. Suppose each vertex in $D$ has 1 charge. We redistribute this charge so that each vertex in $G_H$ has at least $\frac{5}{12}$ charge. Below are the discharging rules:

1. If a vertex, $v$, is not in $D$ and has $k$ neighbors in $D$, then $v$ receives $\frac{5}{12k}$ from each of these neighbors.

2. Let $v_\frac{1}{3}$ be a one-third vertex, and let $a$, $b$ and $c$ be the vertices adjacent to $v_\frac{1}{3}$.

   (a) If $a$ and $b$ are 1-clusters and $c$ is in a $3^+$-cluster, $C$, then each of $a$ and $b$ receives $\frac{1}{72}$ from $C$.

   (b) If $a$ is a 1-cluster and $b$ and $c$ are in $3^+$-clusters, $C_1$ and $C_2$, then $a$ receives $\frac{1}{36}$ from each of $C_1$ and $C_2$. If $C_1 = C_2$, then $a$ receives $2 \cdot \frac{1}{36} = \frac{1}{18}$ from $C_1$.

3. Let $v$ be a poor 1-cluster.

   (a) If $v$ is distance-2 from a closed 3-cluster or $4^+$-cluster, $C$, then $v$ receives $\frac{1}{24}$ from $C$.

   (b) If $v$ is distance-2 from an open 3-cluster, $C$, and has not received charge by previous rules, then $v$ receives $\frac{1}{24}$ from $C$ unless (a) $C$ is type-1 paired.
on top and \( v \) is in a shoulder or arm position, or (b) \( C \) is type-2 paired and \( v \) is in an arm position.

(c) If \( v \) is distance-3 from a closed 3-cluster or \( 4^+ \)-cluster, \( C \), and has not received charge by previous rules, then \( v \) receives \( \frac{1}{24} \) from \( C \) unless \( C \) is a closed 3-cluster and \( v \) is in the arm position of an open 3-cluster that is type-2 paired with \( C \).

(d) Let \( h \) be a non-poor 1-cluster that is not in a group of non-poor 1-clusters. If \( v \) is distance-2 from \( h \) and has not received charge by previous rules, then \( v \) receives \( \frac{1}{24} \) from \( h \).

(e) If \( v \) is distance-2 from a group of non-poor 1-clusters, \( H \), and has not received charge by previous rules, then \( v \) receives \( \frac{1}{24} \) from \( H \).

4. If a closed 3-cluster, \( C_1 \), and an open 3-cluster, \( C_2 \), are type-2 paired and the arm position of \( C_2 \) is in \( D \) and the hand position on the same side is not in \( D \), then \( C_1 \) receives \( \frac{1}{24} \) from \( C_2 \).

5. If a very poor 1-cluster, \( v \), is in a head position of a crowded open 3-cluster, \( C_0 \), then \( v \) receives \( \frac{1}{24} \) from \( C_0 \).

6. Let \( v \) be a very poor 1-cluster in a symmetric orientation, and let \( a \) be a distance-2 poor 1-cluster. If \( a \) is in a shoulder position of an open 3-cluster and distance-3 from a closed 3-cluster or \( 4^+ \)-cluster, \( C \), then \( a \) receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules and \( v \) receives \( \frac{1}{24} \) from \( a \).

7. Let \( v \) be a very poor 1-cluster in an asymmetric orientation, and let \( u \), and \( w \) be poor 1-clusters in the \( u \)-position and \( w \)-position, respectively, of \( v \). The following applies only if \( v \) does not receive charge by Discharging Rule 5.

(a) If \( w \) is distance-3 from a closed 3-cluster or \( 4^+ \)-cluster, \( C \), then \( w \) receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules and \( v \) receives \( \frac{1}{24} \) from \( C \).
\( \frac{1}{24} \) from \( w \).

(b) Let \( C \) be an open 3-cluster, and let the tail position of \( C \) be in \( D \). If \( u, v \) or \( w \) is in the hand position on the finless side of \( C \), then \( u, v \) or \( w \), respectively, receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules. If \( u \) or \( w \) receives this charge, then \( v \) receives \( \frac{1}{24} \) from \( u \) or \( w \), respectively.

(c) If \( u \) distance-2 from a leaf, \( \ell \), of a type-2 paired closed 3-cluster, a closed 3-cluster with at most 6 nearby poor 1-clusters or a 4\(^+\)-cluster, \( C \), such that \( u \) is not in a shoulder or tail position, a one-turn position or a backwards position of a closed 3-cluster, a linear 4-cluster or a curved 4-cluster, respectively, and \( u \) is the only vertex in \( D \setminus C \) at distance-2 from \( \ell \), then \( u \) receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules and \( v \) receives \( \frac{1}{24} \) from \( u \).

(d) Let \( C \) be an open 3-cluster, and let the hand and arm positions on one side of \( C \) be in \( D \). If \( v \) or \( w \) is in the hand position on the other side of \( C \), then \( v \) or \( w \), respectively, receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules. If \( w \) receives this charge, then \( v \) receives \( \frac{1}{24} \) from \( w \).

(e) Let \( C \) be an open 3-cluster that is type-1 paired on top. If \( u \) is in the foot position of \( C \), then \( u \) receives \( \frac{1}{24} \) from \( C \) in addition to any charge received by previous rules and \( v \) receives \( \frac{1}{24} \) from \( u \).

Now we verify that the above discharging rules allow each vertex in \( G_H \) to retain at least \( \frac{5}{12} \) charge. For a given vertex, \( v \), let \( f(v) \) be the final charge of \( v \) and let \( f_n(v) \) be the charge of \( v \) after Discharging Rule \( n \). And for a given \( k \)-cluster, \( C \), where \( k \geq 3 \), let \( f(C) \) be the final charge of \( C \) and let \( f_n(C) \) be the charge of \( C \) after Discharging Rule \( n \); note that \( f(C) \geq \frac{5k}{12} \) immediately implies that each vertex in \( C \) can retain at least \( \frac{5}{12} \) charge.
If \( v \in V(G_H) \), then \( v \notin D \) or \( v \in D_1 \) or \( v \in D_3 \) or \( v \in D_{4^+} \). We consider vertices not in \( D \) in Claim 7.1. We partition \( D_1 \) such that \( D_1 = (D_1^p \setminus D_1^{vp}) \cup D_1^{vp} \cup D_1^{ip} \), and we consider each case separately in Claims 7.3, 7.4 and 7.5, respectively. Rather than considering individual vertices in \( D_3 \), we consider \( K_3 \). We partition \( K_3 \) such that \( K_3 = K_3^o \cup K_3^c \), and we consider each case separately in Claims 7.7 and 7.11, respectively. We defer our discussion of \( 4^+ \)-clusters until after Claim 7.11.

### 7.1 \( v \not\in D \)

**Claim 7.1.** If a vertex, \( v \), is not in \( D \), then \( f(v) = \frac{5}{12} \).

**Proof.** Let \( v \not\in D \), and suppose \( v \) has \( k \) neighbors in \( D \). Then, by Discharging Rule 1, \( v \) receives \( \frac{5}{12k} \) from each of these neighbors. That is, \( f(v) = f_1(v) = k \cdot \frac{5}{12k} = \frac{5}{12} \). \( \square \)

### 7.2 1-Clusters

**Proposition 7.2.** Any poor 1-cluster at distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster receives charge by Discharging Rules 3a-3c.

**Proof.** Let \( v \in D_1^p \) such that \( v \) is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster. Then, by Rules 3a-3c, \( v \) receives \( \frac{1}{24} \) from a nearby \( 3^+ \)-cluster except, potentially, in 2 cases. In the first case, \( v \) is in a shoulder or arm position of an open 3-cluster which is type-1 paired on top. But, by Lemma 6.25, \( v \) is nearby another \( 3^+ \)-cluster, \( C_1 \), such that if \( C_1 \) is an open 3-cluster then \( v \) is distance-2 from \( C_1 \) but not in an arm position and \( C_1 \) is not type-1 paired on top; therefore, \( v \) receives \( \frac{1}{24} \) from \( C_1 \) by Discharging Rules 3a-3c. In the second case, \( v \) is in an arm position of an open 3-cluster which is type-2 paired with a closed 3-cluster. But, by Lemma 6.26, \( v \) is nearby a third \( 3^+ \)-cluster, \( C_2 \), such that if \( C_2 \) is an open 3-cluster
then \( v \) is distance-2 from \( C_2 \) but not in an arm position and \( C_2 \) is not type-1 paired on top; therefore, \( v \) receives \( \frac{1}{24} \) from \( C_2 \) by Discharging Rules 3a-3c.

\[ \square \]

**Claim 7.3.** If a poor 1-cluster, \( v \), is not very poor, then \( f(v) = \frac{5}{12} \).

**Proof.** Let \( v \in D_1^p \setminus D_1^{vp} \). Then, \( v \) must send charge to all 3 neighbors, each of which has exactly 2 neighbors in \( D \). That is, \( f_1(v) = 1 - 3 \cdot \frac{5}{12} = \frac{9}{24} \). But \( v \) is not very poor; therefore, \( v \) is distance-2 from a \( 3^+ \)-cluster or non-poor 1-cluster or distance-3 from a closed 3-cluster or \( 4^+ \)-cluster. If \( v \) is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or \( 4^+ \)-cluster, then \( v \) receives \( \frac{1}{24} \) by Rules 3a-3c (Proposition 7.2); if not, then \( v \) receives charge from a distance-2 non-poor 1-cluster by Rules 3d-3e. Thus, we have shown that \( v \) will receive \( \frac{1}{24} \) from a nearby cluster. Therefore, \( f_3(v) = f_1(v) + \frac{1}{24} = \frac{9}{24} + \frac{1}{24} = \frac{5}{12} \). If \( v \) is distance-2 from a very poor 1-cluster, \( w \), then \( v \) may need to receive charge from a nearby cluster and send charge to \( w \) by Rules 6-7. If \( w \) is in a symmetric orientation, then \( v \) is not distance-2 from any very poor 1-cluster other than \( w \) (Corollary 6.19). If \( w \) is in an asymmetric orientation and \( v \) is in the \( u \)-position or \( w \)-position of \( w \), then \( v \) is not distance-2 from any very poor 1-cluster other than \( w \) (Corollary 6.22). Thus, if Rules 6-7 require \( v \) to receive and send charge, then \( v \) must only send charge to one very poor 1-cluster. Then, if Rule 6 is applicable, \( v \) receives \( \frac{1}{24} \) and sends \( \frac{1}{24} \). The same is true of Rules 7a-7e. Therefore, Rules 6-7 have no effect on the final charge of \( v \). Therefore, \( f(v) = f_7(v) = f_3(v) = \frac{5}{12} \). \[ \square \]

**Claim 7.4.** For every non-poor 1-cluster, \( v \), \( f(v) \geq \frac{5}{12} \).

**Proof.** Let \( v \in D_1^{np} \). Then, \( v \) must send charge to all 3 of its neighbors, at least one of which has 3 neighbors in \( D \). Therefore, \( f_1(v) \geq 1 - (2 \cdot \frac{5}{12} + \frac{5}{12} \cdot \frac{5}{3}) = \frac{4}{9} \).

Suppose \( v \) is in a group of non-poor 1-clusters, \( H \). Then, \( f_1(H) \geq 3 \cdot f_1(v) = \frac{4}{3} \). By Discharging Rule 3e, \( H \) may need to send charge to distance-2 poor 1-clusters that
do not receive charge by Rules 3a-3d; therefore, $H$ must send charge to a distance-2 poor 1-cluster, $u$, only if $u$ is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4-+cluster (Proposition 7.2). Then, $H$ must send charge to at most 2 distance-2 poor 1-clusters (Lemma 6.2). Therefore, $f(H) = f_3(H) \geq f_1(H) - 2 \cdot \frac{1}{24} = \frac{4}{3} - \frac{1}{12} = \frac{15}{12} = 3 \cdot \frac{5}{12}$. Therefore, $f(v) \geq \frac{5}{12}$.

Now, suppose $v$ shares a one-third vertex with a non-poor 1-cluster, $w$, and a 3+-cluster, $C$. By Discharging Rule 2a, each of $v$ and $w$ receives $\frac{1}{72}$ from $C$. And, by Discharging Rule 3d, each of $v$ and $w$ may need to send charge to distance-2 poor 1-clusters that do not receive charge by Rules 3a-3c; therefore, $v$ and $w$ send charge to a poor 1-cluster, $u$, only if $u$ is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4-+cluster (Proposition 7.2). Then, each of $v$ and $w$ has at most one distance-2 poor 1-cluster that does not receive charge by Rules 3a-3c (Lemma 6.3). Therefore, $f(v) = f_3(v) \geq f_2(v) - \frac{1}{24} = (f_1(v) + \frac{1}{72}) - \frac{1}{24} \geq (\frac{4}{9} + \frac{1}{72}) - \frac{1}{12} = \frac{5}{12}$.

Now, suppose $v$ shares a one-third vertex with 3+-clusters only. Then, by Rule 2b, $v$ receives $\frac{1}{18}$ from these 3+-clusters. By Rule 3d, $v$ may need to send charge to distance-2 poor 1-clusters. However, $v$ has at most 2 distance-2 poor 1-clusters (Lemma 6.4). Therefore, $f(v) = f_3(v) \geq f_2(v) - 2 \cdot \frac{1}{24} = (f_1(v) + \frac{1}{18}) - \frac{1}{12} \geq (\frac{4}{9} + \frac{1}{18}) - \frac{1}{12} = \frac{5}{12}$. 

\begin{claim}
For every very poor 1-cluster, $v$, $f(v) \geq \frac{5}{12}$.
\end{claim}

\begin{proof}
Let $v \in D_{1v}^p$. We saw above that $f_1(v) = \frac{9}{24}$.

If $v$ is in a symmetric orientation, then $v$ is in a head position of 3 open 3-clusters, $C_1$, $C_2$ and $C_3$ (Lemma 6.17). If any of $C_1$, $C_2$ and $C_3$ is crowded, say $C_1$, then $v$ receives $\frac{1}{24}$ from $C_1$. Then, $f_3(v) = f_1(v) + \frac{1}{24} = \frac{5}{12}$. Let $a$, $b$ and $c$ be the vertices in $D_1$ at distance-2 from $v$. Since $v \in D_1^p$, we must have $a, b, c \in D_1^p$. However, exactly one of $a$, $b$ and $c$ is in a shoulder position of each of $C_1$, $C_2$ and $C_3$ (Lemma 6.17);
therefore $a, b, c \notin D^{vp}_1$. Then, Rules 6-7 do not require $v$ to send any charge; therefore, $f(v) \geq f_5(v) \geq \frac{5}{12}$. Now, if each of $C_1$, $C_2$ and $C_3$ is uncrowded, then each of $a$, $b$ and $c$ is distance-3 from a closed 3-cluster or $4^+$-cluster (Lemma 6.17). Discharging Rule 5 is not applicable, but, by Discharging Rule 6, each of $a$, $b$ and $c$ receives $\frac{1}{24}$ from a distance-3 closed 3-cluster or $4^+$-cluster and sends $\frac{1}{24}$ to $v$. Rule 7 is not applicable; therefore $f(v) = f_6(v) = f_1(v) + 3 \cdot \frac{1}{24} = \frac{9}{24} + \frac{1}{8} = \frac{1}{2} > \frac{5}{12}$.

Now, suppose $v$ is in an asymmetric orientation. Let $u$, $w$ and $x$ be in the $u$-position, $w$-position and $x$-position, respectively. Now, $u, w \notin D^{vp}_1$ (Corollary 6.22); and if $x \in D^{vp}_1$, then $v$ is in the $x$-position of $x$ (Corollary 6.23). Therefore, none of the Discharging Rules requires $v$ to send charge. Now, by Lemma 6.20, $v$ and $x$ are in the head positions of an open 3-cluster, $C_0$, and one of the following holds:

- $C_0$ is crowded. In this case, $v$ receives $\frac{1}{24}$ from $C_0$ by Rule 5.

- There exists a closed 3-cluster or $4^+$-cluster at distance-3 from $w$. In this case, $v$ receives $\frac{1}{24}$ from $w$ by Rule 7a.

- There exists an open 3-cluster, $C$, such that the tail position of $C$ is in $D$ and $u$, $v$ or $w$ is in the hand position on the finless side of $C$. In this case, $v$ receives $\frac{1}{24}$ from $u$, $w$ or $C$ by Rule 7b.

- There exists a leaf, $\ell$, of a $4^+$-cluster, $C$, at distance-2 from $u$ such that $u$ is not in a one-turn position or a backwards position of a linear 4-cluster or a curved 4-cluster, respectively, and $u$ is the only vertex in $D \setminus C$ at distance-2 from $\ell$. In this case, $v$ receives $\frac{1}{24}$ from $u$ by Rule 7c.

- There exists a leaf, $\ell$, of a closed 3-cluster, $C$, at distance-2 from $u$ such that $u$ is in a foot or arm position and $u$ is the only vertex in $D \setminus C$ at distance-2 from $\ell$. Furthermore, either $C$ is type-2 paired or $C$ has at most 6 nearby poor 1-clusters. In this case, $v$ receives $\frac{1}{24}$ from $u$ by Rule 7c.
There exists an open 3-cluster, \( C \), such that \( v \) or \( w \) is in a hand position of \( C \) and the hand and arm positions on the other side of \( C \) are in \( D \). In this case, \( v \) receives \( \frac{1}{24} \) from \( C \) or \( w \) by Rule 7d.

There exists an open 3-cluster, \( C \), such that \( u \) is in a foot position and \( C \) is type-1 paired on top. In this case, \( v \) receives \( \frac{1}{24} \) from \( u \) by Rule 7e.

Therefore, \( v \) receives at least \( \frac{1}{24} \) from a nearby cluster by Discharging Rules 5 and 7.

Therefore, \( f(v) = f_7(v) \geq f_1(v) + \frac{1}{24} = \frac{9}{24} + \frac{1}{24} = \frac{5}{12} \).

### 7.3 3-Clusters

**Proposition 7.6.** If an open 3-cluster, \( C \), is neither type-1 paired nor type-2 paired and \( f_2(C) \geq \frac{34 + P(C)}{24} \), then \( f(C) \geq 3 \cdot \frac{5}{12} \).

**Proof.** Let \( C \) be an open 3-cluster that is neither type-1 nor type-2 paired. Note that Rule 3b requires \( C \) to send at most \( P(C) \cdot \frac{1}{24} \), Rule 5 requires \( C \) to send at most \( \frac{2}{24} \) and Rules 7b and 7d each require \( C \) to send at most \( \frac{1}{24} \) (Lemma 6.5); therefore, \( C \) sends at most \( \frac{P(C) + 4}{24} \) by Rules 3-7. Therefore, if \( f_2(C) \geq \frac{34 + P(C)}{24} \), then \( f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12} \).

**Claim 7.7.** For every open 3-cluster, \( C \), \( f(C) \geq 3 \cdot \frac{5}{12} \).

**Proof.** Consider an open 3-cluster, \( C \). Then, by Discharging Rule 1, the middle vertex of \( C \) must send \( \frac{5}{12} \). Each of the leaf vertices has at least one distance-2 vertex in \( D \setminus C \) (Proposition 4.6); therefore, each of the leaf vertices must send at most \( \frac{5}{12} + \frac{5}{12} = \frac{15}{24} \) by Rule 1. Thus, \( f_1(C) \geq 3 - \frac{5}{12} - 2 \cdot \frac{15}{24} = \frac{4}{3} \).

First, suppose \( C \) is type-1 paired on top (see Figure 5.3b). Now, the shoulder position of \( C \) is in \( D \) or the arm position is in \( D \) (Proposition 4.6). Suppose exactly one is in \( D \), and let \( v \) be this vertex. Then, \( C \) has exactly 2 distance-2 vertices in \( D \); therefore, \( f_1(C) = \frac{4}{3} \) and Rule 2 does not apply. By Rule 3b, \( C \) does not send
charge to $v$ even if $v \in D^p_1$, but $C$ may need to send charge to the poor 1-cluster, $u$, in the foot position. Therefore, $f_3(C) \geq f_1(C) - \frac{1}{24} = \frac{31}{24}$. Now, $C$ is not type-2 paired (Corollary 5.10); therefore, Rule 4 does not apply. Since at least one of the shoulder positions of $C$ is not in $D$, Rule 5 does not apply (Lemma 6.24). Since the tail position of a paired 3-cluster is not in $D$, Rule 7b does not apply. At least one of the hand positions of $C$ is not in $D$; therefore, Rule 7d does not apply. By Rule 7e, $C$ may need to send $\frac{1}{24}$ to $u$; therefore, $f_3(C) = f_7(C) \geq f_3(C) - \frac{1}{24} \geq \frac{31}{24} - \frac{1}{24} = 3 \cdot \frac{5}{12}$.

Now, suppose $C$ is type-1 paired on top and both the shoulder position and the arm position of $C$ are in $D$. Then, $f_1(C) = 3 - 3 \cdot \frac{5}{12} - \frac{5}{12} = \frac{101}{72}$. Let $v$ and $w$ be the shoulder and arm positions of $C$, respectively. If $v$ and $w$ are 1-clusters, then $C$ sends $\frac{1}{24}$ to both by Discharging Rule 2a. If exactly one of $v$ and $w$ is a 1-cluster, say $v$, then $C$ sends $\frac{1}{36}$ to $v$ by Rule 2b. In both cases, $C$ sends a total of $\frac{1}{36}$; thus, $f_3(v) \geq f_1(v) - \frac{1}{36} = \frac{31}{24}$. By Rule 3b, $C$ may need to send $\frac{1}{24}$ to the poor 1-cluster, $u$, in the foot position; therefore, $f_3 \geq \frac{32}{24}$. Again, Rules 4-7d do not apply. By Rule 7e, $C$ may need to send $\frac{1}{24}$ to $u$; therefore, $f(C) = f_7(C) \geq f_3(C) - \frac{1}{24} \geq \frac{32}{24} - \frac{1}{24} = \frac{31}{24} > 3 \cdot \frac{5}{12}$.

Suppose $C$ is type-2 paired (see Figure 5.3c). One of the shoulder positions of $C$ is in $D$. On the other side of $C$, the shoulder position is in $D$ or the arm position is in $D$ (Proposition 4.6). Let $v$ and $w$ be the shoulder and arm positions of $C$, respectively. Then, $w \in D_{1np}$ or $w \in D^p_1$ or $w \in D_{3+}$ or $w \notin D$. If $w \in D_{1np}$ and $v \notin D$, then $f_1(C) = \frac{32}{24}$. Now, $C$ is not type-1 paired (Corollary 5.10); therefore, $C$ may need to send charge to the shoulder position on the side opposite $w$ by Rule 3b. Then, $f_3(C) \geq \frac{31}{24}$. By Rule 4, $C$ sends $\frac{1}{24}$ to the closed 3-cluster with which $C$ is type-2 paired. Then, $f_4(C) \geq \frac{30}{24}$. Rule 5 does not apply (Lemma 6.24). And Rules 7b, 7c and 7e do not apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \in D_{1np}$ and $v \in D$, then $f_1(C) = \frac{101}{72}$. By Rule 2, $C$ may need to send $\frac{1}{36}$ to distance-2 non-poor 1-clusters. Then, $f_2(C) \geq \frac{33}{24}$. By Rule 3b, $C$ sends at most $\frac{1}{24}$. By Rule 4, $C$ sends $\frac{1}{24}$. Since $v \in D_{1np} \cup D_{3+}$, at least one of the head positions of $C$ is not a very poor
1-cluster; therefore, $C$ sends at most $\frac{1}{24}$ by Rule 5. Rules 6-7 do not apply. Therefore, 
\[
f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}.
\]
Now, if $w \in D_1^p$, then $v \notin D$ (Corollary 5.2) and $f_1(C) = \frac{32}{24}$. Then, $C$ sends at most $\frac{1}{24}$ by Rule 3b, and $C$ sends $\frac{1}{24}$ by Rule 4. Rules 5-7 do not apply. Therefore, 
\[
f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}.
\]
If $w \notin D_3^p$ and $v \notin D$, then $f_1(C) = \frac{32}{24}$. By Rule 3b, $C$ sends at most $\frac{1}{24}$. If the hand position adjacent to $w$ is in $D$, then $C$ does not send charge by Rule 4 but $C$ may need to send charge by Rule 7d. If the hand position adjacent to $w$ is not in $D$, then $C$ must send charge by Rule 4 but not by Rule 7d. Therefore, $C$ sends charge by at most one of Rules 4 and 7d. No other rules apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. Then, $C$ sends at most $\frac{1}{24}$ by Rule 5. Again, $C$ sends charge by at most one of Rules 4 and 7d. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \notin D$, then $v \in D$ (Proposition 4.6). Then, $f_1(C) = \frac{32}{24}$ and both shoulder positions of $C$ are in $D$. If both are poor 1-clusters, then $C$ sends at most $\frac{1}{24}$ by Rule 3b and sends no charge by other rules. Therefore, 
\[
f(C) \geq 3 \cdot \frac{5}{12}.
\]
If one is not a poor 1-cluster, then at most one of the head positions of $C$ is a very poor 1-cluster. Therefore, $C$ sends at most $\frac{1}{24}$ by Rule 5. Again, $C$ sends charge by at most one of Rules 4 and 7d. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \notin D$, then $v \in D$ (Proposition 4.6). Then, 
\[
f_1(C) = \frac{32}{24} \text{ and both shoulder positions of } C \text{ are in } D. \]
If both are poor 1-clusters, then $C$ sends at most $\frac{2}{24}$ by Rule 3b and sends no charge by other rules. Therefore, 
\[
f(C) \geq 3 \cdot \frac{5}{12}.
\]
If neither is a poor 1-cluster, then $C$ does not send charge by any rule; therefore, $f(C) = \frac{32}{24} > 3 \cdot \frac{5}{12}$.

Suppose $C$ is neither type-1 paired nor type-2 paired and $P(C) = 2$. Then, 4 and 7e do not apply and $f_1(C) = \frac{32}{24}$. Since $P(C) = 2$, there exists no one-third vertex adjacent to $C$; therefore, Rule 2 does not apply. Now, $C$ may need to send charge by Rules 3b, 5, 7b, and 7d. If $C$ must send charge by Rule 5, then both shoulder positions are in $D$ (Lemma 6.24); therefore, the arm positions and the tail position of $C$ are not in $D$ and, hence, Rules 7b and 7d do not apply. If $C$ must send charge by Rule 7b, then the tail position of $C$ is in $D$; therefore, the arm and shoulder positions
of $C$ are not in $D$ and, hence, Rules 5 and 7d do not apply. If $C$ must send charge by Rule 7d, then an arm position of $C$ is in $D$. Since each leaf must have at least one distance-2 vertex in $D$ (Proposition 4.6) and $P(C) = 2$, the tail position of $C$ is not in $D$ and at least one of the shoulder positions is not in $D$; therefore, Rules 5 and 7b do not apply. Thus, we have shown that $C$ sends charge by at most one of Rules 5, 7b and 7d. First, suppose $C$ sends by none of Rules 5, 7b and 7d. Then, $C$ is uncrowded and the tail position of $C$ is not in $D$; therefore, $C$ has exactly 2 distance-2 poor 1-clusters. Then, $C$ sends $\frac{2}{21}$ by Rule 3b and no charge by any other rule; therefore, $f(C) = \frac{30}{21}$. Suppose $C$ sends charge by Rule 5. Then, both shoulder positions of $C$ are in $D$ (Lemma 6.24), and $C$ is crowded; therefore, one of the shoulder positions is not a poor 1-cluster. But then one of the head positions of $C$ is distance-2 from a non-poor 1-cluster or $3^+$-cluster; therefore, there exists only one very poor 1-cluster in a head position of $C$. Then, $C$ sends $\frac{1}{21}$ by 3b and $\frac{1}{21}$ by Rule 5; therefore, $f(C) = \frac{30}{21}$. Now suppose $C$ sends charge by Rule 7b. Then, the tail position of $C$ is in $D$. Since $P(C) = 2$, the tail position is the only vertex in $D$ at distance-2 from $C$; therefore, there exists at most one distance-2 poor 1-cluster. Then, $f_3(C) \geq \frac{31}{24}$ and Rule 7b requires $C$ to send at most $\frac{1}{21}$; therefore, $f(C) \geq \frac{30}{21}$. Finally, suppose $C$ sends charge by Rule 7d. Then, the hand and arm positions on one side of $C$ are in $D$; therefore, at least one of the vertices in $D$ at distance-2 from $C$ is not a poor 1-cluster. Then, $f_3(C) \geq \frac{31}{24}$ and, by Rule 7d, $C$ sends at most $\frac{1}{21}$; therefore, $f(C) \geq \frac{30}{24}$.

Suppose $C$ is neither type-1 paired nor type-2 paired and $P(C) = 3$. If $C$ is adjacent to no one-third vertices, then $f_1(C) = 3 - 3 \cdot \frac{5}{24} - 2 \cdot \frac{5}{12} = \frac{37}{24}$; therefore, $f(C) \geq \frac{30}{21}$ (Proposition 7.6). If $C$ is adjacent to a one-third vertex, then $f_1(C) = \frac{101}{72}$ and $f_2(C) \geq \frac{99}{72} = \frac{33}{24}$. Since $C$ is adjacent to a one-third vertex and $P(C) = 3$, there exists at most one distance-2 poor 1-cluster; therefore, $f_3(C) \geq \frac{32}{21}$. First, suppose the tail position of $C$ is in $D$. Then, a foot position is also in $D$ and no other distance-2 vertices are in $D$; therefore, Rules 5 and 7d do not apply. By Rule 7b, $C$ sends at
most $\frac{1}{24}$; therefore, $f(C) \geq \frac{31}{24}$. Now, suppose the tail position of $C$ is not in $D$; therefore, Rule 7b does not apply. If $C$ does not send charge by Rule 5, then $C$ sends at most by Rule 7d and $f(C) \geq \frac{31}{24}$. If $C$ sends charge by Rule 5, then both shoulder positions of $C$ are in $D$ (Lemma 6.24). However, $C$ is adjacent to a one-third vertex; therefore, one of the shoulder positions is not a poor 1-cluster. Then, at most one of the head positions is a very poor 1-cluster. Therefore, $C$ sends at most $\frac{1}{24}$ by Rule 5 and at most $\frac{1}{24}$ by Rule 7d. Therefore, $f(C) \geq \frac{30}{24}$.

Suppose $C$ is neither type-1 paired nor type-2 paired and $P(C) = 4$. First, suppose $C$ is adjacent to no one-third vertices. Then, $f_1(C) = \frac{42}{24}$; therefore, $f(C) \geq \frac{30}{24}$ (Proposition 7.6). Now, suppose $C$ is adjacent to exactly one one-third vertex. Then, $f_1(C) = \frac{116}{72}$. By Rule 2, $C$ sends at most $\frac{1}{36}$; therefore, $f_2(C) = \frac{144}{72} = \frac{38}{24}$. Therefore, $f(C) \geq \frac{30}{24}$ (Proposition 7.6). Now, suppose $C$ is adjacent to exactly 2 one-third vertices. Then, $f_1(C) = \frac{53}{36}$. By Rule 2, $C$ sends at most $2 \cdot \frac{1}{36}$; therefore, $f_2(C) \geq \frac{34}{24}$. Since $C$ is adjacent to 2 one-third vertices and $P(C) = 4$, there exist no distance-2 poor 1-clusters; therefore, $f_3(C) \geq f_2(C) \geq \frac{34}{24}$. In total, Rules 5-7 require $C$ to send at most $\frac{4}{24}$; therefore, $f(C) \geq \frac{30}{24}$.

Suppose $C$ is neither type-1 paired nor type-2 paired and $P(C) \geq 5$. Then, $f_2(C) \geq \frac{30}{24}$. Now, an open 3-cluster has at most 4 distance-2 poor 1-clusters. Therefore, $C$ sends at most $\frac{4}{24}$ by Rule 3b. By Rules 5-7, $C$ sends at most $\frac{4}{24}$; therefore, $f(C) \geq \frac{31}{24}$. □

**Proposition 7.8.** A closed 3-cluster or 4+-cluster sends charge to a distance-3 poor 1-cluster by at most one of Rules 3c, 6 and 7a.

**Proof.** Let $C_1$ be closed 3-cluster or 4+-cluster. If a poor 1-cluster, $v$, is distance-2 from a very poor 1-cluster in a symmetric orientation, then $v$ is distance-2 from exactly one very poor 1-cluster. If $v$ is in the $u$-position or $w$-position of a very poor 1-cluster in an asymmetric orientation, then $v$ is distance-2 from exactly one
very poor 1-cluster. Therefore, $C_1$ sends charge to a poor 1-cluster by at most one of Rules 6 and 7a. If $C_1$ sends charge to a poor 1-cluster, $a$, by Rule 6, then $a$ is distance-2 from a very poor 1-cluster in a symmetric orientation and in a shoulder position of an open 3-cluster, $C_2$ (Lemma 6.17). Now, $C_2$ is not type-1 paired on top (Corollary 6.18); therefore, $a$ receives charge from $C_2$ by Rule 3b and not from $C_1$ by Rule 3c. Therefore, $C_1$ sends charge to $a$ by at most one of Rules 3c and 6. If $C_1$ sends charge to a poor 1-cluster, $w$, by Rule 7a, then $w$ is in the $w$-position of a very poor 1-cluster in an asymmetric orientation; therefore, $w$ is in the shoulder position of an open 3-cluster, $C_0$ (Lemma 6.20). Now, $C_0$ is not type-1 paired on top (Corollary 6.21); therefore, $w$ receives charge from $C_0$ by Rule 3b and not from $C_1$ by Rule 3c. Therefore, $C_1$ sends charge to $w$ by at most one of Rules 3c and 7a.

**Proposition 7.9.** If $C$ is a closed 3-cluster and $f_2(C) \geq \frac{43}{24}$, then $f(C) \geq 3 \cdot \frac{5}{12}$.

**Proof.** By Rules 3c, 6 and 7a, $C$ sends at most $\frac{1}{24}$ to each distance-3 poor 1-cluster (Proposition 7.8). By Rule 3a, $C$ sends at most $\frac{1}{24}$ to each distance-2 poor 1-cluster. Now, $C$ has at most 11 nearby poor 1-clusters (Lemma 6.16); therefore, by Rules 3-7a, $C$ sends at most $\frac{11}{24}$. By Rule 7c, $C$ sends at most $\frac{2}{24}$ to distance-2 poor 1-clusters. Therefore, $C$ sends at most $\frac{13}{24}$ by Rules 3-7. □

**Corollary 7.10.** A closed 3-cluster sends at most $\frac{11}{24}$ by Rules 3-7a and at most $\frac{2}{24}$ by Rule 7c.

**Claim 7.11.** For every closed 3-cluster, $C$, $f(C) \geq 3 \cdot \frac{5}{12}$.

**Proof.** Consider a closed 3-cluster, $C_1$, and let $P(C_1) = 3$. Then, $f_1(C_1) = \frac{37}{24}$. By Lemma 6.6, $C_1$ has at most 8 nearby poor 1-clusters; however, if $C_1$ has 8 such clusters, at least one of the poor 1-clusters at distance-3, $v$, is distance-2 from another $3^+$-cluster, $C_2$, such that
(a) if \( C_2 \) is an open 3-cluster, then \( v \) is not in a shoulder position; 

(b) if \( C_2 \) is an open 3-cluster and \( v \) is in an arm position, then \( C_2 \) is not type-1 paired; if \( C_2 \) is type-2 paired, then \( C_1 \) is type-2 paired with \( C_2 \).

Therefore, \( v \) receives charge from \( C_2 \) by Rules 3a-3b and not from \( C_1 \) by Rule 3c; additionally, \( C_1 \) does not send charge to \( v \) by Rules 6 and 7a. Then, \( C_1 \) sends at most \( \frac{7}{24} \) by Rules 3, 6 and 7a (Proposition 7.8). If \( C_1 \) sends charge by Rule 7c, then \( C_1 \) has at most 6 nearby poor 1-clusters or \( C_1 \) is type-2 paired. A poor 1-cluster, \( u \), receives charge by Rule 7c only if \( u \) is the only vertex in \( D \setminus C_1 \) at distance-2 from a leaf of \( C_1 \) and \( u \) is not in a shoulder or tail position; therefore, \( C_1 \) sends at most \( \frac{2}{24} \) by Rule 7c. If \( C_1 \) sends \( \frac{2}{24} \) by Rule 7c, then the shoulder positions and the tail position of \( C_1 \) are not in \( D \); therefore, \( C_1 \) has at most 5 nearby poor 1-clusters (Lemma 6.6). Then, \( C \) sends at most \( \frac{5}{24} \) by Rules 3, 6 and 7a (Proposition 7.8) and \( \frac{2}{24} \) by Rule 7c, and \( f(C_1) \geq \frac{30}{24} \). If \( C_1 \) sends \( \frac{1}{24} \) by Rule 7c and \( C_1 \) has at most 6 nearby poor 1-clusters, then \( C_1 \) sends at most \( \frac{6}{24} \) by Rules 3, 6 and 7a (Proposition 7.8) and \( \frac{1}{24} \) by Rule 7c, and \( f(C_1) \geq \frac{30}{24} \). If \( C_1 \) sends \( \frac{1}{24} \) by Rule 7c and \( C_1 \) is type-2 paired with the open 3-cluster, \( C_2 \), then the argument is identical to the previous case unless \( C_1 \) has 7 nearby poor 1-clusters. In this case, the arm position of \( C_2 \) is in \( D \) and the hand position on the same side is not in \( D \) (Lemma 6.26); therefore, \( C_1 \) receives \( \frac{1}{24} \) from \( C_2 \) by Rule 4. Then, \( C_1 \) sends at most \( \frac{7}{24} \) by Rules 3, 6 and 7a (Proposition 7.8) and \( \frac{1}{24} \) by Rule 7c; however, \( C_1 \) receives \( \frac{1}{24} \) by Rule 4 and, therefore, \( f(C_1) \geq \frac{30}{24} \).

Consider a closed 3-cluster, \( C \), and let \( P(C) = 4 \). Then, either \( C \) is adjacent to no one-third vertices or \( C \) is adjacent to exactly one one-third vertex. In the former case, \( f_1(C) = \frac{42}{24} \); and since \( C \) is not adjacent to any one-third vertices, \( f_2(C) = \frac{42}{24} \). Now, \( C \) sends at most \( \frac{11}{24} \) by Rules 3-7a and at most \( \frac{2}{24} \) by Rule 7c (Corollary 7.10). Since \( C \) is adjacent to no one-third vertices, \( C \) is closed by a single vertex; therefore, since \( P(C) = 4 \), one of the leaves of \( C \) is distance-2 from 2 vertices in \( D \setminus C \); therefore, \( C \) sends at most \( \frac{1}{24} \) by Rule 7c. Therefore, \( f(C) \geq \frac{30}{24} \). Now suppose \( C \) is adjacent
to a one-third vertex. Then, \( f_2(C) \geq \frac{38}{24} \). Now, \( C \) has at most 8 nearby poor 1-clusters (Lemma 6.7); therefore, \( C \) sends at most \( \frac{8}{24} \) by Rules 3-7a (Proposition 7.8). Therefore, if \( C \) sends no charge by Rule 7c, then \( f(C) \geq \frac{30}{24} \). If \( C \) sends \( \frac{1}{24} \) by Rule 7c, then an arm position or foot position of \( C \) is a poor 1-cluster. But then \( C \) has at most 7 nearby poor 1-clusters (Lemma 6.7) and \( f_7a \geq \frac{31}{24} \); therefore, \( f(C) \geq \frac{30}{24} \). If \( C \) sends \( \frac{2}{24} \) by Rule 7c, then 2 arm or foot positions are poor 1-clusters. But then \( C \) has at most 6 nearby poor 1-clusters (Lemma 6.7) and \( f_7a \geq \frac{32}{24} \); therefore, \( f(C) \geq \frac{30}{24} \).

Let \( P(C) = 5 \). Then, either \( C \) is adjacent to no one-third vertices, one one-third vertex or 2 one-third vertices. In the first case, \( f_2(C) = \frac{47}{24} \); therefore, \( f(C) \geq \frac{30}{24} \) (Proposition 7.9). In the second case, \( f_2(C) \geq \frac{43}{24} \); therefore, \( f(C) \geq \frac{30}{24} \) (Proposition 7.9). In the last case, \( f_2(C) \geq \frac{39}{24} \). Now, \( C \) has at most 11 nearby clusters (Lemma 6.16). Since \( C \) is adjacent to 2 one-third vertices, at least 3 of these clusters are not poor 1-clusters; additionally, at least one of the leaves of \( C \) is distance-2 from more than one vertex in \( D \setminus C \). Therefore, \( C \) sends at most \( \frac{8}{24} \) by Rules 3-7a (Proposition 7.8) and at most \( \frac{1}{24} \) by Rule 7c; therefore, \( f(C) \geq \frac{30}{24} \).

Let \( P(C) \geq 6 \). If \( f_2(C) \geq \frac{43}{24} \), then \( f(C) \geq \frac{30}{24} \) (Proposition 7.9). If \( f_2(C) < \frac{43}{24} \), then \( C \) is adjacent to at least 3 one-third vertices. Therefore, at least 5 of the clusters nearby \( C \) are not poor 1-clusters. Therefore, \( C \) has at most 6 nearby poor 1-clusters (Lemma 6.16). Then, \( C \) sends at most \( \frac{6}{24} \) by Rules 3-7a (Proposition 7.8). By Rule 7c, \( C \) sends at most \( \frac{2}{24} \). Therefore, \( C \) sends at most \( \frac{8}{24} \). But if \( P(C) \geq 6 \), then \( f_2(C) \geq \frac{40}{24} \); therefore, \( f(C) \geq \frac{32}{24} \).

\[ \alpha_i = |\{v \in C : d_C(v) = i\}| \]

\[ \boxed{7.4 \ 4^+\text{-Clusters}} \]

Now we begin our discussion of 4\(^+\)-clusters. For \( k \geq 4 \), let \( C \) be a \( k \)-cluster, and let \( v \) be a vertex in \( C \). Then, \( d_C(v) \in \{1, 2, 3\} \). Let

\[ \alpha_i = |\{v \in C : d_C(v) = i\}| \]
Now, $C$ has at most $k + 8$ nearby poor 1-clusters (Lemma 6.16); therefore, $C$ sends at most $\frac{k + 8}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{1}{24}$ for each leaf of $C$. Now, the number of leaves of $C$ is $\alpha_1$; therefore, $C$ sends at most $\frac{1}{24}\alpha_1$ by Rule 7c. Rules 1 and 2 are the only others by which $C$ may need to send charge; therefore, $f(C) \geq f_2(C) - \frac{1}{24}[(k + 8) + \alpha_1]$. Now, $f_2(C)$ is minimal when $P(C) = 2$; that is, $f_2(C) \geq k - (\frac{5}{12} + \frac{5}{24})\alpha_1 - \frac{5}{12}\alpha_2$. Let $F(C) = f(C) - \frac{5}{12}k$. Then,

$$F(C) \geq \left[ k - \left( \frac{5}{12} + \frac{5}{24} \right)\alpha_1 - \frac{5}{12}\alpha_2 \right] - \frac{1}{24}[(k + 8) + \alpha_1] - \frac{5}{12}k$$

Now, $k = \alpha_1 + \alpha_2 + \alpha_3$. Then, substituting and simplifying,

$$F(C) \geq \frac{1}{24}(-3\alpha_1 + 3\alpha_2 + 13\alpha_3 - 8)$$

Now, $\Delta(C) = 3$; therefore, $\alpha_1 \leq \alpha_3 + 2$. Then,

$$F(C) \geq \frac{1}{24}[-3(\alpha_3 + 2) + 3\alpha_2 + 13\alpha_3 - 8] = \frac{1}{24}(3\alpha_2 + 10\alpha_3 - 14) \quad (7.1)$$

Now, $F(C) < 0$ if, and only if, $f(C) < \frac{5}{12}k$. Let

$$A = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (0, 1), (1, 1)\}$$

If $F(C) < 0$, then $(\alpha_2, \alpha_3) \in A$. That is, for all $(\alpha_2, \alpha_3) \notin A$, Equation 7.1 implies $f(C) \geq \frac{5}{12}k$. Therefore, we have only left to consider the cases in which $(\alpha_2, \alpha_3) \in A$.

If $(\alpha_2, \alpha_3) \in \{(0, 0), (1, 0)\}$, then $C \in K_1 \cup K_3$. But we assumed $C \in K_{4+}$; therefore, we need not consider this case. If $(\alpha_2, \alpha_3) \in \{(2, 0), (0, 1)\}$, then $C \in K_4$; we consider this case in Claim 7.12. If $(\alpha_2, \alpha_3) \in \{(3, 0), (1, 1)\}$, then $C \in K_5$; we consider this case in Claim 7.13. Finally, if $(\alpha_2, \alpha_3) = (4, 0)$, then $C \in \{L \in K_6 : \Delta(L) = 2\}$; we consider this case in Claim 7.14.
Claim 7.12. For every 4-cluster, \( C \), \( f(C) \geq 4 \cdot \frac{5}{12} \).

Proof. First, consider a linear 4-cluster, \( C \), and let \( P(C) = 2 \). Then, \( f_1(C) = \frac{46}{24} \). Rule 2 does not apply; therefore, \( f_2(C) = \frac{46}{24} \). First suppose \( C \) sends no charge by Rule 7c. Then, \( C \) has at most 8 nearby poor 1-clusters; however, if \( C \) has \( k \) nearby poor 1-clusters, where \( k > 6 \), then \( k - 6 \) of the distance-3 poor 1-clusters are stealable (Lemma 6.8) – that is, \( k - 6 \) of the nearby poor 1-clusters will receive charge from other 3+-clusters by Rules 3a-3b and not from \( C \) by Rules 3c, 6 or 7a. Therefore, \( C \) sends charge to at most 6 nearby poor 1-clusters. Then, \( C \) sends at most \( \frac{6}{24} \) (Proposition 7.8) and, therefore, \( f(C) \geq \frac{40}{24} = 4 \cdot \frac{5}{12} \). Now suppose \( C \) sends \( \frac{1}{24} \) by Rule 7c. Then, one one-turn position is not in \( D \); therefore, \( C \) has at most 6 nearby poor 1-clusters and if \( C \) has exactly 6 such clusters then at least one of the distance-3 poor 1-clusters is stealable (Lemma 6.8) – that is, at least one of the distance-3 poor 1-clusters will receive charge by Rules 3a-3b and not from \( C \) by Rules 3c, 6 or 7a. Therefore, \( C \) sends charge to at most 5 nearby poor 1-clusters by Rules 3, 6 and 7a. Then, \( C \) sends at most \( \frac{5}{24} \) by Rules 3, 6 and 7a (Proposition 7.8) and \( \frac{1}{24} \) by Rule 7c; therefore, \( f(C) \geq \frac{40}{24} \). Finally, suppose \( C \) sends \( \frac{2}{24} \) by Rule 7c. Then, neither one-turn position is in \( D \); therefore, \( C \) has at most 4 nearby poor 1-clusters. Then, \( C \) sends at most \( \frac{4}{24} \) by Rules 3, 6 and 7a (Proposition 7.8) and \( \frac{1}{24} \) by Rule 7c; therefore, \( f(C) \geq \frac{40}{24} \).

Let \( P(C) = 3 \). First, suppose \( C \) is adjacent to no one-third vertices. Then, \( f_2(C) = \frac{51}{24} \). Now, \( C \) has at most 9 nearby poor 1-clusters (Lemma 6.9); therefore, \( C \) sends at most \( \frac{9}{24} \) by Rules 3-7a (Proposition 7.8). And \( C \) sends at most \( \frac{2}{24} \) by Rule 7c; therefore, \( f(C) \geq \frac{40}{24} \). Now, suppose \( C \) is adjacent to a one-third vertex, \( v_{\frac{1}{3}} \). Then, \( f_2(C) \geq \frac{47}{24} \). Now, \( C \) has at most 6 nearby poor 1-clusters (Lemma 6.9); therefore, \( C \) sends at most \( \frac{6}{24} \) by Rules 3-7a (Proposition 7.8). Since \( P(C) = 3 \), one of the leaves of \( C \) must be adjacent to \( v_{\frac{1}{3}} \); therefore, one of the leaves of \( C \) has more than one distance-2 vertex in \( D \setminus C \). Then, \( C \) sends at most \( \frac{1}{24} \) by Rule 7c; therefore, \( f(C) \geq \frac{40}{24} \).
Now, consider a curved 4-cluster, $C$, and let $P(C) = 2$. Then, $f_1(C) = \frac{46}{24}$. Rule 2 does not apply; therefore, $f_2(C) = \frac{46}{24}$. First, suppose $C$ sends no charge by Rule 7c. Then, $C$ has at most 8 nearby poor 1-clusters; however, if $C$ has $k$ such clusters, where $k > 6$, then at least $k - 6$ of the distance-3 poor 1-clusters are stealable (Lemma 6.10) – that is, $C$ sends charge to at most 6 nearby poor 1-clusters by Rules 3-7a. Then, $C$ sends no charge by Rule 7c and at most $\frac{6}{24}$ by Rules 3-7a (Proposition 7.8); therefore, $f(C) \geq \frac{40}{24}$. Now, suppose $C$ sends $\frac{1}{24}$ by Rule 7c. Then, one backwards position of $C$ is not in $D$; therefore, $C$ has at most 6 nearby poor 1-clusters, and if $C$ has 6 such clusters then at least one is stealable (Lemma 6.10) – that is, $C$ sends at most $\frac{5}{24}$ by Rules 3-7a (Proposition 7.8). Therefore, $f(C) \geq \frac{40}{24}$. Finally, suppose $C$ sends $\frac{2}{24}$ by Rule 7c. Then, neither backwards position of $C$ is in $D$; therefore, $C$ has at most 2 nearby poor 1-clusters (Lemma 6.10). Then, $C$ sends at most $\frac{2}{24}$ by Rules 3-7a (Proposition 7.8); therefore, $f(C) \geq \frac{42}{24}$.

Let $P(C) = 3$. First, suppose $C$ is adjacent to no one-third vertices. Then, $f_2(C) = \frac{51}{24}$. Now, $C$ has at most 11 nearby poor 1-clusters (Lemma 6.11); therefore, if $C$ sends no charge by Rule 7c, then $f(C) \geq \frac{40}{24}$ (Proposition 7.8). If $C$ sends $\frac{1}{24}$ by Rule 7c, then one backwards position of $C$ is not in $D$; therefore, $C$ has at most 10 nearby poor 1-clusters (Lemma 6.11). Then, $C$ sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 7.8) and $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. If $C$ sends $\frac{2}{24}$, then both backwards positions are not in $D$; therefore, $C$ has at most 9 nearby poor 1-clusters (Lemma 6.11). Then, $C$ sends at most $\frac{9}{24}$ by Rules 3-7a (Proposition 7.8) and $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. Now, suppose $C$ is adjacent to a one-third vertex, $v_{\frac{1}{3}}$. Then, $f_2(C) \geq \frac{47}{24}$. Since $P(C) = 3$ and each leaf of $C$ has at least one distance-2 vertex in $D \setminus C$ (Proposition 4.6), $v_{\frac{1}{3}}$ is adjacent to one of the leaves of $C$; therefore, $C$ sends at most $\frac{1}{24}$ by Rule 7c. Now, $C$ has at most 6 nearby poor 1-clusters (Lemma 6.11). Therefore, $C$ sends at most $\frac{6}{24}$ by Rules 3-7a (Proposition 7.8) and at most $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$.
Consider a linear or curved 4-cluster, $C$, and let $P(C) \geq 4$. First, suppose $C$ is adjacent to no one-third vertices. Then, $f_2(C) \geq \frac{56}{24}$. Now, $C$ has at most 12 nearby poor 1-clusters (Lemma 6.16); therefore, $C$ sends at most $\frac{12}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{2}{24}$; therefore, $f(C) \geq \frac{42}{24}$. Now, suppose $C$ is adjacent to exactly one one-third vertex. Then, $f_2(C) \geq \frac{52}{24}$. Now, $C$ has at most 12 nearby clusters (Lemma 6.16). However, since $C$ is adjacent to a one-third vertex, at least 2 of these clusters are not poor 1-clusters; therefore, $C$ has at most 10 nearby poor 1-clusters. Then, $C$ sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 7.8) and at most $\frac{5}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. Finally, suppose $C$ is adjacent to 2 one-third vertices. Now, if $P(C) = 4$, then each leaf is adjacent to a one-third vertex and $f_2(C) \geq \frac{48}{24}$. Then, at least 4 of the 12 possible nearby clusters are not poor 1-clusters; therefore, $C$ sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 7.8). Since both leaves have more than one distance-2 vertex in $D \setminus C$, no charge is sent by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. If $P(C) \geq 5$ and $C$ is adjacent to 3 one-third vertices, then $f_2(C) \geq \frac{49}{24}$ and at least 5 of the 12 possible nearby clusters are not poor 1-clusters; therefore, $C$ sends at most $\frac{7}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{2}{24}$; therefore, $f(C) \geq \frac{42}{24}$.

Consider a 4-cluster, $C$, and let $H$ have a degree-3 vertex. First, suppose $P(C) = 3$. Then, $f_2(C) = \frac{54}{24}$. Now, $C$ has at most 8 nearby poor 1-clusters; therefore, $C$ sends at most $\frac{5}{24}$ by Rules 3-7a (Proposition 7.8). Since $C$ has 3 leaves, $C$ sends at most $\frac{3}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{40}{24}$. Now, suppose $P(C) \geq 4$. If $C$ is adjacent to a one-third vertex, $v_{\frac{1}{3}}$, then $f_2(C) \geq \frac{52}{24}$ and at least 2 of the 12 possible nearby clusters (Lemma 6.16) are not poor 1-clusters; therefore, $C$ sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 7.8). From the structure of $C$, we see that $v_{\frac{1}{3}}$ must be adjacent to...
a leaf of $C$; therefore, at least one of the leaves of $C$ has more than one distance-2 vertex in $D \setminus C$. Therefore, $C$ sends at most $\frac{2}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{40}{24}$. If $C$ is adjacent to no one-third vertices, then $f_2(C) \geq \frac{56}{24}$. Now, $C$ has at most 12 nearby poor 1-clusters (Lemma 6.16), and $C$ sends at most $\frac{3}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{41}{24}$ (Proposition 7.8).

\begin{claim}
For every 5-cluster, $C$, $f(C) \geq 5 \cdot \frac{5}{12}$.
\end{claim}

\begin{proof}
Consider a 5-cluster, $C$ with $\Delta(C) = 2$. If $C \in \mathcal{K}_5^c$, then $f_2(C) \geq \frac{65}{24}$. Now, $C$ has at most 13 nearby poor 1-clusters (Lemma 6.16); therefore, $C$ sends at most $\frac{13}{24}$ by Rules 3-7a (Proposition 7.8). Since $C$ has exactly 2 leaves, $C$ sends at most $\frac{2}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{50}{24} = 5 \cdot \frac{5}{12}$. If $C \in \mathcal{K}_5^c$, then $f_2(C) \geq \frac{60}{24}$. Now, $C$ has at most 9 nearby poor 1-clusters; furthermore, if $C$ has exactly 9 such clusters, then at least one is stealable (Lemma 6.12) – that is, $C$ sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{63}{24}$.

Now, let $C$ have a degree-3 vertex. Then, $f_2(C) \geq \frac{65}{24}$. Now, $C$ has at most 12 nearby poor 1-clusters (Lemma 6.14); therefore, $C$ sends at most $\frac{12}{24}$ by Rules 3-7a (Proposition 7.8). Since $C$ has 3 leaves, $C$ sends at most $\frac{3}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{60}{24}$.
\end{proof}

\begin{claim}
For every 6-cluster, $C$, with $\Delta(C) = 2$, $f(C) \geq 6 \cdot \frac{5}{12}$.
\end{claim}

\begin{proof}
Consider a 6-cluster, $C$ with $\Delta(C) = 2$. Then, $C$ has exactly 2 leaves. If $C \in \mathcal{K}_6^c$, then $f_2(C) \geq \frac{79}{24}$. Now, $C$ has at most 14 nearby poor 1-clusters; therefore, $C$ sends at most $\frac{14}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{63}{24} > 6 \cdot \frac{5}{12}$. If $C \in \mathcal{K}_6^c$, then $f_2(C) \geq \frac{74}{24}$. Now, $C$ has at most 10 nearby poor 1-clusters (Lemma 6.15); therefore, $C$ sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 7.8). By Rule 7c, $C$ sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{62}{24} > 6 \cdot \frac{5}{12}$.
\end{proof}
Chapter 8

Future Research

This paper is the fourth attempt (see [3], [4] and [5]) to determine the minimum density of a vertex identifying code for the infinite hexagonal grid. And we have not entirely succeeded. Like the three papers preceding this one, we have merely improved the bounds. As such, a section on future research is perhaps justified. We will begin by discussing suggested approaches to improving the upper bound, and we will finish by discussing possible improvements to the lower bound.

8.1 Upper Bound

As discussed in Chapter 2, the upper bound is proved by example. The goal is to find a code with the smallest density possible. Since $G_H$ is infinite, we usually look for repeating patterns. In this way, we can determine the limit of the density as the number of vertices approaches infinity. The best known upper bound is currently $3/7$, and we have five constructions with this density (see Figures 2.1-2.4). The last two of these constructions were generated by computer searches relying on linear integer programming techniques. We found that these techniques allow for much faster searches.

As the upper bound was not the main focus of this research, our linear program-
ming search remained rather limited in scope. There are several relatively simple improvements that can be made.

Notice that the constructions of Figures 2.1 and 2.2 contain brick tiling; that is, there are no “four-corners”. Whereas the constructions of Figures 2.3 and 2.4 – that is, the constructions found using our linear programming search – do contain “four-corners”. This is no accident. Our program was not capable of generating brick tiling patterns. This is relatively easy to fix and simply has to do with how we “tie” the border vertices to each other. If a linear programming search is to be undertaken in the future, the first order of business should be to extend the search to brick tiling patterns. The constructions of Figures 2.1 and 2.2 lead us to believe that this would be a relatively promising improvement to our search capabilities.

Before running a search, we must decide on the dimensions of the tile. This is limiting for two reasons. The first is that it is very difficult to determine beforehand what dimensions will be most conducive to a low density tiling pattern. The second is that the size of the tile – in particular, the number of vertices in the tile – limits the possible densities that can appear. The density of the tile is a ratio of two natural numbers (see Equation 2.1). Once we have decided on the denominator, we have limited the set of possible densities to a discrete set of values. Except in the case of relatively small tiles, it is very difficult to know what denominators are most promising. The solution to these limitations is quite simply to try as many tile sizes and dimensions as possible. We would recommend starting with smaller tiles because the run-time is lower and gradually working up to larger tiles.

We experimented with different kinds of tiles and tiling schemes. For instance, we tried letting each tile be the mirror reflection of its neighboring tiles or perhaps only those neighbors to the right and the left. We tried letting each tile be a rotation of its neighboring tiles. We even tried rotating the tiles themselves to create even larger tiles of tiles. We had a certain amount of luck considering hexagonal tiles. In particular,
we considered relatively large hexagonal tiles with six rotational symmetries. We tried to use these tiles to construct even larger rotationally symmetric hexagonal tiles. Inspired by this, we also considered triangular tiles. There is no reason to suppose that the minimum density is attainable with a rectangular tiling pattern. As such, we would recommend trying as many tiling schemes as possible.

As a sobering last thought, we must mention that all of these labors may be in vain. It is, of course, possible that the minimum density cannot be achieved by any kind of tiling pattern.

### 8.2 Lower Bound

The discharging method was also used by Cranston and Yu [4] in 2009 to prove a lower bound of $12/29$. They required 5 discharging rules to achieve this result. In this paper, we have improved the lower bound to $5/12$ but at the cost of 16 discharging rules and a proof that is about twice as long.

We can compare the relative difficulty of a discharging proof for different lower bounds by considering the open 3-cluster. Let $C$ be an open 3-cluster, and suppose we wish to prove a lower bound of $\tau$. Let us take the usual first discharging rule:

- If a vertex, $v$, is not in $D$ and has $k$ neighbors in $D$, then $v$ receives $\tau/k$ from each of these neighbors.

As in Section 7, let $f_i(C)$ denote the charge of $C$ after Discharging Rule $i$ and $f(C)$ the final charge. Then, $f_1(C) \geq 3 - \left(3\tau + \frac{2\tau}{2}\right) = 3 - 4\tau$. Again, as in Section 7, let $F_i(C) = f_i(C) - 3\tau$. Recall that our goal is to have $f(C) \geq 3\tau$. This occurs if, and only if, $F(C) \geq 0$. From above, we see that $F_1(C) \geq 3 - 7\tau$. Also recall that we are often interested in sending any excess charge to nearby poor 1-clusters. For a poor 1-cluster, $v$, we have $f_1(v) = 1 - \frac{3\tau}{2}$. Equivalently, we have $F_1(v) = 1 - \frac{5\tau}{2}$. Now let us try different values for $\tau$. The results have been summarized in Table 8.1.
Table 8.1: Results of the first discharging rule for an open 3-cluster, $C$, and a poor 1-cluster, $v$, for different attempted lower bounds, $\tau$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$F_1(C)$</th>
<th>$F_1(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/29</td>
<td>$\geq 3/29$</td>
<td>$-1/29$</td>
</tr>
<tr>
<td>5/12</td>
<td>$\geq 2/24$</td>
<td>$-1/24$</td>
</tr>
<tr>
<td>8/19</td>
<td>$\geq 1/19$</td>
<td>$-1/19$</td>
</tr>
<tr>
<td>3/7</td>
<td>$\geq 0$</td>
<td>$-1/14$</td>
</tr>
</tbody>
</table>

Now, $P(C) \geq 2$. This means that $C$ may have 2 or more poor 1-clusters within distance-2. Take $P(C) = 2$ – the other cases are trivial. Then $F_1(C)$ is at a minimum. From Table 8.1, we see that for $\tau = 5/12$ each open 3-cluster can afford to send charge to exactly 2 poor 1-clusters, which is precisely as many as may lie within distance-2 (since we assumed $P(C) = 2$). It is this observation that inspired Discharging Rule 3b in the proof of Theorem 1.2. For $\tau = 12/29$, we see that $C$ can take care of the distance-2 poor 1-clusters with charge to spare. On the other hand, for $\tau = 8/19$ or $\tau = 3/7$ an open 3-cluster might not have enough charge to take care of the distance-2 poor 1-clusters. This analysis seems to point to 5/12 as the most attractive potentially provable lower bound.

Similar analyses of other types of clusters also pointed to 5/12 as the most attractive lower bound. For instance, for a closed 3-cluster, $C$, we have $F_1(C) \geq 7/24$; when $F_1(C)$ is at a minimum there are at most 7 nearby poor 1-clusters that are not nearby other $3^+$-clusters (Lemma 6.6). For most of the clusters that can appear in the hexagonal grid, we found that the choice of $\tau = 5/12$ led to a rather smooth discharging process. The one major exception was the case of the asymmetric very poor 1-cluster. In fact, most of the lemmas and most of the discharging rules are directly or indirectly related to the asymmetric very poor 1-cluster. The length of our proof is mostly due to this one case. For $\tau > 5/12$, many of the nice properties disappear and the case of the asymmetric very poor 1-cluster becomes even more difficult. One should expect that a discharging proof for $\tau > 5/12$ would be rather cumbersome.
For the reasons stated above, we would discourage the use of the discharging method for any potential improvements on the lower bound. It seems that a new approach must be found.

One type of discharging, however, that might prove fruitful is a kind of reverse discharging: we assign \((1 - \tau)\) to each vertex not in \(D\) and then redistribute the charge so that every vertex retains \((1 - \tau)\). It is not clear whether this will be more or less useful than the regular discharging method, but it may be something worth looking into.

One might also consider formulating the problem in terms of matrices. It might be better to begin by considering finite graphs and only later generalize to infinite graphs. Let \(A_G\) be the adjacency matrix associated with the graph, \(G\). Suppose \(V(G) = \{v_1, v_2, \ldots, v_n\}\), and let \(I_n\) be the \(n \times n\) identity matrix. Then define

\[
M_G = A_G + I_n
\]  

Then \(M_G\) is an \(n \times n\) matrix containing only zeros and ones. Now consider the \(i\)th column of \(M_G\). There is a 1 in the \(j\)th entry if, and only if, \(v_j \in N[v_i]\). Then the \(i\)th column indicates which vertices are in \(N[v_i]\). Suppose that \(D\) is a vertex identifying code for \(G\). Let \(T_D\) be the \(n \times n\) diagonal matrix defined by

\[
(T_D)_{ii} = \begin{cases} 
1, & \text{if } v_i \in D \\
0, & \text{if } v_i \notin D
\end{cases}
\]

Then define

\[
M_D = T_D M_G
\]  

Then \(M_D\) is the same as \(M_G\) except that the rows corresponding to vertices not in \(D\) have been set to 0. Consider the \(i\)th column of \(M_D\). There is a 1 in the \(j\)th entry if, and only if, \(v_j \in N[v_i] \cap D\). Then the \(i\)th column indicates which vertices are in...
$N[v_i] \cap D$.

Since $D$ is a vertex identifying code, each $N[v_i] \cap D$ must be unique and non-empty. In the matrix formulation, the equivalent condition is that each column of $M_D$ contains at least one non-zero entry and no two columns are identical, i.e., if we have $i \neq j$ and $|N[v_i] \cap D| = |N[v_j] \cap D|$, then

$$
\left( M_D^{(i)} \right)^T M_D^{(j)} < \left| M_D^{(i)} \right|^2 
$$

(8.3)

where $M_D^{(i)}$ is the $i$th column of $M_D$. If $|N[v_i] \cap D| \neq |N[v_j] \cap D|$, then the sets are definitely distinct, as desired. We can summarize this in the matrix $M_D^T M_D$. Let us call this matrix $Q$. Then $Q$ is a symmetric positive semidefinite $n \times n$ matrix with $Q_{ij} = \left( M_D^{(i)} \right)^T M_D^{(j)}$. Then $Q_{ii} = |N[v_i] \cap D|$. The above condition says that if $Q_{ii} = Q_{jj}$, then $Q_{ij} = Q_{ji} < Q_{ii} = Q_{jj}$. We also note that if $Q_{ii} < Q_{jj}$, then $Q_{ij} = Q_{ji} \leq Q_{ii}$. The previous two statements can be summarized by the condition that all $2 \times 2$ principal minors are positive.

It is not clear whether or not the matrix formulation will prove useful. We include it only as a possible starting point.
Appendix A

Graph Theory Basics

Perhaps the best way to introduce graph theory is by example. Figure A.1 shows a graph. The circles are called vertices, and the lines connecting them are called edges. A vertex can represent any kind of element, and an edge can represent any kind of relationship between two elements. The graph in Figure A.1 is actually a

![Figure A.1: The Seven Bridges of Königsberg](image)

rather famous graph and relates to the problem of the Seven Bridges of Königsberg. The vertices represent land masses along the river Pregel, and the edges represent bridges. The problem is to find a route which crosses each bridge once and only once. It was proved impossible in 1735 by Leonhard Euler. The point to be made is that
once the problem has been reduced to a graph, we can begin to ask questions about the graph itself without reference to the bridges of Königsberg. This is the realm of graph theory.

The set of vertices in a graph, $G$, is denoted by $V(G)$ and the set of edges is denoted by $E(G)$. If $G$ is the graph of Figure A.1, for example, then $V(G) = \{u, v, w, x\}$ and $E(G) = \{a, b, c, d, e, f, g\}$. Each end of an edge must be incident on a vertex. If both ends of an edge, $e$, are incident on the same vertex, then $e$ is called a loop. Some graphs are directed. This means that each edge is associated with a direction. Figure A.2 shows an example of a directed graph. One might decide to represent a problem as a directed graph if, for instance, the relationships between pairs of elements are not always equal and opposite. An undirected graph is simply one in which the edges are not associated with a particular direction, e.g., the graph of Figure A.1. A simple graph is one with no loops and for which every pair of vertices is joined by at most one edge; alternatively, a simple graph is one in which no two edges join the same pair of vertices (and no loops exist). So we see that the graphs shown in Figures A.1 and A.2 are not simple, but every other graph shown in this paper is a simple graph. Two vertices are said to be adjacent if they are joined by an edge; such vertices are also called neighbors.

Any given graph can be drawn in a multitude of ways. The essence of a graph lies in the connections among vertices. The pictorial representation exists solely for the
convenience of the graph theorist. Figure A.3 shows two equivalent representations of the same graph. Actually, this graph is rather special. Notice that each vertex is joined to every other vertex; such a graph is called complete and is denoted by $K_n$ where $n$ is the total number of vertices. Then, using this notation, we have in Figure A.3 two representations of $K_4$. We consider these graphs to be equivalent because both contain the same information regarding connections among vertices.

For two graphs, $G$ and $H$, we call $H$ a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $V_0 \subseteq V(G)$. The subgraph of $G$ induced by $V_0$, denoted by $G[V_0]$, is the graph whose vertex set is $V_0$ and whose edge set contains all the edges in $E(G)$ which join vertices of $V_0$.

For a graph, $G$, a walk is an alternating sequence of vertices and edges where each edge joins the vertices that immediately precede and follow it; the first and last terms of the sequence are vertices. A walk for which no edge is repeated is called a trail. The problem of the Seven Bridges of Königsberg can then be rephrased as the search for a trail which contains every edge. A trail for which no vertex is repeated is called a path. The length of a path is the number of edges in the path. A path which starts and ends at the same vertex is called a cycle. A cycle of length $k$ is called a $k$-cycle.

Two vertices, $v_1$ and $v_2$, are connected if there exists a path which starts at $v_1$ and ends at $v_2$. We say that $v_1$ is distance-$i$ from $v_2$ if the shortest path from $v_1$ to $v_2$ has length $i$. A graph is connected if every pair of vertices is connected. A maximally
A connected subgraph of $G$ is called a component of $G$.

A graph is infinite if its vertex set or its edge set is infinite. Otherwise, it is finite.

The degree of a vertex, $v$, is the number of edges incident on $v$. If all the vertices of a graph, $G$, have the same degree, say $k$, then $G$ is called $k$-regular. A vertex with degree 1 is called a leaf.

Let $G$ be a graph with $V(G)\{v_1, v_2, ..., v_n\}$. The adjacency matrix of $G$, which we will denote by $A_G$, is the $n \times n$ matrix defined by

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Notice that $A_G$ is symmetric, and every diagonal element is 0.
Bibliography


