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Critical exponents: Old and New

A thesis submitted in partial fulfillment of the requirement
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by

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Abstract

Let \mathcal{P} be a class of matrices, and let A be an m -by- n matrix in the class; consider some continuous powering, $A^{\{t\}}$. The critical exponent of \mathcal{P} , if it exists, with respect to the powering is the lowest power $g(\mathcal{P})$ such that for any matrix $B \in \mathcal{P}$, $B^{\{t\}} \in \mathcal{P} \forall t > g(\mathcal{P})$. For powering relative to matrix multiplication in the traditional sense, hereafter referred to as *conventional* multiplication, this means that A^t is in the specified class for all $t > g_C(\mathcal{P})$. For Hadamard multiplication, similarly, $A^{(t)}$ is in the class $\forall t > g_H(\mathcal{P})$. This paper considers two questions for several classes \mathcal{P} (including doubly nonnegative and totally positive): 1) does a critical exponent $g(\mathcal{P})$ exist? and 2) if so, what is it? For those where no exact result has been determined, lower and upper bounds are provided.

1 Introduction

Let \mathcal{P} be a class of m -by- n matrices, and let $A \in \mathcal{P}$. Suppose that a notion of continuous powering, $A^{\{t\}}$ is defined for all $A \in \mathcal{P}$ and all $t \geq 0$. The *critical exponent* of $A \in \mathcal{P}$, with respect to \mathcal{P} and this powering, is the least real number $g(A)$, such that $A^{\{t\}} \in \mathcal{P}$ for all $t > g(A)$, if such exists; otherwise $g(A) = \infty$.

If each element of \mathcal{P} has finite critical exponent, the critical exponent of \mathcal{P} with respect to the powering is $M = g(\mathcal{P}) = \sup_{A \in \mathcal{P}} \{g(A)\}$. If M is finite, \mathcal{P} is said to have critical exponent. We note that, even if each element of \mathcal{P} has critical exponent, the class \mathcal{P} may or may not have critical exponent. Our purpose here is to survey both past and current work on the existence and values of critical exponents for important classes of matrices under both conventional powering A^t (i.e. powering that is consistent with ordinary multiplication of square matrices) and Hadamard (entry-wise) powering $A^{(t)}$. These classes include doubly nonnegative matrices (which are nonnegative positive semidefinite), matrices with nonnegative eigenvalues and nonnegative entries, totally positive matrices (in which all minors are positive), and inverse M-matrices, all of which are discussed in greater detail later in the paper.

To illustrate the concept of a critical exponent, a simple example is provided.

Example 1.1. *The critical exponent for the class of totally positive (TP) Vandermonde matrices \mathcal{V}_{TP} , under Hadamard powering, is $g_H(\mathcal{V}_{TP}) = 0$.*

Proof: Let $\{x_1, x_2, \dots, x_m\}$ be a sequence of strictly increasing positive numbers. The Vandermonde matrix corresponding to this sequence is

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix} \quad (1)$$

It is known that such a Vandermonde matrix is TP if and only if the x_i 's are positive

and increasing [1].

Consider the Hadamard power $V^{(t)}$ of this TP Vandermonde matrix for any $t > 0$:

$$V^{(t)} = \begin{pmatrix} 1 & x_1^t & (x_1^t)^2 & \cdots & (x_1^t)^{n-1} \\ 1 & x_2^t & (x_2^t)^2 & \cdots & (x_2^t)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_m^t & (x_m^t)^2 & \cdots & (x_m^t)^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & x_1^t & (x_1^t)^2 & \cdots & (x_1^t)^{n-1} \\ 1 & x_2^t & (x_2^t)^2 & \cdots & (x_2^t)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_m^t & (x_m^t)^2 & \cdots & (x_m^t)^{n-1} \end{pmatrix}$$

which is itself the TP Vandermonde matrix corresponding to the strictly increasing positive sequence $\{x_1^t, x_2^t, \dots, x_m^t\}$ for all $t > 0$. So the matrix $V^{(t)}$ is in the class $\mathcal{V}_{TP} \forall t > 0$, and the critical exponent of \mathcal{V}_{TP} is $g_H(\mathcal{V}_{TP}) = 0$. \square

Observation. As has been mentioned already, the TP denomination of this class refers to the positivity of every minor of V , a notion that will be discussed later in greater detail. For the purposes of this comment, however, it can quickly be observed that if $\{x_1, x_2, \dots, x_m\}$ is *not* strictly increasing, then there exists at least one minor that is non-positive (consider the determinant of the 2-by-2 submatrix $V(\{i, j\}, \{1, 2\})$ where $i < j$ and $x_i \geq x_j$).

From this, it can be seen that Hadamard powers of a TP Vandermonde matrix

$$V^{(t)} = \begin{pmatrix} 1 & x_1^t & (x_1^t)^2 & \cdots & (x_1^t)^{n-1} \\ 1 & x_2^t & (x_2^t)^2 & \cdots & (x_2^t)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_m^t & (x_m^t)^2 & \cdots & (x_m^t)^{n-1} \end{pmatrix}$$

are not in \mathcal{V}_{TP} for $t \leq 0$ (the sequence $\{x_1^t, x_2^t, \dots, x_m^t\}$ strictly decreases).

If we instead consider the class \mathcal{V} of Vandermonde matrices having the *form* of (1) but without the requirement that $\{x_1^t, x_2^t, \dots, x_m^t\}$ strictly increases, we observe that $V \in \mathcal{V}$ implies $V^{(t)} \in \mathcal{V}$ for all t . Because our definition of critical exponent considers only $t \geq 0$, the critical exponent of \mathcal{V} is formally $g_H(\mathcal{V}) = 0$; the same as \mathcal{V}_{TP} despite the two classes' differing behaviors at $t < 0$. Discussion of the cases in which it is useful to consider negative t and the appropriate generalizations of “critical exponent” that would result is a worthwhile endeavor not covered in this paper.

2 Exponential polynomials

Expressions known as *exponential polynomials* arise repeatedly in our view of critical exponent problems and within a variety of contexts. For the purposes of this paper, an exponential polynomial has the form

$$\alpha_n e^{\beta_n t} + \alpha_{n-1} e^{\beta_{n-1} t} + \cdots + \alpha_1 e^{\beta_1 t},$$

with $\beta_n > \beta_{n-1} > \cdots > \beta_1$ and all α_i real and non-zero.

Lemma 2.1. *The number of zeros an exponential polynomial can experience (counting multiplicity) does not exceed the number of sign changes within the sequence of coefficients $\{\alpha_n, \alpha_{n-1}, \dots, \alpha_1\}$.*

This statement is well-known, but a proof adapted from [2] is reproduced below.

Proof: Suppose there are $Z(f)$ positive real roots of $f(t) = \alpha_n e^{\beta_n t} + \alpha_{n-1} e^{\beta_{n-1} t} + \cdots + \alpha_1 e^{\beta_1 t}$, $\beta_n > \beta_{n-1} > \cdots > \beta_1$. Let w be the number of sign changes in $\{\alpha_n, \alpha_{n-1}, \dots, \alpha_1\}$. The aim of this proof is to show $w \geq Z(f)$ via induction on w .

When $w = 0$, $Z(f)$ must also be zero: f can have no positive real roots. Assume that the induction hypothesis holds, that f has w sign changes in the coefficients and that the first sign change of the sequence occurs between α_{k+1} and α_k . Choose β such that $\beta_{k+1} > \beta > \beta_k$ and define

$$f_0(t) = f(t)e^{-\beta t} = \alpha_n e^{(\beta_n - \beta)t} + \alpha_{n-1} e^{(\beta_{n-1} - \beta)t} + \cdots + \alpha_1 e^{(\beta_1 - \beta)t}.$$

It can be easily seen that the roots of $f_0(t)$ and $f(t)$ are identical. Taking the derivative of $f_0(t)$ yields

$$f'_0(t) = (\beta_n - \beta)\alpha_n e^{(\beta_n - \beta)t} + (\beta_{n-1} - \beta)\alpha_{n-1} e^{(\beta_{n-1} - \beta)t} + \cdots + (\beta_1 - \beta)\alpha_1 e^{(\beta_1 - \beta)t}.$$

As $\beta_{k+1} - \beta > 0$ and $\beta_k - \beta < 0$, $(\beta_{k+1} - \beta)\alpha_{k+1}$ and $(\beta_k - \beta)\alpha_k$ must have the same sign, thereby eliminating the first sign change while preserving all later ones. The function $f'_0(t)$ thus has $w - 1$ sign changes and, by the induction hypothesis, $Z(f'_0) \leq w - 1$. By Rolle's theorem, however, $Z(f_0) \leq Z(f'_0) + 1$; consequently, $f_0(t)$ has at most w zeros. It follows that $f(t)$ has at most w zeros. \square

The β_i are hereafter referred to as the bases of an exponential polynomial, while the α_i are the coefficients. Exponential polynomials appear in both the conventional and Hadamard problems, although in different ways. For conventional powering, exponential polynomials describe the behavior of entries (minors) with the eigenvalues (products of the eigenvalues) of the original matrix as the bases, and coefficients derived from the eigenvectors. For Hadamard powers, the exponential polynomials describing minors (including, trivially, the entries) consist of bases that are products of the entries and integer coefficients.

Example 2.1. *Consider the following 3-by-3 matrix under conventional powering:*

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \sigma(A) = \{3, 6\}.$$

$$A^t = UD^tU^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^t \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Via direct computation, it can be determined that all diagonal entries of A^t are described by the exponential polynomial $(A^t)_{ii} = \frac{1}{3}6^t + \frac{2}{3}3^t$ and all off-diagonal entries are given by the exponential polynomial $(A^t)_{ij} = \frac{1}{3}6^t - \frac{1}{3}3^t$.

Example 2.2. For conventional powers of the 4-by-4 matrix

$$B = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & 5 & 1 & 0 \\ 0 & 1 & 5 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}, \quad \sigma(B) = \{2, 4, 6, 8\},$$

the entries can be written as

$$(B^t)_{ij} = \begin{cases} \frac{1}{4}8^t + \frac{1}{4}6^t + \frac{1}{4}4^t + \frac{1}{4}2^t & \text{for } i = j \\ \frac{1}{4}8^t + \frac{1}{4}6^t - \frac{1}{4}4^t - \frac{1}{4}2^t & (i, j) \in \{(1, 2), (2, 1), (3, 4), (4, 3)\} \\ \frac{1}{4}8^t - \frac{1}{4}6^t - \frac{1}{4}4^t + \frac{1}{4}2^t & (i, j) \in \{(1, 3), (3, 1), (2, 4), (4, 2)\} \\ \frac{1}{4}8^t - \frac{1}{4}6^t + \frac{1}{4}4^t - \frac{1}{4}2^t & (i, j) \in \{(2, 3), (3, 2), (1, 4), (4, 1)\} \end{cases}$$

Additionally, the determinant of the 3-by-3 principal submatrix is equal to

$$\det(A(3, 3)) = \frac{1}{4}6^t 8^t 2^t + \frac{1}{4}6^t 4^t 2^t + \frac{1}{4}8^t 4^t 2^t + \frac{1}{4}8^t 6^t 4^t = \frac{1}{4}96^t + \frac{1}{4}48^t + \frac{1}{4}64^t + \frac{1}{4}192^t$$

with other minors having similar forms. This example and the previous one were both designed to have rational coefficients; in general, because the coefficients of conventional powering are derived from the eigenvectors, this is not the case.

Example 2.3. Consider Hadamard powers of the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad A^{(t)} = \begin{bmatrix} 4^t & 1^t & 1^t \\ 1^t & 4^t & 1^t \\ 1^t & 1^t & 4^t \end{bmatrix},$$

The determinant of this matrix is $\det(A^{(t)}) = (4^3)^t - 3(4)^t + 2 = 64^t - 3(4)^t + 2$.

Example 2.4. *The equation for the determinant of $C^{(t)}$ for*

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is

$$\det(C^{(t)}) = 256^t - 3(144^t) + 81^t + 4(72^t) - 2(64^t) - 2(36^t) - 2(27^t) + 4(24^t) - 2(12^t) + 9^t,$$

an exponential polynomial with 6 sign changes in the ordered coefficient sequence. Because of the six sign changes, the equation has at most six zeros; in fact, this example has no positive roots.

3 Conventional powers

3.1 Doubly nonnegative (DN) matrices

Doubly nonnegative matrices are real symmetric matrices with all entries and eigenvalues nonnegative. Conventional integer powers of a DN matrix are DN because symmetry, nonnegativity of the entries, and nonnegativity of the eigenvalues are each preserved; non-integer conventional powers, however, run the risk of having a pair or more of negative entries (the eigenvalues are nonnegative for all continuous powers, as is clear from the spectral decomposition $A^t = UD^tU^T$).

Let $A \in M_n$ be DN. Because A is, by definition, symmetric (and positive semi-definite), A^t can be decomposed into

$$A^t = x_n x_n^T \lambda_n^t + \cdots + x_1 x_1^T \lambda_1^t$$

in which the λ_i are the eigenvalues of A and x_i the corresponding eigenvectors. Every entry of A^t is therefore an exponential polynomial of the form

$$(A^t)_{ij} = (x_n x_n^T)_{ij} \lambda_n^t + \cdots + (x_1 x_1^T)_{ij} \lambda_1^t$$

that by Lemma 2.1 can have at most $n - 1$ zeros.

3.1.1 Existence

Theorem 3.1. *A conventional critical exponent for DN matrices, $g_C(DN)$, exists.*

Proof: Let $A \in M_n$ be DN. Then A^k is DN for all positive integers k . If A is DN for all $t \in [m, m + 1]$, $m \in \mathbb{Z}$, then repeated multiplication by A demonstrates that A^t is nonnegative for all $t \geq m$ and therefore DN for all $t \geq m$. If, instead, A^t has a negative entry for some $t \in [m, m + 1]$, then the interval $[m, m + 1]$ must contain at

least two roots of that entry's exponential polynomial. As the maximum number of roots allowed to each entry is finite and dependent upon n (by Lemma 2.1), it follows that there exists some value $g_C(\text{DN})$ such that for any A , A^t is DN for all $t > g(\text{DN})$. \square

A refined form of this proof (which appears in a very similar form in [3]) provides a quadratic upper bound for the DN conventional critical exponent; however, several statements must first be verified.

3.1.2 Upper bound

Let A be an n -by- n DN matrix. Define $W = [w_{ij}]$ to be the n -by- n *sign change matrix* corresponding to A and let w_{ij} equal the number of sign changes in the coefficient sequence of entry $(A^t)_{ij}$, when the eigenvalues are arranged in increasing or decreasing order. By Lemma 2.1, the number of positive roots of $(A^t)_{ij}$ is less than or equal to w_{ij} . The assumption that A is DN places several restrictions on the form of W .

Lemma 3.1. *Let A be an n -by- n DN matrix and W be its corresponding sign change matrix. Then every diagonal entry of W is equal to zero and every row and column of W can have at most one entry equal to $n - 1$, with all other entries less than or equal to $n - 2$.*

Proof: A is symmetric, thus there exists an orthogonal matrix U such that $A^t = UD^tU^T$, where $D = \text{diag}(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ and $U = [u_{ij}]$. The (i, j) -entry of A^t can be written as

$$(A^t)_{ij} = u_{in}u_{jn}\lambda_n^t + \dots + u_{i1}u_{j1}\lambda_1^t$$

When expressed this way, it becomes clear that the coefficients of the exponential polynomial are given by the Hadamard product of the i^{th} and j^{th} rows of U . Because the rows of U are orthogonal, no two rows can have the same sign pattern. Only one sign pattern of the coefficient sequence allows the maximum number of sign changes ($n - 1$, sign alternating with every coefficient), and only one ‘‘complementary’’ row sign pattern exists for each row of U such that the Hadamard product of the two yields $n - 1$ sign changes. Therefore, any row or column of W can have at most 1 entry equal to $n - 1$ with all other entries at most $n - 2$. Finally, because the diagonal entries of A^t have no sign changes in their coefficient sequences (every coefficient is positive), $w_{ii} = 0 \forall i$. \square

Lemma 3.2. *Let A be invertible and DN and let W be its associated sign change matrix. Define $T_{ij}^- = \{t > 1 : (A^t)_{ij} < 0\}$. The maximum number of connected components of T_{ij}^- is $\lfloor (w_{ij} - 1)/2 \rfloor$ if $w_{ij} > 0$ and zero if $w_{ij} = 0$.*

Proof: As A is invertible, the exponential polynomials for $t > 0$ are continuous at $t = 0$. Because A^0 is the identity, one root for every entry (excluding the diagonal) occurs at $t = 0$; thus, $(A^t)_{ij}$ ($i \neq j$) has at most $w_{ij} - 1$ roots in $[1, \infty)$.

Since A^k is nonnegative for all positive integers k , the connected components T_{ij}^- are bounded between integers. Each endpoint of a connected component is a root of $(A^t)_{ij}$, and any shared endpoints are roots with multiplicity at least 2. The number of real roots (counting multiplicity) of $(A^t)_{ij}$ for $t \geq 1$ is thus at least twice the number of connected components of T_{ij}^- . Diagonal entries, as discussed before, have no sign changes in their coefficients, no real positive roots, and no connected components of T_{ij}^- . \square

Theorem 3.2. *The conventional critical exponent for DN matrices, $g_C(DN)$, satisfies*

$$g(DN) \leq \begin{cases} \frac{n^2-4n+5}{2} & \text{if } n \text{ is odd;} \\ \frac{n^2-5n+8}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof: Let $A \in M_n$ be DN and assume, temporarily, that A is irreducible and invertible. As in Lemma 3.2, let W denote the sign change matrix of A and $T_{ij}^- = \{t > 1 : (A^t)_{ij} < 0\}$. Consider a single column j , with $T_j^- = \cup_{1 \leq i \leq n} T_{ij}^-$.

If n is odd, then by Lemma 3.1, at most one entry in column j can have $w_{ij} = n-1$ and $(n-2)/2$ connected components in T_{ij}^- , while the remaining entries (excluding the diagonal) can have at most $(n-3)/2$ connected components. (The diagonal entries are always positive.) T_j^- can thus have at most

$$\frac{(n-2)}{2} + \frac{(n-2)(n-3)}{2} = \frac{n^2-5n+6}{2}$$

connected components.

If, on the other hand, n is even, then entries with $w_{ij} = n-1$ or $n-2$ can both have at most $(n-3)/2$ connected components. The number of connected components of T_j^- is therefore at most

$$\frac{(n-1)(n-3)}{2} = \frac{n^2-4n+3}{2}.$$

We have already stated that each connected component of T_j^- is bounded within the open interval $(m, m+1)$ for positive $m \in \mathbb{Z}$ by the fact that such integer powers of A are nonnegative. If $T_j^- \cap (m, m+1) = \emptyset$, then every entry in column j of A^t is nonnegative for $t \in [m, m+1]$ and, by repeated left multiplication of A^t by A , every entry in column j is nonnegative for $t \geq m$. This implies that if the intersection $T_j^- \cap (m, m+1)$ is nonempty for some m , then the intersection must be nonempty for all positive integers less than m . Let $b(n)$ be the number of connected components of T_j^- as calculated above. Then $T_j^- \cap (b(n)+1, \infty) = \emptyset$ and A^t is nonnegative (and therefore DN) for all $t \geq b(n)+1$.

Assume now that A is invertible and reducible. Then there exists a permutation matrix P such that PAP^T is a direct sum of smaller, irreducible DN matrices. The critical exponent of A is therefore the maximum critical exponent of these smaller blocks and the inequality $t \geq b(n)+1$ holds. If A is singular then, by continuity, A^t cannot have a negative entry for any $t > b(n)+1$. The critical exponent must therefore satisfy $g(DN) \leq b(n)+1$. \square

3.1.3 Lower bound

Theorem 3.3. *The conventional critical exponent for DN matrices is at least $n - 2$.*

Proof: Let $A \in M_n$ be an irreducible, invertible, tridiagonal DN matrix. Then the $(1, n)$ and $(n, 1)$ entries of A^t are zero for $t = 0, 1, 2, \dots, n - 2$. By Lemma 2.1, the exponential polynomials $(A^t)_{1n}$ and $(A^t)_{n1}$ each have at most $n - 1$ roots; the integer zeros listed account for all of these. Thus, $(A^t)_{1n} \geq 0$ for all $t \geq n - 2$ and $(A^t)_{1n} < 0$ for $t \in (n - 3, n - 2)$. ($(A^t)_{n1}$ behaves identically). So, $g_C(\text{DN}) \geq n - 2$.

It is worth noting that the entries of DN tridiagonal matrices alternate signs in interesting ways as the matrix is raised to continuous conventional powers. An example of these sign alternations appears in Figure 1.

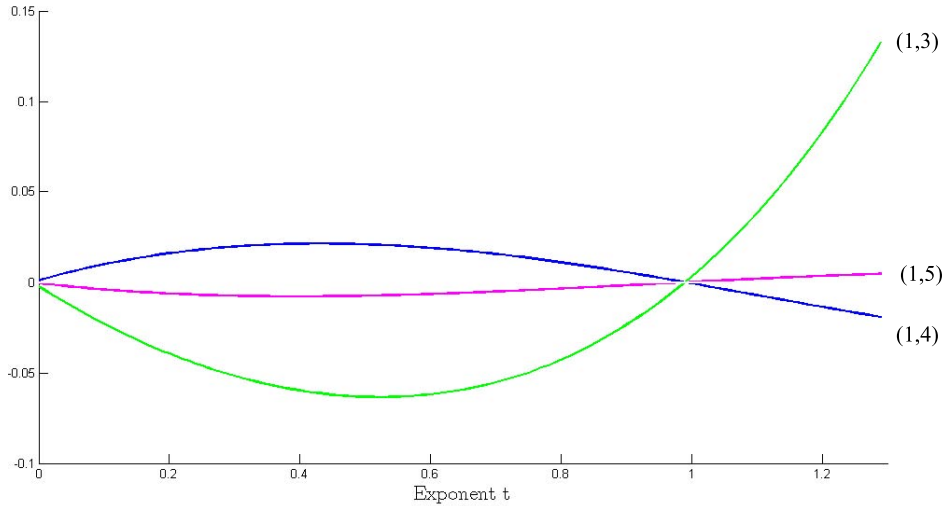


Figure 1: The (1,3) [green], (1,4) [blue], and (1,5) [magenta] entries of a 5-by-5 DN tridiagonal matrix over a small window of possible continuous exponent.

3.1.4 DN matrices for $n < 6$

Theorem 3.4. *The conventional critical exponent for n -by- n DN matrices with $n < 6$ is $g_C(\text{DN}) = n - 2$.*

The cases where $n < 5$ are taken care of by the upper bounds given in Theorem 3.2. The proof of the statement when $n = 5$ is achieved by manually computing all possible 5-by-5 sign change matrices and analyzing each to verify that no entry can be negative after $t = 3$. Two lemmas, provided below, are necessary tools in this analysis.

Lemma 3.3. *Let $A = [a_{ij}] \in M_n$ be invertible and DN, and let $W = [w_{ij}]$ be its affiliated sign change matrix. Define the critical exponent of a_{ij} to be the smallest g_{ij}*

such that for any A , $(A^t)_{ij} \geq 0 \forall t \geq g_{ij}$. If $w_{ij} = 0$ or 1 , then $g_{ij} = 0$. If $w_{ij} = 2$, $g_{ij} = 1$.

Proof: Because A is invertible and continuous at $t = 0$, $A^0 = I$ and all off-diagonal entries exhaust one available zero at $t = 0$. So if $w_{ij} = 0$, the entry must necessarily be a diagonal entry (which are always positive), and if $w_{ij} = 1$, there are no zeros greater than $t = 0$, and the entry is positive for all $t > 0$.

If $w_{ij} = 2$, two cases are possible: either $(A^t)_{ij}$ is 0 at $t = 0$ and positive thereafter, or $(A^t)_{ij} < 0$ for some interval $t \in (0, \epsilon)$ where $\epsilon < 1$ (at $t = 1$, $(A^t)_{ij} = a_{ij} \geq 0$). So g_{ij} is at most 1. \square

Lemma 3.4. *Let $A \in M_n$ be invertible and DN and let $W = [w_{ij}]$ be its sign change matrix. If a column (or row) of W contains no entry greater than 4 and at most M entries greater than 2, then the critical exponent of every entry in the column (or row) is less than or equal to $M + 1$.*

As before, because $A^0 = I$, every off-diagonal entry's exponential polynomial has one of its zeros at $t = 0$. For all integers k , if the entry $(A^t)_{ij}$ is negative for some $t \in (k, k + 1)$, then because $(A^k)_{ij}$ and $(A^{k+1})_{ij}$ are nonnegative, the closed interval $[k, k + 1]$ must contain at least two zeros of the exponential polynomial of $(A^t)_{ij}$. If $w_{ij} = 3$ or 4 , there is at most one $k > 0$ such that $(A^t)_{ij}$ is negative within $(k, k + 1)$; if there were more than one such k , a minimum of 5 zeros, counting multiplicity, would be necessary.

Let s be a real number such that every entry in column j of A^t is nonnegative for all $t \in (s, s + 1)$. Then column j is nonnegative for all $t > s$, as repeated left multiplication of A^t by A defines column j of A^t as the product of a nonnegative vector and a nonnegative matrix for all $t > s$. (The same approach can be used for rows with right multiplication). An equivalent statement is that, if a column (row) of A^t contains an entry that is negative at some real t , then that column (row) of A^{t-k} also contains a negative entry for all integers $0 < k < t$.

Consider column j of W and assume it contains M entries greater than 2 and no entry greater than 4. The statement from the previous paragraph, applied here, says that if column j of A^t has a negative entry at some $t > M$, then it must contain some negative entry within each of the intervals $(0, 1)$, $(1, 2)$, \dots , $(M, M + 1)$. The negative entry in the interval $(0, 1)$ can be accounted for by any entry w_{ij} that equals 1 or 2; all other intervals must contain a negative entry corresponding to $w_{ij} > 2$. Furthermore, because $w_{ij} \leq 4$, every entry such that w_{ij} is greater than 2 can account for a negative entry of column j in at most one interval. From this, we conclude that column j of A^t has no negative entries for $t > M + 1$. \square

Possible 5-by-5 W -matrices:

Lemmas 3.3 and 3.4 apply only to invertible DN matrices, but if Theorem 3.4 holds for any invertible A with distinct eigenvalues and eigenvectors with no zero

entries, the result will hold for all DN matrices by continuity. To verify that the theorem is satisfied by all such matrices for $n = 5$, MATLAB was used to generate all possible sign patterns of 5-by-5 eigenvector matrices under the following conditions: 1. one eigenvector was all positive (the Perron vector), 2. the top entry of every column was positive (via scaling), and 3. no two rows or columns had the same sign pattern (orthogonality).

Because $n = 5$, no entry in the following W matrices can exceed $w_{ij} = 4$. It can be checked that every entry lies in a row or column with no more than two entries greater than $w_{ij} = 2$; that is, $M \leq 2$. By Lemma 3.5, the critical exponent is $g(\text{DN}) = \sup(M) + 1 = 3$.

$$\begin{aligned}
& \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 1 & 0 & 2 & 4 \\ 2 & 3 & 2 & 0 & 2 \\ 2 & 3 & 4 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 3 & 2 \\ 2 & 1 & 0 & 2 & 3 \\ 2 & 3 & 2 & 0 & 3 \\ 3 & 2 & 3 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 4 \\ 3 & 2 & 1 & 0 & 3 \\ 2 & 3 & 4 & 3 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 3 \\ 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 3 & 4 \\ 2 & 1 & 0 & 2 & 3 \\ 2 & 3 & 2 & 0 & 1 \\ 3 & 4 & 3 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 & 4 \\ 2 & 1 & 2 & 0 & 2 \\ 2 & 3 & 4 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 \\ 2 & 1 & 2 & 0 & 1 \\ 3 & 2 & 3 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & 2 & 3 \\ 2 & 1 & 2 & 0 & 3 \\ 3 & 4 & 3 & 3 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 1 & 0 & 2 & 2 \\ 2 & 3 & 2 & 0 & 2 \\ 2 & 3 & 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 1 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 3 \\ 4 & 3 & 2 & 3 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 2 & 3 & 3 \\ 3 & 2 & 0 & 1 & 3 \\ 2 & 3 & 1 & 0 & 2 \\ 2 & 3 & 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 & 2 & 3 \\ 1 & 0 & 2 & 3 & 4 \\ 3 & 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 0 & 1 \\ 3 & 4 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 4 & 3 \\ 1 & 0 & 1 & 3 & 4 \\ 2 & 1 & 0 & 2 & 3 \\ 4 & 3 & 2 & 0 & 1 \\ 3 & 4 & 3 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 1 & 3 & 2 & 3 \\ 1 & 0 & 2 & 3 & 4 \\ 3 & 2 & 0 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 \\ 3 & 4 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 & 3 & 3 \\ 1 & 0 & 2 & 4 & 2 \\ 3 & 2 & 0 & 2 & 2 \\ 3 & 4 & 2 & 0 & 2 \\ 3 & 2 & 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 4 & 2 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 1 & 0 & 2 & 4 \\ 4 & 3 & 2 & 0 & 2 \\ 2 & 3 & 4 & 2 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ 1 & 0 & 1 & 4 & 2 \\ 2 & 1 & 0 & 3 & 3 \\ 3 & 4 & 3 & 0 & 2 \\ 3 & 2 & 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 & 2 & 3 \\ 1 & 0 & 2 & 3 & 2 \\ 3 & 2 & 0 & 3 & 2 \\ 2 & 3 & 3 & 0 & 3 \\ 3 & 2 & 2 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 & 3 & 3 \\ 1 & 0 & 2 & 2 & 2 \\ 3 & 2 & 0 & 2 & 2 \\ 3 & 2 & 2 & 0 & 2 \\ 3 & 2 & 2 & 2 & 0 \end{pmatrix}$$

3.1.5 Three distinct eigenvalues and fewer

Theorem 3.5. *Let A be an invertible DN matrix with three distinct eigenvalues. Then A^t is DN for all $t \geq 1$.*

Proof: Every entry in A^t is an exponential polynomial with at most three distinct bases and, by Lemma 2.1, at most 2 roots. Any connected component of T_{ij}^- requires at least two roots (the endpoints), therefore every T_{ij}^- associated with A can have at most one connected component. Furthermore, because A is invertible, one of these endpoints must occur at 0 for all off-diagonal entries (diagonal entries are always positive). If a connected component does not occur within $[0, 1]$, therefore, the argument from Theorem 3.1 shows that A^t is nonnegative (DN) for all $t \geq 1$. If it does occur within $[0, 1]$, no connected components remain for any greater t and A^t is again DN for $t \geq 1$.

This same argument can be repeated for invertible DN matrices with 2 or 1 distinct eigenvalues and, by continuity, for rank 2 or rank 1 matrices. \square

Theorem 3.5 generalizes the critical exponent for $n = 3$ to higher dimensions when there are at most three distinct eigenvalues. It is worth noting that it can never apply to irreducible, symmetric tridiagonal matrices, which necessarily have n distinct eigenvalues [11].

3.2 Nonnegative matrices with nonnegative eigenvalues

One natural extension of the question of a critical exponent for conventional powers of DN matrices is to consider those diagonalizable matrices with nonnegative entries and nonnegative eigenvalues, without requiring symmetry. Here, the lower bound is the same as in the DN case ($n - 2$), but the method used to find an upper bound must be modified.

Let S be a matrix with nonnegative entries and nonnegative eigenvalues. Without the assumption of symmetry, a slightly different approach must be used to arrive at a formula for the entries of S^t than was used for DN matrices. Specifically, by the spectral decomposition, we have $S^t = VD^tV^{-1}$, where D is the diagonal matrix of the eigenvalues and V is the matrix of eigenvectors. Let v_i be the columns of V and u_i be the rows of V^{-1} . Then the (i, j) -entry of S^t is:

$$(S^t)_{ij} = \lambda_n^t (v_n u_n)_{ij} + \cdots + \lambda_1^t (v_1 u_1)_{ij}$$

Theorem 3.6. *The conventional critical exponent for nonnegative matrices with nonnegative eigenvalues is no greater than $\lfloor \frac{n(n-1)}{2} \rfloor$.*

Proof: As we are unable to make the same assumptions about the number of sign changes in every entry's exponential polynomial as in the DN case, assume the maximum $n - 1$ and consider a single column. Integer powers of S again have nonnegative entries and nonnegative eigenvalues, so all negative entries must occur within the interval of length 1 between two integers. Any entry that becomes negative must exhaust at least two of its available zeros, and if a column is nonnegative for an interval of length 1, the column remains so for all higher powers. Multiplying the number of available zeros by the number of entries in a column and dividing by the minimum number of zeros exhausted in each interval (two) gives the result. \square

3.3 Totally positive (TP) matrices

A totally positive (nonnegative) matrix is one in which all minors are positive (nonnegative). Identifying a critical exponent for TP matrices requires considering not only the signs of the entries as the power is increased, but also the signs of the minors. As with DN matrices, the entries can be taken to be exponential polynomials, but now minors can be as well. The conventional product of two TP matrices is TP (can be seen with Cauchy-Binet); thus, all positive integer conventional powers of a TP matrix are TP.

Lemma 3.5. *Let $R \in M_n$ be a TP matrix. Then each k -by- k minor of R^t can have at most $\binom{n+k-1}{k} - 1$ roots.*

Proof: Each minor is the determinant of a submatrix of R and can be expressed as the sum of $k!$ terms, each the product of k (not necessarily distinct) entries. By k -fold multiplication of the eigenvalues in each entry's exponential polynomial, it can be observed that there are at most $\binom{n+k-1}{k}$ distinct bases in each k -by- k minor's exponential polynomial, each of which has at most $\binom{n+k-1}{k} - 1$ zeros. \square

3.3.1 Existence and upper bound

Once again, a proof nearly identical to that of Theorem 3.2 can be used to show the existence of a conventional critical exponent for TP matrices and to provide an upper limit for that exponent.

Theorem 3.7. *The conventional critical exponent for TP matrices, $g_C(TP)$, exists.*

Proof: Let $R \in M_n$ be TP. Then R^k is TP for all positive integers k . If R is TP for all $t \in [m, m + 1]$, $m \in \mathbb{Z}$, then it follows from Theorem 3.1 and repeated multiplication by R that R^t is TP for all $t \geq m$. If R^t has a negative minor for some $t \in [m, m + 1]$, then the interval $[m, m + 1]$ must contain at least two roots of that minor's exponential polynomial. By Lemma 2.1, the number of total zeros allowed to

the minors is a function of n ; therefore, there exists some constant $g_C(\text{TP})$ such that R^t is TP for all $t > g_C(\text{TP})$. \square

It can be intuited that any upper bound for the conventional critical exponent of a TP matrix arrived at using the techniques employed for DN matrices will increase very quickly with n ; however, several results exist that can be used to improve the bound.

One such result appears below. An initial minor Δ_{IJ} , where I and J are index sets, is one in which I and J consist of consecutive indices, at least one of which is 1. There are n^2 initial minors (each entry is the bottom right hand corner of an initial minor), $2(n - k) + 1$ of which are size k . A proof of Lemma 3.6 can be found in [1].

Lemma 3.6. *A matrix is TP if and only if every initial minor is positive.*

Theorem 3.8. *The conventional critical exponent for TP matrices satisfies*

$$g_C(\text{TP}) \leq \max_{k=1, \dots, n} \left\{ (2(n - k) + 1) \left\lfloor \frac{\binom{n+k-1}{k} - 1}{2} \right\rfloor \right\} + 1.$$

Proof: Let $R \in M_n$ be a TP matrix and define $\Gamma_{ij}^- = \{t : \det(A^t[I, J])\}$, where I and J are the index sets defining the initial minor with the bottom right entry at (i, j) . $A^t[I, J]$ is $k = (\min\{i, j\})$ -dimensional, so it follows from the same reasoning of Lemma 3.2 that Γ_{ij}^- has at most $\lfloor ((\binom{n+k-1}{k}) - 1) / 2 \rfloor$ connected components.

Let $\Gamma_k^- = (\cup_{k \leq i \leq n} \Gamma_{ik}^-) \cup (\cup_{k \leq j \leq n} \Gamma_{kj}^-)$. Then each Γ_k^- can have at most

$$b_k(n) = (2(n - k) + 1) \left\lfloor \frac{\binom{n+k-1}{k} - 1}{2} \right\rfloor$$

connected components.

All minors are positive at positive integers so all connected components of Γ_k^- are contained within two integers. If there exists a positive integer such that $\Gamma_k^- \cap (m, m + 1) = \emptyset$, then Cauchy-Binet and repeated left and right multiplication by R can be used to show that $\Gamma_k^- \cap (m, \infty) = \emptyset$, or, that the k -by- k initial minors of R^t are each positive for $t \geq m$. It follows, via the same argument from the DN proof, that the k -by- k initial minors are nonnegative for all $t \geq b_k(n) + 1$. The maximum b_k therefore determines the upper bound for the critical exponent. Let $b(n) = \max\{b_k(n)\}, k = 1, \dots, n$. Then the conventional critical exponent of R cannot exceed

$$g_C(\text{TP}) \leq b(n) + 1 = \max_{k=1, \dots, n} \left\{ (2(n - k) + 1) \left\lfloor \frac{\binom{n+k-1}{k} - 1}{2} \right\rfloor \right\} + 1.$$

3.3.2 Lower bound

Theorem 3.9. *The critical exponent for conventional powers of TN matrices is at least $n - 2$.*

Proof: Consider

$$R = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

It is quickly clear that most minors of R are zero. Of the k -by- k initial minors, $2(n - k) - 2$ will be zero, two will equal 1, and one will be the k -th principal minor. Every principal submatrix is a tridiagonal Toeplitz matrix, the eigenvalues of which are well known [4]. In the example provided, the eigenvalues of the k -by- k principal submatrix are

$$\lambda_j = 2 + 2 \cos \left(\frac{\pi j}{k + 1} \right),$$

all of which are positive. Every principal minor is thus positive, and the matrix R is TN. Because R is DN, however, it is known (Theorem 3.3) that the $(1, n)$ entry is negative within the interval $(n - 3, n - 2)$. So the critical exponent for conventional powers of TN matrices must be at least $n - 2$.

4 Hadamard powers

The term critical exponent, as it is used here, first arose in addressing the question of continuous Hadamard powers of DN matrices in relation to the Bieberbach conjecture. A detailed discussion of the original proof can be found in [5], although the topic is also covered in [6].

Theorem 4.1. *The Hadamard critical exponent for DN matrices is $n - 2$.*

The proof by induction of this statement appears in [5] and [6] and involves a modified version of the formula for the remainder of a Taylor Series. Specifically, let $A = [a_{ij}] \in M_n$ be DN, define $\gamma = [a_{n1} \cdots a_{nn}] / \sqrt{a_{nn}}$, and assume that the Hadamard critical exponent for all k -by- k matrices is $k - 2$ for all $k < n$ (trivial for $k = 2$). By performing a change of variables on the remainder formula for the Taylor Series expansion of $A^{(t)}$, we arrive at

$$A^{(t)} = (\gamma\gamma^T)^{(t)} + t(A - \gamma\gamma^T) \circ \int_0^1 (x(A - \gamma\gamma^T) + \gamma\gamma^T)^{(t-1)} dx$$

Because $\gamma\gamma^T$ and $A - \gamma\gamma^T$ are both positive semi-definite matrices (verified in [5]), the integral is the limit of Riemann sums of PSD matrices raised to the $(t - 1)$ -th power. Hadamard multiplication by $A - \gamma\gamma^T$ selects the upper-left $(n - 1)$ -by- $(n - 1)$ submatrix of every matrix in this sum which, by the induction hypothesis, is DN for all $t \geq n - 2$. $A^{(t)}$ is thus the sum of two nonnegative PSD matrices. So $A^{(t)}$ is DN for all $t \geq n - 2$ and $g(\text{DN}) = n - 2$ is an upper bound for the critical exponent.

That $n - 2$ is also the lower bound for the critical exponent follows from the following example, provided in [5]. Define $A_\epsilon = (1 + \epsilon ij)$, $1 \leq i, j \leq n$. Analysis in the next section will verify that A_ϵ is DN, and the vectors $v_k = (1^k, 2^k, \dots, n^k)^T$ for $k = 0, 1, 2, \dots$ can be determined to be linearly independent because the determinant of the Vandermonde matrix produced from them is positive. Take $\alpha < n - 2$ and choose a real n vector $x = (x_i)$ such that x is orthogonal to $v_0, v_1, \dots, v_{\lfloor \alpha \rfloor + 1}$ and $\sum i^{\lfloor \alpha \rfloor + 2} x_i = 1$. Expand $x^T A_\epsilon^\alpha x$ (with $\binom{\alpha}{k} = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)/k!$ understood to be the standard binomial coefficient), arriving at:

$$\begin{aligned} x^T A_\epsilon^\alpha x &= \sum_{i,j=1}^n (1 + \epsilon ij)^\alpha x_i x_j = \sum_{i,j=1}^n \sum_{k=0}^{\infty} \binom{\alpha}{k} \epsilon^k i^k j^k x_i x_j \\ &= \sum_{k=0}^{\lfloor \alpha \rfloor + 1} \binom{\alpha}{k} \epsilon^k \left(\sum_{i=1}^n i^k x_i \right)^2 + \binom{\alpha}{\lfloor \alpha \rfloor + 2} \epsilon^{\lfloor \alpha \rfloor + 2} \left(\sum_{i=1}^n i^{\lfloor \alpha \rfloor + 2} x_i \right)^2 + \mathcal{O}(\epsilon^{\lfloor \alpha \rfloor + 3}) \\ &= \binom{\alpha}{\lfloor \alpha \rfloor + 2} \epsilon^{\lfloor \alpha \rfloor + 2} + \mathcal{O}(\epsilon^{\lfloor \alpha \rfloor + 3}) \qquad \text{as} \qquad \epsilon \rightarrow 0 \end{aligned}$$

The binomial coefficient $\binom{\alpha}{\lfloor \alpha \rfloor + 2}$ is negative; therefore, for sufficiently small ϵ , $x^T A_\epsilon^\alpha x < 0$ and A_ϵ^α is *not* DN. □

4.1 Totally positive matrices

The question of a Hadamard critical exponent for TP matrices is of particular interest for the following reason. Let A be TP_2 (that is, assume all entries and 2-by-2 minors are positive). Then there exists some power p , specific to A , such that $A^{(t)}$ is TP for all $t \geq p$ [1]. Thus, there is a Hadamard critical exponent for each TP matrix A , which does not, by itself, imply that there is a critical exponent for the class TP, under Hadamard multiplication. Those subclasses for which a critical exponent is known to exist are documented below.

4.1.1 Existence and upper bound

It is currently unclear whether or not a Hadamard critical exponent exists. Certain classes within TP matrices are known to have a critical exponent and are discussed; however, no clear general picture has emerged. The following condition *would* imply the existence of a critical exponent, if it could be verified to be true.

Theorem 4.2. *If there exists some power l such that, for all n -by- n TP matrices A , $A^{(l)}$ is TP, then there exists a Hadamard critical exponent.*

Proof: Let $l \in \mathbb{R}$ be a number such that $A^{(l)}$ is TP for all TP matrices A . Then $A^{(l^k)}$ is TP $\forall k \in \mathbb{Z}$. If $A^{(t)}$ is TP for all $t \in [l^k, l^{k+1}]$ then it follows from repeated powering of the matrices in the interval that $A^{(t)}$ is TP for all $t \geq l$. If, instead, $A^{(t)}$ has one or more negative minors in the interval $[l^k, l^{k+1}]$, then at least two roots of each of the exponential polynomials defining these minors must occur within the interval. Because the maximum number of roots every minor can experience is a function of n , there exists some $g_H(\text{TP})$ past which no minor can be negative and $A^{(t)}$ is TP for all $t \geq g_H(\text{TP})$.

The only instance in which this observation has been useful is in the case of 3-by-3 TP matrices, discussed later.

4.1.2 Lower bound

While the critical exponent for TP matrices is at least $n - 2$ (discussed in “Sums of two rank one matrices”), higher lower bounds have been found. In fact, a 4-by-4 example of a TP matrix has been found such that its power has negative determinant in the interval $t \in (5, 6)$; this matrix is below.

Example 4.1.

$$\begin{pmatrix} 210 & 48547 & 80633 & 82930 \\ 71539 & 17126755 & 29592586 & 30438643 \\ 84121 & 21345134 & 39294848 & 42461372 \\ 73730 & 20385912 & 40697553 & 46818689 \end{pmatrix}$$

That Hadamard powers past $t = 2$ of this matrix had negative determinant was verified using multiple computing engines (MATLAB, Maple, and WolframAlpha), each with high degrees of precision, to rule out the possibility that rounding errors were responsible for the result.

In addition to the 4-by-4 example, several examples of 5-by-5 TP matrices with critical exponent greater than 3 have been identified, including the following:

Example 4.2.

$$\begin{pmatrix} 16 & 45461 & 44457 & 63707 & 59825 \\ 26881 & 79157731 & 79367653 & 114713863 & 108564715 \\ 11150 & 33282070 & 34695898 & 51108214 & 49202383 \\ 25055 & 75689845 & 81588250 & 122093057 & 119169425 \\ 20130 & 60873512 & 66933031 & 104064830 & 105621231 \end{pmatrix}$$

A 6-by-6 example with critical exponent greater than $n - 2$ is omitted in the interest of space.

4.1.3 3-by-3 TP matrices

Lemma 4.1. *The Hadamard square of a 3-by-3 TP matrix is TP.*

Proof: Let R be a TP matrix scaled so that both the first row and column are all ones. Without loss of generality, assume the (3,2)-entry to be greater than or equal to the (2,3) entry. Then R can be written as:

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + \alpha & 1 + \alpha + \beta \\ 1 & 1 + \alpha + \beta + \gamma & 1 + \alpha + \beta + \gamma + \delta \end{pmatrix}$$

where $\alpha, \beta, \gamma,$ and δ are all positive. It can be quickly checked that this way of writing the matrix is a valid way to preserve the positivity of the 2-by-2 minors and that the 2-by-2 minors of R^t will be positive for all $t > 0$.

Because R is TP, its determinant, $r = -\alpha\beta + \alpha\delta - \beta^2 - \beta\gamma > 0$. It can be shown by explicit calculation that the determinant of $R^{(2)}, r^{(2)}$, is expressible in the following way:

$$r^{(2)} = 4r + 2\alpha^2r + 6\alpha r + \beta^2r + \alpha\delta r + 2\gamma r + \beta\gamma r + 4\beta r + 3\alpha\beta r + 2\alpha\gamma r + 2\alpha\delta^2 + 2\alpha\gamma\delta$$

As every term in the expression on the right hand side of the equation is positive, $r^{(2)}$ is positive. R is therefore TP.

Theorem 4.3. *The Hadamard critical exponent for 3-by-3 TP matrices is 1.*

Proof: We know from Theorem 4.2 and Lemma 4.1 that there exists a critical exponent for 3-by-3 TP matrices, but it must be demonstrated that the critical exponent is 1. Let R be a 3-by-3 TP matrix scaled in the same manner as above. As in Lemma 3.1, it is easy to observe that the 2-by-2 minors of $R^{(t)}$ will remain positive for $t > 0$; it remains to show the same for the determinant. Write R as follows:

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}$$

The determinant of $R^{(t)}$, therefore, is $(ad)^t - d^t - (bc)^t + b^t + c^t - a^t$. The requirements upon the 2-by-2 minors of R that make it TP (for instance, $ad > bc$) define a partial ordering on $a, b, c,$ and d . The terms with ambiguous order, d^t and $(bc)^t$, and b^t and c^t , can be rearranged in the determinant without changing the sign pattern of the coefficients. Because the number of sign changes is $w = 3$, the maximum number of zeros that can be experienced by the determinant is also 3.

Assume $R^{(t)}$ is TP for the entirety of the interval $t \in (0, 1]$. Then by the arguments of Lemma 3.1 and Theorem 3.2, $R^{(t)}$ is TP for all $t > 0$. Suppose the determinant of $R^{(t)}$ is negative at some point in $(0, 1]$. Then at least two of the determinant's three possible roots have occurred. If the remaining available root occurs at any $t > 1$, the determinant will be negative for all t thereafter; contradicting the claim that any TP matrix is eventually permanently TP. So the upper bound for the Hadamard critical exponent of a 3-by-3 TP matrix is $g_H = 1$. As is discussed in the next section, the lower bound is also 1; thus, the Hadamard critical exponent for a 3-by-3 TP matrix is $g_H = 1$.

4.1.4 Sums of two rank one matrices

In their proof that $n - 2$ is the lower bound for the Hadamard critical exponent for DN matrices [5], FitzGerald and Horn use the matrix $A_\epsilon = [1 + \epsilon ij]$. In fact, this is a specific example of a more general type of matrix, discussed here in the context of TP. Consider the sum of two rank one matrices, and WLOG assume one to be J (the matrix of all 1's). Let p and q be two positive vectors.

Theorem 4.4. $A = J + pq^T$ is TP_2 iff p and q are both strictly increasing or both strictly decreasing.

Proof: Any 2-by-2 minor of A has the form $(1 + p_i q_i)(1 + p_j q_j) - (1 + p_i q_j)(1 + p_j q_i)$ which can in turn be written as

$$(1 + p_i q_i)(1 + p_j q_j) - (1 + p_i q_j)(1 + p_j q_i) = p_i q_i + p_j q_j - p_i q_j - p_j q_i = (p_i - p_j)(q_i - q_j)$$

The expression $(p_i - p_j)(q_i - q_j)$ is nonnegative if and only if p and q are either both increasing or both decreasing. \square

Theorem 4.5. *The Hadamard critical exponent for TP matrices is at least $n - 2$.*

By Lemma 4.4, A_ϵ from Theorem 4.1 is TP_2 in addition to being DN. By continuity, this extends to the class TP.

It is conjectured that the critical exponent for TP matrices that are the sums of two rank one matrices is $n - 2$; asymmetry prevents an identical proof to Theorem 4.1 from being used to show this.

4.1.5 Hurwitz matrices

The Hurwitz matrix H of the polynomial $\sum_{i=0}^n a_i x^i$ is given by $h_{ij} = a_{2j-1}$, $1 \leq i, j \leq n$, $0 \leq 2j - 1 \leq n$, and $h_{ij} = 0$ elsewhere. The properties of TP Hurwitz matrices are covered in [1].

Theorem 4.6. *There exists a critical exponent for TP Hurwitz matrices.*

Proof: It is shown in [7] that integer Hadamard powering of Hurwitz TP matrices preserves the property of being TP. By Theorem 3.2, there exists a critical exponent for Hurwitz TP matrices.

4.2 Inverse M-matrices

An M-matrix is an invertible square matrix with positive principal minors and non-positive off-diagonal entries. An inverse M-matrix (IM), unsurprisingly, is a matrix that is the inverse of an M-matrix.

Theorem 4.7. *The Hadamard critical exponent for the class of n -by- n inverse M-matrices is $g(IM) = 1$ for $n > 2$, and $g(IM) = 0$ for $n = 2$.*

Lemma 4.2. *If $A \in M_n$ is inverse-M, the following conditions hold:*

- i. All principal minors are positive.*
- ii. All principal submatrices of A is IM.*
- iii. If P is a permutation matrix of order n , PAP^T is IM.*
- iv. If α is a proper subset of the indices of A , $A(\alpha)^{-1}A(\alpha, \alpha^c) \geq 0$ and $A(\alpha^c, \alpha)A(\alpha)^{-1} \geq 0$, where α^c is the complement of α .*
- v. For all distinct positive integers $i, j, k \leq n$, $a_{ii} > 0$, $a_{ik}a_{kj} \leq a_{kk}a_{ij}$, and $a_{ik}a_{ki} < a_{kk}a_{ii}$.*

The first three statements of Lemma 4.3 are well-known facts about IM matrices. Statement (iv) comes from work by C. R. Johnson [8], and statement (v) appears originally in [9]. Theorem 4.7 originally appeared in [10], from which one additional statement is needed.

Theorem 4.8. [10] *Let $r > 1$ be a real number. Let $A = [a_{ij}] \in M_n$ be IM with columns $\alpha_1, \alpha_2, \dots, \alpha_n$. If $A^{(r)}$ is IM and β is a nonnegative linear combinations of α_i ; that is, $\beta = \sum_{i=1}^n x_i \alpha_i$, where $x_i \geq 0$. Then:*

$$\det \left(\alpha_1^{(r)}, \alpha_2^{(r)}, \dots, \alpha_{n-1}^{(r)}, \beta^{(r)} \right) \geq 0.$$

Proof of Theorem 4.7 When $n = 2$, it is easy to verify that for any $r > 1$ $A^{(r)}$ is IM. Assume $n \geq 3$ and proceed by induction. Let the (ij) entry of $\text{adj}(A^{(r)})$ be

$$b_{ij} = -\det \begin{pmatrix} A(\sigma)^{(r)} & A(\sigma, j)^{(r)} \\ A(i, \sigma)^{(r)} & a_{ij}^{(r)} \end{pmatrix}$$

Define

$$B = \begin{pmatrix} A(\sigma) & A(\sigma, i) & A(\sigma, j) \\ A(i, \sigma) & a_{ii} & a_{ij} \end{pmatrix}$$

with columns $\beta_1, \beta_2, \dots, \beta_n$. The matrix

$$H = \begin{pmatrix} A(\sigma) & A(\sigma, i) & A(\sigma, j) \\ A(i, \sigma) & a_{ii} & a_{ij} \\ A(j, \sigma) & a_{ji} & a_{jj} \end{pmatrix}$$

is also IM by Lemma 4.3 (iii). By Lemma 4.3 (iv), it is known that $(\beta_1, \beta_2, \dots, \beta_{n-1})^{-1} \beta_n = (x_1, x_2, \dots, x_{n-1})^T \geq 0$; or $\beta_n = \sum_{i=1}^{n-1} x_i \beta_i$.

Our assumption includes that every principal submatrix (of size less than n) of $H^{(r)}$ is IM; this includes $(\beta_1^{(r)}, \beta_2^{(r)}, \dots, \beta_{n-1}^{(r)})$. From Theorem 4.8, we find

$$b_{ij} = -\det \left(\beta_1^{(r)}, \beta_2^{(r)}, \dots, \beta_{n-2}^{(r)}, \beta_n^{(r)} \right) \leq 0$$

This is equivalent to the statement $A^{(r)}$ is IM. □

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