7-2012

Fixed Points of Pick and Stieltjes functions: A Linear Algebraic Approach

Nicholas Andrew Woods
College of William and Mary

Fixed Points of Pick and Stieltjes Functions:
A Linear Algebraic Approach

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Mathematics from
The College of William and Mary

by

Nicholas Andrew Woods

Accepted for ______________________
(Honors, High Honors, Highest Honors)

____
Vladimir Bolotnikov, Director

____
Ilya Spitkovsky

____
Jianjun Paul Tian

____
Joshua Gert

Williamsburg, VA
April 25, 2012

FIXED POINTS OF PICK AND STIELTJES FUNCTIONS: A LINEAR ALGEBRAIC APPROACH

NICK WOODS

Abstract. The functions analytic in the upper half-plane $\mathbb{C}^+$ and mapping $\mathbb{C}^+$ into itself (the so-called Pick functions) play a prominent role in several branches of mathematics. In this thesis we study fixed points of such functions. It is known that a Pick-class function different from the identity map can have at most one fixed point in $\mathbb{C}^+$. However, it may have many (even infinitely many) appropriately defined boundary fixed points. We establish relations between the values of the derivative of a Pick function at these fixed points. Similar questions are considered in the context of Stieltjes-class functions which, in addition, are analytic on the positive half-axis and map this half-axis into itself.

Acknowledgements. I would first like to thank Dr. Jianjun Paul Tian, Dr. Ilya Spitkovsky, and Dr. Joshua Gert for their time and their willingness to serve on my honors committee. I would also like to thank my advisor, Dr. Ryan Vinroot, for direction and encouragement throughout my time at William & Mary. Finally, my thesis advisor, Dr. Vladimir Bolotnikov, provided an incredible amount of time, knowledge, and ideas, and I am enormously grateful for his help and for making this experience possible.

Contents

1. Pick class functions 1
2. Stieltjes class functions 5
3. Interior fixed points of Pick class functions 7
4. Interior fixed points of Stieltjes class functions 10
5. Boundary fixed points of Pick class functions 13
6. Boundary fixed points of Stieltjes class functions 27
References 28

1. Pick class functions

A function f is in the Pick class $\mathcal{P}$ if it is analytic on the open upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ and $\text{Im } f(z) \geq 0$ for all $z \in \mathbb{C}^+$. In other words, any Pick function maps $\mathbb{C}^+$ into $\overline{\mathbb{C}^+} = \mathbb{C}^+ \cup \mathbb{R}$. It then follows that the

Date: April 25, 2012.

class $\mathcal{P}$ is closed under addition and under composition. Simple examples of Pick functions are the functions

$$a + bi, \quad a + bz, \quad \frac{b}{c - z} \qquad (b > 0,\ a, c \in \mathbb{R}).$$

The functions of Pick class traditionally play central roles in extention theory of symmetric operators [15, 27], the spectral theory of ordinary differential and difference operators [7, 9, 10, 13, 20, 25, 33], interpolation problems [34], inverse spectral theory [2, 4, 5, 6, 28, 30, 31, 32], inverse scattering [2, 4, 5, 6], and completely integrable hierarchies of non-linear evolution equations [8, 16]. Other areas of application include control theory [26] and Loewner theory of monotone matrix functions [17].

The fundamental result on integral representations of Pick functions is due to Herglotz and Riesz (see [3, Ch. 6] or [23] for the proof).

Theorem 1.1. *A function f is in the Pick class if and only if it can be represented in the form*

$$f(z) = a + bz + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\mu(t), \tag{1.1}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $d\mu$ is a positive measure on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1 + t^2} < \infty. \tag{1.2}$$

The representation (1.1) is called the *Riesz-Herglotz Integral Representation* of f. The constants a and b in this representation are recovered from f by the formulas

$$a = \operatorname{Re} f(i) \quad \text{and} \quad b = \lim_{y \to \infty} \frac{f(iy)}{iy} \geq 0, \tag{1.3}$$

whereas the measure $d\mu$ is recovered from f by the Stieltjes inversion formula: for every $x_1 < x_2 \in \mathbb{R}$,

$$\frac{1}{2}\mu(\{x_1\}) + \frac{1}{2}\mu(\{x_2\}) + \mu((x_1, x_2)) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{x_1}^{x_2} \operatorname{Im} f(t + i\varepsilon)\, dt. \tag{1.4}$$

Remark 1.2. It is quite traditional to extend Pick functions to the lower half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ by reflection, thereby defining the Pick class as the class of functions analytic in $\mathbb{C}\backslash\mathbb{R}$ and such that

$$f(z) = \overline{f(\overline{z})} \quad \text{and} \quad \frac{f(z) - \overline{f(z)}}{z - \overline{z}} \geq 0. \quad \text{for all} \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.5}$$

It is readily seen that the formula (1.1) holds true for any f subject to conditions (1.5) and for any $z \in \mathbb{C}\backslash\mathbb{R}$.

Another fundamental result concerning Pick functions is due to Fatou, Lusin and Privalov (see for example [24, Ch. 6] or [29, Ch. 5]).

Theorem 1.3. *Let f be a Pick function. Then the normal (equivalently, angular) boundary limits*

$$f(x \pm i0) := \lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon) \tag{1.6}$$

exist for almost all $x \in \mathbb{R}$. Furthermore, if f has a zero normal limit on a subset of $\mathbb{R}$ having positive Lebesgue measure, then $f \equiv 0$.

Definition 1.4. *A function $K(z, \zeta)$ is called a positive kernel on the domain $\Omega \subset \mathbb{C}$ if for every $n \in \mathbb{N}$ and for any choice of n points $z_1, \ldots, z_n \in \Omega$ and n complex numbers $c_1, \ldots, c_n \in \mathbb{C}$,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(z_i, z_j) c_i \overline{c_j} \geq 0,$$

or, equivalently, the matrix $[K(z_i, z_j)]_{i,j=1}^{n}$ is positive semidefinite.

With any function f, we may associate the kernel $K_f(z, \zeta) = \dfrac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}}$ defined on the Cartesian square of $\mathrm{Dom}(f)$, the domain of definition of f. This kernel is *Hermitian* in the sense that $K_f(z, \zeta) = \overline{K_f(\zeta, z)}$ for all $z, \zeta \in \mathrm{Dom}(f)$ such that $z \neq \overline{\zeta}$. For Pick-class functions, the kernel K_f plays a particularly important role. Observe that due to the symmetry relation (1.5) the kernel K_f can be extended to the point $(z, \overline{z})$ by continuity as $K(z\overline{z}) = f'(z)$.

Theorem 1.5. *If $f \in \mathcal{P}$, then the kernel*

$$K_f(z, \zeta) = \begin{cases} \dfrac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ f'(z) & \text{if } z = \overline{\zeta}, \end{cases} \tag{1.7}$$

is positive on $\mathbb{C} \backslash \mathbb{R}$.

Proof. Using the Riesz-Herglotz Integral Representation (1.1) for f and recalling that a and b are real, we have

$$\begin{aligned} K_f(z, \zeta) = \frac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} &= \frac{1}{z - \overline{\zeta}} \left(bz - b\overline{\zeta} + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{1}{t - \overline{\zeta}} \right) d\mu(t) \right) \\ &= b + \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t - z)(t - \overline{\zeta})}. \end{aligned} \tag{1.8}$$

We can therefore write for some fixed points $z_1, \dots, z_n \in \mathbb{C}\backslash\mathbb{R}$ such that $z_i \neq \overline{z}_j$ for $i, j \in \{1, \dots, n\}$,

$$[K_f(z_i, z_j)]_{i,j=1}^n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} b \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + \int_{-\infty}^{\infty} \begin{bmatrix} \frac{1}{t-z_1} \\ \vdots \\ \frac{1}{t-z_n} \end{bmatrix} d\mu(t) \begin{bmatrix} \frac{1}{t-\overline{z_1}} & \cdots & \frac{1}{t-\overline{z_n}} \end{bmatrix}. \tag{1.9}$$

If $z_i = \overline{z}_j$ for some i, j, then the equality (1.9) also holds true, since (thanks to (1.1))

$$f'(z) = b + \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-z)^2}.$$

Since $b \geq 0$ and $d\mu(t) \geq 0$, we see that both terms on the right of (1.9) are positive semidefinite matrices so that their sum

$$K^f_{z_1,\dots,z_n} := [K_f(z_i, z_j)]_{i,j=1}^n = \left[\frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \right]_{i,j=1}^n \tag{1.10}$$

is also positive semidefinite, which completes the proof. □

Definition 1.6. *For $f \in \mathcal{P}$, the matrix* (1.10) *is called the Schwarz-Pick matrix of f based on points $z_1, \dots, z_n$.*

We thus proved that for any choice of $z_1, \dots, z_n \in \mathbb{C}\backslash\mathbb{R}$, the Schwarz-Pick matrix (1.10) is positive semidefinite. A natural question to ask is the following: *does the positivity of the kernel K_f on $\mathbb{C}^+$ guarantee the membership of f in the Pick class $\mathcal{P}$?*

The answer is clearly affirmative if we know that f is analytic on $\mathbb{C}^+$. In this case, the positivity of K_f implies in particular that

$$K(z, z) = \frac{f(z) - \overline{f(z)}}{z - \overline{z}} = \frac{\mathrm{Im} f(z)}{\mathrm{Im} z} \geq 0 \quad \text{for all} \quad z \in \mathbb{C}^+$$

and thus $\mathrm{Im} f(z) \geq 0$ for all $z \in \mathbb{C}^+$ so that f belongs to $\mathcal{P}$ by the very definition of this class. It turns out that in fact the positivity of K_f on $\mathbb{C}^+$ *implies* that f is analytic on on $\mathbb{C}^+$ and thus one may conclude that *a function f defined everywhere on $\mathbb{C}^+$ belongs to the Pick class $\mathcal{P}$ if and only if the kernel K_f is positive on $\mathbb{C}^+$*. Equivalently, $f \in \mathcal{P}$ if and only if the Schwarz-Pick matrices (1.10) are positive semidefinite for every choice of a positive integer n and points $z_1, \dots, z_n \in \mathbb{C}^+$. A remarkable result of Hindmarsch [21] states that the membership $f \in \mathcal{P}$ follows from substantially weaker assumptions.

Theorem 1.7. *If f is defined everywhere on $\mathbb{C}^+$ and 3×3 Schwarz-Pick matrices $K^f_{z_1,z_2,z_3}$ are positive semidefinite for all choices of $z_1, z_2, z_3 \in \mathbb{C}^+$, then f is analytic on $\mathbb{C}^+$ and therefore belongs to $\mathcal{P}$.*

To conclude this introductory section, we present a simple corollary of Theorem 1.5. An analytic proof is given in [17, p.18].

Theorem 1.8. *If $f \in \mathcal{P}$ and* $\operatorname{Im} f(z_0) = 0$ *for some* $z_0 \in \mathbb{C}^+$, *then* $f(z) \equiv c \in \mathbb{R}$.

Proof. Since $\operatorname{Im} f(z_0) = 0$, we have $f(z_0) = \overline{f(z_0)}$. Then, for all $z \in \mathbb{C}^+$ such that $z \neq z_0$ we have the Schwarz-Pick matrix

$$K^f_{z_0,z} = \begin{bmatrix} \dfrac{f(z_0) - \overline{f(z_0)}}{z_0 - \overline{z_0}} & \dfrac{f(z) - \overline{f(z_0)}}{z - \overline{z_0}} \\ \\ \dfrac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{f(z) - f(z_0)}{z - \overline{z_0}} \\ \\ \dfrac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix}.$$

Because f belongs to $\mathcal{P}$, the matrix $K^f_{z_0,z}$ is positive semidefinite, so

$$0 \leq \det A = 0 - \frac{f(z) - \overline{f(z_0)}}{z - \overline{z_0}} \cdot \frac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} = -\frac{|f(z) - \overline{f(z_0)}|^2}{|z - \overline{z_0}|^2}. \tag{1.11}$$

We have that $|z - \overline{z_0}|^2$ is always positive and that $|f(z) - \overline{f(z_0)}|^2$ is nonnegative. Therefore we have $f(z) - \overline{f(z_0)} = 0$, so $f(z) = \overline{f(z_0)} = f(z_0)$ for all $z \in \mathbb{C}^+$. □

Corollary 1.9. *Whenever $f \in \mathcal{P}$ is not constant,* $\operatorname{Im} f(z) > 0$ *for every* $z \in \mathbb{C}^+$.

Thus we conclude that the class $\mathcal{P}$ consists of two types of functions: (real) constant functions, which we exclude as trivial, and analytic self-mappings of $\mathbb{C}^+$. Then it makes sense to consider iterations of a given function $f \in \mathcal{P}$, in which case the knowledge of fixed points is of great importance.

2. Stieltjes class functions

A function f is in the Stieltjes class $\mathcal{S}$ if $f \in \mathcal{P}$, and, in addition, is analytic on $\mathbb{R}_+ = \{x \mid x > 0\}$ and $s(x) > 0$ for all $x \in \mathbb{R}_+$. That is, Stieltjes functions are Pick functions which also map $\mathbb{R}_+$ into itself. By the symmetry principle, every Stieltjes function s satisfies the symmetry relation $s(\bar{z}) = \overline{s(z)}$ for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. By Theorem 1.1 every Stieltjes function admits the Herglotz integral representation (1.1). Due to additional property that s is analytic and nonnegative on $\mathbb{R}_+$, the representation (1.1) is quite special. The two next results appear in [18].

Theorem 2.1. *A function s is in the Stieltjes class if and only if it can be represented in the form*

$$s(z) = a + bz + \int_0^\infty \frac{z}{z+t} d\mu(t) \tag{2.1}$$

where $a, b \geq 0$ and $d\mu$ is a positive measure on $\mathbb{R}$ such that

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

Some simple examples of Stieltjes functions and their corresponding measures are

$$\begin{aligned}
&s(z) = z^\alpha,\ 0 < \alpha < 1 && \text{where} \quad \mu(t) = \frac{\sin(\alpha\pi)}{\pi} t^{\alpha-1},\\
&s(z) = \sqrt{z}(1 - e^{-2a\sqrt{z}}),\ a > 0 && \text{where} \quad \mu(t) = \frac{2}{\pi\sqrt{t}} \sin^2(a\sqrt{t}),\\
&s(z) = \log\left(1 + \frac{z}{a}\right),\ a > 0 && \text{where} \quad \mu(t) = \frac{1}{t}\mathbf{1}_{(a,\infty)}(t),\ \text{and}\\
&s(z) = \sqrt{z}\arctan\left(\sqrt{\frac{z}{a}}\right),\ a > 0 && \text{where} \quad \mu(t) = \frac{1}{2\sqrt{t}}\mathbf{1}_{(a,\infty)}(t).
\end{aligned}$$

Here $\mathbf{1}_A$ is the indicator function; that is, $\mathbf{1}(t) = 1$ if $t \in A$ and $\mathbf{1}(t) = 0$ otherwise.

The next theorem establishes more precise connection between Pick and Stieltjes classes.

Theorem 2.2. *A function s is in the Stieltjes class if and only if $s \in \mathcal{P}$ and $-\frac{s(z)}{z} \in \mathcal{P}$.*

The latter theorem allows us to associate with any Stieltjes-class function s two kernels

$$K_s(z,\zeta) = \begin{cases} \dfrac{s(z) - \overline{s(\zeta)}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ s'(z) & \text{if } z = \overline{\zeta}, \end{cases} \tag{2.2}$$

$$\widetilde{K}_s(z,\zeta) = \begin{cases} \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(\zeta)}}{\overline{\zeta}}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ \dfrac{s(z)}{z^2} - \dfrac{s'(z)}{z} & \text{if } z = \overline{\zeta}, \end{cases} \tag{2.3}$$

which are positive on $\mathbb{C}^+$. We next express the kernel $\widetilde{K}_s$ in terms of the integral representation (2.1) (the expression for the kernel K_s is the same as in formula (1.8)):

$$\begin{aligned}\widetilde{K}_s(z,\zeta) &= \frac{1}{z-\overline{\zeta}}\left(-\frac{a}{z} - \int_0^\infty \frac{d\mu(t)}{z+t} + \frac{a}{\overline{\zeta}} + \int_0^\infty \frac{d\mu(t)}{\overline{\zeta}+t}\right) \\ &= \frac{a}{z\overline{\zeta}} + \frac{1}{z-\overline{\zeta}} \cdot \int_0^\infty \left(\frac{1}{\overline{\zeta}+t} - \frac{1}{z+t}\right) d\mu(t) \\ &= \frac{a}{z\overline{\zeta}} + \int_0^\infty \frac{d\mu(t)}{(z+t)(\overline{\zeta}+t)}. \end{aligned} \tag{2.4}$$

In fact, the latter formula combined with Theorem 2.1 demonstrates the positivity of the kernel $\widetilde{K}_s$.

Theorem 2.2 allows us to characterize Stieltjes functions in terms of two Schwartz-Pick matrices.

Corollary 2.3. *A function s belongs to the Stieltjes class if and only if, for all $z_1,\dots,z_n \in \mathbb{C}^+$,*

$$K^s_{z_1,\dots,z_n} = \left[\frac{s(z_i) - \overline{s(z_j)}}{z_i - \overline{z}_j}\right]_{i,j=1}^n \geq 0 \tag{2.5}$$

and

$$\widetilde{K}^s_{z_1,\dots,z_n} = \left[\frac{z_i\overline{s(z_j)} - \overline{z}_j s(z_i)}{z_i\overline{z}_j(z_i - \overline{z}_j)}\right]_{i,j=1}^n \geq 0. \tag{2.6}$$

Proof. The first matrix is simply the Pick matrix of s and so is positive semidefinite for all choices of $z_1,\dots,z_n$ if and only if $f \in \mathcal{P}$. For the second matrix, if $g(z) = -s(z)/z$ we have

$$[\widetilde{K}^s_{z_1,\dots,z_n}]_{ij} = \frac{-s(z_i)/z_i + \overline{s(z_j)/z_j}}{z_i - \overline{z}_j} = \frac{z_i\overline{s(z_j)} - \overline{z}_j s(z_i)}{z_i\overline{z}_j(z_i - \overline{z}_j)},$$

so the matrix being positive semidefinite is equivalent to $g \in \mathcal{P}$.

3. Interior fixed points of Pick class functions

Given a function $f \in \mathcal{P}$, we say that a point $x \in \mathbb{C}^+$ is a fixed point if $f(z_0) = z_0$. As the next theorem shows, a Pick function may have at most one fixed point in $\mathbb{C}^+$.

Theorem 3.1. *Any Pick class function $f \in \mathcal{P}$ different from the identity map has at most one interior fixed point.*

Proof. Let $z_1, z_2 \in \mathbb{C}^+$ be two distinct points such that

$$f(z_1) = z_1 \quad \text{and} \quad f(z_2) = z_2, \tag{3.1}$$

and let z be an arbitrary point in $\mathbb{C}^+$. The Schwarz-Pick matrix $K^f_{z_1,z_2,z}$ is positive semidefinite. According to (1.10) and (3.1) this matrix takes the form

$$K^f_{z_1,z_2,z} = \begin{bmatrix} 1 & 1 & \dfrac{z_1-\overline{f(z)}}{z_1-\overline{z}} \\ 1 & 1 & \dfrac{z_2-\overline{f(z)}}{z_2-\overline{z}} \\ \dfrac{f(z)-\overline{z}_1}{z-\overline{z}_1} & \dfrac{f(z)-\overline{z}_2}{z-\overline{z}_2} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \geq 0. \tag{3.2}$$

Then we also have

$$\begin{aligned} 0 &\leq \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} K^f_{z_1,z_2,z} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \dfrac{z_1-\overline{f(z)}}{z_1-\overline{z}} - \dfrac{z_2-\overline{f(z)}}{z_2-\overline{z}} \\ \dfrac{f(z)-\overline{z}_1}{z-\overline{z}_1} - \dfrac{f(z)-\overline{z}_2}{z-\overline{z}_2} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \det \begin{bmatrix} 0 & \dfrac{z_1-\overline{f(z)}}{z_1-\overline{z}} - \dfrac{z_2-\overline{f(z)}}{z_2-\overline{z}} \\ \dfrac{f(z)-\overline{z}_1}{z-\overline{z}_1} - \dfrac{f(z)-\overline{z}_2}{z-\overline{z}_2} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \\ &= -\left| \dfrac{f(z)-\overline{z}_1}{z-\overline{z}_1} - \dfrac{f(z)-\overline{z}_2}{z-\overline{z}_2} \right|^2, \end{aligned}$$

from which we conclude that for every $z \in \mathbb{C}_+ \setminus \{z_1, z_2\}$,

$$0 = \frac{f(z)-\overline{z}_1}{z-\overline{z}_1} - \frac{f(z)-\overline{z}_2}{z-\overline{z}_2} = \frac{(f(z)-z)(\overline{z_1}-\overline{z_2})}{(z-\overline{z}_1)(z-\overline{z}_2)}.$$

Therefore, $f(z) \equiv z$ which contradicts the assumption of the theorem. □

A fixed point z_0 is called *attractive* of a function f if for any point z that is close enough to z_0, the sequence of iterates $z,\ f(z),\ f(f(z)),\ f(f(f(z))), \dots$ converges to z_0. It is known that a fixed point z_0 of an analytic function f is attractive if $|f'(z_0)| < 1$.

Theorem 3.2. *If $z_0 \in \mathbb{C}^+$ is a fixed point of a Pick function f, then $|f'(z_0)| \leq 1$.*

Proof. Take z_0 to be the fixed point of f and let $z \in \mathbb{C}^+$. Then we have the positive semidefinite Schwarz-Pick matrix $K^f_{z_0,\overline{z_0}}$:

$$K^f_{z_0,\overline{z_0}} = \begin{bmatrix} \dfrac{f(z_0)-\overline{f(z_0)}}{z_0-\overline{z}_0} & f'(z_0) \\ \overline{f'(z_0)} & \dfrac{\overline{f(z_0)}-f(z_0)}{\overline{z_0}-z_0} \end{bmatrix} = \begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix} \geq 0.$$

Therefore, the determinant of the matrix, $1-|f'(z_0)|^2$, is greater than or equal to zero, so $|f'(z_0)| \leq 1$, which completes the proof. □

It is worth noting that, if f has an interior fixed point z_0 such that $|f'(z_0)| = 1$, then f is a linear fractional function which is real on $\mathbb{R}$.

Remark 3.3. *Let z_0 be an interior fixed point of $f \in \mathcal{P}$ such that $|f'(z_0)| = 1$. Then*

$$f(z) = \frac{(f'(z_0)-1)|z_0|^2 + (z_0 - \overline{z_0}f'(z_0))z}{(z_0 f'(z_0) - \overline{z_0}) + (1-f'(z_0))z}. \tag{3.3}$$

Proof. As in Theorem 3.1 above, use a 3×3 Schwarz-Pick matrix, where the first point is an interior fixed point and the second approaches the same point. Let z_0 be the interior fixed point of f and consider the positive semidefinite Schwartz-Pick matrix $K^f_{z_0,\overline{z_0},z}$:

$$K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & \dfrac{z_0-\overline{f(z)}}{z_0-\overline{z}} \\ \overline{f'(z_0)} & 1 & \dfrac{\overline{z}_0-\overline{f(z)}}{\overline{z}_0-\overline{z}} \\ \dfrac{\overline{z}_0-f(z)}{\overline{z}_0-z} & \dfrac{z_0-f(z)}{z_0-z} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \geq 0. \tag{3.4}$$

Keeping in mind that $|f'(z_0)| = 1$, we know that $\begin{bmatrix} f'(z_0) \\ -1 \end{bmatrix}$ is in the null space of $\begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix} \begin{bmatrix} f'(z_0) \\ -1 \end{bmatrix} = \begin{bmatrix} f'(z_0) - f'(z_0) \\ 1-1 \end{bmatrix}.$$

Therefore,

$$0 \leq \begin{bmatrix} \overline{f'(z_0)} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} K^f_{z_0,\overline{z}_0,z} \begin{bmatrix} f'(z_0) & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \overline{f'(z_0)} \left(\dfrac{z_0 - \overline{f(z)}}{z_0 - \overline{z}} \right) - \dfrac{\overline{z}_0 - \overline{f(z)}}{\overline{z}_0 - \overline{z}} \\ f'(z_0) \left(\dfrac{\overline{z}_0 - f(z)}{\overline{z}_0 - z} \right) - \dfrac{z_0 - f(z)}{z_0 - z} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix}.$$

Since the matrix is positive semidefinite, its determinant must be non-negative, so

$$0 - \left| f'(z_0) \left(\frac{\overline{z}_0 - f(z)}{\overline{z}_0 - z} \right) - \frac{z_0 - f(z)}{z_0 - z} \right|^2 \geq 0.$$

However, this can only occur when $f'(z_0) \left(\dfrac{\overline{z}_0 - f(z)}{\overline{z}_0 - z} \right) - \dfrac{z_0 - f(z)}{z_0 - z} = 0$. Then we have

$$0 = f'(z_0)(\overline{z_0} - f(z))(z_0 - z) - (z_0 - f(z))(\overline{z_0} - z)$$

which being solved for $f(z)$ leads us to formula (3.3). If $f'(z_0) = 1$, then (3.3) implies that $f(z) \equiv z$. If $f'(z_0) \neq 1$, we may rewrite (3.3) as

$$f(z) = \frac{|z_0|^2 + \alpha z}{\beta - z}, \quad \text{where} \quad \alpha = \frac{z_0 - \overline{z}_0 f'(z_0)}{f'(z_0) - 1}, \quad \beta = \frac{z_0 f'(z_0) - \overline{z}_0}{f'(z_0) - 1}.$$

Since $|f'(z_0)| = 1$, we have

$$\alpha - \overline{\alpha} = \frac{z_0 - \overline{z_0} f'(z_0)}{f'(z_0) - 1} - \frac{\overline{z}_0 - z_0 \overline{f'(z_0)}}{\overline{f'(z_0)} - 1} = \frac{(\overline{z}_0 - z_0)(1 - |f'(z_0)|^2}{|f'(z_0) - 1|^2} = 0$$

and thus α is real. Similarly, β is real, thus f takes real values on $\mathbb{R}$. □

4. Interior fixed points of Stieltjes class functions

We will now examine the fixed points of Stieltjes class functions inside the domain of analyticity, that is, in $\mathbb{C} \setminus \mathbb{R}_-$.

Theorem 4.1. *A Stieltjes class function s (not the identity map) may have at most one fixed point in $\mathbb{C} \setminus \mathbb{R}_-$. This point necessarily belongs to $\mathbb{R}_+$.*

Proof. We first show that s cannot have fixed points in $\mathbb{C}_+$. Let us assume that $s(z_0) = z_0$ for some $z_0 \in \mathbb{C}_+$. Then

$$\frac{z_0 \overline{s(z_0)} - \overline{z}_0 s(z_0)}{|z_0|^2 (z_0 - \overline{z}_0)} = \frac{|z_0|^2 - |z_0|^2}{|z_0|^2 (z_0 - \overline{z}_0)} = 0.$$

The Schwarz-Pick matrix $\widetilde{K}^s_{z_0,z}$ is positive semidefinite and its leftmost diagonal entry equals zero. Then its non-diagonal entries, which are complex conjugates, are also zero:

$$\frac{z\overline{s(z_0)}-\overline{z}_0 s(z)}{z\overline{z}_0(z-\overline{z}_0)}=\frac{z\overline{z_0}-\overline{z}_0 s(z)}{z\overline{z}_0(z-\overline{z}_0)}=\frac{z-s(z)}{z(z-\overline{z}_0)}=0.$$

Since the latter equality holds for all $z\in\mathbb{C}_+$, we conclude that $s(z)\equiv z$ which contradicts the assumption.

Assuming that $s(z_0)=z_0$ for some $z_0\in\mathbb{C}_-$, then by the symmetry relation we have $s(\overline{z}_0)=\overline{s(z_0)}=\overline{z}_0$ and thus the point $\overline{z}_0\in\mathbb{C}_+$ is a fixed point for s which is impossible. Therefore, s cannot have fixed points in $\mathbb{C}_+\cup\mathbb{C}_-$.

Let us assume that s has two fixed points in $\mathbb{R}_+$, i.e., that

$$s(x_1)=x_1 \quad \text{and} \quad s(x_2)=x_2 \quad \text{for some} \quad x_1,\ x_2>0. \tag{4.1}$$

The associated Schwarz-Pick matrices $K^s_{x_1,x_2}$ and $\widetilde{K}^s_{x_1,x_2}$ are positive semidefinite. According to (2.2), (2.3), (2.5),(2.6) and (4.1),

$$\begin{aligned}
&[K^s_{x_1,x_2}]_{ii}=s'(x_i) \quad \text{for} \quad i=1,2;\\
&[K^s_{x_1,x_2}]_{12}=\frac{s(x_1)-s(x_2)}{x_1-x_2}=\frac{x_1-x_2}{x_1-x_2}=1;\\
&[\widetilde{K}^s_{x_1,x_2}]_{ii}=\frac{s(x_i)}{x_i}-\frac{s'(x_i)}{x_i^2}=\frac{1-s'(x_i)}{x_i} \quad \text{for} \quad i=1,2;\\
&[\widetilde{K}^s_{x_1,x_2}]_{12}=\frac{-\frac{s(x_1)}{x_1}+\frac{s(x_2)}{x_2}}{x_1-x_2}=\frac{1-1}{x_1-x_2}=0.
\end{aligned}$$

We thus have

$$K^s_{x_1,x_2}=\begin{bmatrix} s'(x_1) & 1\\ 1 & s'(x_2)\end{bmatrix} \quad \text{and} \quad \widetilde{K}^s_{x_1,x_2}=\begin{bmatrix} \frac{1-s'(x_1)}{x_1} & 0\\ 0 & \frac{1-s'(x_2)}{x_2}\end{bmatrix}. \tag{4.2}$$

Since both matrices in (4.2) are positive semidefinite and since x_1, x_2 are positive numbers, we conclude that

$$0<s'(x_1),\ s'(x_2)\le 1 \quad \text{and} \quad s'(x_1)\cdot s'(x_2)\ge 1. \tag{4.3}$$

The latter may happen only if $s'(x_1)=s'(x_2)=1$. We then consider the Schwarz-Pick matrix $\widetilde{K}^s_{x_1,z}$ based on the fixed point x_1 and an arbitrary point

$z \in \mathbb{C} \setminus \mathbb{R}$:

$$\widetilde{K}^s_{x_1,z} = \begin{bmatrix} \dfrac{1-s'(x_1)}{x_1} & \dfrac{-\dfrac{s(x_1)}{x} + \dfrac{\overline{s(z)}}{\overline{z}}}{x_1 - \overline{z}} \\ \dfrac{-\dfrac{s(z)}{z} + \dfrac{s(x_1)}{x_1}}{z - x_1} & \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(z)}}{\overline{z}}}{z - \overline{z}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1 + \dfrac{\overline{s(z)}}{\overline{z}}}{x_1 - \overline{z}} \\ \dfrac{-\dfrac{s(z)}{z} + 1}{z - x_1} & \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(z)}}{\overline{z}}}{z - \overline{z}} \end{bmatrix}$$

where the last equality follows since $s(x_1) = x_1$ and $s'(x_1) = 1$. Since the matrix $\widetilde{K}^s_{x_1,z}$ is positive senidefinite for every z, we conclude that $-\dfrac{s(z)}{z} + 1 = 0$ for every z so that $s(z) \equiv z$ which contradicts the assumption of the theorem. □

The next result contains the Stieltjes-class analogues of Theorem 3.2 and of Remark 3.3.

Theorem 4.2. *If $x_0 \in \mathbb{R}^+$ is a fixed point of a Stieltjes function s, then $0 \leq s'(x_0) \leq 1$. Moreover, if $s'(x_0) = 0$, then $s(z) \equiv z_0$. If $s'(x_0) = 1$, then $s(z) \equiv z$.*

Proof. Consider that $\widetilde{K}^f_{x_0}$, as a 1×1 positive semidefinite matrix, must be a nonnegative real number. Then

$$\widetilde{K}^f_{x_0} = \frac{s(x_0)}{x_0^2} - \frac{s'(x_0)}{x_0} = \frac{x_0}{x_0^2} - \frac{s'(x_0)}{x_0} = \frac{1 - s'(x_0)}{x_0} \geq 0.$$

Then, since x_0 is non-negative, we have that $s'(x_0) \leq 1$.

Now consider that if $s'(x_0) = 0$, we have the positive semidefinite Schwartz-Pick Matrix

$$K^f_{x_0,z} = \begin{bmatrix} 0 & \dfrac{x_0 - \overline{s(z)}}{x_0 - \overline{z}} \\ \dfrac{s(z) - x_0}{z - x_0} & \dfrac{s(z) - \overline{s(z)}}{z - \overline{(z)}} \end{bmatrix}. \tag{4.4}$$

Taking the determinant, then, we have $\left| \dfrac{x_0 - \overline{s(z)}}{x_0 - \overline{z}} \right|^2 \geq 0$, which is true only if $x_0 - \overline{s(z)} = 0$. Therefore, $\overline{s(z)} = x_0$ for all $z \in \mathbb{C}^+$; since x_0 is real, we have $s(z) \equiv x_0$.

Now let $s'(x_0) = 1$. Taking the matrix $\widetilde{K}^s_{x_0,z}$, we know from (4.2) that

$$[\widetilde{K}^s_{x_0,z}]_{11} = \frac{1 - s'(x_o)}{x_0} = 0$$

and, since $s(x_0) = x_0$,

$$[\widetilde{K}^s_{x_0,z}]_{21} = \frac{-\dfrac{s(z)}{z} + \dfrac{s(x_0)}{x_0}}{z - x_0} = \frac{-\dfrac{s(z)}{z} + 1}{z - x_0}.$$

Therefore, the determinant of $\widetilde{K}^s_{x_0,z}$ is

$$-[\widetilde{K}^s_{x_0,z}]_{12}[\widetilde{K}^s_{x_0,z}]_{21} = -\overline{[\widetilde{K}^s_{x_0,z}]_{21}}[\widetilde{K}^s_{x_0,z}]_{21} = -\left|\frac{-\dfrac{s(z)}{z} + 1}{z - x_0}\right|^2,$$

since $\widetilde{K}^s_{x_0,z}$ is positive semidefinite and therefore Hermitian. Also, since $\widetilde{K}^s_{x_0,z}$ is positive semidefinite, $\det \widetilde{K}^s_{x_0,z} \geq 0$, which only occurs when $-\dfrac{s(z)}{z} + 1 = 0$. We can then conclude that $s(z) = z$ for all z.

5. Boundary fixed points of Pick class functions

Let us say that a function f analytic on $\mathbb{C}^+$ admits the angular limit at a boundary point $x_0 \in \mathbb{R}$ if the limit

$$f(x_0) := \lim_{z \to x_0} f(z) \tag{5.1}$$

exists whenever $z \in \mathbb{C}^+$ tends to $x_0 \in \mathbb{R}$ staying inside the angle $\alpha < \arg(z - x_0) < \pi - \alpha$ for some fixed $\alpha \in (0, \pi/2)$. A celebrated result of Pierre Fatou [19] asserts that if f is bounded on $\mathbb{C}^+$ in the sense that $\sup_{z \in \mathbb{C}^+} |f(z)| < \infty$, then the angular boundary limit (5.1) exists at almost every $x \in \mathbb{R}$.

Proposition 5.1. *Every non-constant Pick-class function $f \in \mathcal{P}$ admits a representation*

$$f(z) = i \cdot \frac{1 + g(z)}{1 - g(z)} \tag{5.2}$$

for some non-constant function g analytic on $\mathbb{C}^+$ and such that $\sup_{z \in \mathbb{C}^+} |g(z)| \leq 1$.

Proof: To see that $|g(z)| \leq 1$, we solve $f(z) = i \cdot \dfrac{1 + g(z)}{1 - g(z)}$ for $g(z)$, resulting in $g(z) = \dfrac{f(z) - i}{f(z) + i}$ for $f(z) \neq -i$. Then

$$1 - |g(z)|^2 = 1 - \left|\frac{f(z) - i}{f(z) + i}\right|^2 = \frac{|f(z) + 1|^2 - |f(z) - 1|^2}{|f(z) + 1|^2}. \tag{5.3}$$

Since $|f(z)+i|^2 > 0$, to show that $1-|g(z)|^2$ is positive we must show that $|f(z)+i|^2 - |f(z)-i|^2 > 0$. Expand the terms of the left hand side to

$$\begin{aligned}|f(z)+i|^2 - |f(z)-i|^2 &= (f(z)+i)\overline{(f(z)+i)} - (f(z)-i)\overline{(f(z)-i)}\\ &= (f(z)+i)(\overline{f(z)}-i) - (f(z)-i)(\overline{f(z)}+i)\\ &= |f(z)|^2 + i\overline{f(z)} - if(z) - 1 - |f(z)|^2 + i\overline{f(z)} - if(z) + 1\\ &= -2i(f(z)-\overline{f(z)}).\end{aligned}$$

Then, since $\operatorname{Im} f(z) = \dfrac{f(z)-\overline{f(z)}}{2i}$, we have that $-2i(f(z)-\overline{f(z)}) = 4\operatorname{Im} f(z)$, which is positive since f is a Pick-class function. Therefore $1-|g(z)|^2 > 0$, and thus $g(z)$ is positive. □

Combining the latter proposition with the classic Fatou's theorem we conclude that any Pick function admits angular boundary limits almost everywhere on $\mathbb{R}$. The next theorem is due G. Julia [22] and C. Carathéodory [12].

Theorem 5.2. *Let f be a Pick-class function and let us assume that*

$$f(x_0) := \lim_{z\to x_0} f(z) = a \in \mathbb{R}.$$

Then the following limits exist (finitely or infinitely) in $\overline{\mathbb{R}_+}$:

$$f'(x_0) := \lim_{z\to x_0} f'(z) = \lim_{z\to x_0} \frac{f(z)-a}{z-x_0} = \lim_{z\to x_0} \frac{\operatorname{Im} f(z)}{\operatorname{Im} z}.$$

Definition 5.3. *A point $x_0 \in \mathbb{R}$ is called a boundary fixed point if $f(x_0) := \lim_{z\to x_0} f(z) = x_0$.*

By Theorem 5.8, for every fixed point x_0 of a Pick function f, the limit $f'(x_0) := \lim_{z\to x_0} f'(z)$ exists and is nonnegative or infinite.

Remark 5.4. *A Pick function $f \in \mathcal{P}$ can have many fixed boundary points.*

As an example of a Pick function with multiple boundary fixed points, we let

$$f(z) = \frac{z(z-75)}{15z^2-125}.$$

To see that $f \in \mathcal{P}$, we first note that f is analytic everywhere except its poles at $\pm 5/\sqrt{3}$, so its domain of analyticity covers $\mathbb{C}^+$. By Definition 1.2, we still need to show that $\dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \geq 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since

$$\begin{aligned}\frac{f(z)-\overline{f(z)}}{z-\overline{z}} &= \left(\frac{z(z-75)}{15z^2-125} - \frac{\overline{z}(\overline{z}-75)}{15\overline{z}^2-125}\right)\left(\frac{1}{z-\overline{z}}\right) \\ &= \frac{z(z-75)(15\overline{z}^2-125) - \overline{z(z-75)(15\overline{z}^2-125)}}{(z-\overline{z})|15z^2-125|}\end{aligned}$$

and $|15z^2-125| > 0$, we still need that $\dfrac{\text{Im } z(z-75)(15\overline{z}^2-125)}{\text{Im } z} \geq 0$. Solving algebraically and letting $z = a+bi$, we have

$$\text{Im } z(z-75)(15\overline{z}^2-125) = 1125|z|^2 b - 250ab + 9375b.$$

Dividing by b and flipping the inequality if b is negative, then, we need to show that $1125|z|^2 - 250a + 9375 > 0$. If $a \leq 1$, then $9375 > 250a$ and so this expression is positive. If $a > 1$, then $|z|^2 = a^2 + b^2 > a$ and so $1125|z|^2 > 250a$ and the expression is still positive. Therefore $f \in \mathcal{P}$.

It is immediately apparent that $x_0 = 0$ is a fixed point. Dividing each side by 0 and then rearranging terms, we have $0 = 15z^2 - z - 50$, so the two other fixed points are $x_1 = (1+\sqrt{3001})/30$ and $x_2 = (1-\sqrt{3001})/30$. Taking the derivative $f'(z) = \dfrac{(15z^2-125)(2z-75) - 30z^2(z-75)}{(15z^2-125)^2}$, we have $f'(0) = 3/5$. For the other two fixed points, we have $f'(x_1) \approx 2.3$ and $f'(x_2) \approx 2.28$. Note that there is only one boundary fixed point x such that $|f'(x)| \leq 1$; as we will see below, there is exactly one fixed point, whether internal or boundary, with this characteristic for all Pick functions. We call this point the *Denjoy-Wolff point.*

Theorem 5.5. *Let us assume that a Pick function f which is not the identity function has an interior fixed point $z_0 \in \mathbb{C}^+$. Then for every boundary fixed point x (if such points exist), $f'(x) > 1$.*

Proof. From (3.4), we know that

$$K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & \dfrac{z_0-\overline{f(z)}}{z_0-\overline{z}} \\ \overline{f'(z_0)} & 1 & \dfrac{\overline{z}_0-\overline{f(z)}}{\overline{z}_0-\overline{z}} \\ \dfrac{\overline{z}_0-f(z)}{\overline{z}_0-z} & \dfrac{z_0-f(z)}{z_0-z} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \geq 0. \tag{5.4}$$

Letting $z = x+iy \to x$ from above (so $y \to 0$), we have

$$\lim_{z\to x} K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & f'(x) \end{bmatrix} \geq 0. \tag{5.5}$$

Since the above matrix is positive semidefinite, all of its principal submatrices are also positive semidefinite, so

$$0 \le \det \begin{bmatrix} 1 & 1 \\ 1 & f'(x) \end{bmatrix} = f'(x) - 1$$

and therefore $f'(x) \ge 1$.

Now assume for the sake of a contradiction that $f'(x) = 1$, so from (5.5),

$$\begin{bmatrix} \frac{1}{f'(z_0)} & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ge 0.$$

Then we have that

$$0 \le \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{f'(z_0)} & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{f'(z_0)} - 1 & f'(z_0) - 1 \\ \overline{f'(z_0)} - 1 & 0 \end{bmatrix}.$$

Taking the determinant of the above matrix and considering that it is positive semidefinite, we have that $-|f'(z_0) - 1|^2 \ge 0$ and so $f'(z_0) = 1$. Therefore, by formula (3.3), we have

$$f(z) = \frac{(f'(z_0) - 1)|z_0|^2 + (z_0 - \overline{z_0} f'(z_0))z}{(z_0 f'(z_0) - \overline{z_0}) + (1 - f'(z_0))z} = \frac{(z_0 - \overline{z_0})z}{z_0 - \overline{z_0}} = z$$

for all $z \in \mathbb{C}^+$. Then f is the identity function, which was disallowed, so $f'(x) > 1$. □

Theorem 5.5 tells us that a necessary condition for the existence of an attractive boundary fixed point is the absence of interior fixed points. The next theorem (see [1] for the proof) shows that this condition is almost sufficient.

Theorem 5.6. *If $f \in \mathcal{P}$ does not have an interior fixed point, then either it has a boundary fixed point x_0 with $f'(x_0) \le 1$ or*

$$\frac{\operatorname{Im} f(z)}{\operatorname{Im} z} \ge 1 \quad \text{for every} \quad z \in \mathbb{C}^+.$$

In the latter case $f'(\infty) := \lim_{y \to +\infty} f'(x + iy) \ge 1$.

We now ask how many attractive fixed boundary points a Pick-class function can have.

Theorem 5.7. *A Pick-class function f cannot have two boundary fixed points x_0, x_1 such that $f'(x_0) < 1$ and $f'(x_1) \le 1$.*

Proof. Take two points z_0 and z_1 in $\mathbb{C}^+$ and onsider the Pick matrix

$$K^f_{z_0,z_1} = \begin{bmatrix} \frac{f(z_0) - \overline{f(z_0)}}{z_0 - \overline{z_0}} & \frac{f(z_0) - \overline{f(z_1)}}{z_0 - \overline{z_1}} \\ \frac{f(z_1) - \overline{f(z_0)}}{z_1 - \overline{z_0}} & \frac{f(z_1) - \overline{f(z_1)}}{z_1 - \overline{z_1}} \end{bmatrix} \ge 0. \tag{5.6}$$

Letting $z_0 \to x_0$ and $z_1 \to x_1$ from directly above, we have

$$\begin{bmatrix} f'(x_0) & 1 \\ 1 & f'(x_1) \end{bmatrix} \geq 0. \tag{5.7}$$

Therefore, the determinant of the above matrix $f'(x_0)f'(x_1)-1$ is non-negative, so that $f'(x_0)f'(x_1) \geq 1$. Since $f'(x_0)$ and $f'(x_1)$ are both non-negative, they cannot exceed one simultaneously. □

In the last theorem we actually proved that for two fixed boundary points z_0 and x_1 of a Pick function f, we always have $f'(x_0)f'(x_1) \geq 1$. The next theorem makes this statement more precise.

Theorem 5.8. *Let x_0 and x_1 be two fixed boundary points of a function $f \in \mathcal{P}$. Then $f'(x_0)f'(x_1) = 1$ if and only if*

$$f(z) = \frac{(f'(x_0)-1)x_0x_1 + (x_0 - f'(x_0)x_1)z}{(x_0f'(x_0) - x_1) + (1 - f'(x_0))z}. \tag{5.8}$$

Proof. Consider that

$$\lim_{\substack{z_0\to x_0\\ z_1\to x_1}} K^f_{z_0,z_1,z} = \begin{bmatrix} f'(x_0) & 1 & \dfrac{x_0 - \overline{f(z)}}{x_0 - \overline{z}} \\ 1 & f'(x_1) & \dfrac{x_1 - \overline{f(z)}}{x_1 - \overline{z}} \\ \dfrac{f(z)-x_0}{z-x_0} & \dfrac{f(z)-x_1}{z-x_1} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix}. \tag{5.9}$$

Then, because the above matrix is positive semidefinite,

$$\begin{aligned} 0 &\leq \begin{bmatrix} -1 & f'(x_0) & 0 \\ 0 & 0 & 1 \end{bmatrix} \lim_{\substack{z_0\to x_0\\ z_1\to x_1}} K^f_{z_0,z_1,z} \begin{bmatrix} -1 & 0 \\ f'(x_0) & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & f'(x_0)\dfrac{x_1 - \overline{f(z)}}{x_1 - \overline{z}} - \dfrac{x_0 - \overline{f(z)}}{x_0 - \overline{z}} \\ f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0} & 1 \end{bmatrix}. \end{aligned}$$

The determinant of this matrix is $-\left|f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0}\right|^2$, which is greater than or equal to zero. Consequently, $f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0} = 0$, so

$$f'(x_0)(f(z)-x_1)(z-x_0) = (f(z)-x_0)(z-x_1)$$

and

$$f(z)(z - x_1 - zf'(x_0) + x_0f'(x_0)) = zx_0 - x_0x_1 - zf'(x_0)x_1 + f'(x_0)x_0x_1,$$

yielding (5.8). Differentiating (5.8) gives

$$f'(z) = \frac{f'(x_0)(x_0 - x_1)^2}{((x_0 f'(x_0) - x_1) + (1 - f'(x_0))z)^2}$$

and evaluating the latter formula at $z = x_1$ gives

$$f'(x_1) = \frac{f'(x_0)(x_0 - x_1)^2}{(x_0 f'(x_0) - x_1 + (1 - f'(x_0))x_1)^2} = \frac{1}{f'(x_0)},$$

so that $f'(x_0)f'(x_1) = 1$. □

The next theorem establishes a relation between the values of the derivative of a Pick-class function at fixed points. Using different methods, such a result was established in [14] for analytic self-mappings of the unit disk.

Theorem 5.9. *Let $f \in \mathcal{P}$, $z_0 \in \mathbb{C}^+$, and $x_1, x_2, \ldots, x_n$ be fixed points of f. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{1 - |f'(z_0)|^2}{|1 - f'(z_0)|^2}. \tag{5.10}$$

Proof. We use the Schwartz-Pick matrix $K^f_{z_0, \overline{z_0}, \zeta_1, \zeta_2, \ldots, \zeta_n}$ and take the limit as $\zeta_i \to x_i$ for all $1 \leq i \leq n$, resulting in the matrix

$$\begin{bmatrix} \frac{1}{\overline{f'(z_0)}} & f'(z_0) & 1 & \cdots & 1 \\ \overline{f'(z_0)} & 1 & 1 & \cdots & 1 \\ 1 & 1 & f'(x_1) & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & f'(x_n) \end{bmatrix} \geq 0. \tag{5.11}$$

Exchanging the first and second columns, we have that the matrix

$$\begin{bmatrix} f'(z_0) & \frac{1}{\overline{f'(z_0)}} & 1 & \cdots & 1 \\ 1 & \overline{f'(z_0)} & 1 & \cdots & 1 \\ 1 & 1 & f'(x_1) & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & f'(x_n) \end{bmatrix} \tag{5.12}$$

has a non-postive determinant. Letting $\mu_i = f'(x_i) - 1$, we then subtract the first row from all other rows in the matrix to attain

$$\det \begin{bmatrix} f'(z_0) & \frac{1}{\overline{f'(z_0)}} & 1 & \cdots & 1 \\ 1 - f'(z_0) & \overline{f'(z_0)} - 1 & 0 & \cdots & 0 \\ 1 - f'(z_0) & 0 & \mu_1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 - f'(z_0) & 0 & 0 & & \mu_n \end{bmatrix} \leq 0. \tag{5.13}$$

Computing the determinant along the first column, we then have

$$f'(z_0)(\overline{f'(z_0)}-1)\prod_{i=1}^{n}\mu_i-(1-f'(z_0))\prod_{i=1}^{n}\mu_i \\ +\sum_{i=1}^{n}(-1)^{i-1}(1-f'(z_0))\det A_i\leq 0 \tag{5.14}$$

where A_i is the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \overline{f'(z_0)}-1 & & & & & & & 0 \\ & \mu_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \mu_{i-1} & 0 & & & \\ & & & & 0 & \mu_{i+1} & & \\ & & & & & & \ddots & \\ 0 & & & & & & & \mu_n \end{bmatrix}.$$

because we have an upper diagonal matrix after exchanging columns i times, we have that $\det A_i=(-1)^i(\overline{f'(z_0)}-1)\dfrac{\mu_1\cdots\mu_n}{\mu_i}$. Then, from (5.14),

$$f'(z_0)(\overline{f'(z_0)}-1)-(1-f'(z_0))-\sum_{i=1}^{n}(1-f'(z_0))(\overline{f'(z_0)}-1)\frac{1}{\mu_i}\leq 0.$$

Simplifying, we have

$$|f'(z_0)|^2-1+|1-f'(z_0)|^2\left(\frac{1}{\mu_1}+\cdots+\frac{1}{\mu_n}\right)\leq 0$$

and therefore

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1}=\sum_{i=1}^{n}\frac{1}{\mu_i}\leq\frac{1-|f'(z_0)|^2}{|1-f'(z_0)|^2}.$$

□

In the next theorem, the Denjoy-Wolff point is on the boundary.

Theorem 5.10. *Let $f\in\mathcal{P}$ and $x_0,x_1,\ldots,x_n\in\mathbb{R}$ be fixed points, with $f'(x_0)<1$. Then*

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1}\leq\frac{f'(x_0)}{1-f'(x_0)}. \tag{5.15}$$

Proof. Take the Schwartz-Pick matrix $K^f_{\zeta_0,\zeta_1,\cdots,\zeta_n}$. Letting $\zeta_i \to x_i$ for all $x = 0, 1, \cdots, n$, we have the positive semidefinite matrix

$$\begin{bmatrix} f'(x_0) & 1 & \cdots & 1 \\ 1 & f'(x_1) & & 1 \\ \vdots & & \ddots & \\ 1 & 1 & & f'(x_n) \end{bmatrix}. \tag{5.16}$$

Subtracting the first row from all other rows, which has no effect on the determinant of the matrix, we have

$$\begin{bmatrix} f'(x_0) & 1 & \cdots & 1 \\ 1-f'(x_0) & \mu_1 & & 0 \\ \vdots & & \ddots & \\ 1-f'(x_0) & 0 & & \mu_n \end{bmatrix} \geq 0 \tag{5.17}$$

where $\mu_i = f'(x_i) - 1$ for $i = 1, 2, \ldots, n$. Taking the determinant of (5.17) with respect to the first column, we have

$$f'(x_0)\mu_1 \cdots \mu_n + \sum_{i=1}^{n} (-1)^i (1 - f'(x_0)) A_i \geq 0 \tag{5.18}$$

where A_i is equal to

$$\begin{bmatrix} 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ \mu_1 & 0 & & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & & \\ & \ddots & \mu_{i-1} & 0 & 0 & & \vdots \\ \vdots & & 0 & 0 & \mu_{i+1} & \ddots & \\ & & & & \ddots & \ddots & 0 \\ 0 & & \cdots & & & 0 & \mu_n \end{bmatrix}.$$

Also, consider that

$$\det A_i = (-1)^{i-1} \det \begin{bmatrix} 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \mu_1 & 0 & & \cdots & & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \mu_{i-1} & 0 & & \vdots \\ \vdots & & & 0 & \mu_{i+1} & \ddots & \\ & & & & \ddots & \ddots & 0 \\ 0 & & \cdots & & & 0 & \mu_n \end{bmatrix} \tag{5.19}$$

since the second matrix can be obtained by transposing $i-1$ columns. Because the matrix is upper triangular, the above is equal to $(-1)^{i-1}\frac{\mu_1 \cdots \mu_n}{\mu_i}$. Then from

(5.18) we have

$$\mu_1\cdots\mu_n(f'(x_0)-(1-f'(x_0))\left(\frac{1}{\mu_1}+\cdots+\frac{1}{\mu_n}\right)\geq 0$$

and therefore

$$\sum_{i=1}^n \frac{1}{f'(x_i)-1}=\sum_{i=1}^n\frac{1}{\mu_i}\leq\frac{f'(x_0)}{1-f'(x_0)}.$$

□

The estimate (5.10) does not provide much in the case of $f'(x_0)=1$. A natural question to ask is: *is it possible to establish a nontrivial estimate for the expression on the left side of* (5.10) *in terms of higher order boundary derivatives of the function f?*

Let us assume that the limits

$$f''(x_0)=\lim_{y\to 0}f''(x_0+iy)\quad\text{and}\quad f'''(x_0)=\lim_{y\to 0}f'''(x_0+iy) \tag{5.20}$$

exist finitely and are real. If $x_1,\dots,x_n\in\mathbb{R}$ are other fixed points of f, then the following matrix is positive semidefinite:

$$\begin{bmatrix} f'(x_1) & 1 & \dots & 1 & 1 & \frac{1-f'(x_0)}{x_1-x_0}\\ 1 & f'(x_2) & \dots & 1 & 1 & \frac{1-f'(x_0)}{x_2-x_0}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 1 & 1 & \dots & f'(x_n) & 1 & \frac{1-f'(x_0)}{x_n-x_0}\\ 1 & 1 & \dots & 1 & f'(x_0) & \frac{f''(x_0)}{2}\\ \frac{1-f'(x_0)}{x_1-x_0} & \frac{1-f'(x_0)}{x_2-x_0} & \dots & \frac{1-f'(x_0)}{x_n-x_0} & \frac{f''(x_0)}{2} & \frac{f'''(x_0)}{6}\end{bmatrix}\geq 0. \tag{5.21}$$

The proof is similar to that of Theorem 1.5. Let b and $d\mu$ are taken from the Herglotz representation (1.1) of f. Then the matrix

$$\begin{bmatrix}1\\ \vdots\\ 1\\ 0\end{bmatrix} b\begin{bmatrix}1 & \cdots & 1 & 0\end{bmatrix}+\int_{-\infty}^{\infty}\begin{bmatrix}\frac{1}{t-z_1}\\ \vdots\\ \frac{1}{t-z_n}\\ \frac{1}{t-z_0}\\ \frac{1}{(t-z_n)^2}\end{bmatrix} d\mu(t)\begin{bmatrix}\frac{1}{t-\bar z_1} & \cdots & \frac{1}{t-\bar z_n} & \frac{1}{t-\bar z_0} & \frac{1}{(t-\bar z_0)^2}\end{bmatrix}. \tag{5.22}$$

is positive semidefinite. Taking the limit as $z_k=x_k+iy\to x_k$ we get the matrix as in (5.21) which is positive semidefinite as the limit of positive semidefinite matrices. Since $f'(x_0) = 1$, the matrix (5.21) simplifies to

$$\begin{bmatrix} f'(x_1) & 1 & \dots & 1 & 1 & 0 \\ 1 & f'(x_2) & \ddots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & f'(x_n) & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & \frac{f''(x_0)}{2} \\ 0 & 0 & \dots & 0 & \frac{f''(x_0)}{2} & \frac{f'''(x_0)}{6} \end{bmatrix} \geq 0.$$

Taking the determinant of this matrix along the the bottom row, we have that

$$-\frac{f''(x_0)}{2} \det \begin{bmatrix} f'(x_1) & & 1 & 0 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 0 \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix} + \frac{f'''(x_0)}{6} \det \begin{bmatrix} f'(x_1) & & 1 & 1 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 1 \\ 1 & \dots & 1 & 1 \end{bmatrix} \tag{5.23}$$

is nonnegative. Focusing on the first matrix above and again defining $\mu_i = f'(x_i) - 1$,

$$\det \begin{bmatrix} f'(x_1) & & 1 & 0 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 0 \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix} = \det \begin{bmatrix} \mu_1 & & 0 & -\frac{f''(x_0)}{2} \\ & \ddots & & \vdots \\ 0 & & \mu_n & -\frac{f''(x_0)}{2} \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix}.$$

We then evaluate with respect to the last column, attaining

$$\sum_{i=1}^{n}\left((-1)^{i+n+1}\left(-\frac{f''(x_0)}{2}\right)\det\begin{bmatrix}\mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1\end{bmatrix}\right)$$
$$+\frac{f''(x_0)}{2}\det\begin{bmatrix}\mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_3\end{bmatrix}. \quad (5.24)$$

Transposing columns $n-i$ times so that the column not containing any μ_j is on the far right side, we have

$$\det\begin{bmatrix}\mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1\end{bmatrix} = (-1)^{n-i}\det\begin{bmatrix}\mu_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mu_n & 0 \\ 1 & \cdots & 1 & 1\end{bmatrix}$$
$$= (-1)^{n-i}\frac{\mu_1\cdots\mu_n}{\mu_i}$$

and thus the expression in (5.24) is equal to

$$\frac{f''(x_0)}{2}\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_n}\right).$$

Plugging into (5.23), we have

$$-\left(\frac{f''(x_0)}{2}\right)^2\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_n}\right)+\frac{f'''(x_0)}{6}\mu_1\cdots\mu_n\geq 0,$$

which simplifies to

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1}\leq\frac{2f'''(x_0)}{3(f''(x_0))^2}-1.$$

We thus arrive at the following result:

Theorem 5.11. *Let $f\in\mathbb{P}$, let $x_0,\ldots,x_n\in\mathbb{R}$ be fixed points. Let x_0 be the Denjoy-Wolff point such that $f'(x_0)=1$ and the limits* (5.20) *exist and are* *real. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)-1} \leq \frac{2f'''(x_0)}{3(f''(x_0))^2} - 1. \tag{5.25}$$

The latter theorem does not provide any meaningful information in case

$$f'(x_0) = 1 \quad \text{and} \quad f''(x_0) = 0, \tag{5.26}$$

that is, in case x_0 is the boundary fixed point of order two. In this case we assume that the limits

$$f_3 = \lim_{y\to 0} \frac{f'''(x_0+iy)}{6}, \quad f_4 = \lim_{y\to 0} \frac{f^{(4)}(x_0+iy)}{24} \quad \text{and} \quad f_5 = \lim_{y\to 0} \frac{f^{(5)}(x_0+iy)}{120} \tag{5.27}$$

exist finitely and are real. Then the following matrix is positive semidefinite:

$$\begin{bmatrix} K & B^* \\ B & D \end{bmatrix} \geq 0 \tag{5.28}$$

where K is the matrix on the left hand side of (5.21), where $D = \dfrac{f^{(5)}(x_0)}{120}$ and where

$$B = \begin{bmatrix} \frac{1-f'(x_0)}{(x_1-x_0)^2} - \frac{f''(x_0)}{x_1-x_0} & \dots & \frac{1-f'(x_0)}{(x_n-x_0)^2} - \frac{f''(x_0)}{x_n-x_0} & \frac{f^{(4)}(x_0)}{24} \end{bmatrix}.$$

For the proof we should consider the matrix similar to that in (5.22) but with

$$\begin{bmatrix} 1 & \dots 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{t-\bar{z}_1} & \cdots & \frac{1}{t-\bar{z}_n} & \frac{1}{t-\bar{z}_0} & \frac{1}{(t-\bar{z}_0)^2} \end{bmatrix}$$

replaced by the extended rows

$$\begin{bmatrix} 1 & \dots 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{t-\bar{z}_1} & \cdots & \frac{1}{t-\bar{z}_n} & \frac{1}{t-\bar{z}_0} & \frac{1}{(t-\bar{z}_0)^2} & \frac{1}{(t-\bar{z}_0)^3} \end{bmatrix},$$

respectively, and then pass to the limits as as $z_k = x_k + iy \to x_k$. Due to conditions (5.26), the matrix in (5.28) takes the form

$$\begin{bmatrix} f'(x_1) & 1 & \dots & 1 & 1 & 0 & 0 \\ 1 & f'(x_2) & \dots & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & f'(x_n) & 1 & 0 & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & f_3 \\ 0 & 0 & \dots & 0 & 0 & f_3 & f_4 \\ 0 & 0 & \dots & 0 & f_3 & f_4 & f_5 \end{bmatrix} \geq 0.$$

Computing the determinant of this matrix with respect to the last row, we have

$$f_3 \det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & 0 & f_3 \\ 0 & \cdots & 0 & f_3 & f_4 \end{bmatrix} - f_4 \det \begin{bmatrix} f'(x_1) & & 1 & 1 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 1 & 0 \\ 1 & \cdots & 1 & 1 & f_3 \\ 0 & \cdots & 0 & 0 & f_4 \end{bmatrix}$$
$$+ f_5 \det \begin{bmatrix} f'(x_1) & & 1 & 1 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 1 & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & f_3 \end{bmatrix} \leq 0. \tag{5.29}$$

The first matrix in the above expression can be simplified using row operations:

$$\det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & 0 & f_3 \\ 0 & \cdots & 0 & f_3 & f_4 \end{bmatrix} = -\det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & f_3 & 0 \\ 0 & \cdots & 0 & f_4 & f_3 \end{bmatrix}$$
$$= -\det \begin{bmatrix} \mu_1 & & 0 & -f_3 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & \mu_n & -f_3 & 0 \\ 1 & \cdots & 1 & f_3 & 0 \\ 0 & \cdots & 0 & f_4 & f_3 \end{bmatrix} = -f_3 \det \begin{bmatrix} \mu_1 & & 0 & -f_3 \\ & \ddots & & \vdots \\ 0 & & \mu_n & -f_3 \\ 1 & \cdots & 1 & f_3 \end{bmatrix}.$$

We then take the determinant with respect to the last column, arriving at

$$(-1)^{i+n+1}(-f_3) \left(\sum_{i=1}^{n} -f_3 \det \begin{bmatrix} \mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+i} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} + \mu_1 \cdots \mu_n f_3 \right).$$

Then, transposing $n-i$ columns, the above is equal to

$$-f_3^2\mu_1\cdots\mu_n + f_3^2\sum_{i=1}^{n}(-1)^{n+i+1}\det\begin{bmatrix}\mu_1 & & & & & & 0\\ & \ddots & & & 0 & & \vdots\\ & & \mu_{i-1} & & & & 0\\ & & & \mu_{i+1} & & & 0\\ & 0 & & & \ddots & & \vdots\\ & & & & & \mu_n & 0\\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1\end{bmatrix}$$

$$= -f_3^2\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_i}\right).$$

As for the second matrix in (5.29), we have

$$\det\begin{bmatrix}f'(x_1) & & 1 & 1 & 0\\ & \ddots & & \vdots & \vdots\\ 1 & & f'(x_n) & 1 & 0\\ 1 & \cdots & 1 & 1 & f_3\\ 0 & \cdots & 0 & 0 & f_4\end{bmatrix}$$

$$= -f_3\det\begin{bmatrix}f'(x_1) & & 1 & 1\\ & \ddots & & \vdots\\ 1 & & f'(x_n) & 1\\ 0 & \cdots & 0 & 0\end{bmatrix} + f_4\det\begin{bmatrix}f'(x_1) & & 1 & 1\\ & \ddots & & \vdots\\ 1 & & f'(x_n) & 1\\ 1 & \cdots & 1 & 1\end{bmatrix}$$

$$= 0 + f_4\det\begin{bmatrix}\mu_1 & & 0 & 0\\ & \ddots & & \vdots\\ 0 & & \mu_n & 0\\ 1 & \cdots & 1 & 1\end{bmatrix} = f_4\mu_1\cdots\mu_n,$$

and finding the determinant of the third matrix in (5.29), we have

$$\det\begin{bmatrix}f'(x_1) & & 1 & 1 & 0\\ & \ddots & & \vdots & \vdots\\ 1 & & f'(x_n) & 1 & 0\\ 1 & \cdots & 1 & 1 & 0\\ 0 & \cdots & 0 & 0 & f_3\end{bmatrix} = \det\begin{bmatrix}\mu_1 & & 0 & 0 & 0\\ & \ddots & & \vdots & \vdots\\ 0 & & \mu_n & 0 & 0\\ 1 & \cdots & 1 & 1 & 0\\ 0 & \cdots & 0 & 0 & f_3\end{bmatrix} = f_3\mu_1\cdots\mu_n.$$

Therefore, the inequality in (5.29) is equivalent to

$$-f_3^3\left(1+\sum_{i=1}^{n}\right)\mu_1\cdots\mu_n - f_4^2\mu_1\cdots\mu_n + f_3f_5\mu_1\cdots\mu_n \le 0,$$

which simplifies to

$$\sum_{i=1}^{n} \frac{1}{\mu_i} \leq \frac{f_5}{f_3^2} - \frac{f_4^2}{f_3^3} - 1.$$

We thus arrive at the following result:

Theorem 5.12. *Let $f \in \mathbb{P}$, let $x_0, \ldots, x_n \in \mathbb{R}$ be fixed points. Let x_0 be the Denjoy-Wolff point of f satisfying conditions* (5.26) *and such that the limits* (5.27) *exist and are real. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{f_5}{f_3^2} - \frac{f_4^2}{f_3^3} - 1. \tag{5.30}$$

Theorems 5.11 and 5.12 suggest that in case $f'''(x_0) = 0$, the sum on the left side of (5.10) can be estimated in terms of derivatives of f of higher orders. However, it is not so. The Burns-Crantz theorem [11] implies that in case $f(x_0) = x_0$, $f'(x_0) = 1$ and $f''(x_0) = f'''(x_0) = 0$ for a Pick class function f and a boundary point $x_0 \in \mathbb{R}$, then necessarily $f(z) \equiv z$.

6. Boundary fixed points of Stieltjes class functions

A point $x_0 \in \mathbb{R}_-$ is called a boundary fixed point of a Stieltjes function $s(z)$ if $s(x_0) = \lim_{y \to 0} s(x_0 + iy) = x_0$. By Theorem 5.8, for every boundary fixed point x_0, the boundary derivative $f'(x_0) := \lim_{y\to 0} s'(x_0 + iy)$ exists (though it can be infinite). Since $\widetilde{K}_z^s = \dfrac{z\overline{s(z)} - \overline{z}s(z)}{|z|^2(z - \overline{z})}$ is nonnegative for every non-real z, we let $z = x_0 + iy \to x_0$ to conclude that $\dfrac{1 - s'(x_0)}{x_0} \geq 0$ so that $s'(x_0) \geq 1$. On the other hand, if $s'(x_0) = 1$, then $s(z) \equiv z$ (the proof is the same as that in Theorem 4.2). Thus, except for the trivial case $s(z) \equiv z$, the boundary derivative of s at any boundary fixed point is greater than 1. The next theorem is the Stieltjes-class analogue of Theorem 6.1.

Theorem 6.1. *Let $f \in \mathcal{S}$, $x_0 \in \mathbb{R}_+$, and $x_1, x_2, \ldots, x_n \in \mathbb{R}_-$ be fixed points of f. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{1 + f'(x_0)}{1 - f'(x_0)}. \tag{6.1}$$

We have shown above that $f'(x) > 1$ for any boundary fixed point $x < 0$ and $f'(x_0) < 1$ for an interior fixed point $x_0 > 0$. The most interesting case is when $x = 0$ is a fixed point for f. The derivative of f at 0 can be equal any positive number. For example, the fumction

$$f(z) = \ln\left(1 + \frac{z}{a}\right)$$

belongs to the Stieltjes class for any $a > 0$ and its derivative equals $\frac{1}{z+a}$ so that $f'(0) = \frac{1}{a}$. Inequalities involving the fixed point $x = 0$ will be studied in future.

References

[1] M. Abate, *Iteration theory of holomorphic maps on taut manifolds*, Mediterranean Press, Cosenza, 1989.

[2] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York, 1963.

[3] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space, Dover, New York, 1993.*

[4] D. Alpay and I. Gohberg, *Inverse spectral problem for difference operators with rational scattering matrix function*, Integral Eqs. Oper. Theory **20** (1994), 125–170.

[5] D. Alpay and I. Gohberg, *Inverse spectral problem for differential operators with rational scattering matrix functions*, J. Diff. Eqs. **118** (1995), 1–19.

[6] D. Alpay, I. Gohberg, and L. Sakhnovich, *Inverse scattering problem for continuous transmission lines with rational reflection coefficient function*, Operator Theory: Advances and Applications, **87**, Birkhäuser, Basel, 1996, pp. 1–16.

[7] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm-Liouville equations*, Am. J. Math. **79** (1957), 597–610.

[8] N. Asano and Y. Kato, *Algebraic and Spectral Methods for Nonlinear Wave Equations*, Longman, New York, 1990.

[9] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[10] Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, 1968.

[11] Burns, D., and Krantz, S. G., *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), no. 3, 661–676.

[12] C. Carathéodory, *Über die Winkelderivierten von beschränkten Funktionen*, Sitzungber. Preuss. Akad. Wiss. (1929), 39–52.

[13] S. L. Clark, *A criterion for absolute continuity of the continuous spectrum of a Hamiltonian system*, J. Math. Anal. Appl. **151** (1990), 108–128.

[14] C. C. Cowen and Ch. Pommerenke, *Inequalities for the angular derivative of an analytic function in the unit disk,*. J. London Math. Soc. **26** (1982), no. 2, 271–289.

[15] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73** (1995), 141–242.

[16] L. A. Dickey, *Soliton Equations and Hamiltonian Systems, World Scientific*, Singapore, 1991.

[17] W. F. Donoghue, *Monotone matrix functions and analytic continuation*, Springer-Verlag, New York-Heidelberg, 1974.

[18] Y. M. Dyukarev and V. E. Katsnelson, *Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them*, Teor. Funktsi Funktsional. Anal. i Prilozhen. **36** (1981), 13–27.

[19] P. Fatou, *Séries trigonomtriques et séries de Taylor*, Acta Math. **30** (1906), 335-400.

[20] M. Fukushima, *A spectral representation of ordinary linear difference equation with operatorvalued coefficients of the second order*, J. Math. Phys **17** (1976), 1064-1072.

[21] A. C. Hindmarsh, *Pick's conditions and analyticity*, Pacific J. Math. **27** (1968) 527–531.

[22] G. Julia, *Extension d'un lemma de Schwartz*, Acta Math. **42** (1920), 349–355.

[23] I. S. Kac and M. G. Krein, *R-functionsanalytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl. **103** (1974), 1–18.

[24] P. Koosis, *Introduction to H_p Spaces*, London Math. Soc. Lecture Note Series 40, Cambridge Univ. Press, Cambridge, 1980.

[25] S. Kotani and B. Simon, *Stochastic Schrödinger operators and Jacobi matrices on the strip*, Commun. Math. Phys. **119** (1988), 403–429.

[26] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.

[27] M. G. Krein, *The theory of self-adjoint extensions of semibounded Hermitian transformations and its applications*, I. Mat. Sb. **20** (1947), 431–495.

[28] A. V. Megretskii, V. V. Peller, and S. R. Treil, *The inverse spectral problem for self-adjoint Hankel operators*, Acta Math. **174** (1995), 241–309.

[29] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser, Basel, 1994.

[30] L. A. Sakhnovich, *Inverse problems for equations systems, in Matrix and Operator-Valued Functions*. Operator Theory: Advances and Applications, **72**, Birkhäuser, Basel, 1994, pp. 202–211.

[31] L. A. Sakhnovich, *Nonlinear equations and inverse spectral problems on the axis*, Operator Theory: Advances and Apllications, **80**, Birkhäuser, Basel, 1995, pp. 244–255.

[32] A. Sakhnovich, *Iterated Backlund transform for canonical systems*, J. Funct. Anal. **144** (1997), 359–370.

[33] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Math. 1258, Springer, Berlin, 1987.

[34] H. Woracek, *Multiple point interpolation in Nevanlinna classes*, Integral Equations Operator Theory **28** (1997), no. 1, 97–109