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Fixed Points of Pick and Stieltjes Functions:
A Linear Algebraic Approach

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science in Mathematics from The College of William and Mary

by

Nicholas Andrew Woods

Accepted for ______________________________
(Honors, High Honors, Highest Honors)

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April 25, 2012

FIXED POINTS OF PICK AND STIELTJES FUNCTIONS: A LINEAR ALGEBRAIC APPROACH

NICK WOODS

Abstract. The functions analytic in the upper half-plane $\mathbb{C}^+$ and mapping $\mathbb{C}^+$ into itself (the so-called Pick functions) play a prominent role in several branches of mathematics. In this thesis we study fixed points of such functions. It is known that a Pick-class function different from the identity map can have at most one fixed point in $\mathbb{C}^+$. However, it may have many (even infinitely many) appropriately defined boundary fixed points. We establish relations between the values of the derivative of a Pick function at these fixed points. Similar questions are considered in the context of Stieltjes-class functions which, in addition, are analytic on the positive half-axis and map this half-axis into itself.

Acknowledgements. I would first like to thank Dr. Jianjun Paul Tian, Dr. Ilya Spitkovsky, and Dr. Joshua Gert for their time and their willingness to serve on my honors committee. I would also like to thank my advisor, Dr. Ryan Vinroot, for direction and encouragement throughout my time at William & Mary. Finally, my thesis advisor, Dr. Vladimir Bolotnikov, provided an incredible amount of time, knowledge, and ideas, and I am enormously grateful for his help and for making this experience possible.

Contents

1. Pick class functions

A function f is in the Pick class $\mathcal{P}$ if it is analytic on the open upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $\operatorname{Im} f(z) \geq 0$ for all $z \in \mathbb{C}^+$. In other words, any Pick function maps $\mathbb{C}^+$ into $\overline{\mathbb{C}^+} = \mathbb{C}^+ \cup \mathbb{R}$. It then follows that the

Date: April 25, 2012.

class $\mathcal{P}$ is closed under addition and under composition. Simple examples of Pick functions are the functions

$$a + bi, \quad a + bz, \quad \frac{b}{c - z} \qquad (b > 0,\ a, c \in \mathbb{R}).$$

The functions of Pick class traditionally play central roles in extention theory of symmetric operators [15, 27], the spectral theory of ordinary differential and difference operators [7, 9, 10, 13, 20, 25, 33], interpolation problems [34], inverse spectral theory [2, 4, 5, 6, 28, 30, 31, 32], inverse scattering [2, 4, 5, 6], and completely integrable hierarchies of non-linear evolution equations [8, 16]. Other areas of application include control theory [26] and Loewner theory of monotone matrix functions [17].

The fundamental result on integral representations of Pick functions is due to Herglotz and Riesz (see [3, Ch. 6] or [23] for the proof).

Theorem 1.1. *A function f is in the Pick class if and only if it can be represented in the form*

$$f(z) = a + bz + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\mu(t), \tag{1.1}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $d\mu$ is a positive measure on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1 + t^2} < \infty. \tag{1.2}$$

The representation (1.1) is called the *Riesz-Herglotz Integral Representation* of f. The constants a and b in this representation are recovered from f by the formulas

$$a = \operatorname{Re} f(i) \quad \text{and} \quad b = \lim_{y \to \infty} \frac{f(iy)}{iy} \geq 0, \tag{1.3}$$

whereas the measure $d\mu$ is recovered from f by the Stieltjes inversion formula: for every $x_1 < x_2 \in \mathbb{R}$,

$$\frac{1}{2}\mu(\{x_1\}) + \frac{1}{2}\mu(\{x_2\}) + \mu((x_1, x_2)) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{x_1}^{x_2} \operatorname{Im} f(t + i\varepsilon)\, dt. \tag{1.4}$$

Remark 1.2. It is quite traditional to extend Pick functions to the lower half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ by reflection, thereby defining the Pick class as the class of functions analytic in $\mathbb{C}\backslash\mathbb{R}$ and such that

$$f(z) = \overline{f(\overline{z})} \quad \text{and} \quad \frac{f(z) - \overline{f(z)}}{z - \overline{z}} \geq 0. \quad \text{for all} \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.5}$$

It is readily seen that the formula (1.1) holds true for any f subject to conditions (1.5) and for any $z \in \mathbb{C}\backslash\mathbb{R}$.

Another fundamental result concerning Pick functions is due to Fatou, Lusin and Privalov (see for example [24, Ch. 6] or [29, Ch. 5]).

Theorem 1.3. *Let f be a Pick function. Then the normal (equivalently, angular) boundary limits*

$$f(x \pm i0) := \lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon) \tag{1.6}$$

exist for almost all $x \in \mathbb{R}$. Furthermore, if f has a zero normal limit on a subset of $\mathbb{R}$ having positive Lebesgue measure, then $f \equiv 0$.

Definition 1.4. *A function $K(z,\zeta)$ is called a positive kernel on the domain $\Omega \subset \mathbb{C}$ if for every $n \in \mathbb{N}$ and for any choice of n points $z_1, \dots, z_n \in \Omega$ and n complex numbers $c_1, \dots, c_n \in \mathbb{C}$,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(z_i, z_j) c_i \overline{c_j} \geq 0,$$

or, equivalently, the matrix $[K(z_i, z_j)]_{i,j=1}^{n}$ is positive semidefinite.

With any function f, we may associate the kernel $K_f(z,\zeta) = \dfrac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}}$ defined on the Cartesian square of $\mathrm{Dom}(f)$, the domain of definition of f. This kernel is *Hermitian* in the sense that $K_f(z,\zeta) = \overline{K_f(\zeta, z)}$ for all $z, \zeta \in \mathrm{Dom}(f)$ such that $z \neq \overline{\zeta}$. For Pick-class functions, the kernel K_f plays a particularly important role. Observe that due to the symmetry relation (1.5) the kernel K_f can be extended to the point $(z, \overline{z})$ by continuity as $K(z\overline{z}) = f'(z)$.

Theorem 1.5. *If $f \in \mathcal{P}$, then the kernel*

$$K_f(z,\zeta) = \begin{cases} \dfrac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ f'(z) & \text{if } z = \overline{\zeta}, \end{cases} \tag{1.7}$$

is positive on $\mathbb{C} \backslash \mathbb{R}$.

Proof. Using the Riesz-Herglotz Integral Representation (1.1) for f and recalling that a and b are real, we have

$$\begin{aligned} K_f(z,\zeta) = \frac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} &= \frac{1}{z - \overline{\zeta}} \left(bz - b\overline{\zeta} + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{1}{t - \overline{\zeta}} \right) d\mu(t) \right) \\ &= b + \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t - z)(t - \overline{\zeta})}. \end{aligned} \tag{1.8}$$

We can therefore write for some fixed points $z_1, \dots, z_n \in \mathbb{C}\backslash\mathbb{R}$ such that $z_i \neq \overline{z}_j$ for $i, j \in \{1, \dots, n\}$,

$$[K_f(z_i, z_j)]_{i,j=1}^n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} b \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + \int_{-\infty}^{\infty} \begin{bmatrix} \frac{1}{t-z_1} \\ \vdots \\ \frac{1}{t-z_n} \end{bmatrix} d\mu(t) \begin{bmatrix} \frac{1}{t-\overline{z_1}} & \cdots & \frac{1}{t-\overline{z_n}} \end{bmatrix}. \tag{1.9}$$

If $z_i = \overline{z}_j$ for some i, j, then the equality (1.9) also holds true, since (thanks to (1.1))

$$f'(z) = b + \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-z)^2}.$$

Since $b \geq 0$ and $d\mu(t) \geq 0$, we see that both terms on the right of (1.9) are positive semidefinite matrices so that their sum

$$K_{z_1,\dots,z_n}^f := [K_f(z_i, z_j)]_{i,j=1}^n = \left[\frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \right]_{i,j=1}^n \tag{1.10}$$

is also positive semidefinite, which completes the proof. □

Definition 1.6. *For $f \in \mathcal{P}$, the matrix* (1.10) *is called the Schwarz-Pick matrix of f based on points $z_1, \dots, z_n$.*

We thus proved that for any choice of $z_1, \dots, z_n \in \mathbb{C}\backslash\mathbb{R}$, the Schwarz-Pick matrix (1.10) is positive semidefinite. A natural question to ask is the following: *does the positivity of the kernel K_f on $\mathbb{C}^+$ guarantee the membership of f in the Pick class $\mathcal{P}$?*

The answer is clearly affirmative if we know that f is analytic on $\mathbb{C}^+$. In this case, the positivity of K_f implies in particular that

$$K(z, z) = \frac{f(z) - \overline{f(z)}}{z - \overline{z}} = \frac{\mathrm{Im} f(z)}{\mathrm{Im} z} \geq 0 \quad \text{for all} \quad z \in \mathbb{C}^+$$

and thus $\mathrm{Im} f(z) \geq 0$ for all $z \in \mathbb{C}^+$ so that f belongs to $\mathcal{P}$ by the very definition of this class. It turns out that in fact the positivity of K_f on $\mathbb{C}^+$ *implies* that f is analytic on on $\mathbb{C}^+$ and thus one may conclude that *a function f defined everywhere on $\mathbb{C}^+$ belongs to the Pick class $\mathcal{P}$ if and only if the kernel K_f is positive on $\mathbb{C}^+$*. Equivalently, $f \in \mathcal{P}$ if and only if the Schwarz-Pick matrices (1.10) are positive semidefinite for every choice of a positive integer n and points $z_1, \dots, z_n \in \mathbb{C}^+$. A remarkable result of Hindmarsch [21] states that the membership $f \in \mathcal{P}$ follows from substantially weaker assumptions.

Theorem 1.7. *If f is defined everywhere on $\mathbb{C}^+$ and 3×3 Schwarz-Pick matrices K_{z_1,z_2,z_3}^f are positive semidefinite for all choices of $z_1, z_2, z_3 \in \mathbb{C}^+$, then f is analytic on $\mathbb{C}^+$ and therefore belongs to $\mathcal{P}$.*

To conclude this introductory section, we present a simple corollary of Theorem 1.5. An analytic proof is given in [17, p.18].

Theorem 1.8. *If $f \in \mathcal{P}$ and* $\operatorname{Im} f(z_0) = 0$ *for some* $z_0 \in \mathbb{C}^+$*, then* $f(z) \equiv c \in \mathbb{R}$.

Proof. Since $\operatorname{Im} f(z_0) = 0$, we have $f(z_0) = \overline{f(z_0)}$. Then, for all $z \in \mathbb{C}^+$ such that $z \neq z_0$ we have the Schwarz-Pick matrix

$$K^f_{z_0,z} = \begin{bmatrix} \dfrac{f(z_0) - \overline{f(z_0)}}{z_0 - \overline{z_0}} & \dfrac{f(z) - \overline{f(z_0)}}{z - \overline{z_0}} \\ \\ \dfrac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{f(z) - f(z_0)}{z - \overline{z_0}} \\ \\ \dfrac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix}.$$

Because f belongs to $\mathcal{P}$, the matrix $K^f_{z_0,z}$ is positive semidefinite, so

$$0 \leq \det A = 0 - \frac{f(z) - \overline{f(z_0)}}{z - \overline{z_0}} \cdot \frac{f(z_0) - \overline{f(z)}}{z_0 - \overline{z}} = -\frac{|f(z) - \overline{f(z_0)}|^2}{|z - \overline{z_0}|^2}. \tag{1.11}$$

We have that $|z - \overline{z_0}|^2$ is always positive and that $|f(z) - \overline{f(z_0)}|^2$ is nonnegative. Therefore we have $f(z) - \overline{f(z_0)} = 0$, so $f(z) = \overline{f(z_0)} = f(z_0)$ for all $z \in \mathbb{C}^+$. $\square$

Corollary 1.9. *Whenever $f \in \mathcal{P}$ is not constant,* $\operatorname{Im} f(z) > 0$ *for every* $z \in \mathbb{C}^+$.

Thus we conclude that the class $\mathcal{P}$ consists of two types of functions: (real) constant functions, which we exclude as trivial, and analytic self-mappings of $\mathbb{C}^+$. Then it makes sense to consider iterations of a given function $f \in \mathcal{P}$, in which case the knowledge of fixed points is of great importance.

2. Stieltjes class functions

A function f is in the Stieltjes class $\mathcal{S}$ if $f \in \mathcal{P}$, and, in addition, is analytic on $\mathbb{R}_+ = \{x \mid x > 0\}$ and $s(x) > 0$ for all $x \in \mathbb{R}_+$. That is, Stieltjes functions are Pick functions which also map $\mathbb{R}_+$ into itself. By the symmetry principle, every Stieltjes function s satisfies the symmetry relation $s(\bar{z}) = \overline{s(z)}$ for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. By Theorem 1.1 every Stieltjes function admits the Herglotz integral representation (1.1). Due to additional property that s is analytic and nonnegative on $\mathbb{R}_+$, the representation (1.1) is quite special. The two next results appear in [18].

Theorem 2.1. *A function s is in the Stieltjes class if and only if it can be represented in the form*

$$s(z) = a + bz + \int_0^\infty \frac{z}{z+t} d\mu(t) \tag{2.1}$$

where $a, b \geq 0$ and $d\mu$ is a positive measure on $\mathbb{R}$ such that

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

Some simple examples of Stieltjes functions and their corresponding measures are

$$\begin{aligned}
&s(z) = z^\alpha,\ 0 < \alpha < 1 && \text{where} && \mu(t) = \frac{\sin(\alpha\pi)}{\pi} t^{\alpha-1},\\
&s(z) = \sqrt{z}(1 - e^{-2a\sqrt{z}}),\ a > 0 && \text{where} && \mu(t) = \frac{2}{\pi\sqrt{t}} \sin^2(a\sqrt{t}),\\
&s(z) = \log\left(1 + \frac{z}{a}\right),\ a > 0 && \text{where} && \mu(t) = \frac{1}{t}\mathbf{1}_{(a,\infty)}(t),\ \text{and}\\
&s(z) = \sqrt{z}\arctan\left(\sqrt{\frac{z}{a}}\right),\ a > 0 && \text{where} && \mu(t) = \frac{1}{2\sqrt{t}}\mathbf{1}_{(a,\infty)}(t).
\end{aligned}$$

Here $\mathbf{1}_A$ is the indicator function; that is, $\mathbf{1}(t) = 1$ if $t \in A$ and $\mathbf{1}(t) = 0$ otherwise.

The next theorem establishes more precise connection between Pick and Stieltjes classes.

Theorem 2.2. *A function s is in the Stieltjes class if and only if $s \in \mathcal{P}$ and $-\dfrac{s(z)}{z} \in \mathcal{P}$.*

The latter theorem allows us to associate with any Stieltjes-class function s two kernels

$$K_s(z,\zeta) = \begin{cases} \dfrac{s(z) - \overline{s(\zeta)}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ s'(z) & \text{if } z = \overline{\zeta}, \end{cases} \tag{2.2}$$

$$\widetilde{K}_s(z,\zeta) = \begin{cases} \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(\zeta)}}{\overline{\zeta}}}{z - \overline{\zeta}} & \text{if } z \neq \overline{\zeta}, \\ \dfrac{s(z)}{z^2} - \dfrac{s'(z)}{z} & \text{if } z = \overline{\zeta}, \end{cases} \tag{2.3}$$

which are positive on $\mathbb{C}^+$. We next express the kernel $\widetilde{K}_s$ in terms of the integral representation (2.1) (the expression for the kernel K_s is the same as in formula (1.8)):

$$\begin{aligned}\widetilde{K}_s(z,\zeta) &= \frac{1}{z-\overline{\zeta}}\left(-\frac{a}{z} - \int_0^\infty \frac{d\mu(t)}{z+t} + \frac{a}{\overline{\zeta}} + \int_0^\infty \frac{d\mu(t)}{\overline{\zeta}+t}\right) \\ &= \frac{a}{z\overline{\zeta}} + \frac{1}{z-\overline{\zeta}} \cdot \int_0^\infty \left(\frac{1}{\overline{\zeta}+t} - \frac{1}{z+t}\right) d\mu(t) \\ &= \frac{a}{z\overline{\zeta}} + \int_0^\infty \frac{d\mu(t)}{(z+t)(\overline{\zeta}+t)}. \end{aligned} \tag{2.4}$$

In fact, the latter formula combined with Theorem 2.1 demonstrates the positivity of the kernel $\widetilde{K}_s$.

Theorem 2.2 allows us to characterize Stieltjes functions in terms of two Schwartz-Pick matrices.

Corollary 2.3. *A function s belongs to the Stieltjes class if and only if, for all $z_1, \dots, z_n \in \mathbb{C}^+$,*

$$K^s_{z_1,\dots,z_n} = \left[\frac{s(z_i) - \overline{s(z_j)}}{z_i - \overline{z}_j}\right]_{i,j=1}^n \geq 0 \tag{2.5}$$

and

$$\widetilde{K}^s_{z_1,\dots,z_n} = \left[\frac{z_i\overline{s(z_j)} - \overline{z}_j s(z_i)}{z_i\overline{z}_j(z_i - \overline{z}_j)}\right]_{i,j=1}^n \geq 0. \tag{2.6}$$

Proof. The first matrix is simply the Pick matrix of s and so is positive semidefinite for all choices of $z_1, \dots, z_n$ if and only if $f \in \mathcal{P}$. For the second matrix, if $g(z) = -s(z)/z$ we have

$$[\widetilde{K}^s_{z_1,\dots,z_n}]_{ij} = \frac{-s(z_i)/z_i + \overline{s(z_j)/z_j}}{z_i - \overline{z}_j} = \frac{z_i\overline{s(z_j)} - \overline{z}_j s(z_i)}{z_i\overline{z}_j(z_i - \overline{z}_j)},$$

so the matrix being positive semidefinite is equivalent to $g \in \mathcal{P}$.

3. Interior fixed points of Pick class functions

Given a function $f \in \mathcal{P}$, we say that a point $x \in \mathbb{C}^+$ is a fixed point if $f(z_0) = z_0$. As the next theorem shows, a Pick function may have at most one fixed point in $\mathbb{C}^+$.

Theorem 3.1. *Any Pick class function $f \in \mathcal{P}$ different from the identity map has at most one interior fixed point.*

Proof. Let $z_1, z_2 \in \mathbb{C}^+$ be two distinct points such that

$$f(z_1) = z_1 \quad \text{and} \quad f(z_2) = z_2, \tag{3.1}$$

and let z be an arbitrary point in $\mathbb{C}^+$. The Schwarz-Pick matrix $K^f_{z_1,z_2,z}$ is positive semidefinite. According to (1.10) and (3.1) this matrix takes the form

$$K^f_{z_1,z_2,z} = \begin{bmatrix} 1 & 1 & \dfrac{z_1 - \overline{f(z)}}{z_1 - \overline{z}} \\ 1 & 1 & \dfrac{z_2 - \overline{f(z)}}{z_2 - \overline{z}} \\ \dfrac{f(z) - \overline{z}_1}{z - \overline{z}_1} & \dfrac{f(z) - \overline{z}_2}{z - \overline{z}_2} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix} \geq 0. \tag{3.2}$$

Then we also have

$$\begin{aligned} 0 &\leq \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} K^f_{z_1,z_2,z} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \dfrac{z_1 - \overline{f(z)}}{z_1 - \overline{z}} - \dfrac{z_2 - \overline{f(z)}}{z_2 - \overline{z}} \\ \dfrac{f(z) - \overline{z}_1}{z - \overline{z}_1} - \dfrac{f(z) - \overline{z}_2}{z - \overline{z}_2} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \det \begin{bmatrix} 0 & \dfrac{z_1 - \overline{f(z)}}{z_1 - \overline{z}} - \dfrac{z_2 - \overline{f(z)}}{z_2 - \overline{z}} \\ \dfrac{f(z) - \overline{z}_1}{z - \overline{z}_1} - \dfrac{f(z) - \overline{z}_2}{z - \overline{z}_2} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix} \\ &= -\left| \dfrac{f(z) - \overline{z}_1}{z - \overline{z}_1} - \dfrac{f(z) - \overline{z}_2}{z - \overline{z}_2} \right|^2, \end{aligned}$$

from which we conclude that for every $z \in \mathbb{C}_+ \setminus \{z_1, z_2\}$,

$$0 = \frac{f(z) - \overline{z}_1}{z - \overline{z}_1} - \frac{f(z) - \overline{z}_2}{z - \overline{z}_2} = \frac{(f(z) - z)(\overline{z_1} - \overline{z_2})}{(z - \overline{z}_1)(z - \overline{z}_2)}.$$

Therefore, $f(z) \equiv z$ which contradicts the assumption of the theorem. □

A fixed point z_0 is called *attractive* of a function f if for any point z that is close enough to z_0, the sequence of iterates z, $f(z)$, $f(f(z))$, $f(f(f(z)))$, … converges to z_0. It is known that a fixed point z_0 of an analytic function f is attractive if $|f'(z_0)| < 1$.

Theorem 3.2. *If $z_0 \in \mathbb{C}^+$ is a fixed point of a Pick function f, then $|f'(z_0)| \leq 1$.*

Proof. Take z_0 to be the fixed point of f and let $z \in \mathbb{C}^+$. Then we have the positive semidefinite Schwarz-Pick matrix $K^f_{z_0,\overline{z_0}}$:

$$K^f_{z_0,\overline{z_0}} = \begin{bmatrix} \dfrac{f(z_0)-\overline{f(z_0)}}{z_0-\overline{z}_0} & f'(z_0) \\ \overline{f'(z_0)} & \dfrac{\overline{f(z_0)}-f(z_0)}{\overline{z_0}-z_0} \end{bmatrix} = \begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix} \geq 0.$$

Therefore, the determinant of the matrix, $1-|f'(z_0)|^2$, is greater than or equal to zero, so $|f'(z_0)| \leq 1$, which completes the proof. ☐

It is worth noting that, if f has an interior fixed point z_0 such that $|f'(z_0)| = 1$, then f is a linear fractional function which is real on $\mathbb{R}$.

Remark 3.3. *Let z_0 be an interior fixed point of $f \in \mathcal{P}$ such that $|f'(z_0)| = 1$. Then*

$$f(z) = \frac{(f'(z_0)-1)|z_0|^2 + (z_0 - \overline{z_0}f'(z_0))z}{(z_0 f'(z_0) - \overline{z_0}) + (1-f'(z_0))z}. \tag{3.3}$$

Proof. As in Theorem 3.1 above, use a 3×3 Schwarz-Pick matrix, where the first point is an interior fixed point and the second approaches the same point. Let z_0 be the interior fixed point of f and consider the positive semidefinite Schwartz-Pick matrix $K^f_{z_0,\overline{z_0},z}$:

$$K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & \dfrac{z_0-\overline{f(z)}}{z_0-\overline{z}} \\ \overline{f'(z_0)} & 1 & \dfrac{\overline{z}_0-\overline{f(z)}}{\overline{z}_0-\overline{z}} \\ \dfrac{\overline{z}_0-f(z)}{\overline{z}_0-z} & \dfrac{z_0-f(z)}{z_0-z} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \geq 0. \tag{3.4}$$

Keeping in mind that $|f'(z_0)| = 1$, we know that $\begin{bmatrix} f'(z_0) \\ -1 \end{bmatrix}$ is in the null space of $\begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & f'(z_0) \\ \overline{f'(z_0)} & 1 \end{bmatrix} \begin{bmatrix} f'(z_0) \\ -1 \end{bmatrix} = \begin{bmatrix} f'(z_0) - f'(z_0) \\ 1-1 \end{bmatrix}.$$

Therefore,

$$0 \le \begin{bmatrix} \overline{f'(z_0)} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} K^f_{z_0,\overline{z}_0,z} \begin{bmatrix} f'(z_0) & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \overline{f'(z_0)}\left(\dfrac{z_0 - \overline{f(z)}}{z_0 - \overline{z}}\right) - \dfrac{\overline{z}_0 - \overline{f(z)}}{\overline{z}_0 - \overline{z}} \\ f'(z_0)\left(\dfrac{\overline{z}_0 - f(z)}{\overline{z}_0 - z}\right) - \dfrac{z_0 - f(z)}{z_0 - z} & \dfrac{f(z) - \overline{f(z)}}{z - \overline{z}} \end{bmatrix}.$$

Since the matrix is positive semidefinite, its determinant must be non-negative, so

$$0 - \left| f'(z_0)\left(\frac{\overline{z}_0 - f(z)}{\overline{z}_0 - z}\right) - \frac{z_0 - f(z)}{z_0 - z} \right|^2 \ge 0.$$

However, this can only occur when $f'(z_0)\left(\dfrac{\overline{z}_0 - f(z)}{\overline{z}_0 - z}\right) - \dfrac{z_0 - f(z)}{z_0 - z} = 0$. Then we have

$$0 = f'(z_0)(\overline{z_0} - f(z))(z_0 - z) - (z_0 - f(z))(\overline{z_0} - z)$$

which being solved for $f(z)$ leads us to formula (3.3). If $f'(z_0) = 1$, then (3.3) implies that $f(z) \equiv z$. If $f'(z_0) \ne 1$, we may rewrite (3.3) as

$$f(z) = \frac{|z_0|^2 + \alpha z}{\beta - z}, \quad \text{where} \quad \alpha = \frac{z_0 - \overline{z}_0 f'(z_0)}{f'(z_0) - 1}, \quad \beta = \frac{z_0 f'(z_0) - \overline{z}_0}{f'(z_0) - 1}.$$

Since $|f'(z_0)| = 1$, we have

$$\alpha - \overline{\alpha} = \frac{z_0 - \overline{z_0} f'(z_0)}{f'(z_0) - 1} - \frac{\overline{z}_0 - z_0 \overline{f'(z_0)}}{\overline{f'(z_0)} - 1} = \frac{(\overline{z}_0 - z_0)(1 - |f'(z_0)|^2}{|f'(z_0) - 1|^2} = 0$$

and thus α is real. Similarly, β is real, thus f takes real values on $\mathbb{R}$. □

4. Interior fixed points of Stieltjes class functions

We will now examine the fixed points of Stieltjes class functions inside the domain of analyticity, that is, in $\mathbb{C} \setminus \mathbb{R}_-$.

Theorem 4.1. *A Stieltjes class function s (not the identity map) may have at most one fixed point in $\mathbb{C} \setminus \mathbb{R}_-$. This point necessarily belongs to $\mathbb{R}_+$.*

Proof. We first show that s cannot have fixed points in $\mathbb{C}_+$. Let us assume that $s(z_0) = z_0$ for some $z_0 \in \mathbb{C}_+$. Then

$$\frac{z_0 \overline{s(z_0)} - \overline{z}_0 s(z_0)}{|z_0|^2 (z_0 - \overline{z}_0)} = \frac{|z_0|^2 - |z_0|^2}{|z_0|^2 (z_0 - \overline{z}_0)} = 0.$$

The Schwarz-Pick matrix $\widetilde{K}^s_{z_0,z}$ is positive semidefinite and its leftmost diagonal entry equals zero. Then its non-diagonal entries, which are complex conjugates, are also zero:

$$\frac{z\overline{s(z_0)} - \overline{z}_0 s(z)}{z\overline{z}_0(z-\overline{z}_0)} = \frac{z\overline{z_0} - \overline{z}_0 s(z)}{z\overline{z}_0(z-\overline{z}_0)} = \frac{z-s(z)}{z(z-\overline{z}_0)} = 0.$$

Since the latter equality holds for all $z \in \mathbb{C}_+$, we conclude that $s(z) \equiv z$ which contradicts the assumption.

Assuming that $s(z_0) = z_0$ for some $z_0 \in \mathbb{C}_-$, then by the symmetry relation we have $s(\overline{z}_0) = \overline{s(z_0)} = \overline{z}_0$ and thus the point $\overline{z}_0 \in \mathbb{C}_+$ is a fixed point for s which is impossible. Therefore, s cannot have fixed points in $\mathbb{C}_+ \cup \mathbb{C}_-$.

Let us assume that s has two fixed points in $\mathbb{R}_+$, i.e., that

$$s(x_1) = x_1 \quad \text{and} \quad s(x_2) = x_2 \quad \text{for some} \quad x_1,\ x_2 > 0. \tag{4.1}$$

The associated Schwarz-Pick matrices $K^s_{x_1,x_2}$ and $\widetilde{K}^s_{x_1,x_2}$ are positive semidefinite. According to (2.2), (2.3), (2.5),(2.6) and (4.1),

$$\begin{aligned}
[K^s_{x_1,x_2}]_{ii} &= s'(x_i) \quad \text{for} \quad i=1,2;\\
[K^s_{x_1,x_2}]_{12} &= \frac{s(x_1)-s(x_2)}{x_1-x_2} = \frac{x_1-x_2}{x_1-x_2} = 1;\\
[\widetilde{K}^s_{x_1,x_2}]_{ii} &= \frac{s(x_i)}{x_i} - \frac{s'(x_i)}{x_i^2} = \frac{1-s'(x_i)}{x_i} \quad \text{for} \quad i=1,2;\\
[\widetilde{K}^s_{x_1,x_2}]_{12} &= \frac{-\frac{s(x_1)}{x_1} + \frac{s(x_2)}{x_2}}{x_1-x_2} = \frac{1-1}{x_1-x_2} = 0.
\end{aligned}$$

We thus have

$$K^s_{x_1,x_2} = \begin{bmatrix} s'(x_1) & 1 \\ 1 & s'(x_2) \end{bmatrix} \quad \text{and} \quad \widetilde{K}^s_{x_1,x_2} = \begin{bmatrix} \frac{1-s'(x_1)}{x_1} & 0 \\ 0 & \frac{1-s'(x_2)}{x_2} \end{bmatrix}. \tag{4.2}$$

Since both matrices in (4.2) are positive semidefinite and since x_1, x_2 are positive numbers, we conclude that

$$0 < s'(x_1),\ s'(x_2) \le 1 \quad \text{and} \quad s'(x_1)\cdot s'(x_2) \ge 1. \tag{4.3}$$

The latter may happen only if $s'(x_1) = s'(x_2) = 1$. We then consider the Schwarz-Pick matrix $\widetilde{K}^s_{x_1,z}$ based on the fixed point x_1 and an arbitrary point

$z \in \mathbb{C} \setminus \mathbb{R}$:

$$\widetilde{K}^s_{x_1,z} = \begin{bmatrix} \dfrac{1-s'(x_1)}{x_1} & \dfrac{-\dfrac{s(x_1)}{x} + \dfrac{\overline{s(z)}}{\overline{z}}}{x_1 - \overline{z}} \\ \dfrac{-\dfrac{s(z)}{z} + \dfrac{s(x_1)}{x_1}}{z - x_1} & \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(z)}}{\overline{z}}}{z - \overline{z}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1 + \dfrac{\overline{s(z)}}{\overline{z}}}{x_1 - \overline{z}} \\ \dfrac{-\dfrac{s(z)}{z} + 1}{z - x_1} & \dfrac{-\dfrac{s(z)}{z} + \dfrac{\overline{s(z)}}{\overline{z}}}{z - \overline{z}} \end{bmatrix}$$

where the last equality follows since $s(x_1) = x_1$ and $s'(x_1) = 1$. Since the matrix $\widetilde{K}^s_{x_1,z}$ is positive senidefinite for every z, we conclude that $-\dfrac{s(z)}{z} + 1 = 0$ for every z so that $s(z) \equiv z$ which contradicts the assumption of the theorem. $\square$

The next result contains the Stieltjes-class analogues of Theorem 3.2 and of Remark 3.3.

Theorem 4.2. *If $x_0 \in \mathbb{R}^+$ is a fixed point of a Stieltjes function s, then $0 \leq s'(x_0) \leq 1$. Moreover, if $s'(x_0) = 0$, then $s(z) \equiv z_0$. If $s'(x_0) = 1$, then $s(z) \equiv z$.*

Proof. Consider that $\widetilde{K}^f_{x_0}$, as a 1×1 positive semidefinite matrix, must be a nonnegative real number. Then

$$\widetilde{K}^f_{x_0} = \frac{s(x_0)}{x_0^2} - \frac{s'(x_0)}{x_0} = \frac{x_0}{x_0^2} - \frac{s'(x_0)}{x_0} = \frac{1 - s'(x_0)}{x_0} \geq 0.$$

Then, since x_0 is non-negative, we have that $s'(x_0) \leq 1$.

Now consider that if $s'(x_0) = 0$, we have the positive semidefinite Schwartz-Pick Matrix

$$K^f_{x_0,z} = \begin{bmatrix} 0 & \dfrac{x_0 - \overline{s(z)}}{x_0 - \overline{z}} \\ \dfrac{s(z) - x_0}{z - x_0} & \dfrac{s(z) - \overline{s(z)}}{z - \overline{(z)}} \end{bmatrix}. \tag{4.4}$$

Taking the determinant, then, we have $\left| \dfrac{x_0 - \overline{s(z)}}{x_0 - \overline{z}} \right|^2 \geq 0$, which is true only if $x_0 - \overline{s(z)} = 0$. Therefore, $\overline{s(z)} = x_0$ for all $z \in \mathbb{C}^+$; since x_0 is real, we have $s(z) \equiv x_0$.

Now let $s'(x_0) = 1$. Taking the matrix $\widetilde{K}^s_{x_0,z}$, we know from (4.2) that

$$[\widetilde{K}^s_{x_0,z}]_{11} = \frac{1 - s'(x_o)}{x_0} = 0$$

and, since $s(x_0) = x_0$,

$$[\widetilde{K}^s_{x_0,z}]_{21} = \frac{-\dfrac{s(z)}{z} + \dfrac{s(x_0)}{x_0}}{z - x_0} = \frac{-\dfrac{s(z)}{z} + 1}{z - x_0}.$$

Therefore, the determinant of $\widetilde{K}^s_{x_0,z}$ is

$$-[\widetilde{K}^s_{x_0,z}]_{12}[\widetilde{K}^s_{x_0,z}]_{21} = -\overline{[\widetilde{K}^s_{x_0,z}]_{21}}[\widetilde{K}^s_{x_0,z}]_{21} = -\left|\frac{-\dfrac{s(z)}{z} + 1}{z - x_0}\right|^2,$$

since $\widetilde{K}^s_{x_0,z}$ is positive semidefinite and therefore Hermitian. Also, since $\widetilde{K}^s_{x_0,z}$ is positive semidefinite, $\det \widetilde{K}^s_{x_0,z} \geq 0$, which only occurs when $-\dfrac{s(z)}{z} + 1 = 0$. We can then conclude that $s(z) = z$ for all z.

5. Boundary fixed points of Pick class functions

Let us say that a function f analytic on $\mathbb{C}^+$ admits the angular limit at a boundary point $x_0 \in \mathbb{R}$ if the limit

$$f(x_0) := \lim_{z \to x_0} f(z) \tag{5.1}$$

exists whenever $z \in \mathbb{C}^+$ tends to $x_0 \in \mathbb{R}$ staying inside the angle $\alpha < \arg(z - x_0) < \pi - \alpha$ for some fixed $\alpha \in (0, \pi/2)$. A celebrated result of Pierre Fatou [19] asserts that if f is bounded on $\mathbb{C}^+$ in the sense that $\sup_{z \in \mathbb{C}^+} |f(z)| < \infty$, then the angular boundary limit (5.1) exists at almost every $x \in \mathbb{R}$.

Proposition 5.1. *Every non-constant Pick-class function $f \in \mathcal{P}$ admits a representation*

$$f(z) = i \cdot \frac{1 + g(z)}{1 - g(z)} \tag{5.2}$$

for some non-constant function g analytic on $\mathbb{C}^+$ and such that $\sup_{z \in \mathbb{C}^+} |g(z)| \leq 1$.

Proof: To see that $|g(z)| \leq 1$, we solve $f(z) = i \cdot \dfrac{1 + g(z)}{1 - g(z)}$ for $g(z)$, resulting in $g(z) = \dfrac{f(z) - i}{f(z) + i}$ for $f(z) \neq -i$. Then

$$1 - |g(z)|^2 = 1 - \left|\frac{f(z) - i}{f(z) + i}\right|^2 = \frac{|f(z) + 1|^2 - |f(z) - 1|^2}{|f(z) + 1|^2}. \tag{5.3}$$

Since $|f(z)+i|^2 > 0$, to show that $1-|g(z)|^2$ is positive we must show that $|f(z)+i|^2 - |f(z)-i|^2 > 0$. Expand the terms of the left hand side to

$$\begin{aligned} |f(z)+i|^2 - |f(z)-i|^2 &= (f(z)+i)\overline{(f(z)+i)} - (f(z)-i)\overline{(f(z)-i)} \\ &= (f(z)+i)(\overline{f(z)}-i) - (f(z)-i)(\overline{f(z)}+i) \\ &= |f(z)|^2 + i\overline{f(z)} - if(z) - 1 - |f(z)|^2 + i\overline{f(z)} - if(z) + 1 \\ &= -2i(f(z)-\overline{f(z)}). \end{aligned}$$

Then, since $\operatorname{Im} f(z) = \dfrac{f(z)-\overline{f(z)}}{2i}$, we have that $-2i(f(z)-\overline{f(z)}) = 4\operatorname{Im} f(z)$, which is positive since f is a Pick-class function. Therefore $1-|g(z)|^2 > 0$, and thus $g(z)$ is positive. □

Combining the latter proposition with the classic Fatou's theorem we conclude that any Pick function admits angular boundary limits almost everywhere on $\mathbb{R}$. The next theorem is due G. Julia [22] and C. Carathéodory [12].

Theorem 5.2. *Let f be a Pick-class function and let us assume that*

$$f(x_0) := \lim_{z\to x_0} f(z) = a \in \mathbb{R}.$$

Then the following limits exist (finitely or infinitely) in $\overline{\mathbb{R}_+}$:

$$f'(x_0) := \lim_{z\to x_0} f'(z) = \lim_{z\to x_0} \frac{f(z)-a}{z-x_0} = \lim_{z\to x_0} \frac{\operatorname{Im} f(z)}{\operatorname{Im} z}.$$

Definition 5.3. *A point $x_0 \in \mathbb{R}$ is called a boundary fixed point if $f(x_0) := \lim_{z\to x_0} f(z) = x_0$.*

By Theorem 5.8, for every fixed point x_0 of a Pick function f, the limit $f'(x_0) := \lim_{z\to x_0} f'(z)$ exists and is nonnegative or infinite.

Remark 5.4. *A Pick function $f \in \mathcal{P}$ can have many fixed boundary points.*

As an example of a Pick function with multiple boundary fixed points, we let

$$f(z) = \frac{z(z-75)}{15z^2-125}.$$

To see that $f \in \mathcal{P}$, we first note that f is analytic everywhere except its poles at $\pm 5/\sqrt{3}$, so its domain of analyticity covers $\mathbb{C}^+$. By Definition 1.2, we still need to show that $\dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \geq 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since

$$\frac{f(z)-\overline{f(z)}}{z-\overline{z}} = \left(\frac{z(z-75)}{15z^2-125} - \frac{\overline{z}(\overline{z}-75)}{15\overline{z}^2-125}\right)\left(\frac{1}{z-\overline{z}}\right)$$
$$= \frac{z(z-75)(15\overline{z}^2-125) - \overline{z(z-75)(15\overline{z}^2-125)}}{(z-\overline{z})|15z^2-125|}$$

and $|15z^2-125| > 0$, we still need that $\dfrac{\text{Im } z(z-75)(15\overline{z}^2-125)}{\text{Im } z} \geq 0$. Solving algebraically and letting $z = a+bi$, we have

$$\text{Im } z(z-75)(15\overline{z}^2-125) = 1125|z|^2 b - 250ab + 9375b.$$

Dividing by b and flipping the inequality if b is negative, then, we need to show that $1125|z|^2 - 250a + 9375 > 0$. If $a \leq 1$, then $9375 > 250a$ and so this expression is positive. If $a > 1$, then $|z|^2 = a^2 + b^2 > a$ and so $1125|z|^2 > 250a$ and the expression is still positive. Therefore $f \in \mathcal{P}$.

It is immediately apparent that $x_0 = 0$ is a fixed point. Dividing each side by 0 and then rearranging terms, we have $0 = 15z^2 - z - 50$, so the two other fixed points are $x_1 = (1+\sqrt{3001})/30$ and $x_2 = (1-\sqrt{3001})/30$. Taking the derivative $f'(z) = \dfrac{(15z^2-125)(2z-75)-30z^2(z-75)}{(15z^2-125)^2}$, we have $f'(0) = 3/5$. For the other two fixed points, we have $f'(x_1) \approx 2.3$ and $f'(x_2) \approx 2.28$. Note that there is only one boundary fixed point x such that $|f'(x)| \leq 1$; as we will see below, there is exactly one fixed point, whether internal or boundary, with this characteristic for all Pick functions. We call this point the *Denjoy-Wolff point*.

Theorem 5.5. *Let us assume that a Pick function f which is not the identity function has an interior fixed point $z_0 \in \mathbb{C}^+$. Then for every boundary fixed point x (if such points exist), $f'(x) > 1$.*

Proof. From (3.4), we know that

$$K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & \dfrac{z_0-\overline{f(z)}}{z_0-\overline{z}} \\ \overline{f'(z_0)} & 1 & \dfrac{\overline{z}_0-\overline{f(z)}}{\overline{z}_0-\overline{z}} \\ \dfrac{\overline{z}_0-f(z)}{\overline{z}_0-z} & \dfrac{z_0-f(z)}{z_0-z} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix} \geq 0. \tag{5.4}$$

Letting $z = x+iy \to x$ from above (so $y \to 0$), we have

$$\lim_{z\to x} K^f_{z_0,\overline{z}_0,z} = \begin{bmatrix} 1 & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & f'(x) \end{bmatrix} \geq 0. \tag{5.5}$$

Since the above matrix is positive semidefinite, all of its principal submatrices are also positive semidefinite, so

$$0 \le \det \begin{bmatrix} 1 & 1 \\ 1 & f'(x) \end{bmatrix} = f'(x) - 1$$

and therefore $f'(x) \ge 1$.

Now assume for the sake of a contradiction that $f'(x) = 1$, so from (5.5),

$$\begin{bmatrix} \frac{1}{f'(z_0)} & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ge 0.$$

Then we have that

$$0 \le \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{f'(z_0)} & f'(z_0) & 1 \\ \overline{f'(z_0)} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{f'(z_0)} - 1 & f'(z_0) - 1 \\ & 0 \end{bmatrix}.$$

Taking the determinant of the above matrix and considering that it is positive semidefinite, we have that $-|f'(z_0) - 1|^2 \ge 0$ and so $f'(z_0) = 1$. Therefore, by formula (3.3), we have

$$f(z) = \frac{(f'(z_0) - 1)|z_0|^2 + (z_0 - \overline{z_0} f'(z_0))z}{(z_0 f'(z_0) - \overline{z_0}) + (1 - f'(z_0))z} = \frac{(z_0 - \overline{z_0})z}{z_0 - \overline{z_0}} = z$$

for all $z \in \mathbb{C}^+$. Then f is the identity function, which was disallowed, so $f'(x) > 1$. $\square$

Theorem 5.5 tells us that a necessary condition for the existence of an attractive boundary fixed point is the absence of interior fixed points. The next theorem (see [1] for the proof) shows that this condition is almost sufficient.

Theorem 5.6. *If $f \in \mathcal{P}$ does not have an interior fixed point, then either it has a boundary fixed point x_0 with $f'(x_0) \le 1$ or*

$$\frac{\operatorname{Im} f(z)}{\operatorname{Im} z} \ge 1 \quad \text{for every} \quad z \in \mathbb{C}^+.$$

In the latter case $f'(\infty) := \lim_{y \to +\infty} f'(x + iy) \ge 1$.

We now ask how many attractive fixed boundary points a Pick-class function can have.

Theorem 5.7. *A Pick-class function f cannot have two boundary fixed points x_0, x_1 such that $f'(x_0) < 1$ and $f'(x_1) \le 1$.*

Proof. Take two points z_0 and z_1 in $\mathbb{C}^+$ and onsider the Pick matrix

$$K^f_{z_0,z_1} = \begin{bmatrix} \frac{f(z_0) - \overline{f(z_0)}}{z_0 - \overline{z_0}} & \frac{f(z_0) - \overline{f(z_1)}}{z_0 - \overline{z_1}} \\ \frac{f(z_1) - \overline{f(z_0)}}{z_1 - \overline{z_0}} & \frac{f(z_1) - \overline{f(z_1)}}{z_1 - \overline{z_1}} \end{bmatrix} \ge 0. \tag{5.6}$$

Letting $z_0 \to x_0$ and $z_1 \to x_1$ from directly above, we have

$$\begin{bmatrix} f'(x_0) & 1 \\ 1 & f'(x_1) \end{bmatrix} \geq 0. \tag{5.7}$$

Therefore, the determinant of the above matrix $f'(x_0)f'(x_1)-1$ is non-negative, so that $f'(x_0)f'(x_1) \geq 1$. Since $f'(x_0)$ and $f'(x_1)$ are both non-negative, they cannot exceed one simultaneously. □

In the last theorem we actually proved that for two fixed boundary points z_0 and x_1 of a Pick function f, we always have $f'(x_0)f'(x_1) \geq 1$. The next theorem makes this statement more precise.

Theorem 5.8. *Let x_0 and x_1 be two fixed boundary points of a function $f \in \mathcal{P}$. Then $f'(x_0)f'(x_1) = 1$ if and only if*

$$f(z) = \frac{(f'(x_0)-1)x_0x_1 + (x_0 - f'(x_0)x_1)z}{(x_0f'(x_0) - x_1) + (1 - f'(x_0))z}. \tag{5.8}$$

Proof. Consider that

$$\lim_{\substack{z_0 \to x_0 \\ z_1 \to x_1}} K^f_{z_0,z_1,z} = \begin{bmatrix} f'(x_0) & 1 & \dfrac{x_0 - \overline{f(z)}}{x_0 - \overline{z}} \\ 1 & f'(x_1) & \dfrac{x_1 - \overline{f(z)}}{x_1 - \overline{z}} \\ \dfrac{f(z)-x_0}{z-x_0} & \dfrac{f(z)-x_1}{z-x_1} & \dfrac{f(z)-\overline{f(z)}}{z-\overline{z}} \end{bmatrix}. \tag{5.9}$$

Then, because the above matrix is positive semidefinite,

$$0 \leq \begin{bmatrix} -1 & f'(x_0) & 0 \\ 0 & 0 & 1 \end{bmatrix} \lim_{\substack{z_0 \to x_0 \\ z_1 \to x_1}} K^f_{z_0,z_1,z} \begin{bmatrix} -1 & 0 \\ f'(x_0) & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & f'(x_0)\dfrac{x_1 - \overline{f(z)}}{x_1 - \overline{z}} - \dfrac{x_0 - \overline{f(z)}}{x_0 - \overline{z}} \\ f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0} & 1 \end{bmatrix}.$$

The determinant of this matrix is $-\left| f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0} \right|^2$, which is greater than or equal to zero. Consequently, $f'(x_0)\dfrac{f(z)-x_1}{z-x_1} - \dfrac{f(z)-x_0}{z-x_0} = 0$, so

$$f'(x_0)(f(z)-x_1)(z-x_0) = (f(z)-x_0)(z-x_1)$$

and

$$f(z)(z - x_1 - zf'(x_0) + x_0f'(x_0)) = zx_0 - x_0x_1 - zf'(x_0)x_1 + f'(x_0)x_0x_1,$$

yielding (5.8). Differentiating (5.8) gives

$$f'(z) = \frac{f'(x_0)(x_0 - x_1)^2}{((x_0 f'(x_0) - x_1) + (1 - f'(x_0))z)^2}$$

and evaluating the latter formula at $z = x_1$ gives

$$f'(x_1) = \frac{f'(x_0)(x_0 - x_1)^2}{(x_0 f'(x_0) - x_1 + (1 - f'(x_0))x_1)^2} = \frac{1}{f'(x_0)},$$

so that $f'(x_0)f'(x_1) = 1$. $\square$

The next theorem establishes a relation between the values of the derivative of a Pick-class function at fixed points. Using different methods, such a result was established in [14] for analytic self-mappings of the unit disk.

Theorem 5.9. *Let $f \in \mathcal{P}$, $z_0 \in \mathbb{C}^+$, and $x_1, x_2, \ldots, x_n$ be fixed points of f. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{1 - |f'(z_0)|^2}{|1 - f'(z_0)|^2}. \tag{5.10}$$

Proof. We use the Schwartz-Pick matrix $K^f_{z_0, \overline{z_0}, \zeta_1, \zeta_2, \ldots, \zeta_n}$ and take the limit as $\zeta_i \to x_i$ for all $1 \leq i \leq n$, resulting in the matrix

$$\begin{bmatrix} \frac{1}{\overline{f'(z_0)}} & f'(z_0) & 1 & \cdots & 1 \\ \overline{f'(z_0)} & 1 & 1 & \cdots & 1 \\ 1 & 1 & f'(x_1) & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & f'(x_n) \end{bmatrix} \geq 0. \tag{5.11}$$

Exchanging the first and second columns, we have that the matrix

$$\begin{bmatrix} f'(z_0) & \frac{1}{\overline{f'(z_0)}} & 1 & \cdots & 1 \\ 1 & \overline{f'(z_0)} & 1 & \cdots & 1 \\ 1 & 1 & f'(x_1) & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & f'(x_n) \end{bmatrix} \tag{5.12}$$

has a non-postive determinant. Letting $\mu_i = f'(x_i) - 1$, we then subtract the first row from all other rows in the matrix to attain

$$\det \begin{bmatrix} f'(z_0) & \frac{1}{\overline{f'(z_0)}} & 1 & \cdots & 1 \\ 1 - f'(z_0) & \overline{f'(z_0)} - 1 & 0 & \cdots & 0 \\ 1 - f'(z_0) & 0 & \mu_1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 - f'(z_0) & 0 & 0 & & \mu_n \end{bmatrix} \leq 0. \tag{5.13}$$

Computing the determinant along the first column, we then have

$$\begin{aligned} f'(z_0)(\overline{f'(z_0)}-1)\prod_{i=1}^{n}\mu_i - (1-f'(z_0))\prod_{i=1}^{n}\mu_i \\ + \sum_{i=1}^{n}(-1)^{i-1}(1-f'(z_0))\det A_i \le 0 \end{aligned} \tag{5.14}$$

where A_i is the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \overline{f'(z_0)}-1 & & & & & & & 0 \\ & \mu_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \mu_{i-1} & 0 & & & \\ & & & & 0 & \mu_{i+1} & & \\ & & & & & & \ddots & \\ 0 & & & & & & & \mu_n \end{bmatrix}.$$

because we have an upper diagonal matrix after exchanging columns i times, we have that $\det A_i = (-1)^i(\overline{f'(z_0)}-1)\dfrac{\mu_1\cdots\mu_n}{\mu_i}$. Then, from (5.14),

$$f'(z_0)(\overline{f'(z_0)}-1) - (1-f'(z_0)) - \sum_{i=1}^{n}(1-f'(z_0))(\overline{f'(z_0)}-1)\frac{1}{\mu_i} \le 0.$$

Simplifying, we have

$$|f'(z_0)|^2 - 1 + |1-f'(z_0)|^2\left(\frac{1}{\mu_1}+\cdots+\frac{1}{\mu_n}\right) \le 0$$

and therefore

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1} = \sum_{i=1}^{n}\frac{1}{\mu_i} \le \frac{1-|f'(z_0)|^2}{|1-f'(z_0)|^2}.$$

$\square$

In the next theorem, the Denjoy-Wolff point is on the boundary.

Theorem 5.10. *Let $f \in \mathcal{P}$ and $x_0, x_1, \ldots, x_n \in \mathbb{R}$ be fixed points, with $f'(x_0) < 1$. Then*

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1} \le \frac{f'(x_0)}{1-f'(x_0)}. \tag{5.15}$$

Proof. Take the Schwartz-Pick matrix $K^f_{\zeta_0,\zeta_1,\cdots,\zeta_n}$. Letting $\zeta_i \to x_i$ for all $x = 0, 1, \cdots, n$, we have the positive semidefinite matrix

$$\begin{bmatrix} f'(x_0) & 1 & \cdots & 1 \\ 1 & f'(x_1) & & 1 \\ \vdots & & \ddots & \\ 1 & 1 & & f'(x_n) \end{bmatrix}. \tag{5.16}$$

Subtracting the first row from all other rows, which has no effect on the determinant of the matrix, we have

$$\begin{bmatrix} f'(x_0) & 1 & \cdots & 1 \\ 1 - f'(x_0) & \mu_1 & & 0 \\ \vdots & & \ddots & \\ 1 - f'(x_0) & 0 & & \mu_n \end{bmatrix} \geq 0 \tag{5.17}$$

where $\mu_i = f'(x_i) - 1$ for $i = 1, 2, \ldots, n$. Taking the determinant of (5.17) with respect to the first column, we have

$$f'(x_0)\mu_1 \cdots \mu_n + \sum_{i=1}^{n} (-1)^i (1 - f'(x_0)) A_i \geq 0 \tag{5.18}$$

where A_i is equal to

$$\begin{bmatrix} 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ \mu_1 & 0 & & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & & \\ & \ddots & \mu_{i-1} & 0 & 0 & & \vdots \\ \vdots & & 0 & 0 & \mu_{i+1} & \ddots & \\ & & & & \ddots & \ddots & 0 \\ 0 & & \cdots & & & 0 & \mu_n \end{bmatrix}.$$

Also, consider that

$$\det A_i = (-1)^{i-1} \det \begin{bmatrix} 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \mu_1 & 0 & & \cdots & & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \mu_{i-1} & 0 & & \vdots \\ \vdots & & & 0 & \mu_{i+1} & \ddots & \\ & & & & \ddots & \ddots & 0 \\ 0 & & \cdots & & & 0 & \mu_n \end{bmatrix} \tag{5.19}$$

since the second matrix can be obtained by transposing $i-1$ columns. Because the matrix is upper triangular, the above is equal to $(-1)^{i-1}\frac{\mu_1 \cdots \mu_n}{\mu_i}$. Then from

(5.18) we have

$$\mu_1 \cdots \mu_n (f'(x_0) - (1 - f'(x_0)) \left(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}\right) \geq 0$$

and therefore

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} = \sum_{i=1}^{n} \frac{1}{\mu_i} \leq \frac{f'(x_0)}{1 - f'(x_0)}.$$

□

The estimate (5.10) does not provide much in the case of $f'(x_0) = 1$. A natural question to ask is: *is it possible to establish a nontrivial estimate for the expression on the left side of* (5.10) *in terms of higher order boundary derivatives of the function f?*

Let us assume that the limits

$$f''(x_0) = \lim_{y \to 0} f''(x_0 + iy) \quad \text{and} \quad f'''(x_0) = \lim_{y \to 0} f'''(x_0 + iy) \tag{5.20}$$

exist finitely and are real. If $x_1, \ldots, x_n \in \mathbb{R}$ are other fixed points of f, then the following matrix is positive semidefinite:

$$\begin{bmatrix} f'(x_1) & 1 & \ldots & 1 & 1 & \frac{1-f'(x_0)}{x_1 - x_0} \\ 1 & f'(x_2) & \ldots & 1 & 1 & \frac{1-f'(x_0)}{x_2 - x_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & f'(x_n) & 1 & \frac{1-f'(x_0)}{x_n - x_0} \\ 1 & 1 & \ldots & 1 & f'(x_0) & \frac{f''(x_0)}{2} \\ \frac{1-f'(x_0)}{x_1 - x_0} & \frac{1-f'(x_0)}{x_2 - x_0} & \ldots & \frac{1-f'(x_0)}{x_n - x_0} & \frac{f''(x_0)}{2} & \frac{f'''(x_0)}{6} \end{bmatrix} \geq 0. \tag{5.21}$$

The proof is similar to that of Theorem 1.5. Let b and $d\mu$ are taken from the Herglotz representation (1.1) of f. Then the matrix

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix} b \begin{bmatrix} 1 & \cdots & 1 & 0 \end{bmatrix} + \int_{-\infty}^{\infty} \begin{bmatrix} \frac{1}{t-z_1} \\ \vdots \\ \frac{1}{t-z_n} \\ \frac{1}{t-z_0} \\ \frac{1}{(t-z_n)^2} \end{bmatrix} d\mu(t) \begin{bmatrix} \frac{1}{t-\bar{z}_1} & \cdots & \frac{1}{t-\bar{z}_n} & \frac{1}{t-\bar{z}_0} & \frac{1}{(t-\bar{z}_0)^2} \end{bmatrix}. \tag{5.22}$$

is positive semidefinite. Taking the limit as $z_k = x_k + iy \to x_k$ we get the matrix as in (5.21) which is positive semidefinite as the limit of positive semidefinite matrices. Since $f'(x_0) = 1$, the matrix (5.21) simplifies to

$$\begin{bmatrix} f'(x_1) & 1 & \dots & 1 & 1 & 0 \\ 1 & f'(x_2) & \ddots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & f'(x_n) & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & \frac{f''(x_0)}{2} \\ 0 & 0 & \dots & 0 & \frac{f''(x_0)}{2} & \frac{f'''(x_0)}{6} \end{bmatrix} \geq 0.$$

Taking the determinant of this matrix along the the bottom row, we have that

$$\begin{aligned} &-\frac{f''(x_0)}{2} \det \begin{bmatrix} f'(x_1) & & 1 & 0 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 0 \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix} \\ &\qquad +\frac{f'''(x_0)}{6} \det \begin{bmatrix} f'(x_1) & & 1 & 1 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 1 \\ 1 & \dots & 1 & 1 \end{bmatrix} \end{aligned} \tag{5.23}$$

is nonnegative. Focusing on the first matrix above and again defining $\mu_i = f'(x_i) - 1$,

$$\det \begin{bmatrix} f'(x_1) & & 1 & 0 \\ & \ddots & & \vdots \\ 1 & & f'(x_n) & 0 \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix} = \det \begin{bmatrix} \mu_1 & & 0 & -\frac{f''(x_0)}{2} \\ & \ddots & & \vdots \\ 0 & & \mu_n & -\frac{f''(x_0)}{2} \\ 1 & \dots & 1 & \frac{f''(x_0)}{2} \end{bmatrix}.$$

We then evaluate with respect to the last column, attaining

$$\sum_{i=1}^{n}\left((-1)^{i+n+1}\left(-\frac{f''(x_0)}{2}\right)\det\begin{bmatrix}\mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1\end{bmatrix}\right)$$
$$+\frac{f''(x_0)}{2}\det\begin{bmatrix}\mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_3\end{bmatrix}. \quad (5.24)$$

Transposing columns $n-i$ times so that the column not containing any μ_j is on the far right side, we have

$$\det\begin{bmatrix}\mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1\end{bmatrix} = (-1)^{n-i}\det\begin{bmatrix}\mu_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mu_n & 0 \\ 1 & \cdots & 1 & 1\end{bmatrix}$$
$$= (-1)^{n-i}\frac{\mu_1\cdots\mu_n}{\mu_i}$$

and thus the expression in (5.24) is equal to

$$\frac{f''(x_0)}{2}\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_n}\right).$$

Plugging into (5.23), we have

$$-\left(\frac{f''(x_0)}{2}\right)^2\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_n}\right)+\frac{f'''(x_0)}{6}\mu_1\cdots\mu_n \geq 0,$$

which simplifies to

$$\sum_{i=1}^{n}\frac{1}{f'(x_i)-1} \leq \frac{2f'''(x_0)}{3(f''(x_0))^2}-1.$$

We thus arrive at the following result:

Theorem 5.11. *Let $f \in \mathbb{P}$, let $x_0, \ldots, x_n \in \mathbb{R}$ be fixed points. Let x_0 be the Denjoy-Wolff point such that $f'(x_0) = 1$ and the limits (5.20) exist and are* *real. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)-1} \le \frac{2f'''(x_0)}{3(f''(x_0))^2} - 1. \tag{5.25}$$

The latter theorem does not provide any meaningful information in case

$$f'(x_0) = 1 \quad \text{and} \quad f''(x_0) = 0, \tag{5.26}$$

that is, in case x_0 is the boundary fixed point of order two. In this case we assume that the limits

$$f_3 = \lim_{y\to 0} \frac{f'''(x_0+iy)}{6}, \quad f_4 = \lim_{y\to 0} \frac{f^{(4)}(x_0+iy)}{24} \quad \text{and} \quad f_5 = \lim_{y\to 0} \frac{f^{(5)}(x_0+iy)}{120} \tag{5.27}$$

exist finitely and are real. Then the following matrix is positive semidefinite:

$$\begin{bmatrix} K & B^* \\ B & D \end{bmatrix} \ge 0 \tag{5.28}$$

where K is the matrix on the left hand side of (5.21), where $D = \dfrac{f^{(5)}(x_0)}{120}$ and where

$$B = \begin{bmatrix} \dfrac{1-f'(x_0)}{(x_1-x_0)^2} - \dfrac{f''(x_0)}{x_1-x_0} & \dots & \dfrac{1-f'(x_0)}{(x_n-x_0)^2} - \dfrac{f''(x_0)}{x_n-x_0} & \dfrac{f^{(4)}(x_0)}{24} \end{bmatrix}.$$

For the proof we should consider the matrix similar to that in (5.22) but with

$$\begin{bmatrix} 1 & \dots 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{t-\bar{z}_1} & \cdots & \frac{1}{t-\bar{z}_n} & \frac{1}{t-\bar{z}_0} & \frac{1}{(t-\bar{z}_0)^2} \end{bmatrix}$$

replaced by the extended rows

$$\begin{bmatrix} 1 & \dots 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{t-\bar{z}_1} & \cdots & \frac{1}{t-\bar{z}_n} & \frac{1}{t-\bar{z}_0} & \frac{1}{(t-\bar{z}_0)^2} & \frac{1}{(t-\bar{z}_0)^3} \end{bmatrix},$$

respectively, and then pass to the limits as as $z_k = x_k + iy \to x_k$. Due to conditions (5.26), the matrix in (5.28) takes the form

$$\begin{bmatrix} f'(x_1) & 1 & \dots & 1 & 1 & 0 & 0 \\ 1 & f'(x_2) & \dots & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & f'(x_n) & 1 & 0 & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & f_3 \\ 0 & 0 & \dots & 0 & 0 & f_3 & f_4 \\ 0 & 0 & \dots & 0 & f_3 & f_4 & f_5 \end{bmatrix} \ge 0.$$

Computing the determinant of this matrix with respect to the last row, we have

$$f_3 \det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & 0 & f_3 \\ 0 & \cdots & 0 & f_3 & f_4 \end{bmatrix} - f_4 \det \begin{bmatrix} f'(x_1) & & 1 & 1 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 1 & 0 \\ 1 & \cdots & 1 & 1 & f_3 \\ 0 & \cdots & 0 & 0 & f_4 \end{bmatrix}$$
$$+ f_5 \det \begin{bmatrix} f'(x_1) & & 1 & 1 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 1 & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & f_3 \end{bmatrix} \leq 0. \tag{5.29}$$

The first matrix in the above expression can be simplified using row operations:

$$\det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & 0 & f_3 \\ 0 & \cdots & 0 & f_3 & f_4 \end{bmatrix} = -\det \begin{bmatrix} f'(x_1) & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 1 & & f'(x_n) & 0 & 0 \\ 1 & \cdots & 1 & f_3 & 0 \\ 0 & \cdots & 0 & f_4 & f_3 \end{bmatrix}$$
$$= -\det \begin{bmatrix} \mu_1 & & 0 & -f_3 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & \mu_n & -f_3 & 0 \\ 1 & \cdots & 1 & f_3 & 0 \\ 0 & \cdots & 0 & f_4 & f_3 \end{bmatrix} = -f_3 \det \begin{bmatrix} \mu_1 & & 0 & -f_3 \\ & \ddots & & \vdots \\ 0 & & \mu_n & -f_3 \\ 1 & \cdots & 1 & f_3 \end{bmatrix}.$$

We then take the determinant with respect to the last column, arriving at

$$(-1)^{i+n+1}(-f_3)\left(\sum_{i=1}^{n} -f_3 \det \begin{bmatrix} \mu_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \mu_{i-1} & 0 & & & \\ & & & 0 & \mu_{i+i} & & \\ & 0 & & & & \ddots & \\ & & & & & & \mu_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} + \mu_1 \cdots \mu_n f_3 \right).$$

Then, transposing $n-i$ columns, the above is equal to

$$-f_3^2\mu_1\cdots\mu_n + f_3^2\sum_{i=1}^{n}(-1)^{n+i+1}\det\begin{bmatrix}\mu_1 & & & & & & 0\\ & \ddots & & & 0 & & \vdots\\ & & \mu_{i-1} & & & & 0\\ & & & \mu_{i+1} & & & 0\\ & 0 & & & \ddots & & \vdots\\ & & & & & \mu_n & 0\\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1\end{bmatrix}$$
$$= -f_3^2\mu_1\cdots\mu_n\left(1+\sum_{i=1}^{n}\frac{1}{\mu_i}\right).$$

As for the second matrix in (5.29), we have

$$\det\begin{bmatrix}f'(x_1) & & 1 & 1 & 0\\ & \ddots & & \vdots & \vdots\\ 1 & & f'(x_n) & 1 & 0\\ 1 & \cdots & 1 & 1 & f_3\\ 0 & \cdots & 0 & 0 & f_4\end{bmatrix}$$
$$= -f_3\det\begin{bmatrix}f'(x_1) & & 1 & 1\\ & \ddots & & \vdots\\ 1 & & f'(x_n) & 1\\ 0 & \cdots & 0 & 0\end{bmatrix} + f_4\det\begin{bmatrix}f'(x_1) & & 1 & 1\\ & \ddots & & \vdots\\ 1 & & f'(x_n) & 1\\ 1 & \cdots & 1 & 1\end{bmatrix}$$
$$= 0 + f_4\det\begin{bmatrix}\mu_1 & & 0 & 0\\ & \ddots & & \vdots\\ 0 & & \mu_n & 0\\ 1 & \cdots & 1 & 1\end{bmatrix} = f_4\mu_1\cdots\mu_n,$$

and finding the determinant of the third matrix in (5.29), we have

$$\det\begin{bmatrix}f'(x_1) & & 1 & 1 & 0\\ & \ddots & & \vdots & \vdots\\ 1 & & f'(x_n) & 1 & 0\\ 1 & \cdots & 1 & 1 & 0\\ 0 & \cdots & 0 & 0 & f_3\end{bmatrix} = \det\begin{bmatrix}\mu_1 & & 0 & 0 & 0\\ & \ddots & & \vdots & \vdots\\ 0 & & \mu_n & 0 & 0\\ 1 & \cdots & 1 & 1 & 0\\ 0 & \cdots & 0 & 0 & f_3\end{bmatrix} = f_3\mu_1\cdots\mu_n.$$

Therefore, the inequality in (5.29) is equivalent to

$$-f_3^3\left(1+\sum_{i=1}^{n}\right)\mu_1\cdots\mu_n - f_4^2\mu_1\cdots\mu_n + f_3f_5\mu_1\cdots\mu_n \le 0,$$

which simplifies to

$$\sum_{i=1}^{n} \frac{1}{\mu_i} \leq \frac{f_5}{f_3^2} - \frac{f_4^2}{f_3^3} - 1.$$

We thus arrive at the following result:

Theorem 5.12. *Let $f \in \mathbb{P}$, let $x_0, \dots, x_n \in \mathbb{R}$ be fixed points. Let x_0 be the Denjoy-Wolff point of f satisfying conditions* (5.26) *and such that the limits* (5.27) *exist and are real. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{f_5}{f_3^2} - \frac{f_4^2}{f_3^3} - 1. \tag{5.30}$$

Theorems 5.11 and 5.12 suggest that in case $f'''(x_0) = 0$, the sum on the left side of (5.10) can be estimated in terms of derivatives of f of higher orders. However, it is not so. The Burns-Crantz theorem [11] implies that in case $f(x_0) = x_0$, $f'(x_0) = 1$ and $f''(x_0) = f'''(x_0) = 0$ for a Pick class function f and a boundary point $x_0 \in \mathbb{R}$, then necessarily $f(z) \equiv z$.

6. Boundary fixed points of Stieltjes class functions

A point $x_0 \in \mathbb{R}_-$ is called a boundary fixed point of a Stieltjes function $s(z)$ if $s(x_0) = \lim\limits_{y \to 0} s(x_0 + iy) = x_0$. By Theorem 5.8, for every boundary fixed point x_0, the boundary derivative $f'(x_0) := \lim\limits_{y \to 0} s'(x_0 + iy)$ exists (though it can be infinite). Since $\widetilde{K}_z^s = \dfrac{z\overline{s(z)} - \overline{z}s(z)}{|z|^2(z - \overline{z})}$ is nonnegative for every non-real z, we let $z = x_0 + iy \to x_0$ to conclude that $\dfrac{1 - s'(x_0)}{x_0} \geq 0$ so that $s'(x_0) \geq 1$. On the other hand, if $s'(x_0) = 1$, then $s(z) \equiv z$ (the proof is the same as that in Theorem 4.2). Thus, except for the trivial case $s(z) \equiv z$, the boundary derivative of s at any boundary fixed point is greater than 1. The next theorem is the Stieltjes-class analogue of Theorem 6.1.

Theorem 6.1. *Let $f \in \mathcal{S}$, $x_0 \in \mathbb{R}_+$, and $x_1, x_2, \dots, x_n \in \mathbb{R}_-$ be fixed points of f. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(x_i) - 1} \leq \frac{1 + f'(x_0)}{1 - f'(x_0)}. \tag{6.1}$$

We have shown above that $f'(x) > 1$ for any boundary fixed point $x < 0$ and $f'(x_0) < 1$ for an interior fixed point $x_0 > 0$. The most interesting case is when $x = 0$ is a fixed point for f. The derivative of f at 0 can be equal any positive number. For example, the fumction

$$f(z) = \ln\left(1 + \frac{z}{a}\right)$$

belongs to the Stieltjes class for any $a > 0$ and its derivative equals $\frac{1}{z+a}$ so that $f'(0) = \frac{1}{a}$. Inequalities involving the fixed point $x = 0$ will be studied in future.

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