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Existence of positive solutions to Kirchhoff type problems with zero mass ☆

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ABSTRACT

The existence of positive solutions depending on a nonnegative parameter λ to Kirchhoff type problems with zero mass is proved by using variational method, and the new result does not require usual compactness conditions. A priori estimate and a Pohozaev type identity are used to obtain the bounded Palais–Smale sequences for constant coefficient nonlinearity, while a cut-off functional and Pohozaev type identity are utilized to obtain the bounded Palais–Smale sequences for the variable-coefficient case.

1. Introduction

In this paper, we consider positive solutions to the following nonlinear Kirchhoff type problem

$$\begin{cases} -\left(a+\lambda \displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u = K(x)f(u), & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$, a is a positive constant, $\lambda \geqslant 0$ is a parameter, and K is a potential function. Kirchhoff type problem on a bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -\left(a+b \displaystyle\int_{\Omega} |\nabla u|^2\right)\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has been studied by many authors, for example [5–7,10,18–20,24,27,28]. Many solvability conditions on the nonlinearity f near zero and infinity for the problem (1.2) have been considered, such as the superlinear case [19]; and asymptotical linear case [24]. In addition, the following growth condition on f is often assumed:

(f) $f(t)t \geqslant 4F(t)$ for $|t|$ large, where $F(t) = \int_0^t f(s)\,ds$,

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which assures the boundedness of any Palais–Smale (PS) or Cerami sequence. Indeed the condition (f) may appear in different forms as follows:

(f_0) there exists $\theta \geqslant 1$ such that $\theta G(t) \geqslant G(st)$ for all $t \in \mathbb{R}$ and $s \in [0,1]$, where $G(t) = tf(t) - 4F(t)$ (see [24]);
(f_1) $\lim_{|t|\to\infty} G(t) = \infty$ (see [27]); or
(f_2) $\lim_{|t|\to\infty} G(t) = \infty$ and there exists $\sigma > \max\{1, N/2\}$ such that $|f(t)|^\sigma \leqslant CG(t)|t|^\sigma$ for $|t|$ large (see [19]).

In the papers above, each of the conditions (f_0)–(f_2) implies that the condition (f) holds. On the other hand, the condition (f) is sufficient to show the boundedness of any (PS) or Cerami sequence, which has been proved in [26].

There are few papers considering Kirchhoff type problems on $\mathbb{R}^N$ except [8,9,12,17,25,26]. In [26], the author studied the problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u + V(x)u = f(u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

The existence of nontrivial solutions was proved in [26] under the condition (f) and

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and for each $M > 0$, meas $\{x \in \mathbb{R}^N\colon V(x) \leqslant M\} < \infty$;
(f_3) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|f(t)| \leqslant C(|t| + |t|^{p-1})$ for all $t \in \mathbb{R}_+ = [0,\infty)$ and some $p \in (1, 2^*)$, where $2^* = 2N/(N-2)$ for $N \geqslant 3$;
(f_4) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
(f_5) $\lim_{t\to\infty} \frac{F(t)}{t^4} = \infty$.

In [26], the space $\{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N}[|\nabla u|^2 + V(x)u^2] < \infty\}$ is compactly embedded into $L^p(\mathbb{R}^N)$, which makes the existence problem easier. In [8,9,25], the existence, multiplicity and concentration behavior of positive solutions of (1.3) were considered by relating the number of solutions with the topology of the set where V attains its minimum.

In [17], we proved the existence of a positive solution to the problem

$$\left(a + \lambda\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b\int_{\mathbb{R}^N} u^2\right)[-\Delta u + bu] = f(u), \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

We assumed that $a > 0$, $b > 0$, and f satisfies (f_3), (f_4) and the following condition

(f_6) $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$.

The result in [17] does not assume the condition (f) (or any of (f_0)–(f_2)), as the space $H_r^1(\mathbb{R}^N)$ of radial functions can be compactly imbedded into $L^p(\mathbb{R}^N)$, which provides some compactness in the problem for the convergence. However, for the case of $b = 0$ in (1.4) (that is (1.1) with $K \equiv 1$), one has to search for a positive solution in the space $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$, which does not possess compactness as $H_r^1(\mathbb{R}^N)$. Because of this difficulty, there are very few works up to now studying Kirchhoff problems with zero mass, i.e. the problem (1.1).

In this paper we consider the existence of positive solutions to (1.1), and we assume the following conditions which are considerably weaker than the ones in the previous works:

(K_0) $K(x) \equiv 1$ for $x \in \mathbb{R}^N$;
(K_1) let $K : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function and $K \in [L^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)] \setminus \{0\}$ for some $s \geqslant 2N/(N+2)$;
(K_2) $|x \cdot \nabla K(x)| \leqslant \alpha K(x)$ for a.e. $x \in \mathbb{R}^N$ and some $\alpha \in (0,2)$;
(H_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\lim_{t\to 0^+} \frac{f(t)}{t^{2^*-1}} = 0$;
(H_2) $\lim_{t\to\infty} \frac{f(t)}{t^{2^*-1}} = 0$;
(H_3) $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$.

Our first result is for (1.1) with a constant potential function $K(x)$, or equivalently for Eq. (1.4) with $b = 0$.

Theorem 1.1. *Assume that $N \geqslant 3$, a is a positive constant, and $\lambda \geqslant 0$ is a parameter. If the conditions* (K_0), (H_1), (H_2) *and* (H_3) *hold, then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$,* (1.1) *has at least one positive solution.*

Theorem 1.1 appears to be the first existence result for the problem (1.1). We remark also that the condition (H_3) is weaker than the ones in the papers mentioned above, in which $\lim_{t\to\infty} f(t)/t^3 = \infty$ or a positive constant (which implies (H_3)) was assumed. From Theorem 1.1, one can also have the following classical result (see [4]) if we let $\lambda = 0$.

Corollary 1.2. *Assume that $N \geqslant 3$. If the conditions* (H_1), (H_2) *and* (H_3) *hold, then the equation $-\Delta u = f(u)$ in $\mathbb{R}^N$ has at least one positive solution.*

In our second result, we consider the case of non-constant potential function $K(x)$, and we obtain the following result:

Theorem 1.3. *Assume that $N \geqslant 3$, a is a positive constant, and $\lambda \geqslant 0$ is a parameter. If the conditions* (K_1), (K_2), (H_1), (H_2) *and* (H_3) *hold, then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$, (1.1) has at least one positive solution.*

An example of K satisfying (K_1) and (K_2) is

$$K(x) = \frac{1}{1 + |x|^\alpha},$$

where $\alpha \in (1, 2)$. One can easily verify that $|x \cdot \nabla K(x)| = \frac{\alpha|x|^\alpha}{(1+|x|^\alpha)^2}$. Therefore, K satisfies the conditions (K_1) for $s \geqslant N$ and (K_2).

The case of $\lambda = 0$ in (1.1) corresponds to the following well-known nonlinear Poisson equation:

$$-\Delta u = K(x) f(u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

Positive solutions of (1.5) with radially symmetric potential function have been investigated in, for example, [1,3,2,14–16]. For example, in [15], the case of $K(x) = 1/(1 + |x|^2)$ and $f(u) = u^p$ was studied and asymptotic expansions of solutions were obtained. In [14], the author assumed that $f(u) = u^p$ and K satisfies some conditions similar to (K_1) and (K_2). The zero mass case for (1.5) was considered in [3,2]. Our result here implies the following corollary for (1.5):

Corollary 1.4. *Assume that $N \geqslant 3$. If the conditions* (K_1), (K_2), (H_1), (H_2) *and* (H_3) *hold, then the equation $-\Delta u = K(x) f(u)$ in $\mathbb{R}^N$ has at least one positive solution.*

In [1], the existence of a positive solution was proved for $K \in L^s(\mathbb{R}^N)$ with an additional condition

(H_4) $H(t) = tf(t) - 2F(t)$ is increasing in t and $H(0) = 0$.

In Corollary 1.4, we do not assume this monotonicity condition.

As in [17], we prove in this paper the existence of positive solutions to (1.1) without the condition (f) (or any of (f_0)–(f_2)). We use *a priori* estimate, a cut-off functional and a variable-coefficient Pohozaev type identity to obtain bounded (PS) sequences, and then we apply some known variational techniques to prove the existence of a positive solution. The Pohozaev identity with variable-coefficient proved in Lemma 2.2 seems to be the first of this kind for (1.1), which is of independent interest. Similar Pohozaev identities with variable-coefficient have also been obtained in [21,22] for p-Laplace equations with singular weight. Another difficulty in (1.1) is caused by the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2$, which leads to some convergence difficulties, that is, if u_n converges weakly to u, then one cannot conclude that u is a weak solution of (1.1). So we must obtain strong convergence of the (PS) sequence, and this makes (1.1) more difficult to deal with than other similar elliptic equations.

In this paper the problem (1.1) is considered in the Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N)\colon\ |\nabla u| \in L^2(\mathbb{R}^N)\}$. The space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is equipped with the standard inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \qquad \|u\| = (u, u)^{1/2}.$$

When K is a constant, for obtaining the convergence, we consider the problem (1.1) in the subspace $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ consisting of radial functions. Then we have that $\mathcal{D}_r^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ continuously. We denote by $|\cdot|_q$ the usual $L^q(\mathbb{R}^N)$ norm. In this paper, we consider only positive solutions to (1.1), so we assume that $f(t) = 0$ for $t < 0$. We recall some preliminaries and prove some lemmas in Section 2, and we give proofs of Theorems 1.1 and 1.3 in Sections 3 and 4, respectively.

2. Preliminaries

Define a functional J_λ on the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ by

$$J_\lambda(u) = \frac{1}{2} a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} KF(u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

It follows from (K_0) or (K_1), (H_1) and (H_2) that there exists $C > 0$ such that

$$\left|K(x)f(t)t\right| \leqslant C|t|^{2^*}, \quad \left|K(x)F(t)\right| \leqslant C|t|^{2^*}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.1}$$

Then we have that J_λ is well defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, it is of C^1 class for all $\lambda \geqslant 0$, and

$$\left\langle J'_\lambda(u), v\right\rangle = a(u,v) + \lambda\|u\|^2(u,v) - \int_{\mathbb{R}^N} Kf(u)v, \quad u, v \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right).$$

It is standard to verify that weak solutions of (1.1) correspond to critical points of the functional J_λ.

Next we recall a monotonicity method due to Struwe [23] and Jeanjean [10], which will be used in our proof. The version here is from [10].

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\mu(0) = 0$.

For any $\mu \in I$ we set

$$\Gamma_\mu = \left\{\gamma \in C\left([0,1], X\right):\ \gamma(0) = 0,\ J_\mu\left(\gamma(1)\right) < 0\right\}. \tag{2.2}$$

If for every $\mu \in I$ the set Γ_μ is nonempty and

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} J_\mu\left(\gamma(t)\right) > 0, \tag{2.3}$$

then for almost every $\mu \in I$ there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*
(ii) *$J_\mu(u_n) \to c_\mu$ as $n \to \infty$;*
(iii) *$J'_\mu(u_n) \to 0$ as $n \to \infty$, in the dual space X^{-1} of X.*

To prove the boundedness of sequence of critical points in the proof later, we next introduce a Pohozaev type identity with variable-coefficient as follows.

Lemma 2.2. *Assume that K satisfies (K_0) or (K_1), and f satisfies (H_1) and (H_2). If $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of*

$$-\left(a + \lambda\|u\|^2\right)\Delta u = \mu K(x) f(u), \quad x \in \mathbb{R}^N, \tag{2.4}$$

then the following Pohozaev type identity holds

$$\frac{N-2}{2}\left(a + \lambda\|u\|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 = \mu N \int_{\mathbb{R}^N} KF(u) + \mu \int_{\mathbb{R}^N} F(u)(x \cdot \nabla K). \tag{2.5}$$

Proof. Since $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of (2.4), by (K_0) or (K_1), (H_1), (H_2), and the standard regularity results, then $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [1,\infty)$. Hence by the L^p estimate of elliptic equations, we know that $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [1,\infty)$. Thus $u \in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\beta \in (0,1)$. It follows from (2.4) that

$$-\left(a + \lambda\|u\|^2\right)\Delta u(x \cdot \nabla u) = \mu K f(u)(x \cdot \nabla u).$$

We can calculate that

$$Kf(u)(x \cdot \nabla u) = \operatorname{div}\left(xKF(u)\right) - NKF(u) - F(u)(x \cdot \nabla K),$$

$$\begin{aligned}\Delta u(x \cdot \nabla u) &= \operatorname{div}\left(\nabla u(x \cdot \nabla u)\right) - |\nabla u|^2 - x \cdot \nabla\left(\frac{|\nabla u|^2}{2}\right)\\ &= \operatorname{div}\left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2}\right) + \frac{N-2}{2}|\nabla u|^2.\end{aligned}$$

Therefore, for any $R > 0$,

$$\int\limits_{B(0,R)} \left[\Delta u(x\cdot\nabla u) - \frac{N-2}{2}|\nabla u|^2\right] = \int\limits_{\partial B(0,R)} \left[\nabla u(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2}\right]\cdot\frac{x}{R}$$
$$= \int\limits_{\partial B(0,R)} \left[\frac{1}{R}|x\cdot\nabla u|^2 - \frac{R}{2}|\nabla u|^2\right],$$
$$\int\limits_{B(0,R)} Kf(u)(x\cdot\nabla u) = \int\limits_{\partial B(0,R)} xKF(u)\cdot\frac{x}{R} - \int\limits_{B(0,R)} \big[NKF(u) + F(u)x\cdot\nabla K\big]$$
$$= R\int\limits_{\partial B(0,R)} KF(u) - \int\limits_{B(0,R)} \big[NKF(u) + F(u)x\cdot\nabla K\big].$$

Following [4], we show that the boundary terms above converge to 0 for at least a suitably chosen sequence $R_n \to \infty$. Note that

$$\int\limits_{\mathbb{R}^N} \big[KF(u) + |\nabla u|^2\big]\,\mathrm{d}x = \int_0^\infty \left\{\int\limits_{\partial B(0,R)} \big[KF(u) + |\nabla u|^2\big]\,\mathrm{d}S\right\}\mathrm{d}R < \infty,$$

hence there exists a sequence $\{R_n\}$ with $R_n \to \infty$ such that

$$R_n \int\limits_{\partial B(0,R_n)} \big[KF(u) + |\nabla u|^2\big]\,\mathrm{d}S \to 0.$$

Therefore, we can get the conclusion with the choice $R = R_n$ and $n \to \infty$. $\square$

We conclude this section by proving a nonexistence result for larger λ for either $N \geqslant 5$ or $N \geqslant 6$. This shows that the existence results in Theorems 1.1 and 1.3 for $\lambda \in [0, \lambda_0)$ are somehow optimal. For the nonexistence result, we first observe the following lower bound for a positive solution:

Lemma 2.3. *If u is a nontrivial weak solution of (1.1), then $\|u\| \geqslant r$ for some $r > 0$.*

Proof. Since u is a weak solution of (1.1), then from (2.1),

$$a\|u\|^2 \leqslant \big(a + \lambda\|u\|^2\big)\|u\|^2 = \int\limits_{\mathbb{R}^N} Kf(u)u \leqslant C\|u\|^{2^*}.$$

The proof is completed. $\square$

The nonexistence result is as follows:

Theorem 2.4.

1. *If K satisfies (K_0) and $N \geqslant 5$, then there exists a $\lambda_1 > 0$ such that (1.1) has no positive solutions with nonnegative energy for $\lambda \in (\lambda_1, \infty)$.*
2. *If K satisfies (K_1) and (K_2) and $N \geqslant 6$, then there exists a $\lambda_2 > 0$ such that (1.1) has no positive solutions with nonnegative energy for $\lambda \in (\lambda_2, \infty)$.*

Proof. We may assume that u is a positive solution of (1.1) and $J_\lambda(u) = c \geqslant 0$. Then

$$\frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int\limits_{\mathbb{R}^N} KF(u) = c.$$

If $K \equiv 1$ and $N \geqslant 5$, then according to Lemma 2.2,

$$\frac{N-2}{2}\big(a + \lambda\|u\|^2\big)\int\limits_{\mathbb{R}^N} |\nabla u|^2 = N\int\limits_{\mathbb{R}^N} F(u).$$

Hence

$$a\|u\|^2 = \frac{1}{4}\lambda(N-4)\|u\|^4 + cN \geqslant \frac{1}{4}\lambda\|u\|^4.$$

Therefore, we have $\lambda r^2 \leqslant 4a$, where r is defined in Lemma 2.3. If $N \geqslant 6$ and K satisfies (K_1) and (K_2), then according to Lemma 2.2,

$$\frac{N-2}{2}\big(a+\lambda\|u\|^2\big)\int_{\mathbb{R}^N}|\nabla u|^2 = N\int_{\mathbb{R}^N}KF(u) + \int_{\mathbb{R}^N}F(u)x\cdot\nabla K \leqslant (N+\alpha)\int_{\mathbb{R}^N}KF(u).$$

So

$$\frac{2+\alpha}{2}a\|u\|^2 \geqslant \frac{1}{4}\lambda(N-4-\alpha)\|u\|^4 + c(N+\alpha) \geqslant \frac{1}{4}\lambda(2-\alpha)\|u\|^4,$$

thus we obtain $\lambda r^2 \leqslant 2(2+\alpha)/(2-\alpha)$, where r is defined in Lemma 2.3. □

3. The case that K is a constant

In this section, we consider the case that $K(x) \equiv 1$, and assume that conditions (H_1)–(H_3) are satisfied. First we recall the following estimate of the decay rate of radial functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (see [4]).

Lemma 3.1. *Suppose that $N \geqslant 3$. Then every radial function u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is almost everywhere equal to an even function $U : \mathbb{R}^N \to \mathbb{R}$, continuous for $x \neq 0$, such that*

$$\big|U(x)\big| \leqslant C_N|x|^{(2-N)/2}\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad |x| \geqslant 1,$$

where C_N only depends on N.

In this section, for the notation in Theorem 2.1, the space $X = \mathcal{D}_r^{1,2}(\mathbb{R}^N)$, and related functionals on $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ are

$$A(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4, \qquad B(u) = \int_{\mathbb{R}^N}F(u).$$

So the perturbed functional which we will study is

$$J_{\lambda,\mu}(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \mu\int_{\mathbb{R}^N}F(u),$$

and

$$\big\langle (J_{\lambda,\mu})'(u), v\big\rangle = a(u,v) + \lambda\|u\|^2(u,v) - \mu\int_{\mathbb{R}^N}f(u)v. \tag{3.1}$$

To overcome the problem of lacking compactness, we need to consider the functional $J_{\lambda,\mu}$ in the radial function space $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. We shall prove that $J_{\lambda,\mu}$ satisfies the conditions of Theorem 2.1 in the next several lemmas. In the following two lemmas, we assume that (H_1)–(H_3) are satisfied.

We choose a radial function $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$ with $\|\phi\| = 1$ and $\mathrm{supp}(\phi) \subset B(0,R)$ for some $R > 0$. By (H_3), we have that for any $C_1 > 0$ with $C_1\int_{B(0,R)}\phi^2 > a$, there exists $C_2 > 0$ such that

$$F(t) \geqslant C_1|t|^2 - C_2, \quad t \in \mathbb{R}_+. \tag{3.2}$$

Let

$$\lambda_0 = \frac{(C_1\int_{B(0,R)}\phi^2 - a)^2}{4C_3}, \tag{3.3}$$

where $C_3 = C_2|B(0,R)|$. Then we have the following lemma.

Lemma 3.2. *Let Γ_μ be the set of paths defined in (2.2). Then for $\lambda \in [0, \lambda_0)$, $\Gamma_\mu \neq \emptyset$ for $\mu \in I = [1/2, 1]$.*

Proof. According to (3.2) and the definition of λ_0,

$$J_{\lambda,\mu}(t\phi) = \frac{1}{2}at^2 + \frac{1}{4}\lambda t^4 - \mu \int_{\mathbb{R}^N} F(t\phi)$$
$$\leqslant \frac{1}{2}at^2 - \frac{1}{2}C_1 t^2 \int_{B(0,R)} \phi^2 + C_3 + \frac{1}{4}\lambda t^4.$$

If $\lambda = 0$, we can choose $t_0 > 0$ large such that $J_{\lambda,\mu}(t_0\phi) < 0$. If $\lambda \in (0, \lambda_0)$, then by using

$$\left(C_1 \int_{B(0,R)} \phi^2 - a\right)^2 - 4C_3\lambda > 0,$$

we can choose $t_0 > 0$ properly so that $J_{\lambda,\mu}(t_0\phi) < 0$. The proof is completed. □

Lemma 3.3. *Let c_μ be defined as in (2.3). Then there exists a constant $c > 0$ such that $c_\mu \geqslant c$ for all $\mu \in I = [1/2, 1]$.*

Proof. For any $\mu \in I$ and $u \in \mathcal{D}_r^{1,2}(\mathbb{R}^N)$, by using (H_1) and (H_2), we have

$$J_{\lambda,\mu}(u) \geqslant \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - C\int_{\mathbb{R}^N} |u|^{2^*}$$
$$\geqslant \frac{1}{2}a\|u\|^2 - C\int_{\mathbb{R}^N} |u|^{2^*}.$$

From Sobolev's embedding theorem, we conclude that there exists $\rho > 0$ such that $J_{\lambda,\mu}(u) > 0$ for any $\mu \in I$ and $u \in \mathcal{D}_r^{1,2}(\mathbb{R}^N)$ with $\|u\| \in (0, \rho]$. In particular, for $\|u\| = \rho$, we have $J_{\lambda,\mu}(u) \geqslant c > 0$.

Fix $\mu \in I$ and for any $\gamma \in \Gamma_\mu$, by the definition of Γ_μ, we have $\|\gamma(1)\| > \rho$. Since $\gamma(0) = 0$, then from intermediate value theorem we deduce that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\mu \in I$,

$$c_\mu \geqslant \inf_{\gamma \in \Gamma_\mu} J_{\lambda,\mu}\big(\gamma(t_\gamma)\big) \geqslant c. \qquad \square$$

Next we prove that the functional $J_{\lambda,\mu}$ can achieve the critical value at c_μ for any $\mu \in I$.

Lemma 3.4. *For any $\mu \in I$, each bounded (PS) sequence of the functional $J_{\lambda,\mu}$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ admits a convergent subsequence.*

Proof. For any given $\mu \in I$, let $\{u_n\}$ be a bounded (PS) sequence of $J_{\lambda,\mu}$, that is, $\{u_n\}$ and $\{J_{\lambda,\mu}(u_n)\}$ are bounded, $(J_{\lambda,\mu})'(u_n) \to 0$ in $\mathcal{D}'$, where $\mathcal{D}'$ is the dual space of $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. Since $\{u_n\}$ is bounded, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), $u \in \mathcal{D}_r^{1,2}(\mathbb{R}^N)$ such that as $n \to \infty$,

$$u_n \rightharpoonup u, \quad \text{in } \mathcal{D}_r^{1,2}\big(\mathbb{R}^N\big),$$
$$u_n \to u, \quad \text{in } L^p_{\mathrm{loc}}\big(\mathbb{R}^N\big),\ p \in \big(1, 2^*\big),$$
$$u_n(x) \to u(x), \quad \text{a.e. } x \in \mathbb{R}^N.$$

According to Lemma 3.1, we may assume that

$$\big|u_n(x)\big| \leqslant C|x|^{(2-N)/2}, \quad |x| \geqslant 1,\ n \geqslant 1. \tag{3.4}$$

From the conditions (H_1) and (H_2), for any $\varepsilon > 0$, there exist $\delta > 0$ and $C_\varepsilon > 0$ such that

$$\big|f(t)\big| \leqslant \varepsilon|t|^{2^*-1}, \quad |t| \leqslant \delta, \tag{3.5}$$

and

$$\big|f(t)\big| \leqslant \varepsilon|t|^{2^*-1} + C_\varepsilon|t|^{2^*/2}, \quad t \in \mathbb{R}. \tag{3.6}$$

By (3.4), there exists an $R > 0$ such that $|u_n(x)| \leqslant \delta$ for all $|x| \geqslant R$ and all n. Therefore, from (3.5),

$$\big|f\big(u_n(x)\big)\big| \leqslant \varepsilon\big|u_n(x)\big|^{2^*-1}, \quad |x| \geqslant R,\ n \geqslant 1. \tag{3.7}$$

So it follows from (3.6) and (3.7) that

$$\left|\int_{\mathbb{R}^N} f(u_n)(u_n-u)\right| \leqslant \varepsilon \int_{|x|<R} |u_n|^{2^*-1}|u_n-u| + C_\varepsilon \int_{|x|<R} |u_n|^{2^*/2}|u_n-u| + \varepsilon \int_{|x|\geqslant R} |u_n|^{2^*-1}|u_n-u|$$
$$\leqslant C_\varepsilon |u_n|_{2^*}^{2^*/2}|u_n-u|_{L^2(B(0,R))} + \varepsilon |u_n|_{2^*}^{2^*-1}|u_n-u|_{2^*}.$$

Hence we obtain that

$$\int_{\mathbb{R}^N} f(u_n)(u_n-u) \to 0, \quad n\to\infty.$$

Thus,

$$\big\langle (J_{\lambda,\mu})'(u_n), u_n-u\big\rangle = a(u_n,u_n-u) + \lambda\|u_n\|^2(u_n,u_n-u) - \mu\int_{\mathbb{R}^N} f(u_n)(u_n-u)$$
$$= \big[a+\lambda\|u_n\|^2\big](u_n,u_n-u) + o(1),$$

and then

$$\big[a+\lambda\|u_n\|^2\big](u_n,u_n-u)\to 0.$$

It follows that $\|u_n\|\to\|u\|$. This together with $u_n \rightharpoonup u$ shows that $u_n\to u$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. The proof is completed. □

Now we are in the position to show that the modified functional $J_{\lambda,\mu}$ has a nontrivial critical point.

Lemma 3.5. *Let $\lambda\in[0,\lambda_0)$. For almost every $\mu\in I$, there exists $u^\mu\in\mathcal{D}_r^{1,2}(\mathbb{R}^N)\setminus\{0\}$ such that $(J_{\lambda,\mu})'(u^\mu)=0$ and $J_{\lambda,\mu}(u^\mu)=c_\mu$.*

Proof. From Theorem 2.1, for almost every $\mu\in I$, there exists a bounded sequence $\{u_n^\mu\}\subset\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ such that $J_{\lambda,\mu}(u_n^\mu)\to c_\mu$ and $(J_{\lambda,\mu})'(u_n^\mu)\to 0$ as $n\to\infty$. According to Lemma 3.4, we may assume that there exists $u^\mu\in\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ such that $u_n^\mu\to u^\mu$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. Then it follows that $(J_{\lambda,\mu})'(u^\mu)=0$, $J_{\lambda,\mu}(u^\mu)=c_\mu$ and $u^\mu\neq 0$ from Lemma 3.3. □

According to Lemma 3.5, there exist a sequence $\{\mu_n\}\subset I$ with $\mu_n\to 1^-$ and an associated sequence $\{u_n\}\subset\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ such that

$$J_{\lambda,\mu_n}(u_n)=c_{\mu_n}, \qquad (J_{\lambda,\mu_n})'(u_n)=0. \tag{3.8}$$

The following lemma shows that $\{u_n\}$ is bounded, which is a key for this paper.

Lemma 3.6. *Let u_n be a critical point of J_{λ,μ_n} at the level c_{μ_n} as defined in (3.8).*

1. *If $\lambda=0$, then there exists a constant $C>0$ such that $\|u_n\|\leqslant C$ for all n.*
2. *There exists $C>0$ such that for every $\lambda\in(0,\lambda_0)$, we have $\|u_n\|\leqslant C/\sqrt{\lambda}$ for all n.*

Proof. Firstly, since $(J_{\lambda,\mu_n})'(u_n)=0$, from Lemma 2.2, u_n satisfies the following Pohozaev type identity

$$\big(a+\lambda\|u_n\|^2\big)\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 = \mu_n N\int_{\mathbb{R}^N} F(u_n). \tag{3.9}$$

Since also $J_{\lambda,\mu_n}(u_n)=c_{\mu_n}$, we have that

$$\frac{1}{2}aN\|u_n\|^2+\frac{1}{4}\lambda N\|u_n\|^4-\mu_n N\int_{\mathbb{R}^N}F(u_n)=c_{\mu_n}N. \tag{3.10}$$

Therefore, by (3.9) and (3.10), we obtain that

$$a\int_{\mathbb{R}^N}|\nabla u_n|^2 = c_{\mu_n}N+\frac{1}{4}\lambda(N-4)\|u_n\|^4. \tag{3.11}$$

We now estimate the right hand side of (3.11). By the min–max definition of the mountain pass level c_{μ_n}, Lemma 3.2 and (3.2), we have that

$$\begin{aligned} c_{\mu_n} &\leqslant \max_{t\in[0,t_0]} J_{\lambda,\mu_n}(t\phi) \\ &\leqslant \max_{t\in[0,t_0]}\left\{\frac{1}{2}at^2-\mu_n\int_{\mathbb{R}^N}F(t\phi)\right\}+\max_{t\in[0,t_0]}\frac{1}{4}\lambda t^4 \\ &\leqslant \max_{t\in[0,t_0]}\left\{\frac{1}{2}at^2-\frac{1}{2}C_1t^2\int_{B(0,R)}\phi^2+\frac{1}{2}C_3\right\}+\max_{t\in[0,t_0]}\frac{1}{4}\lambda t^4 \\ &=\frac{1}{2}C_3+\frac{1}{4}\lambda t_0^4. \end{aligned}$$

If $\lambda=0$ then it follows from (3.11) that $2a\|u_n\|^2\leqslant C_3N$. If $\lambda\in(0,\lambda_0)$ and $N\leqslant 4$, then $4a\|u_n\|^2\leqslant 2C_3N+\lambda_0t_0^4$. If $N\geqslant 5$, then by (3.11) we have

$$\|u_n\|^2\leqslant 4a/\lambda.$$

Then the conclusion holds. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let λ_0 be defined as in (3.3), and let u_n be a critical point for J_{λ,μ_n} at the level c_{μ_n}. Then we may assume from Lemma 3.6 that for all n,

$$\|u_n\|\leqslant C_\lambda,$$

where C_λ is defined by

$$C_\lambda=\begin{cases} C, & \lambda=0,\\ C/\sqrt{\lambda}, & \lambda\in(0,\lambda_0).\end{cases}$$

Since $\mu_n\to 1$, we can show that $\{u_n\}$ is a (PS) sequence of J_λ. Indeed, the boundedness of $\{u_n\}$ implies that $\{J_\lambda(u_n)\}$ is bounded. Also

$$\langle J_\lambda'(u_n),v\rangle=\langle (J_{\lambda,\mu_n})'(u_n),v\rangle+(\mu_n-1)\int_{\mathbb{R}^N}f(u_n)v,\quad v\in\mathcal{D}_r^{1,2}(\mathbb{R}^N).$$

Thus $J_\lambda'(u_n)\to 0$, and consequently $\{u_n\}$ is a bounded (PS) sequence of J_λ. By Lemma 3.4, $\{u_n\}$ has a convergent subsequence, hence without loss of generality we may assume that $u_n\to u$. Consequently $J_\lambda'(u)=0$. According to Lemma 3.3, we have that $J_\lambda(u)=\lim_{n\to\infty}J_\lambda(u_n)=\lim_{n\to\infty}J_{\lambda,\mu_n}(u_n)\geqslant c>0$ and u is a positive solution by the condition (H_1). The proof is completed. □

4. The case $K\in L^s(\mathbb{R}^N)$

In this section, we assume that K satisfies conditions (K_1) and (K_2), and f satisfies (H_1)–(H_3). For the non-constant K case, we need to use a cut-off functional to obtain the boundedness of $\{u_n\}$. So following [11,13], we choose a cut-off function $\psi\in C^\infty(\mathbb{R}_+,[0,1])$ satisfying

$$\begin{cases} \psi(t)=1, & t\in[0,1],\\ \psi(t)=0, & t\in[2,\infty),\\ |\psi'|_\infty\leqslant 2, & t\in[0,\infty),\end{cases}$$

and study the following modified functional $J_\lambda^T:\mathcal{D}^{1,2}(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$J_\lambda^T(u)=\frac{1}{2}a\|u\|^2+\frac{1}{4}\lambda h_T(u)\|u\|^4-\int_{\mathbb{R}^N}KF(u),\quad u\in\mathcal{D}^{1,2}(\mathbb{R}^N),$$

where for every $T>0$,

$$h_T(u)=\psi\left(\frac{\|u\|^2}{T^2}\right).$$

With this penalization, for T sufficiently large and λ sufficiently small, we are able to find a critical point u of J_λ^T such that $\|u\| \leqslant T$ and so u is also a critical point of J_λ.

In this section, we consider the problem (1.1) in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ because that the function K is not assumed to be radial. For the setting of Theorem 2.1, the space $X = \mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$A(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4, \qquad B(u) = \int_{\mathbb{R}^N} KF(u).$$

So the perturbed functional which we will study is

$$J_{\lambda,\mu}^T(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \mu \int_{\mathbb{R}^N} KF(u),$$

and

$$\big\langle \big(J_{\lambda,\mu}^T\big)'(u), v\big\rangle = a(u,v) + \lambda h_T(u)\|u\|^2(u,v) + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4(u,v) - \mu \int_{\mathbb{R}^N} Kf(u)v. \tag{4.1}$$

Again we establish parallel steps as Lemmas 3.2 and 3.3 as follows.

Lemma 4.1. *Let $\Gamma_{\lambda,\mu}^T$ be defined by (2.2) for the functional $J_{\lambda,\mu}^T$. Then $\Gamma_{\lambda,\mu}^T \neq \emptyset$ for all $\mu \in I = [1/2, 1]$ and $\lambda, T > 0$.*

Proof. By the condition (K_1), we may assume that $\int_{B(0,R)} K > 0$ for some $R > 0$. We now choose a function $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$ with $\|\phi\| = 1$, $\operatorname{supp}(\phi) \subset B(0,R)$ and $\int_{B(0,R)} K\phi^2 > 0$. By the condition ($H_3$), for $C_1 = 2a(\int_{B(0,R)} K\phi^2)^{-1} > 0$, there exists $C_2 > 0$ such that

$$F(t) \geqslant C_1|t|^2 - C_2, \quad t \in \mathbb{R}_+. \tag{4.2}$$

For $t^2 \geqslant 2T^2$, we have from (4.2) that

$$\begin{aligned} J_{\lambda,\mu}^T(t\phi) &= \frac{1}{2}at^2 + \frac{1}{4}\lambda\psi\left(\frac{t^2}{T^2}\right)t^4 - \mu \int_{\mathbb{R}^N} KF(t\phi) \\ &\leqslant \frac{1}{2}at^2 - \frac{1}{2}C_1t^2 \int_{B(0,R)} K\phi^2 + C_3, \end{aligned}$$

where $C_3 = C_2 \int_{B(0,R)} K > 0$. Thus we can choose $t > 0$ large such that $J_{\lambda,\mu}^T(t\phi) < 0$. The proof is completed. $\square$

Lemma 4.2. *Let $c_{\lambda,\mu}^T$ be defined by (2.3) for the functional $J_{\lambda,\mu}^T$. Then there exists a constant $c > 0$ such that $c_{\lambda,\mu}^T \geqslant c$ for all $\mu \in I$ and $\lambda, T > 0$.*

Proof. Let $\mu \in I$ and $\lambda, T > 0$. For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, by (2.1), we have that

$$\begin{aligned} J_{\lambda,\mu}^T(u) &\geqslant \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - C\int_{\mathbb{R}^N} |u|^{2^*} \\ &\geqslant \frac{1}{2}a\|u\|^2 - C\int_{\mathbb{R}^N} |u|^{2^*}. \end{aligned}$$

By Sobolev's embedding theorem, we conclude that there exists $\rho > 0$ such that $J_{\lambda,\mu}^T(u) > 0$ for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $0 < \|u\| \leqslant \rho$. In particular, for $\|u\| = \rho$, it follows $J_{\lambda,\mu}^T(u) \geqslant c > 0$. For every $\gamma \in \Gamma_{\lambda,\mu}^T$, by the definition of $\Gamma_{\lambda,\mu}^T$, we have $\|\gamma(1)\| > \rho$. Then from the intermediate value theorem, there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore,

$$c_{\lambda,\mu}^T \geqslant \inf_{\gamma \in \Gamma_{\lambda,\mu}^T} J_{\lambda,\mu}^T\big(\gamma(t_\gamma)\big) \geqslant c > 0.$$

The proof is completed. $\square$

The next lemma shows that the critical level defined in Lemma 4.2 can be achieved if λ is small.

Lemma 4.3. *Let $4\lambda T^2 < a$ and $\mu \in I = [1/2, 1]$. Then each bounded (PS) sequence of the functional $J^T_{\lambda,\mu}$ admits a convergent subsequence.*

Proof. Let $4\lambda T^2 < a$ and $\mu \in I$. Let $\{u_n\}$ be a bounded (PS) sequence of $J^T_{\lambda,\mu}$, that is, $\{u_n\}$ and $\{J^T_{\lambda,\mu}(u_n)\}$ are bounded, and $(J^T_{\lambda,\mu})'(u_n) \to 0$ in $\mathcal{D}'$, where $\mathcal{D}'$ is the dual space of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $\{u_n\}$ is bounded, we may assume that there exists $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\begin{aligned} &u_n \rightharpoonup u, \quad \text{in } \mathcal{D}^{1,2}\big(\mathbb{R}^N\big), \\ &u_n \to u, \quad \text{in } L^p_{\text{loc}}\big(\mathbb{R}^N\big),\ p \in \big(1, 2^*\big), \\ &u_n(x) \to u(x), \quad \text{a.e. } x \in \mathbb{R}^N. \end{aligned}$$

By the conditions (H_1) and (H_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\big|f(u)\big| \leqslant \varepsilon|u|^{2^*-1} + C_\varepsilon \chi_{(\tau_1 \leqslant |u| \leqslant \tau_2)}, \quad u \in D^{1,2}\big(\mathbb{R}^N\big), \tag{4.3}$$

where τ_1 and τ_2 are two positive constants. Let $E_n = \{x \in \mathbb{R}^N\colon\ \tau_1 \leqslant |u_n(x)| \leqslant \tau_2\}$. Then

$$\tau_1^{2^*}|E_n| \leqslant \int_{E_n} |u_n|^{2^*} \leqslant \int_{\mathbb{R}^N} |u_n|^{2^*} \leqslant C,$$

where $|E_n|$ is Lebesgue's measure of E_n. This implies that $|E_n| \leqslant C\tau_1^{-2^*}$. So it follows from (4.3) and (K_1) that for $r > 0$,

$$\begin{aligned} \bigg|\int_{|x| \geqslant r} Kf(u_n)(u_n - u)\bigg| &\leqslant \varepsilon|K|_\infty \int_{|x| \geqslant r} |u_n|^{2^*-1}|u_n - u| + C_\varepsilon \int_{E_n \setminus B(0,r)} K|u_n - u| \\ &\leqslant \varepsilon|K|_\infty |u_n|^{2^*-1}_{2^*}|u_n - u|_{2^*} + C_\varepsilon |E_n|^{1/s'}\bigg(\int_{E_n \setminus B(0,r)} K^s\bigg)^{1/s}|u_n - u|_{2^*}, \end{aligned}$$

where $s' \in (0, \infty]$ with $1/s' + 1/s + 1/2^* = 1$. Hence,

$$\limsup_{r \to \infty}\bigg|\int_{|x| \geqslant r} Kf(u_n)(u_n - u)\bigg| \leqslant \varepsilon C|K|_\infty.$$

On the other hand, we have that

$$\int_{|x| < r} Kf(u_n)(u_n - u) \to 0, \quad \text{as } n \to +\infty$$

for every $r > 0$. This implies that

$$\int_{\mathbb{R}^N} Kf(u_n)(u_n - u) \to 0, \quad \text{as } n \to +\infty. \tag{4.4}$$

Thus, by (4.4), we get that

$$\begin{aligned} \big\langle \big(J^T_{\lambda,\mu}\big)'(u_n), u_n - u\big\rangle &= a(u_n, u_n - u) + \lambda h_T(u_n)\|u_n\|^2(u_n, u_n - u) \\ &\quad + \frac{\lambda}{2T^2}\psi'\bigg(\frac{\|u_n\|^2}{T^2}\bigg)\|u_n\|^4(u_n, u_n - u) - \mu\int_{\mathbb{R}^N} Kf(u_n)(u_n - u) \\ &= \bigg(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\bigg(\frac{\|u_n\|^2}{T^2}\bigg)\|u_n\|^4\bigg)(u_n, u_n - u) + o(1), \end{aligned}$$

and then as $n \to \infty$,

$$\bigg(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\bigg(\frac{\|u_n\|^2}{T^2}\bigg)\|u_n\|^4\bigg)(u_n, u_n - u) \to 0.$$

If $\|u_n\|^2 > 2T^2$, then $|\psi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4| = 0$. If $\|u_n\|^2 \leqslant 2T^2$, then $|\psi'(\frac{\|u_n\|^2}{T^2})| \leqslant 2$ and $|\psi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4| \leqslant 8T^4$. According to $4\lambda T^2 < a$, then it follows that $\|u_n\| \to \|u\|$. This together with $u_n \rightharpoonup u$ shows that $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The proof is completed. $\square$

Now we are in the position to show that the modified functional $J^T_{\lambda,\mu}$ has a nontrivial critical point.

Lemma 4.4. *Let $4\lambda T^2 < a$. Then for almost every $\mu \in I$, there exists $u^\mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $(J^T_{\lambda,\mu})'(u^\mu) = 0$ and $J^T_{\lambda,\mu}(u^\mu) = c^T_{\lambda,\mu}$.*

Proof. By Theorem 2.1, for almost every $\mu \in I$, there exists a bounded sequence $\{u^\mu_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $J^T_{\lambda,\mu}(u^\mu_n) \to c^T_{\lambda,\mu}$ and $(J^T_{\lambda,\mu})'(u^\mu_n) \to 0$ as $n \to \infty$. According to Lemma 4.3, we can suppose that there exists $u^\mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u^\mu_n \to u^\mu$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then the assertion follows from Lemma 4.2. $\square$

According to Lemma 4.4, there exist a sequence $\{\mu_n\} \subset I$ with $\mu_n \to 1^-$ and a sequence $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$J^T_{\lambda,\mu_n}(u_n) = c_{\mu_n}, \qquad \big(J^T_{\lambda,\mu_n}\big)'(u_n) = 0. \tag{4.5}$$

The following lemma shows that $\|u_n\| \leqslant T$ for all n, which is a key step for the proof.

Lemma 4.5. *Let u_n be a critical point of J^T_{λ,μ_n} at the level c_{μ_n} as defined in (4.5). Then there exist positive constants*

$$\lambda_0 = \frac{(2-\alpha)^2 a^2}{16(N-\alpha)(6N-10)C_3}, \tag{4.6}$$

and

$$T_0^2 = \frac{4(N-\alpha)C_3}{(2-\alpha)a}, \tag{4.7}$$

which satisfy

$$4(6N-10)(2-\alpha)^{-1}\lambda_0 T_0^2 = a, \tag{4.8}$$

such that for any $\lambda \in [0,\lambda_0]$ and any $T \geqslant T_0$, $\|u_n\| \leqslant T$.

Proof. Since $(J^T_{\lambda,\mu_n})'(u_n) = 0$, then from Lemma 2.2, u_n satisfies the following Pohozaev type identity

$$\begin{aligned}\left(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 &= \mu_n N\int_{\mathbb{R}^N} KF(u_n) + \mu_n\int_{\mathbb{R}^N} F(u_n)(x\cdot\nabla K)\\ &\geqslant \mu_n(N-\alpha)\int_{\mathbb{R}^N} KF(u_n).\end{aligned}$$

On the other hand, by using $J^T_{\lambda,\mu_n}(u_n) = c_{\mu_n}$, we have that

$$\frac{1}{2}a(N-\alpha)\|u_n\|^2 + \frac{1}{4}\lambda(N-\alpha)h_T(u_n)\|u_n\|^4 - \mu_n(N-\alpha)\int_{\mathbb{R}^N} KF(u_n) = c_{\mu_n}(N-\alpha). \tag{4.9}$$

Thus, we can obtain that

$$\frac{2-\alpha}{2}a\int_{\mathbb{R}^N}|\nabla u_n|^2 \leqslant c_{\mu_n}(N-\alpha) + \frac{1}{4}\lambda(N-4+\alpha)h_T(u_n)\|u_n\|^4 + \frac{\lambda(N-2)}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6. \tag{4.10}$$

We will estimate the right hand side of (4.10). By the min–max definition of the mountain pass level c_{μ_n}, Lemma 4.1 and (4.2), we have that

$$\begin{aligned} c_{\mu_n} &\leqslant \max_{t\in[0,\infty)} J^T_{\lambda,\mu_n}(t\phi)\\ &\leqslant \max_{t\in[0,\infty)}\left\{\frac{1}{2}at^2 - \mu_n\int_{\mathbb{R}^N} KF(t\phi)\right\} + \max_{t\in[0,\infty)}\frac{1}{4}\lambda\psi\left(\frac{t^2}{T^2}\right)t^4\\ &\leqslant \max_{t\in[0,\infty)}\left\{\frac{1}{2}at^2 - \frac{1}{2}C_1t^2\int_{B(0,R)} K\phi^2 + C_3\right\} + \max_{t\in[0,\infty)}\frac{1}{4}\lambda\psi\left(\frac{t^2}{T^2}\right)t^4\\ &\leqslant C_3 + \lambda T^4.\end{aligned}$$

We have also that

$$\frac{|N-4+\alpha|}{4} h_T(u_n)\|u_n\|^4 \leqslant (N-2+\alpha)T^4,$$
$$\frac{N-2}{4T^2}\left|\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right|\|u_n\|^6 \leqslant 4(N-2)T^4.$$

Then it follows that

$$\frac{2-\alpha}{2} a \int_{\mathbb{R}^N} |\nabla u_n|^2 \leqslant (N-\alpha)C_3 + \lambda(6N-10)T^4,$$

or equivalently

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \leqslant \frac{2(N-\alpha)}{(2-\alpha)a} C_3 + \lambda \frac{2(6N-10)}{(2-\alpha)a} T^4. \tag{4.11}$$

Now for λ_0 and T_0 defined in (4.6) and (4.7), since the equality (4.8) holds, we obtain that

$$2(N-\alpha)(2-\alpha)^{-1}a^{-1}C_3 + 2\lambda(2-\alpha)^{-1}(6N-10)a^{-1}T_0^4 \leqslant 2(N-\alpha)(2-\alpha)^{-1}a^{-1}C_3 + T_0^2/2 = T_0^2 \leqslant T^2.$$

Thus it follows from (4.11) that $\|u_n\| \leqslant T$ for all n, and the stated conclusion holds. □

Now we can complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let λ_0 and T_0 be defined as in Lemma 4.5, and let u_n be a critical point for J^T_{λ,μ_n} at the level c_{μ_n}. Then we may assume that for all n, $\|u_n\| \leqslant T$ holds. Hence

$$J^T_{\lambda,\mu_n}(u_n) = \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}\lambda\|u_n\|^4 - \mu_n \int_{\mathbb{R}^N} KF(u_n).$$

Since $\mu_n \to 1$, we will show that $\{u_n\}$ is a (PS) sequence of J_λ. Indeed, the boundedness of $\{u_n\}$ implies that $\{J_\lambda(u_n)\}$ is bounded. Also

$$\langle J'_\lambda(u_n), v\rangle = \langle (J^T_{\lambda,\mu_n})'(u_n), v\rangle + (\mu_n - 1)\int_{\mathbb{R}^N} Kf(u_n)v, \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Thus $J'_\lambda(u_n) \to 0$, and $\{u_n\}$ is a bounded (PS) sequence of J_λ. By Lemma 4.3, $\{u_n\}$ has a convergent subsequence, and without loss of generality, we may assume that $u_n \to u$. Consequently $J'_\lambda(u) = 0$. According to Lemma 4.2, we have that $J_\lambda(u) = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty} J^T_{\lambda,\mu_n}(u_n) \geqslant c > 0$ and u is a positive solution by the condition (H_1). The proof is completed. □

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