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Product of two positive contractions

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In memory of Professor Robert Thompson.

Abstract

Several characterizations are given for a square matrix that can be written as the product of two positive (semidefinite) contractions. Based on one of these characterizations, and the theory of alternating projections, a Matlab program is written to check the condition and construct the two positive contractions whose product equal to the given matrix, if they exist.

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1 Introduction

Let M_n be the set of $n \times n$ complex matrices. It is known that every matrix $A \in M_n$ with nonnegative determinant can be written as the product of k positive semidefinite matrices with $k \leq 5$; see [1, 2, 5] and their references. Moreover, characterizations are given of matrices that can be written as the product of k positive semidefinite matrices but not fewer for $k = 2, \ldots, 5$. In particular, a matrix A is the product of two positive semidefinite matrices if it is similar to a diagonal matrix with nonnegative diagonal entries.

In this paper, characterizations are given to $A \in M_n$ which is a product of two positive contractions, i.e., positive semidefinite matrices with norm not larger than one. Evidently, if a matrix is the product of two positive contractions, then it is a contraction similar to a diagonal matrix with nonnegative diagonal entries. However, the converse is not true. For example, $A = \frac{1}{25}\begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}$ is a contraction similar to diag $(9, 16)/25$ that is not a product of two positive contractions as shown in [4]. In fact, the result in [4] implies that if $A \in M_n$ is similar to a diagonal matrix with nonzero eigenvalues $a, b \in (0, 1]$ then a necessary and sufficient condition for A to be the product of two positive contractions is:

$$\{\|A\|^2 - (a^2 + b^2) + (ab/\|A\|)^2\}^{1/2} \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)};$$

see Corollary 2.6. In particular, a matrix $A = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix} \in M_2$ is the product of two positive contractions if and only if $a, b \in [0, 1]$ and $|p| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}$.

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In Section 2, we will present several characterizations of a square matrix that can be written as the product of two positive (semidefinite) contractions. In Section 3, based on one of the characterizations in Section 2, we use alternating projection method to check the condition and construct the two positive contractions whose product equal to the given matrix if they exist. Some numerical examples generated by Matlab are presented.

2 Characterizations

If A is a product of two positive semidefinite contractions, then A is similar to a diagonal matrix with nonnegative eigenvalues with magnitudes bounded by $\|A\| \le 1$. We will focus on such matrices in our characterization theorem.

It is known that a matrix A is the product of two orthogonal projections if and only if it is unitarily similar to a matrix which is the direct sum of $I_p \oplus 0_q$ and matrices of the form

$$\begin{pmatrix} a_j & \sqrt{a_j - a_j^2} \\ 0 & 0 \end{pmatrix} \in M_2, \qquad 0 < a_j < 1 \text{ for all } j = 1, \ldots, m;$$

see [3]. Here we give another characterization which will be useful for our study.

Proposition 2.1 *Suppose A is similar to $I_p \oplus 0_q \oplus \operatorname{diag}(a_1, \ldots, a_m)$ with $a_1, \ldots, a_m \in (0,1)$. Then A is the product of two orthogonal projections in M_n if and only if A is unitarily similar to $I_p \oplus A_1$ and there is an $(n-p) \times m$ matrix S of rank m such that $A_1 A_1^* S = A_1 S = S\operatorname{diag}(a_1, \ldots, a_m)$.*

Proof. For simplicity, we assume that I_p is vacuous. Suppose A is the product of two orthogonal projections in M_n. Let $D = \operatorname{diag}(a_1, \ldots, a_m)$. We may assume that $a_1 \ge \cdots \ge a_m$. There is a unitary U such that $U^*AU = \begin{pmatrix} D & \sqrt{D - D^2} \\ 0 & 0_m \end{pmatrix} \oplus 0_{q-m}$. Let $U = [u_1 \cdots u_n]$ and $U_m = [u_1 \cdots u_m]$. Hence, we have $AA^*S = AS = SD$ with $S = U_m$.

Conversely, suppose S satisfies $AA^*S = AS = S\operatorname{diag}(a_1, \ldots, a_m)$, and has linearly independent columns $v_1, \ldots, v_m$. We may assume that $\|v_j\| = 1$ for $1 \le j \le m$ and $\langle v_i, v_j\rangle = 0$ if $a_i = a_j$ and $i \ne j$. Since AA^* is normal and v_i is an eigenvector of AA^* corresponding to the eigenvalue a_i, $\langle v_i, v_j \rangle = 0$ for $a_i \ne a_j$. Hence $S^*S = I_m$. Now, we can find an orthonormal set $\{v_{m+1}, \ldots, v_n\}$ such that $V = [v_1, \ldots, v_n]$ and $V^*AA^*V = D \oplus 0_q$. Then V^*AV is of the form $\begin{pmatrix} D & B \\ 0 & 0_q \end{pmatrix}$, where B is an $m \times q$ matrix with $BB^* = D - D^2$. From the QR factorization, B can be written as RQ with Q unitary and R lower triangular. Let $V_1 = I_m \oplus Q^*$. Then $V_1^*V^*AVV_1 = \begin{pmatrix} D & R \\ 0 & 0_q \end{pmatrix}$ and $RR^* = BQ^*QB^* = D - D^2$. Hence $R = [\sqrt{D - D^2}\ 0_{m,(q-m)}]$, and we see that A is unitarily similar to the direct sum of 0_q and matrices of the form

$$\begin{pmatrix} a_j & \sqrt{a_j - a_j^2} \\ 0 & 0 \end{pmatrix} \in M_2, \qquad j = 1, \ldots, m.$$

Hence A is the product of two orthogonal projections. $\square$

Recall that $A \in M_n$ has a dilation $B \in M_N$ with $n < N$ if there is a unitary $V \in M_N$ such that A is the leading principal submatrix of V^*BV. For two Hermitian matrices $X, Y \in M_n$, we write

$X \geq Y$ if $X - Y$ is positive semidefinite. In the next theorem, we present two characterizations for matrices which can be written as the product of two positive contractions in terms of dilation and matrix inequalities. We begin with the following observation.

Lemma 2.2 *Suppose $A \in M_n$ is the product of two positive contractions. Then A is unitarily similar to a matrix of the form*

$$I_p \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-m} \end{pmatrix},$$

where $A_{11} \in M_m$ is similar to a diagonal matrix with the eigenvalues in $(0,1)$.

Proof. Obviously, the eigenvalues of A are in $[0,1]$. From [2, Proposition 3.1(d)], we have

$$A \cong \begin{pmatrix} I_p & B_1 & B_2 \\ 0 & A_{11} & A_{12} \\ 0 & 0 & 0_{n-p-m} \end{pmatrix},$$

where $A_{11} \in M_m$ is an upper block triangular matrix such that the diagonal blocks are scalar matrices corresponding to distinct scalars, $1 > \lambda_1 > \cdots > \lambda_k > 0$. Since $\|A\| \leq 1$, B_1 and B_2 are zero matrices. By [2, Proposition 3.1(c) and (d)], A_{11} is similar to a diagonal matrix, and the desired conclusion follows. □

Theorem 2.3 *Suppose $A = I_p \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-m} \end{pmatrix} \in M_n$ such that $A_{11} \in M_m$ is similar to $D \equiv \operatorname{diag}(a_1, \ldots, a_m)$ with $1 > a_1 \geq \cdots \geq a_m > 0$. The following conditions are equivalent.*

(a) *A is the product of two positive contractions.*

(b) *A has a dilation $\tilde{T} \in M_{n+2m}$, which is the product of two orthogonal projections and has the same rank and eigenvalues of A. Equivalently, there are matrices $R, C \in M_m$ such that*

$$\tilde{T} = I_p \oplus \begin{pmatrix} A_{11} & A_{12} & 0 & A_{11}C \\ 0 & 0_{n-p-m} & 0 & 0 \\ RA_{11} & RA_{12} & 0_m & RA_{11}C \\ 0 & 0 & 0 & 0_m \end{pmatrix} \in M_{n+2m}$$

is the product of two orthogonal projections.

(c) *There is an invertible contraction $U_{11} \in M_m$ satisfying*

$$A_{11}U_{11} = U_{11}D \quad \text{and} \quad U_{11}DU_{11}^* \geq A_{11}A_{11}^* + A_{12}A_{12}^*.$$

Moreover, if condition (c) *holds, we have $A = (I_p \oplus P)(I_p \oplus Q)$ for the positive contractions*

$$P = \begin{pmatrix} U_{11}U_{11}^* & 0 \\ 0 & 0_{n-p-m} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} (U_{11}^*)^{-1}DU_{11}^{-1} & (U_{11}U_{11}^*)^{-1}A_{12} \\ A_{12}^*(U_{11}U_{11}^*)^{-1} & A_{12}^*(U_{11}DU_{11}^*)^{-1}A_{12} \end{pmatrix}.$$

Proof. For simplicity, we can assume that I_p is vacuous because the matrix A is the product of two positive contractions if and only if each of the two positive contractions is a direct sum of I_p and a positive contraction in M_{n-p}.

First we establish the equivalence of (a) and (b). If (a) holds, then $A = PQ$, where P, Q are two positive contractions. Then

$$\tilde{P} = \begin{pmatrix} P & \sqrt{P-P^2} & 0 \\ \sqrt{P-P^2} & I_n - P & 0 \\ 0 & 0 & 0_n \end{pmatrix} \quad \text{and} \quad \tilde{Q} = \begin{pmatrix} Q & 0 & \sqrt{Q-Q^2} \\ 0 & 0_n & 0 \\ \sqrt{Q-Q^2} & 0 & I_n - Q \end{pmatrix}$$

are orthogonal projections such that

$$\tilde{P}\tilde{Q} = \begin{pmatrix} PQ & 0 & P\sqrt{Q-Q^2} \\ \sqrt{P-P^2}Q & 0_n & \sqrt{(P-P^2)(Q-Q^2)} \\ 0 & 0 & 0_n \end{pmatrix}.$$

Let $Y = \sqrt{Q^+ - Q^+Q}$ and $X = \sqrt{P^+ - P^+P}$, where P^+, Q^+ is the Moore-Penrose inverses of P and Q. (Recall that for a Hermitian matrix $H = \sum_{j=1}^{\ell} \lambda_j \xi_j \xi_j^* \in M_n$ with nonzero eigenvalues $\lambda_1, \ldots, \lambda_\ell$ and orthonormal eigenvectors $\xi_1, \ldots, \xi_\ell$, its Moore-Penrose inverse H^+ is $\sum_{j=1}^{\ell} \lambda_j^{-1} \xi_j \xi_j^*$.) Let

$$T = \begin{pmatrix} A & 0 & AY \\ X^*A & 0_n & X^*AY \\ 0 & 0 & 0_n \end{pmatrix}.$$

The rows of the matrix X^*A lie in the row space of $[A_{11}A_{12}]$ and the columns of AY lie in the column space of A_{11}. So, there is unitary matrix of the form $U = I_n \oplus U_1 \oplus U_2$ with $U_1, U_2 \in M_n$ such that

$$U^*TU = \begin{pmatrix} A_{11} & A_{12} & 0_m & 0_{m,n-m} & A_{11}C & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m} & 0_{n-m,m} & 0_{n-m} & 0_{n-m,m} & 0_{n-m} \\ RA_{11} & RA_{12} & 0_m & 0_{m,n-m} & RA_{11}C & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m} & 0_{n-m,m} & 0_{n-m} & 0_{n-m,m} & 0_{n-m} \\ 0_{n,m} & 0_{n,n-m} & 0_{n,m} & 0_{n,n-m} & 0_{n,m} & 0_{n,n-m} \end{pmatrix}.$$

Thus,

$$\tilde{T} = \begin{pmatrix} A_{11} & A_{12} & 0 & A_{11}C \\ 0 & 0_{n-m} & 0 & 0 \\ RA_{11} & RA_{12} & 0_m & RA_{11}C \\ 0 & 0 & 0 & 0_m \end{pmatrix} \in M_{n+2m}$$

has the same rank and eigenvalues as the leading submatrix A. Thus, condition (b) holds.

Conversely, suppose (b) holds. and $\tilde{T}$ is the product of two orthogonal projections $\tilde{P} = VV^*$ and $\tilde{Q} = WW^*$ with $V \in M_{n+2m,r}, W \in M_{n+2m,s}$ such that $V^*V = I_r$ and $W^*W = I_s$. Evidently, $\tilde{T}$ has rank m. So,

$$V^*W = Y \begin{pmatrix} K & 0 \\ 0 & 0_{(r-m),(s-m)} \end{pmatrix} Z^*$$

such that $Y \in M_r$, $Z \in M_s$ are unitary and $K \in M_m$ is a diagonal matrix with positive diagonal entries. Let $Y = [Y_1|Y_2], Z = [Z_1|Z_2]$ be such that $Y_1 \in M_{r,m}, Z_1 \in M_{s,m}$. Note that

$$Y_1^*V^*WZ_1 = Y_1^*[Y_1|Y_2] \begin{pmatrix} K & 0 \\ 0 & 0_{(r-m),(s-m)} \end{pmatrix} [Z_1|Z_2]^*Z_1 = K.$$

Furthermore,

$$\tilde{V} = VY_1 = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad \textit{and} \quad \tilde{W} = WZ_1 = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix},$$

where V_1, W_1 are $n \times m$, $V_2, V_3, W_2, W_3 \in M_m$. Then

$$\tilde{V}\tilde{V}^*\tilde{W}\tilde{W}^* = VY_1Y_1^*V^*WZ_1Z_1^*W^* = VY_1KZ_1^*W^* = VV^*WW^* = \tilde{T}.$$

Now, the last m rows of $\tilde{T}$ and the $(n+1)$st, $\ldots, (n+m)$th columns of $\tilde{T}$ are zero. Thus,

$$V_3\tilde{V}^*\tilde{W}\tilde{W}^* = V_3K\tilde{W}^* = 0_{m,(n+2m)} \quad \text{and} \quad \tilde{V}\tilde{V}^*\tilde{W}W_2^* = \tilde{V}KW_2^* = 0_{(n+2m),m}.$$

Because $K\tilde{W}^*$ has full row rank and $\tilde{V}K$ has full column rank, we see that $V_3 = 0_m$ and $W_2 = 0_m$. Consequently, $A = V_1V_1^*W_1W_1^*$ is the product of two positive contractions $V_1V_1^*$ and $W_1W_1^*$.

Next, we prove the equivalence of conditions (b) and (c). Suppose (b) holds, and

$$\tilde{T} = \begin{pmatrix} A_{11} & A_{12} & 0 & A_{11}C \\ 0 & 0_{n-m} & 0 & 0 \\ RA_{11} & RA_{12} & 0_m & RA_{11}C \\ 0 & 0 & 0 & 0_m \end{pmatrix} \in M_{n+2m}$$

has the same rank and eigenvalues as the leading submatrix A.

Now, assume that $U = (U_{ij})_{1\le i\le 4, 1\le j\le 3} \in M_{n+2m}$ is unitary with $U_{11}, U_{12} \in M_m, U_{13} \in M_{m,n}$ and $U_{31}, U_{41} \in M_m, U_{21} \in M_{n-m,m}$ such that

$$U^*\tilde{T}U = \begin{pmatrix} D & \sqrt{D-D^2} & 0 \\ 0 & 0_m & 0 \\ 0 & 0 & 0_n \end{pmatrix}.$$

Now,

$$\begin{pmatrix} A_{11}U_{11} + A_{12}U_{21} \\ 0_{n-m,m} \\ RA_{11}U_{11} + RA_{12}U_{21} \\ 0_m \end{pmatrix} = \tilde{T}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{41} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{41} \end{pmatrix} D.$$

It follows that U_{21}, U_{41} are zero matrices. Furthermore,

$$A_{11}U_{11} = U_{11}D, \quad RA_{11}U_{11} = U_{31}D.$$

Thus, $RU_{11}D = U_{31}D$ so that $RU_{11} = U_{31}$. If $x \in \mathbf{C}^m$ satisfies $U_{11}x = 0$, then

$$x = (U_{11}^* \ U_{31}^*)\begin{pmatrix} U_{11} \\ U_{31} \end{pmatrix} x = U_{11}^*(I_m + R^*R)U_{11}x = 0.$$

Hence, $U_{11} \in M_m$ has linearly independent columns, i.e., U_{11} is invertible.

Next, observe that

$$\tilde{T}\tilde{T}^*U = U\begin{pmatrix} D & 0 & 0 \\ 0 & 0_m & 0 \\ 0 & 0 & 0_n \end{pmatrix}.$$

So,

$$(A_{11}A_{11}^* + A_{12}A_{12}^* + A_{11}CC^*A_{11}^*)(I_m + R^*R)U_{11} = U_{11}D,$$

and hence

$$(A_{11}A_{11}^* + A_{12}A_{12}^* + A_{11}CC^*A_{11}^*) = U_{11}DU_{11}^*, \tag{1}$$

because

$$I_m = U_{11}^* U_{11} + U_{31}^* U_{31} = U_{11}^*(I_m + R^*R)U_{11} = (I_m + R^*R)U_{11}U_{11}^*. \qquad (2)$$

So, R and C exist if and only if there is a contraction $U_{11} \in M_m$ satisfying

$$A_{11}U_{11} = U_{11}D \quad \text{and} \quad U_{11}DU_{11}^* \geq A_{11}A_{11}^* + A_{12}A_{12}^*.$$

Conversely, suppose (c) holds. Then there exist R and C satisfying (1) and (2). Let

$$\tilde{U} = \begin{pmatrix} U_{11} \\ 0_{n-m,m} \\ RU_{11} \\ 0_m \end{pmatrix}.$$

Then $\tilde{U}$ has rank m and the matrix $\tilde{T}$ in condition (b) satisfies $\tilde{T}\tilde{T}^*\tilde{U} = \tilde{T}\tilde{U} = \tilde{U}D$. By Proposition 2.1, we see that $\tilde{T}$ is the product of two orthogonal projections.

To verify the last statement, note that $A_{11}U_{11} = U_{11}D$ so that $A_{11} = U_{11}DU_{11}^{-1}$. Hence,

$$PQ = \begin{pmatrix} U_{11}DU_{11}^{-1} & A_{12} \\ 0 & 0_{n-m} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-m} \end{pmatrix},$$

and $Q = ZZ^*$ with $Z = \begin{pmatrix} (U_{11}^*)^{-1}D^{1/2} \\ A_{12}^*(U_{11}^*)^{-1}D^{-1/2} \end{pmatrix}$ so that

$$\begin{aligned} Z^*Z &= D^{1/2}U_{11}^{-1}(U_{11}^*)^{-1}D^{1/2} + D^{-1/2}U_{11}^{-1}A_{12}A_{12}^*(U_{11}^*)^{-1}D^{-1/2} \\ &= D^{-1/2}U_{11}^{-1}(A_{11}A_{11}^* + A_{12}A_{12}^*)(U_{11}^*)^{-1}D^{-1/2} \\ &\leq D^{-1/2}U_{11}^{-1}(U_{11}DU_{11}^*)(U_{11}^*)^{-1}D^{-1/2} = I_m. \end{aligned}$$

This shows that Z is a contraction and hence so is Q. □

As pointed out by the referee, from Theorem 2.3 one can deduce the following corollary, which can be viewed as a 2-variable generalization of the fact that every positive contraction can be dilated to an orthogonal projection; see [6, Problem 222(b)].

Corollary 2.4 *If $A \in M_n$ is the product of two positive contractions, then A can be dilated to a product of two projections on $\mathbb{C}^{n+2m}$, where m equals the number of eigenvalues of A which are not equal to 0 or 1.*

It is not easy to check the existence of the matrices $R, C \in M_m$ in condition (b), and the existence of U_{11} in condition (c) of Theorem 2.3. We refine condition (c) to get Theorem 2.5 below so that one can use computational techniques such as positive semidefinite programming or alternating projection methods to check the condition. In Section 3, we will develop Matlab programs using an alternating projection method based on Theorem 2.5 to check whether a matrix can be written as the product of two positive semidefinite contractions, and construct them if they exist.

Theorem 2.5 *Let $A \in M_n$ be unitarily similar to $I_p \oplus 0_q \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-q-m} \end{pmatrix}$, where $A_{11} \in M_m$ such that A_{11} is diagonalizable with distinct eigenvalues $\alpha_1 > \cdots > \alpha_k$ in $(0,1)$ with multiplicities*

$m_1, \ldots, m_k$, respectively. Suppose $V = [V_1 \ \cdots \ V_k] \in M_m$ is an invertible matrix such that the columns of the $n \times m_j$ matrix V_j form an orthonormal basis for the null space of $A_{11} - \alpha_j I_m$, for $j = 1, \ldots, k$, i.e., $A_{11}V = VD$, where $D = \alpha_1 I_{m_1} \oplus \cdots \oplus \alpha_k I_{m_k}$ and $V_j^* V_j = I_{m_j}$ for $j = 1, \ldots, k$. Then A is the product of two positive contractions if and only if there is a block diagonal matrix $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \in M_{m_1} \oplus \cdots \oplus M_{m_k}$ satisfying

$$D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} \geq \Gamma \geq V^*V. \tag{3}$$

Proof. Suppose $A_{11}V = VD$ as asserted. Then U satisfies $A_{11}U = UD$ if and only if $U = VL$ for some block matrix $L = L_1 \oplus \cdots \oplus L_k \in M_{m_1} \oplus \cdots \oplus M_{m_k}$. One readily checks that condition (c) in Theorem 2.3 reduces to the existence of $\Gamma = (LL^*)^{-1}$. □

By Theorem 2.5, we can deduce the following corollary. The first part of the corollary was obtained in [4, Lemma 2.1] by some rather involved arguments. The second part of the corollary is a proof of a comment in our introduction.

Corollary 2.6 *Let $A = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix}$ with $a, b \in [0, 1]$. Then A is the product of two positive contractions if and only if*

$$|p| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}. \tag{4}$$

Consequently, if $B \in M_n$ is similar to a diagonal matrix with nonzero eigenvalues $a, b \in (0, 1]$ then a necessary and sufficient condition for A to be the product of two positive contractions is:

$$\{\|B\|^2 - (a^2 + b^2) + (ab/\|B\|)^2\}^{1/2} \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}.$$

Proof. Case 1. $a = b$. If A is the product of two positive contractions, then A is similar to a diagonal matrix so that $p = 0$, and inequality (4) holds. If inequality (4) holds, then $p = 0$, and $A = aI_2$ is the product of positive contractions I_2 and aI_2.

Case 2. $a \neq b$. We focus on the non-trivial case that $a, b \in (0, 1)$, $a \neq b$ and $p \neq 0$. One sees that V in Theorem 2.5 can be chosen to be $\begin{pmatrix} 1 & p/\gamma \\ 0 & (b-a)/\gamma \end{pmatrix}$ with $\gamma = \sqrt{(a-b)^2 + p^2}$ so that up to diagonal congruence we have

$$V^*V = \begin{pmatrix} 1 & p/\gamma \\ p/\gamma & 1 \end{pmatrix}.$$

We need to find a diagonal matrix $\Gamma = \mathrm{diag}\,(d_1, d_2)$ with $d_1, d_2 \geq 0$ such that $\Gamma - V^*V \geq 0$ and $VV^* - \mathrm{diag}\,(ad_1, bd_2) \geq 0$. Thus, we want

$$(d_1 - 1)(d_2 - 1) \geq p^2/\gamma^2, \qquad (1 - d_1 a)(1 - d_2 b) \geq p^2/\gamma^2.$$

We consider the maximum values for

$$f(d_1, d_2) = (d_1 - 1)(d_2 - 1)$$

subject to the condition of

$$g(d_1, d_2) = (d_1 - 1)(d_2 - 1) - (1 - d_1 a)(1 - d_2 b) = 0.$$

Consider the Lagrangian function $L(d_1, d_2, \mu) = f(d_1, d_2) - \mu g(d_1, d_2)$.

$$0 = L_{d_1}(d_1, d_2, \mu) = (d_2 - 1) - \mu[(d_2 - 1) + a(1 - d_2 b)]$$

and

$$0 = L_{d_2}(d_1, d_2, \mu) = (d_1 - 1) - \mu[(d_1 - 1) + b(1 - d_1 a)].$$

Thus,

$$(1-\mu)^2(d_1 - 1)(d_2 - 1) = \mu^2 ab(1 - d_1 a)(1 - d_2 b).$$

Because $(d_1-1)(d_2-1) = (1-d_1a)(1-d_2b)$, we see that $(1-\mu)^2 = \mu^2 ab$, and thus, $\mu = (1+\sqrt{ab})^{-1}$. Here, we use the root satisfying $1 - \mu > 0$. Solving d_1 and d_2, we get

$$(d_1 - 1)(d_2 - 1) = (1 - a)(1 - b)/(1 + \sqrt{ab})^2.$$

Furthermore, $(d_1 - 1)(d_2 - 1) \geq p^2/\gamma^2$ if and only if

$$p^2 \leq (a - b)^2(1 - a)(1 - b)/(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a} - \sqrt{b})^2(1 - a)(1 - b).$$

For the last assertion, note that if B satisfies the given assumption, then $(B - aI)(B - bI) = 0$, and B is unitarily similar to the direct sum of $aI_p \oplus bI_l$ and matrices of the form $B_j = \begin{pmatrix} a & p_j \\ 0 & b \end{pmatrix}$, where $p_1 \geq \cdots \geq p_k > 0$, for $j = 1, \ldots, k$. By Theorem 1.1 in [4], B is a product of two positive contractions if and only if

$$\|\mathrm{diag}\,(p_1, \ldots, p_k)\| = |p_1| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1 - a)(1 - b)}.$$

It is easy to check that $\|B\| = \|B_1\|$ and

$$\|B_1\|^2 + (ab/\|B_1\|)^2 - (a^2 + b^2) = \mathrm{tr}\,(B_1^* B_1) - (a^2 + b^2) = p_1^2.$$

The assertion follows. $\square$

3 Alternating projections and numerical examples

In Theorem 2.5, if A_{11} has distinct eigenvalues, then one only needs to search for a diagonal matrix satisfying the condition. However, there is no guarantee that there is a diagonal matrix Γ satisfying the condition in general as shown in the following example.

Example 3.1 Let $D = \mathrm{diag}\,(0.15, 0.15, 0.2)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ 0_3 & 0_3 \end{pmatrix}$ with

$$A_{11} = \begin{pmatrix} 0.1500 & 0 & 0 \\ 0 & 0.1500 & 0.0375 \\ 0 & 0 & 0.2000 \end{pmatrix},$$

and

$$A_{12} = \{UDU^* - A_{11}A_{11}^*\}^{1/2} = \begin{pmatrix} 0.3571 & 0 & 0 \\ 0 & 0.3215 & 0.1070 \\ 0 & 0.1070 & 0.1689 \end{pmatrix},$$

where

$$U = VR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/\sqrt{40} & 3/\sqrt{40} \\ 0 & 0 & 4/\sqrt{40} \end{pmatrix},$$

with

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 3/5 \\ 0 & 0 & 4/5 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/\sqrt{40} & 0 \\ 0 & 0 & 5/\sqrt{40} \end{pmatrix}.$$

Then $A_{11}V = VD$, $A_{11}U = UD$, and U is a contraction such that $UDU^* = A_{11}A_{11}^* + A_{12}A_{12}^*$. There is no $\Gamma = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$ such that

$$M = D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} = \begin{pmatrix} 1.3 & -0.3 & 0 \\ -0.3 & 1.3 & 0 \\ 0 & 0 & 1.6 \end{pmatrix} \geq \Gamma$$

and

$$\Gamma \geq V^*V = \begin{pmatrix} 1.0000 & 0 & 0.4243 \\ 0 & 1.0000 & -0.4243 \\ 0.4243 & -0.4243 & 1.000 \end{pmatrix}$$

because $\mu_1, \mu_2 \in (1, 1.3)$ so that the leading 2×2 principal submatrix $M - \Gamma$ cannot be positive semidefinite. Hence, A is not the product of two positive contractions. □

By Theorem 2.5, one can use positive semidefinite (PSD) programming to check whether there exists Γ satisfying (3). However, standard PSD programming uses dual program to check the feasibility, and does not seem to be effective in checking the result. For example, we use the the SDP mode of cvx program from `http://cvxr.com/cvx/`, and it fails to detect the result even for $A \in M_2$.

We turn to alternating projection method; for example see [7]. Suppose $A \in M_n$ is a contraction matrix unitarily similar to $I_p \oplus 0_q \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-q-m} \end{pmatrix}$ and $V \in M_m$ is an invertible matrix with unit columns $v_1, \ldots, v_m$ satisfying $A_{11}V = VD$ with $D = \alpha_1 I_{m_1} \oplus \cdots \oplus \alpha_k I_{m_k}$ with $\alpha_1 > \cdots > \alpha_k > 0$ the distinct eigenvalues of A_{11}. Let

$$\Omega_0 = \{\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \in M_{m_1} \oplus \cdots \oplus M_{m_k} : \Gamma \text{ is positive semidefinite}\},$$

$$\Omega_1 = \{\Gamma \in M_m : D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} \geq \Gamma \geq 0\},$$

and

$$\Omega_2 = \{\Gamma \in M_m : \Gamma \geq V^*V\}.$$

The following proposition can be readily verified. Here we use the notation X^+ for the positive semidefinite part of a Hermitian matrix X, i.e., $X^+ = (X + \sqrt{X^2})/2$.

Proposition 3.2 *Let $G = [G_{ij}]$ be a Hermitian matrix, where $G_{ii} \in M_{m_i}$.*

1. *The projection of G onto Ω_0 is $G_{11}^+ \oplus \cdots \oplus G_{kk}^+$.*
2. *The projection of G onto Ω_1 is $M - (M-G)^+$, where $M = D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2}$.*
3. *The projection of G onto Ω_2 is $(G - V^*V)^+ + V^*V$.*

In the following algorithm, we create a sequence

$$\Gamma_0 \longrightarrow \hat{\Gamma}_1 \longrightarrow \Gamma_1 \longrightarrow \hat{\Gamma}_2 \longrightarrow \Gamma_2 \longrightarrow \cdots$$

where $\Gamma_k \in \Omega_0$, $\hat{\Gamma}_{2k-1} \in \Omega_1$ and $\hat{\Gamma}_{2k} \in \Omega_2$ for all $k \geq 1$. This sequence converges to a solution $\Gamma \in \Omega_0 \cap \Omega_1 \cap \Omega_2$, provided $\Omega_0 \cap \Omega_1 \cap \Omega_2 \neq \emptyset$; see [8].

Algorithm 3.3 *For checking the existence of* $\Gamma \in \Omega_0 \cap \Omega_1 \cap \Omega_2$.

Step 0. Set $k=0$. *Let* $X = D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2}$ *and* $Y = V^*V$.

Partition X *into* $[X_{ij}]$ *and* Y *into* $[Y_{ij}]$, *both conformed to* D.

Set $\Gamma_0 = \frac{1}{2}\Big((X_{11}+Y_{11}) \oplus \cdots \oplus (X_{kk}+Y_{kk})\Big)$. *Go to Step 1.*

Step 1. Change k *to* $k+1$, *and set*

$$\hat{\Gamma}_k = \begin{cases} X - (X-\Gamma_{k-1})^+ & \text{if } k \text{ is odd,} \\ (\Gamma_{k-1} - Y)^+ + Y & \text{if } k \text{ is even,} \end{cases}$$

where M_+ *denotes the positive part of* M.

Partition $\hat{\Gamma}_k$ *into* $[G_{ij}]$ *conformed to* D *and let* $\Gamma_k = G_{11}^+ \oplus \cdots \oplus G_{kk}^+$.

If error $= \max(0, -\lambda_{\min}(\Gamma_k - Y)) + \max(0, -\lambda_{\min}(X-\Gamma_k)) \approx 0$, *stop.*

Otherwise, go to step 1.

Once we have Γ, we can set $U = V\Gamma^{-1/2}$, and construct the two projections as shown in Theorem 2.3. In particular, we can set $A = (I_p \oplus P)(I_p \oplus Q)$ with

$$P = \begin{pmatrix} UU^* & 0 \\ 0 & 0_{n-p-m} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} (U^*)^{-1}DU^{-1} & (UU^*)^{-1}A_{12} \\ A_{12}^*(UU^*)^{-1} & A_{12}^*(UDU^*)^{-1}A_{12} \end{pmatrix}. \tag{5}$$

We illustrate our Matlab program (available at http://cklixx.wm.edu/mathlib/Twoposcon.txt) for checking whether a given matrix $A \in M_n$ is the product of two positive contractions in the following. Note that all numerical experiments were performed using Matlab 2015a on a Intel(R) Core(TM) i7-5500U CPU 2.4GHz with 8GB RAM and a 64-bit OS.

Example 3.4 *Suppose* $A = \begin{bmatrix} A_{11} & A_{12} \\ 0_5 & 0_5 \end{bmatrix}$, *where*

$$A_{11} = \begin{bmatrix} 0.125 & 0.0126 & 0.0033 & 0.024 & -0.0006 \\ 0 & 0.0625 & 0 & 0.012 & 0.0152 \\ 0 & 0 & 0.0625 & 0.0025 & 0.0453 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.2 \end{bmatrix} \text{ and } A_{12} = \begin{bmatrix} 0.0658 & 0.0218 & 0.0031 & 0.05 & -0.0033 \\ 0.0218 & 0.113 & -0.0107 & -0.0120 & 0.0098 \\ 0.0031 & -0.0107 & 0.0418 & 0.0048 & -0.0409 \\ 0.0500 & -0.012 & 0.0048 & 0.1103 & 0.0037 \\ -0.0033 & 0.0098 & -0.0409 & 0.0037 & 0.128 \end{bmatrix}.$$

We set

$$V \approx \begin{bmatrix} 1 & -0.1976 & -0.0507 & -0.3169 & -0.0169 \\ 0 & 0.9803 & -0.0102 & -0.0824 & -0.1026 \\ 0 & 0 & 0.9987 & -0.0172 & -0.3108 \\ 0 & 0 & 0 & -0.9447 & 0.0203 \\ 0 & 0 & 0 & 0 & -0.9445 \end{bmatrix},$$

which has unit columns and satisfies $A_{11}V = V\mathrm{diag}(0.125, 0.0625, 0.0625, 0.2, 0.2)$; *the second and third columns of* V *are orthogonal and the fourth and fifth columns are orthogonal.*

Using our Matlab program, we obtain $U = V\Gamma^{-\frac{1}{2}}$, where

$$\Gamma = \begin{bmatrix} 3.4737 & 0 & 0 & 0 & 0 \\ 0 & 2.3344 & 0.0216 & 0 & 0 \\ 0 & 0.0216 & 2.9472 & 0 & 0 \\ 0 & 0 & 0 & 2.1257 & -0.2132 \\ 0 & 0 & 0 & -0.2132 & 1.6425 \end{bmatrix}.$$

Defining P and Q as in equation (5), we get that $\lambda_1(P) = s_1^2(U) = 0.7024$ and $\lambda_1(Q) = 1$. Note that Γ is obtained using alternating projection method after 79 iterations done in approximately 0.085 seconds with *errror* $= ||PQ - A|| = 4.3774 \times 10^{-14}$.

Example 3.5 *Suppose*

$$A_{11} = \begin{bmatrix} 0.1 & 0.0244 & 0.026 & 0.0167 & 0.0114 & 0.0014 & 0.0674 \\ 0 & 0.2 & 0.0176 & 0.0251 & 0.0345 & 0.0122 & 0.0088 \\ 0 & 0 & 0.3 & 0 & 0.0072 & 0.0119 & 0.0166 \\ 0 & 0 & 0 & 0.3 & 0.0093 & 0.0007 & 0.0099 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 \end{bmatrix}$$

and

$$A_{12} = \begin{bmatrix} 0.098 & 0.0157 & -0.0315 & 0.0033 & -0.04 & -0.0196 & 0.0171 \\ 0.0157 & 0.0545 & -0.0366 & 0.0302 & 0.0081 & 0.0003 & 0.004 \\ -0.0315 & -0.0366 & 0.1246 & -0.0449 & -0.0005 & 0.0232 & -0.0047 \\ 0.0033 & 0.0302 & -0.0449 & 0.1025 & -0.0193 & -0.031 & 0.0191 \\ -0.04 & 0.0081 & -0.0005 & -0.0193 & 0.1285 & 0.0038 & -0.0504 \\ -0.0196 & 0.0003 & 0.0232 & -0.031 & 0.0038 & 0.07790 & -0.0192 \\ 0.0171 & 0.004 & -0.0047 & 0.0191 & -0.0504 & -0.0192 & 0.0895 \end{bmatrix}.$$

We let

$$V = \begin{bmatrix} 1 & -0.2373 & -0.1475 & -0.1015 & -0.0632 & -0.0196 & -0.2348 \\ 0 & -0.9714 & -0.1713 & -0.2329 & -0.1858 & -0.0673 & -0.0569 \\ 0 & 0 & -0.9741 & 0.0563 & -0.0702 & -0.1162 & -0.1512 \\ 0 & 0 & 0 & -0.9656 & -0.0910 & -0.0052 & -0.0896 \\ 0 & 0 & 0 & 0 & -0.9738 & 0.023 & 0.0454 \\ 0 & 0 & 0 & 0 & 0 & -0.9905 & 0.0278 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.9528 \end{bmatrix}.$$

Using our Matlab program, we obtain

$$\Gamma = [2.9099] \oplus [2.592] \oplus \begin{bmatrix} 1.9048 & 0.1063 \\ 0.1063 & 1.866 \end{bmatrix} \oplus \begin{bmatrix} 1.6447 & 0.0046 & 0.0768 \\ 0.0046 & 1.6923 & 0.0215 \\ 0.0768 & 0.0215 & 1.5846 \end{bmatrix}$$

after 59 iterations (approximately 0.075 seconds) with a 1.227×10^{-16} error. The positive semidefinite matrices P and Q defined in equation (5) will have largest eigenvalues 0.8309 and 1, respectively.

Example 3.6 *Let*

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } A_{11} = \begin{bmatrix} 0.5 & 0.09429 \\ 0 & 0.3 \end{bmatrix} \text{ and } \quad B_{11} = \begin{bmatrix} 0.5 & 0.0943 \\ 0 & 0.3 \end{bmatrix}.$$

It follows from [4] that A is a product of two contractions and B is not. Notice that A and B are very close to each other.

For A, we ran the alternating projection algorithm and obtained $\Gamma = \mathrm{diag}(1.2759, 1.6591)$ after 66321 iterations (48.26 seconds). We also get $||PQ - A|| \approx 1.4778 \times 10^{-16}$ and $\lambda_1(Q), \lambda_1(P) \approx 1$. Meanwhile, for B, after running 100,000 iterations (69.06 seconds) of the algorithm, we see that the values $\max(0, -\min(\mathrm{eig}(M - \Gamma)))$ and $\max(0, -\min(\mathrm{eig}(\Gamma - V^*V)))$ starts to alternate back and forth from 8.5×10^{-5} to 8.52925×10^{-5}.

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