

5-2010

Preservers of Eigenvalue Inclusion Sets

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Preservers of Eigenvalue Inclusion Sets


A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelors of Science in Mathematics from
The College of William and Mary

by

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Accepted for

Honors



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April 20, 2010

Abstract

Denote by M_n the set of $n \times n$ complex matrices. Let $\mathcal{S}(A)$ be one of the following: the Gershgorin Set, Braeur's Set of Cassini Ovals, and the Ostrowki set. Characterization is obtained for maps $\Phi : M_n \rightarrow M_n$ such that $\mathcal{S}(\Phi(A) - \Phi(B)) = \mathcal{S}(A - B)$ for all $A, B \in M_n$. It is shown that such maps can be decomposed as the composition of two or three simple maps such as permutation similarity transforms, permutation of off-diagonal entries in each row, and norm preserving maps on individual diagonal entries. Similar results can be obtained for maps $\Phi : M_n \rightarrow M_n$ satisfying $\mathcal{S}(\Phi(A) + \Phi(B)) = \mathcal{S}(A + B)$ for all $A, B \in M_n$. Moreover, using these results, one can readily determine the structure of additive maps and linear maps $\Psi : M_n \rightarrow M_n$ satisfying $\mathcal{S}(\Psi(A)) = \mathcal{S}(A)$ for all $A \in M_n$. Characterization is also obtained for maps $\Phi : M_n \rightarrow M_n$ satisfying $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all $A, B \in M_n$. In most cases, the maps are composition of similarity transformations by permutation and diagonal unitary matrices followed by a multiplication by a scalar $\mu \in \{1, -1\}$. Related results and problems are also described.

Acknowledgements

We thank members of the Honors Thesis Examining Committee: David Armstrong, Robert Lewis, and Chi-Kwong Li. We also thank the coauthors of the papers upon which the content of the thesis is based: Jim Hartman, Virginia Forstall, Nung-Sing Sze, and Vincent Yannello. Special thanks is given to Chi-Kwong Li, the advisor on this project. Research was partially supported by NSF CSUMS grant DMS 0703532.

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Chapter 1

Introduction

This chapter contains an explanation of Eigenvalue Inclusion Sets and motivations behind their study, then discusses the field of preserver problems in matrix research, and ends with a description of our research and notation used.

1.1 Eigenvalue Inclusion Sets

An eigenvalue inclusion set is a region in the complex plane which gives an estimate of the location of a matrix's eigenvalues. The calculation of eigenvalues of a matrix is an operation whose computational cost increases rapidly with matrix size. In many applications of matrices, it is not practical or efficient to attempt an exact calculation of a matrix's eigenvalues. Instead, researchers use simpler estimation techniques to understand the eigenvalue structure of a matrix.

Consider an arbitrarily large square matrix with eigenvalue inclusion regions like those in the following two figures:

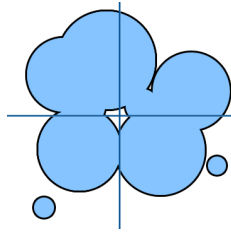


Figure 1.1: Inclusion Set for an Invertible Matrix.

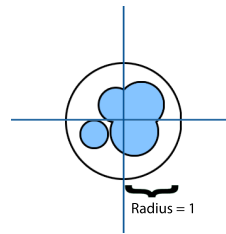


Figure 1.2: Inclusion Set for a Convergent Matrix.

As $\{0\}$ is not in the eigenvalue inclusion set shown in 1.1, any matrix with this inclusion region would be invertible. For a matrix with the inclusion set shown in 1.2, all eigenvalues lie in the unit circle of the complex plane and as such

$\lim_{k \rightarrow +\infty} A^k = 0$. In both examples, useful information about the eigenvalue structure of the matrices can be obtained without performing complicated eigenvalue calculations.

In this paper we study preservers of three Eigenvalue Inclusion Sets: the Gershgorin Set, Brauer's Set of Cassini Ovals, and the Ostrowski set. As the definition of the Ostrowski set will make clear, the Gershgorin set can be viewed as a specific case of the Ostrowski set.

The Gershgorin Set: For a matrix $A = (a_{ij}) \in M_n$ (the set of $n \times n$ complex matrices) the Gershgorin set is defined by:

$$G(A) = \cup_{k=1}^n G_k(A),$$

where for $k = 1, \dots, n$,

$$G_k(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k\} \quad \text{with} \quad R_k = \sum_{j \neq k} |a_{kj}|$$

is a *Gershgorin disc*. R_k is frequently referred to as the "deleted row sum" of row k .

For example, consider the following matrix:

$$A = \begin{pmatrix} 2 & \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & .5 \\ 2 & -1 + i & 0 \\ 0 & 1 & -1 + i \end{pmatrix} \quad \begin{array}{l} G_1(A) = \{\mu \in \mathbb{C} : |\mu - 2| \leq 1.5\}, \\ G_2(A) = \{\mu \in \mathbb{C} : |\mu - (-1 + i)| \leq 2\}, \\ G_3(A) = \{\mu \in \mathbb{C} : |\mu - (-1 + i)| \leq 1\}. \end{array}$$

This gives us an inclusion region that can be seen in the following figure:

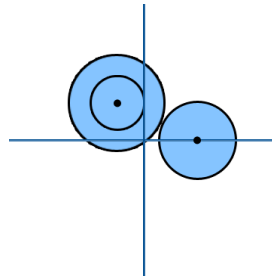


Figure 1.3: $G(A)$.

Brauer's Set of Cassini Ovals: For a matrix $A \in M_n$, Brauer's set is defined by:

$$C(A) = \cup_{1 \leq i < j \leq n} C_{ij}(A),$$

where

$$C_{ij}(A) = \{\mu \in \mathbb{C} : |(\mu - a_{ii})(\mu - a_{jj})| \leq R_i R_j\}$$

is a *Cassini oval* and R_i is defined as before.

Several typical structures of Cassini Ovals are depicted in Figure 1.4:



Figure 1.4: Cassini Oval Examples.

The following facts will be used in the discussion in Chapters 2 and 3:

(1) The set $C(A)$ for a matrix A consists of a collection of isolated points if and only if A has at most one row with nonzero off diagonal entries, in which case $C(A)$ coincides with the spectrum of A . For example, the matrix

$$A = \begin{pmatrix} 1 & 10 \\ 0 & 2 \end{pmatrix}$$

has $C(A) = \{1, 2\}$. This matrix illustrates how Brauer's set reveals more about a matrix's eigenvalue structure than the Gershgorin region. The eigenvalues of A are $\{1, 2\}$, but the Gershgorin region would consist of more than two singletons.

(2) Suppose a_{kk} is an isolated point of $C(A)$. Then either $C(A)$ is a collection of isolated points or the k th row of A has no off diagonal entries. For example, consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 10 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

Then $C(A)$ has an isolated point at $\{10\}$.

The Ostrowski Set: For a matrix $A \in M_n$, Ostrowski's set is defined by:

$$O_\varepsilon(A) = \cup_{k=1}^n O_{\varepsilon,k}(A),$$

for a given $\varepsilon \in (0, 1)$, where

$$O_{\varepsilon,k}(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k(A)^\varepsilon R_k(A^t)^{1-\varepsilon}\}, \quad k \in \{1, \dots, n\}.$$

$R_k(A^t)$ will frequently be referred to as the deleted column sum of the k th row of A . In some cases, this inclusion set can more about the eigenvalue structure of a matrix than the previous sets, as can be seen with the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The only eigenvalue of A is 0. Both the Brauer's Set and the Gershgorin Set of this matrix would include values other than 0, but as no diagonal entry has an associated nonzero off diagonal deleted row sum and deleted column sum, $O_\varepsilon(A) = \{0\}$.

1.2 Preserver Problems

Matrix preserver problems concern the characterization of maps on matrices leaving invariant a certain function, a certain subset, or a certain relation. Past study has focussed on linear maps with these properties. This class of problems has been and continues to be an active area of research interest, with results dating back to Frobenius's 1897 result for the determinant preserver.

Determinant Preserver (Frobenius[1]): A linear operator $\Phi : M_n \mapsto M_n$ satisfies $\det(A) = \det(\Phi(A))$ for all $A \in M_n$ if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that Φ has the form

$$A \mapsto MAN \text{ or } A \mapsto MA^tN.$$

Spectrum Preserver (Marcus and Purves [2]): A linear map $\Phi : M_n \mapsto M_n$ satisfies $\sigma(\Phi(A)) = \sigma(A)$, where $\sigma(A)$ is the spectrum of A , if and only if there exists an invertible $S \in M_n$ such that either

$$\Phi(A) = SAS^{-1} \text{ for all } A \in M_n, \text{ or } \Phi(A) = SA^tS^{-1} \text{ for all } A \in M_n.$$

See [3] for further discussion and examples of linear preserver problems.

More recently, researchers have studied preserver problems under milder constraints. In particular, for a given function f on a matrix set M with a binary operator $A \circ B$, maps $\Phi : M \mapsto M$ have been studied that satisfy $f(\Phi(A) \circ \Phi(B)) = f(A \circ B)$ for all $A, B \in M$. The following result concerns the spectral radius of matrices and M_n^+ , the set of $n \times n$ real matrices with nonnegative entries.

Spectral Radius Preserver of Nonnegative Matrices

(Clark, Li, Rodman[4]): A surjective mapping $\Phi : M_n^+ \mapsto M_n^+$ satisfies $r(AB) = r(\Phi(A)\Phi(B))$ for all $A, B \in M_n^+$, where $r(A)$ is the spectral radius of A , if and only if there exists a permutation matrix P and diagonal matrix D with positive entries such that Φ has the form

$$\Phi(A) = D^{-1}P^tAPD \text{ for all } A \in M_n^+ \text{ or } \Phi(A) = D^{-1}P^tA^tPD \text{ for all } A \in M_n^+.$$

1.3 Our Study

In our study, we study preservers of an eigenvalue inclusion set $\mathcal{S}(A)$ for a matrix $A \in M_n$. If one just assumes that a map Φ satisfies $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix A , the structure of Φ can be quite arbitrary. For instance, one can partition the set of matrices into equivalence classes so that two matrices A and B belong to the same class if $\mathcal{S}(A) = \mathcal{S}(B)$. If Φ sends each of these classes back to itself, then Φ satisfies $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix A . So, it is reasonable to impose some condition on the map Φ relating the eigenvalue value inclusion sets of a pair of matrices.

In Chapter 2, characterizations are obtained for maps Φ satisfying $\mathcal{S}(A - B) = \mathcal{S}(\Phi(A) - \Phi(B))$ for any $A, B \in M_n$. It is shown that such maps have tractable structure. Similar results can be obtained for maps Φ satisfying $\mathcal{S}(A + B) = \mathcal{S}(\Phi(A) + \Phi(B))$ for all matrices $A, B \in M_n$. In applications, one often needs to consider the product or powers of matrices, and estimate their eigenvalues. This observation leads to further study involving multiplicative maps. In Chapter 3, we characterize maps Φ satisfying $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for any two matrices A and B . Different techniques are utilized to characterize multiplicative maps than those difference mappings. A crucial step used in our study of multiplicative maps is to extract information of the eigenvalues of $\Phi(A)$ using $\mathcal{S}(A)$ and $\mathcal{S}(A^2) = \mathcal{S}(\Phi(A)^2)$. To achieve this, we use a basic result in matrix theory (see [5, Theorem 3.2.4.2]).

1.4 Notations

The following notation and definitions will be used in our discussion.

M_n : the set of $n \times n$ complex matrices.

$\{E_{11}, E_{12}, \dots, E_{nn}\}$: the standard basis for M_n .

$\{e_1, \dots, e_n\}$: the standard basis for \mathbb{C}^n .

\mathbf{P}_n : the set of $n \times n$ permutation matrices.

$\mathbf{GP}_n = \{DP : D \in \mathbf{DU}_n, P \in \mathbf{P}_n\}$: the group of generalized permutation matrices in M_n , where \mathbf{DU}_n is the set of diagonal unitary matrices.

$D(a, r) = \{\mu \in \mathbb{C} : |\mu - a| \leq r\}$, where $a \in \mathbb{C}$ and $r \geq 0$.

\oplus : The direct sum of matrices, i.e. $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

$\text{diag}(x_1, x_2, \dots, x_n)$: the diagonal matrix (d_{ij}) with diagonal entries $d_{ii} = x_i$.

Chapter 2

Difference of Matrices

In this chapter, characterization is obtained for maps Φ on $n \times n$ matrices satisfying $\mathcal{S}(\Phi(A) - \Phi(B)) = \mathcal{S}(A - B)$ for all matrices A and B , \mathcal{S} is the Gershgorin region, the Brauer region, or the Ostrowski region.¹ From these results, one can deduce the structure of additive or (real) linear maps satisfying $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix A . We use the following strategy to show that the preservers have a certain asserted form. First, we focus on a small class \mathcal{R} of matrices in M_n and show that $\Phi(X)$ has the form asserted in the result. Then replace Φ by $\tilde{\Phi}$ so that $\tilde{\Phi}$ still has the preserving properties and $\tilde{\Phi}(X) = X$ for $X \in \mathcal{R}$. Then we show that $\tilde{\Phi}(X)$ has the asserted form for a wider class of matrices $\tilde{\mathcal{R}}$. We then modify $\tilde{\Phi}$ again and extend the conclusion to an even wider class of matrices. After several rounds of iterations, we show that the modified map fixed every matrix in M_n , and the modifications can be represented by the composition of some simple maps as asserted.

2.1 Gershgorin Sets

In this section we consider preservers of the Gershgorin Set.

Theorem 2.1.1 *A map $\Phi : M_n \rightarrow M_n$ satisfies*

$$G(\Phi(A) - \Phi(B)) = G(A - B) \quad \text{for all } A, B \in M_n \quad (2.1)$$

if and only if Φ is a composition of three maps of the following forms:

$$(1) \quad A \mapsto \begin{pmatrix} A_1 Q_1 \\ \vdots \\ A_n Q_n \end{pmatrix} \quad \text{for } A \in M_n \text{ with rows } A_1, \dots, A_n, \text{ where } Q_1, \dots, Q_n \in \mathbf{GP}_n$$

are such that the (j, j) entry of Q_j is 1 for $j = 1, \dots, n$.

¹The material of this chapter is included in [6]

(2) $A \mapsto PAP^t + S$, where $P \in \mathbf{P}_n$ and $S \in M_n$.

(3) $A = (a_{rs}) \mapsto (\psi_{rs}(a_{rs}))$, where ψ_{rs} has the form $z \mapsto z$ or $z \mapsto \bar{z}$ such that ψ_{rs} is the identity map if $r = s$.

It is easy to verify that maps of the form (1), (2), and (3) in the theorem indeed satisfy (2.1.1). Hence the sufficiency of the theorem is clear. To prove the necessity part of the theorem, we need some lemmas.

Lemma 2.1.2 *Let $w = e^{i2\pi/n}$ and*

$$\Gamma = \left\{ \frac{1 - w^k}{1 - w^j} : j \in \{1, \dots, n-1\}, k \in \{1, \dots, n\} \right\}. \quad (2.2)$$

Suppose $\mu \in \mathbf{C} \setminus \Gamma$. Then $R \in \mathbf{P}_n$ is such that the set of entries of the vector

$$\mu(w, w^2, \dots, w^n) - (w, w^2, \dots, w^n)R$$

equals $\{(\mu - 1)w^j : 1 \leq j \leq n\}$ if and only if $R = I_n$.

Proof. The sufficiency is clear. To verify the necessity part, assume that the k th entry of the vector $\mu(w, w^2, \dots, w^n) - (w, w^2, \dots, w^n)R$ equals $\mu w^k - w^i$ with $k \neq i$. Then $\mu w^k - w^i = \mu w^j - w^j$ for some $j \in \{1, \dots, n\}$ so that $\mu(w^j - w^k) = w^j - w^i$. If $j = i$, then $j \neq k$ and hence $\mu = 0 = (1 - w^n)/(1 - w)$; if $j \neq i$, then $\mu = (1 - w^{k-j})/(1 - w^{i-j})$. This contradicts the fact that $\mu \notin \Gamma$. \square

The next lemma is standard in the study of distance preserving maps. We include a proof for completeness.

Lemma 2.1.3 *Let $\mathbf{V} = \mathbf{C}^{1 \times m}$. A map $f : \mathbf{V} \rightarrow \mathbf{V}$ satisfies $\ell_1(f(x) - f(y)) = \ell_1(x - y)$ for all $x, y \in \mathbf{V}$, where ℓ_1 is the norm defined by $\ell_1(x) = \sum_{i=1}^n |x_i|$, if and only if there is $z \in \mathbf{V}$ and $Q \in \mathbf{GP}_n$ such that f has the form*

$$(x_1, \dots, x_m) \mapsto (f_1(x_1), \dots, f_m(x_m))Q + z,$$

where f_j is either the identity map $\mu \mapsto \mu$ or the complex conjugation $\mu \mapsto \bar{\mu}$.

Proof. The sufficiency is clear. We consider the converse. We divide the proof into two assertions.

Assertion 1 Let $\mathcal{B} = \{x \in \mathbf{V} : \ell_1(x) \leq 1\}$ and

$$\mathcal{E} = \{x \in \mathcal{B} : x \neq (u + v)/2 \text{ with different } u, v \in \mathcal{B}\}$$

be the set of extreme points of \mathcal{B} . Then $x \in \mathcal{E}$ if and only if x has only one nonzero entry with modulus 1.

Proof. Let $x = (x_1, \dots, x_m) \in \mathcal{B}$. We consider three possible cases.

Case 1 Suppose x has a single nonzero entry at the j th position with $|x_j| = 1$. If $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathcal{B}$ satisfy $x = (u+v)/2$, then $x_j = (u_j+v_j)/2$. Since $|x_j| = 1 \geq |u_j|, |v_j|$, we see that $u_j = v_j = x_j$. Because $u, v \in \mathcal{B}$, all other entries of u and v must be zero. Thus, $x = u = v$. Hence $x \in \mathcal{E}$.

Case 2 Suppose x has a single nonzero entry at the j th position such that $|x_j| < 1$. Then there exists a $\delta \neq 0$ such that $|x_j + \delta|, |x_j - \delta| \leq 1$. We can then let $u = (0, \dots, 0, x_j + \delta, 0, \dots, 0), v = (0, \dots, 0, x_j - \delta, 0, \dots, 0) \in \mathcal{B}$ such that $x = (u + v)/2$. Thus, $x \notin \mathcal{E}$.

Case 3 Suppose $x \in \mathcal{B}$ has at least two nonzero entries, say, at the j th and k th positions such that $x_j = \rho_j e^{it_j}, x_k = \rho_k e^{it_k}$ and $\rho_j, \rho_k \in (0, 1)$, where $\rho_j + \rho_k \leq 1$. Then there exists $u = (u_1, \dots, u_m)$ with two nonzero entries, namely, $u_j = \delta e^{it_j}$ and $u_k = -\delta e^{it_k}$ with $\min\{\rho_j, \rho_k\} > \delta > 0$ so that $x + u, x - u \in \mathcal{B}$ are different vectors and $x = [(x + u) + (x - u)]/2 \notin \mathcal{E}$.

Combining these three cases we see that $x \in \mathcal{E}$ if and only if x has only one nonzero entry with modulus 1.

Assertion 2 The map $g(x) = f(x) - f(0)$ is real linear and satisfies $g(\mathcal{E}) = \mathcal{E}$, and f has the asserted form with $f(0) = z$.

Proof. By the result in [7], we know that $g(x) = f(x) - f(0)$ is real linear. Note that $g(x) = 0$ if and only if $0 = \ell_1(g(x)) = \ell_1(x)$, i.e., $x = 0$. Hence g is a real linear injective map, and therefore is bijective. As a result, $g(\mathcal{B}) = \mathcal{B}$. Clearly, $x = (u + v)/2$ for distinct $u, v \in \mathcal{B}$ if and only if $g(x) = (g(u) + g(v))/2$ with distinct $g(u), g(v) \in \mathcal{B}$. Thus, $g(\mathcal{E}) = \mathcal{E}$.

Let $\{e_1, e_2, \dots, e_m\}$ be the standard basis for $\mathbf{C}^{1 \times m}$. Because $g(\mathcal{E}) = \mathcal{E}$, we see that $g(e_j) = \mu_j e_{r_j}$ for some $r_j \in \{1, \dots, m\}$ and $\mu_j \in \mathbf{C}$ with $|\mu_j| = 1$. Since $\ell_1(e_j + \gamma e_k) = \ell_1(g(e_j) + \gamma g(e_k))$ for all $\gamma \in \mathbf{R}$ and $j \neq k$, we see that (r_1, \dots, r_m) is a permutation of $(1, \dots, m)$. Now, $g(i e_j) = \nu_j e_{s_j}$ for some $s_j \in \{1, \dots, m\}$ and $\nu_j \in \mathbf{C}$ with $|\nu_j| = 1$. Since $\ell_1(e_j + \gamma i e_j) = \ell_1(g(e_j) + \gamma g(i e_j))$ for all $\gamma \in \mathbf{R}$, we see that $s_j = r_j$ and $\nu_j \in \{i\mu_j, -i\mu_j\}$. Thus, g has the form $(x_1, \dots, x_m) \mapsto (f_1(x_1), \dots, f_m(x_m))Q$, where f_1, \dots, f_m and Q satisfy the conclusion of the lemma. Thus, f has the asserted form. \square

Proof of Theorem 2.1.1. The sufficiency is clear, as remarked before. We consider the necessity. Replacing Φ by the map $X \mapsto \Phi(X) - \Phi(0)$, we may assume that $\Phi(0) = 0$ and $G(\Phi(X)) = \Phi(X)$ for all $X \in M_n$ in addition to the assumption that $G(\Phi(A) - \Phi(B)) = G(A - B)$ for all $A, B \in M_n$. Let Γ be defined as in Lemma 2.1.2.

Assertion 1 Let $D = \text{diag}(w, w^2, \dots, w^{n-1}, 1)$ for $w = e^{i2\pi/n}$. There is a permutation matrix P such that $P\Phi(D)P^t = D$. Furthermore, if $\mu \in \mathbf{C} \setminus \Gamma$, then $P\Phi(\mu D)P^t = \mu D$.

To verify the assertion, note that $G(\Phi(\mu D)) = G(\mu D) = \{\mu w^j : 1 \leq j \leq n\}$. Hence, there is a permutation matrix P such that $P\Phi(D)P^t = D$. We may assume that $\Phi(D) = D$. Otherwise, replace Φ by the map $A \mapsto P^t\Phi(A)P$. Suppose $\mu \in \mathbf{C} \setminus \Gamma$. Then

$$G(\Phi(\mu D) - \Phi(D)) = G(\mu D - D) = \{(\mu - 1)w^j : j = 1, \dots, n\}.$$

Thus, the vector of diagonal entries of $\Phi(\mu D) - \Phi(D)$ equals

$$\mu(w, w^2, \dots, w^n) - (w, w^2, \dots, w^n)R$$

for some $R \in \mathbf{P}_n$, and the entries constitute the set $\{(\mu - 1)w^j : 1 \leq j \leq n\}$. By Lemma 2.1.2, the (j, j) entry of $\Phi(\mu D)$ is μw^j for each $j = 1, \dots, n$, i.e., $\Phi(\mu D) = \mu D$.

Assertion 2 Assume that P in the conclusion in Assertion 1 is I_n . Then $\Phi(W_k) \subseteq W_k$ for $k = 1, \dots, n$, where

$$W_k = \left\{ \sum_{j \neq k} a_j E_{kj} : a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n \in \mathbf{C} \right\}.$$

Moreover, define $\Phi_k : \mathbf{C}^{1 \times (n-1)} \rightarrow \mathbf{C}^{1 \times (n-1)}$ by

$$\Phi_k(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n) = (b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n)$$

if $\Phi(\sum_{j \neq k} a_j E_{kj}) = \sum_{j \neq k} b_j E_{kj}$. Then Φ_k satisfies the conclusion of Lemma 2.1.3 with $z = 0$.

To prove the assertion, let $A = \sum_{j \neq k} a_j E_{kj} \in W_k$ and $\Phi(A) = B = (b_{ij})$. Let $\mu \in (0, \infty) \setminus \Gamma$ satisfy $\mu|1 - w| > \sum_{j \neq k} |a_{kj}| = R_k$. Then $\Phi(\mu D) = \mu D$ by Assertion 1. Moreover, since $G(B) = G(A) = \{z \in \mathbf{C} : |z| \leq R_k\}$,

$$\min\{\mu|w^i - w^j| : 1 \leq i < j \leq n\} = \mu|1 - w| > R_k \geq \max\{|b_{jj}| : 1 \leq j \leq n\}. \quad (2.3)$$

Thus,

$$\begin{aligned} G(\mu D - B) &= G(\Phi(\mu D) - \Phi(A)) = G(\mu D - A) \\ &= \{\mu w^j : j \neq k\} \cup \{\gamma : |\gamma - \mu w^k| \leq R_k\}. \end{aligned}$$

For $i = 1, \dots, n$, the (i, i) entry of $\mu D - B$ equals $\mu w^i - b_{ii}$ is the center of one of the disks in $G(\mu D - A)$. Thus, $\mu w^i - b_{ii} = \mu w^j$ implies that $\mu|w^i - w^j| = |b_{ii}| < \mu|1 - w|$ by (2.3). It follows that $i = j$ and $b_{ii} = 0$. Thus, $\mu D - B$ has diagonal entries $\mu w, \mu w^2, \dots, \mu w^{n-1}, \mu$. Since $G(\mu D - B) = G(\mu D - A)$, we see that only the k th row of B can have nonzero off diagonal entries, and $\sum_{j \neq k} |b_{kj}| = R_k$.

Now, define Φ_k as in the assertion. Since $G(\Phi(A_1) - \Phi(A_2)) = G(A_1 - A_2)$ for any $A_1, A_2 \in W_k$ and $\Phi(0) = 0$, we see that $\ell_1(\Phi_k(u) - \Phi_k(v)) = \ell_1(u - v)$ for all $u, v \in \mathbf{C}^{1 \times (n-1)}$. Thus, the last statement of the assertion follows.

Assertion 3 The map Φ has the asserted form.

By Assertion 2, we may compose the map Φ with maps of the form (1) – (3) described in the theorem and assume that

(I) $\Phi(\mu D) = \mu D$ whenever $\mu \in \mathbf{C} \setminus \Gamma$, and (II) $\Phi(X) = X$ whenever $X \in \cup_{k=1}^n W_k$.

Under conditions (I) and (II), we will show that $\Phi(A) = A$ for each $A \in M_n$. First, we consider the special case when $A = \mu D + B$ with $\mu \in \mathbf{C} \setminus \Gamma$ and $B \in W_k$ where $G(A)$ consists of n disjoint disks with at most one of them having positive radius. Suppose $\Phi(A) = C = (c_{ij})$. Let $\nu > 0$ be such that $\nu, \nu\mu \notin \Gamma$ and $G(\nu\mu D - A) = G(\mu(\nu - 1)D - B)$ consists of n connected components. Then $\Phi(\nu\mu D) = \nu\mu D$ by (I), and

$$G(\nu\mu D - C) = G(\Phi(\nu\mu D) - \Phi(A)) = G(\nu\mu D - A) = G(\mu(\nu - 1)D - B)$$

consists of n circular disks with only one of them having positive radius. Considering the centers of the n disks, we conclude that the diagonal entries of $\nu\mu D - C$ equal $\mu(\nu - 1)w^j$ for $j = 1, \dots, n$. Similarly, since $G(C) = G(A)$ consists of n disks, we see that C has diagonal entries lying in $\{\mu w^j : j = 1, \dots, n\}$. Suppose the $c_{jj} = \mu w^k$ such that $k \neq j$. Then the vector of diagonal entries of $\Phi(\nu\mu D) - \Phi(A)$ equals $\nu\mu(w, w^2, \dots, w^n) - \mu(w, w^2, \dots, w^n)R$ for some $R \in \mathbf{P}_n$, and has entries which constitute the set $\{\mu(\nu - 1)w^j : 1 \leq j \leq n\}$. It follows that the vector $\nu(w, w^2, \dots, w^n) - (w, w^2, \dots, w^n)R$ has entries in $\{(\nu - 1)w^j : 1 \leq j \leq n\}$. By Lemma 2.1.2, we see that $c_{jj} = \mu w^j$. Since $G(C) = G(A)$, we see that only the k th row of C can have off-diagonal entries. Since $G(\Phi(A) - \Phi(B)) = G(A - B) = \{\mu w^j : 1 \leq j \leq n\}$, we see that $\Phi(A) - \Phi(B)$ is a diagonal matrix, which is μD . It follows that $\Phi(A) = \mu D + \Phi(B) = \mu D + B$.

Next, we consider a general matrix $A = (a_{ij}) \in M_n$. Suppose $\Phi(A) = C = (c_{ij})$. Let $\mu \in (0, \infty) \setminus \Gamma$ be such that $G(\mu D - A)$ consists of n disjoint disks. Moreover, we can choose $\mu > 0$ such that $\mu|1 - w| > \sum_{j=1}^n (|a_{jj}| + |c_{jj}|)$ so that $\mu w^j - c_{jj} \neq \mu w^k - a_{kk}$ for any $j \neq k$. Since $\Phi(\mu D) = \mu D$, $G(\mu D - C) = G(\Phi(\mu D) - \Phi(A)) = G(\mu D - A)$. By our choice of μ , we see that $\mu w^j - c_{jj} = \mu w^j - a_{jj}$ so that $c_{jj} = a_{jj}$ for $j = 1, \dots, n$. Furthermore, since

$$\begin{aligned} G\left(\mu D + \sum_{j \neq k} a_{kj} E_{kj} - C\right) &= G\left(\Phi\left(\mu D + \sum_{j \neq k} a_{kj} E_{kj}\right) - \Phi(A)\right) \\ &= G\left(\mu D + \sum_{j \neq k} a_{kj} E_{kj} - A\right), \end{aligned}$$

we see that

$$G_k\left(\mu D + \sum_{j \neq k} a_{kj} E_{kj} - C\right) = \{\mu w^k - c_{kk}\}.$$

Thus, $a_{kj} = c_{kj}$ for all $j \neq k$. Since k is arbitrary, we see that $C = A$ as asserted. \square

Corollary 2.1.4 *Let $\Phi : M_n \rightarrow M_n$. The following conditions are equivalent.*

(a) *The map Φ is real linear and satisfies*

$$G(\Phi(A)) = G(A) \quad \text{for all } A \in M_n.$$

(b) *The map Φ is additive and satisfies*

$$G(\Phi(A)) = G(A) \quad \text{for all } A \in M_n.$$

(c) *The map Φ has the form in Theorem 2.1.1 with $S = 0$.*

Proof. The implications (c) \Rightarrow (a) \Rightarrow (b) are clear. We focus on (b) \Rightarrow (c). Clearly, $X \in M_n$ satisfies $G(X) = \{0\}$ if and only if $X = 0$ and thus $\Phi(X) = 0$ if and only if $X = 0$ since $G(\Phi(A)) = G(A)$ for all $A \in M_n$. For any $B \in M_n$, since $G(\Phi(B) + \Phi(-B)) = G(\Phi(B - B)) = \{0\}$, it follows that $\Phi(-B) = -\Phi(B)$. Therefore, $G(\Phi(A - B)) = G(\Phi(A) + \Phi(-B)) = G(\Phi(A) - \Phi(B))$ for all $A, B \in M_n$. Thus Φ has the asserted form in Theorem 2.1.1. Since $\Phi(0) = \{0\}$, we see that $S = 0$ in case condition (2) in Theorem 2.1.1 holds. The conclusion follows. \square

Corollary 2.1.5 *A (complex) linear map $\Phi : M_n \rightarrow M_n$ satisfies*

$$G(\Phi(A)) = G(A) \quad \text{for all } A \in M_n$$

if and only if it is a composition of maps of the forms in (1) or (2) described in Theorem 2.1.1 with $S = 0$.

It is clear that one can have the column vector version of Gershgorin disks. This is the same as considering $G(A^t)$, and we can prove analogous results on preservers.

2.2 Brauer's Sets

In this section we consider preservers of Brauer's set.

Theorem 2.2.1 *A map $\Phi : M_n \rightarrow M_n$ satisfies*

$$C(\Phi(A) - \Phi(B)) = C(A - B) \quad \text{for all } A, B \in M_n$$

if and only if one of the following holds.

(a) $n = 2$ and Φ has the form

$$A \mapsto P \begin{pmatrix} a_{11} & u\tau_1(a_{12}) \\ v\tau_2(a_{21}) & a_{22} \end{pmatrix} P^t + S \quad \text{or} \quad A \mapsto P \begin{pmatrix} a_{11} & u\tau_1(a_{21}) \\ v\tau_2(a_{12}) & a_{22} \end{pmatrix} P^t + S,$$

where $S \in M_2$, $P \in \mathbf{P}_2$, $u, v \in \mathbf{C}$ satisfy $|uv| = 1$, and τ_j is the identity map or the conjugation map for $j = 1, 2$.

(b) Φ has the form described in Theorem 2.1.1.

Proof. The sufficiency is clear. We consider the necessity. We may replace Φ by the map $A \mapsto \Phi(A) - \Phi(0)$ and assume that $C(A) = C(\Phi(A))$ for any $A \in M_n$.

Case 1 Suppose $n = 2$. For any $\mu \in \mathbf{C}$, we have $C(\Phi(\mu E_{12})) = C(\mu E_{12}) = \{0\}$. It follows that $\Phi(\mu E_{12}) = f_1(\mu)E_{12} + f_2(\mu)E_{21}$ with $f_1(\mu)f_2(\mu) = 0$. Similarly, for any $\nu \in \mathbf{C}$, $C(\Phi(\nu E_{21})) = C(\nu E_{21}) = \{0\}$, so we see that $\Phi(\nu E_{21}) = g_1(\nu)E_{12} + g_2(\nu)E_{21}$ with $g_1(\nu)g_2(\nu) = 0$. Since $C(\Phi(\mu E_{12}) - \Phi(\nu E_{21})) = C(\mu E_{12} - \nu E_{21})$ is a circular disk centered at the origin with radius $\sqrt{|\mu\nu|}$, $\Phi(\mu E_{12}) \neq 0$ and $\Phi(\nu E_{21}) \neq 0$ whenever $\mu\nu \neq 0$. Now for any $\mu_1, \mu_2 \in \mathbf{C}$,

$$C(\Phi(\mu_1 E_{12}) - \Phi(\mu_2 E_{12})) = C(\mu_1 E_{12} - \mu_2 E_{12}) = \{0\}.$$

We see that

(1) $\Phi(\mu E_{12}) = f(\mu)E_{12}$ for all $\mu \in \mathbf{C}$, or (2) $\Phi(\mu E_{12}) = f(\mu)E_{21}$ for all $\mu \in \mathbf{C}$.

We may assume that (1) holds. Otherwise, replace Φ by the map $A \mapsto \Phi(A)^t$. It will then follow that $\Phi(\nu E_{21}) = g(\nu)E_{21}$ for all $\nu \in \mathbf{C}$.

Note that $C(\Phi(\mu E_{12}) - \Phi(\nu E_{21})) = C(\mu E_{12} - \nu E_{21}) = \{z \in \mathbf{C} : |z^2| \leq |\mu\nu|\}$. Thus, we see that $|f(\mu)g(\nu)| = |\mu\nu|$. In particular, $f(\mu) = 0$ if and only if $\mu = 0$, and $g(\nu) = 0$ if and only if $\nu = 0$.

Since $C(\Phi(E_{11})) = C(E_{11}) = \{1, 0\}$, we see that $\Phi(E_{11})$ has diagonal entries 1, 0 and at most one nonzero off diagonal entry. Since $C(\Phi(E_{11}) - \Phi(X)) = \Phi(E_{11} - X) = \{1, 0\}$ for $X \in \{E_{12}, E_{21}\}$, we see that $\Phi(E_{11}) = E_{11}$ or $\Phi(E_{11}) = E_{22}$. We may assume the former case holds. Otherwise, we may replace Φ by the map $A \mapsto P\Phi(A^t)P^t$ with $P = E_{12} + E_{21}$. Then we have $\Phi(E_{11}) = E_{11}$, $\Phi(\mu E_{12}) = f(\mu)E_{12}$ and $\Phi(\nu E_{21}) = g(\nu)E_{21}$. Now one may use the facts that $C(\Phi(\mu E_{11}) - \Phi(E_{11})) = \{\mu - 1, 0\}$, $C(\Phi(\mu E_{11}) - \Phi(E_{12})) = \{\mu, 0\}$, and $C(\Phi(\mu E_{11}) - \Phi(E_{21})) = \{\mu, 0\}$ to conclude that $\Phi(\mu E_{11}) = \mu E_{11}$. Similarly, we can argue that $\Phi(\mu E_{22}) = \mu E_{22}$.

Up to this point, we may assume that for $j \in \{1, 2\}$ and $\mu \in \mathbf{C}$, $\Phi(\mu E_{jj}) = \mu E_{jj}$ and there are two functions $f, g \in \mathbf{C} \mapsto \mathbf{C}$ such that $\Phi(\mu E_{12}) = f(\mu)E_{12}$ and $\Phi(\nu E_{21}) = g(\nu)E_{21}$, with $\nu \in \mathbf{C}$. Now suppose $\Phi(A) = (b_{ij})$ for $A = (a_{ij})$. Since $C(\Phi(A) - \Phi(a_{ij}E_{ij})) = C(A - a_{ij}E_{ij}) = \{a_{11}, a_{22}\}$ for $(i, j) \in \{(1, 2), (2, 1)\}$, we see that $b_{12} = f(a_{12})$ and $b_{21} = g(a_{21})$. For $i \in \{1, 2\}$, since $C(\Phi(A) - \Phi(\mu E_{ii})) = C(A - \mu E_{ii})$ for all $\mu \in \mathbf{C}$, we see that $b_{ii} = a_{ii}$. Thus,

$$\Phi(A) = \begin{pmatrix} a_{11} & f(a_{12}) \\ g(a_{21}) & a_{22} \end{pmatrix}$$

such that $|f(a_{12})g(a_{21})| = |a_{12}a_{21}|$.

Now, for any $\mu_1, \mu_2 \in \mathbf{C}$, since $C(\Phi(E_{21} + \mu_1 E_{12}) - \Phi(\mu_2 E_{12})) = C(E_{21} + \mu_1 E_{12} - \mu_2 E_{12})$, we see that $|uf(\mu_1) - uf(\mu_2)| = |\mu_1 - \mu_2|$ if $u = g(1)$. By Lemma 2.1.3, we see that $uf = \tau_1$ is the identity map or the conjugation map.

Similarly, using the fact that $C(\Phi(E_{12} + \nu_1 E_{21}) - \Phi(\nu_2 E_{21})) = C(E_{12} + \nu_1 E_{21} - \nu_2 E_{21})$, we can show that $vg = \tau_2$ is the identity map or the conjugation map, where $v = f(1)$.

Combining the above arguments, we get condition (a).

Case 2 Suppose $n > 2$. We prove condition (b) holds by establishing several assertions. We will assume Γ is defined as in Lemma 2.1.2.

Assertion 1 Let $D = \text{diag}(w, w^2, \dots, w^{n-1}, 1)$ for $w = e^{i2\pi/n}$. Then there is a permutation matrix P such that $P\Phi(\mu D)P^t$ has (j, j) entry equal to μw^j for $j = 1, \dots, n$, for any $\mu \in \mathbf{C} \setminus \Gamma$.

The verification to this assertion is identical to Assertion 1 in the proof of Theorem 2.1.

In the following, we will always assume that D is defined as in Assertion 1, and the matrix P in the conclusion is the identity matrix.

Assertion 2 For $k \in \{1, \dots, n\}$, if

$$W_k = \left\{ \sum_{j \neq k} a_j E_{kj} : a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n \in \mathbf{C} \right\},$$

then $\Phi(W_k) \subseteq W_k$.

For any $\mu > 0$ and $k \in \{2, \dots, n\}$, since

$$C(\Phi(\mu D + E_{k1})) = C(\mu D + E_{k1}) = \{\mu w^j : 1 \leq j \leq n\},$$

we see that $\Phi(\mu D + E_{k1})$ has diagonal entries $\mu w, \dots, \mu w^{n-1}, \mu$, and there is at most one row of $\Phi(\mu D + E_{k1})$ with nonzero off diagonal entries by Lemma 3.1(a).

Note that

$$C(\Phi(\mu D + E_{k1}) - \Phi(D)) = C(\mu D + E_{k1} - D) = \{(\mu - 1)w^j : 1 \leq j \leq n\}.$$

Suppose $A = \sum_{j=2}^n a_j E_{1j} \neq 0$. Since $C(\Phi(A)) = C(A) = \{0\}$, we see that all diagonal entries of $\Phi(A)$ equal zero and there is at most one row of $\Phi(A)$ with nonzero off diagonal entries. Let $\mu > 0$ be sufficiently large so that the (j, j) entry of $\Phi(\mu D + E_{k1})$ is μw^j for sufficiently large $\mu > 0$. Then

$$C(\Phi(\mu D + E_{k1}) - \Phi(A)) = C(\mu D + E_{k1} - A)$$

is the union of $\{\mu w^j : 1 < j \leq n, j \neq k\}$ and the Cassini oval with two connected regions containing the foci μw and μw^k , which is two disconnected regions. Thus, $\Phi(\mu D + E_{k1}) - \Phi(A)$ has nonzero off diagonal entries in row 1 and row k . Since this is true for all $k \in \{2, \dots, n\}$, we see that the only nonzero row of $\Phi(A)$ is row 1. Furthermore, we see that for sufficiently large μ , $\Phi(\mu D + E_{k1})$ has nonzero off diagonal entries at row k for $k \in \{2, \dots, n\}$.

Now, for $k \in \{2, \dots, n\}$, let $A = \sum_{j \neq k} a_j E_{kj} \neq 0$. Since $C(\Phi(A)) = C(A) = \{0\}$, we see that $\Phi(A)$ has zero diagonal entries and has at most one nonzero row. Since

$$C(\Phi(\mu D + E_{m1}) - \Phi(A)) = C(\mu D + E_{m1} - A)$$

for $m \in \{1, \dots, n\} \setminus \{k\}$ and sufficiently large μ , we see that the k th row of $\Phi(A)$ is nonzero.

Combining the above arguments, we see that Assertion 2 holds.

Assertion 3 Suppose $A = (a_{ij})$ and $\Phi(A) = (b_{ij})$. Then for $k \in \{1, \dots, n\}$, we have

$$(i) \ b_{kk} = a_{kk} \quad \text{and} \quad (ii) \ \sum_{j \neq k} b_{kj} E_{kj} = \Phi \left(\sum_{j \neq k} a_{kj} E_{kj} \right).$$

Step 1 We prove (i). Since $C(\mu D - A) = C(\Phi(\mu D) - \Phi(A)) = C(\mu D - \Phi(A))$ for all sufficiently large $\mu > 0$, we see that $\Phi(A)$ has (j, j) entry equal to a_{jj} . See Assertion 3 in the proof of Theorem 2.1.

Step 2 We prove (ii) for the special case when $A = (a_{ij})$ has two rows with nonzero diagonal entries such that $C(A)$ consists of $n - 2$ distinct points and a Cassini oval consisting of two disconnected components. Without loss of generality, we assume that the first two rows of A have nonzero off diagonal entries. Since $\Phi(A) = (b_{ij})$ satisfies $b_{jj} = a_{jj}$ for $j \in \{1, \dots, n\}$, $C(A) = C(\Phi(A))$ and $C(\Phi(\mu D + E_{m1}) - \Phi(A)) = C(\mu D + E_{m1} - A)$ for $m \in \{1, \dots, n\}$ and sufficiently large μ , we see that only the first two rows of $\Phi(A)$ have nonzero off diagonal entries. Moreover, for $k = 1, 2$, $C(\Phi(A) - \Phi(\sum_{j \neq k} a_{kj} E_{kj})) = C(A - \sum_{j \neq k} a_{kj} E_{kj})$. Thus by Assertion 2, the k th row of $\Phi(A) - \Phi(\sum_{j \neq k} a_{kj} E_{kj})$ equals zero, i.e., $\sum_{j \neq k} b_{kj} E_{kj} = \Phi(\sum_{j \neq k} a_{kj} E_{kj})$.

Step 3 We prove (ii) for the case when $A = (a_{ij}) \in M_n$ has a single row with nonzero off diagonal entries. Without loss of generality, we may assume the first row of A has non zero off diagonal entries. Let $X \in M_n$ have two nonzero rows such that $X = \mu D + \sum_{j \neq 1} a_{1j} E_{1j} + E_{kj}$, $\mu \in \mathbf{C} \setminus \Gamma$, $k \neq 1$, and $\Phi(X) = (y_{ij})$. By Steps 1 and 2, we see that $y_{ii} = x_{ii}$ for all $i \in \{1, \dots, n\}$, only rows 1 and k of $\Phi(X)$ have nonzero off diagonal entries, and that $\sum_{j \neq 1} y_{1j} E_{1j} = \Phi(\sum_{j \neq 1} a_{1j} E_{1j})$. Because

$$C(\Phi(A) - \Phi(X)) = C\left((b_{ij}) - \left(\mu D + \sum_{j \neq 1} y_{1j} E_{1j} + \sum_{j \neq k} y_{kj} E_{kj}\right)\right) = C(A - X)$$

consists of n distinct points for sufficiently large $\mu > 0$, we can conclude that $\Phi(A) - \Phi(X)$ has only 1 row with nonzero off diagonal entries. Since $C(\Phi(A)) = C(A)$ implies that $\Phi(A)$ has only 1 row with nonzero off diagonal entries, it follows that this row must be row 1 or row k . As this result holds for any $k \in \{2, \dots, n\}$, we see that only row 1 of $\Phi(A)$ has nonzero off diagonal entries, and hence only row k of $\Phi(A) - \Phi(X)$ has nonzero off diagonal entries. Therefore the first row of $\Phi(A) - \Phi(X)$ equals zero, i.e.,

$$\sum_{j \neq 1} b_{1j} E_{1j} = \sum_{j \neq 1} y_{1j} E_{1j} = \Phi\left(\sum_{j \neq 1} a_{1j} E_{1j}\right)$$

Step 4 We prove (ii) for a diagonal matrix $A = (a_{ij})$. If $\Phi(A) = B = (b_{ij})$, then we can show $a_{jj} = b_{jj}$ for all $j \in \{1, \dots, n\}$ as in the previous two cases. Since $C(B) = C(A)$ does not contain a disk with positive radius, we see that B has at most one row with nonzero off diagonal entries. If B has such a row, then we can find $X \in W_k$ so that $B - \Phi(X)$ has two rows with nonzero off diagonal

entries. But then $C(A - X) = C(B - \Phi(X))$ contains a disk with positive radius, which is a contradiction. Hence, we see that $\Phi(A) = A$ if A is a diagonal matrix.

Step 5 We prove (ii) for a matrix $A = (a_{ij}) \in M_n$ with at least three rows having nonzero off diagonal entries.. Let $\Phi(A) = B = (b_{ij})$ and $k \in \{1, \dots, n\}$. We are going to show that $\sum_{j \neq k} b_{kj} = \Phi(\sum_{j \neq k} a_{kj} E_{kj})$. To this end, let $X = (x_{ij}) \in M_n$ be such that X has exactly two nonzero rows indexed by k and k' , where the off diagonal entries in these two rows are the same as those of $A = (a_{ij})$. Moreover, the diagonal entries x_{11}, \dots, x_{nn} are chosen so that

(1) $C(X)$ consists of $n - 2$ distinct points and a Cassini oval consisting of two connected components,

(2) $C(\Phi(X) - \Phi(A)) = C(X - A)$ consists of $n - 2$ distinct points and a Cassini oval consisting of two connected components which include $\{x_{kk} - a_{kk}\}$ and $\{x_{k'k'} - a_{k'k'}\}$.

Since we have proved the result for the special cases covering the case for X , if $\Phi(X) = Y = (y_{ij})$, then

$$\sum_{j \neq k} y_{kj} E_{kj} = \Phi \left(\sum_{j \neq k} x_{kj} E_{kj} \right) = \Phi \left(\sum_{j \neq k} a_{kj} E_{kj} \right).$$

By (2), we see that

$$\sum_{j \neq k} y_{kj} E_{kj} = \sum_{j \neq k} b_{kj} E_{kj}.$$

It follows that

$$\sum_{j \neq k} b_{kj} E_{kj} = \Phi \left(\sum_{k \neq j} a_{kj} E_{kj} \right).$$

Since this is true for any $k \in \{1, \dots, n\}$, the conclusion of the Assertion 3 holds.

Assertion 4 Condition (b) in the theorem holds.

By Assertion 2, we may define $\Phi_k : \mathbf{C}^{1 \times (n-1)} \rightarrow \mathbf{C}^{1 \times (n-1)}$ by

$$\Phi_k(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n) = (b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n)$$

if $\Phi(\sum_{j \neq k} a_j E_{kj}) = \sum_{j \neq k} b_j E_{kj}$. We claim that $\ell_1(\Phi_1(a) - \Phi_1(a')) = \ell_1(a - a')$ for any $a = (a_2, \dots, a_n), a' = (a'_2, \dots, a'_n) \in \mathbf{C}^{1 \times (n-1)}$. To see this, consider $A_k = \sum_{j=2}^n a_j E_{1j} + E_{k1}$ and $A' = \sum_{j=2}^n a'_j E_{1j}$. By Assertion 3 and the fact that $C(\Phi(A_k)) = C(A_k)$,

$$R_1(\Phi(A_k))R_k(\Phi(A_k)) = R_1(A_k)R_k(A_k) = \ell_1(a).$$

Also, since $C(\Phi(A_k) - \Phi(A')) = C(A_k - A')$, by Assertion 3 we have

$$\begin{aligned} \ell_1(a - a') &= R_1(A_k - A')R_k(A_k) \\ &= R_1(\Phi(A_k) - \Phi(A'))R_k(\Phi(A_k)) \\ &= \ell_1(\Phi_1(a) - \Phi_1(a'))\ell_1(\Phi_k(1, 0, \dots, 0)). \end{aligned}$$

By Assertion 3, $R_k(\Phi(A_k)) = R_k(\Phi(E_{k1}))$. Thus, for $k \in \{2, \dots, n\}$,

$$\ell_1(\Phi_k(1, 0, \dots, 0)) = R_k(\Phi(E_{k1})) = \ell_1(a - a') / \ell_1(\Phi_1(a) - \Phi_1(a')) = \nu > 0.$$

Since $C(\Phi(E_{21}) - \Phi(E_{k1})) = C(E_{21} - E_{k1}) = \{z \in \mathbf{C} : |z| \leq 1\}$, we have

$$1 = R_2(\Phi(E_{21}))R_k(\Phi(E_{k1}))$$

for $k \in \{3, \dots, n\}$. (Here we use the fact that $n \geq 3$.) Thus, $\nu = 1$, and

$$\ell_1(a - a') = \ell_1(\Phi_1(a) - \Phi_1(a')).$$

So, Φ_1 satisfies the conclusion of Lemma 2.1.3. Similarly, we can show that Φ_k satisfies the conclusion of Lemma 2.1.3 for $k \in \{2, \dots, n\}$. Consequently, Φ has the asserted form. \square

As with the Gershgorin discs, it is clear that one can have the column version of Cassini ovals by considering $C(A^t)$. However, there is no extension for

$$C_{ijk}(A) = \{\mu \in \mathbf{C} : |(\mu - a_{ii})(\mu - a_{jj})(\mu - a_{kk})| \leq R_i R_j R_k\}$$

or higher dimensions.

Note that we cannot deduce the structure of additive preservers of $C(A)$ using Theorem 3.2.1 as in Corollary 2.1.4 because $C(A) = \{0\}$ does not imply that $A = 0$. Nevertheless, we can apply a similar proof to characterize additive preservers of $C(A)$ and then deduce the results on (real or complex) linear preservers of $C(A)$.

2.3 Inclusion Sets of Ostrowski

In this section we consider preservers of the Ostrowski set.

Theorem 2.3.1 *Let $\varepsilon \in (0, 1)$. A map $\Phi : M_n \rightarrow M_n$ satisfies*

$$O_\varepsilon(\Phi(A) - \Phi(B)) = O_\varepsilon(A - B) \quad \text{for all } A, B \in M_n$$

if and only if there exist $P \in \mathbf{P}_n$ and maps $\psi_{ij} : \mathbf{C} \rightarrow \mathbf{C}$, where ψ_{jj} is the identity map and for $i \neq j$, ψ_{ij} has the form $z \mapsto \nu_{ij}z$ or $z \mapsto \nu_{ij}\bar{z}$ for a norm one complex number ν_{ij} , such that one of the following holds.

- (a) $(\varepsilon, n) = (1/2, 2)$ and the map Φ has the form in Theorem 2.2.1 (a).
- (b) There is $S \in M_n$ such that the map Φ has the form

$$A = (a_{ij}) \mapsto P(\psi_{ij}(a_{ij}))P^t + S.$$

- (c) $\varepsilon = 1/2$ and there is $S \in M_n$ such that Φ has the form

$$A = (a_{ij}) \mapsto P(\psi_{ij}(a_{ij}))^t P^t + S.$$

Proof. The sufficiency is clear. We consider the proof of the necessity. The proof for the case when $n = 2$ is similar to that of Theorem 2.2.1. So, we assume that $n > 2$ and $O_\varepsilon(\Phi(A) - \Phi(B)) = O_\varepsilon(A - B)$ for all $A, B \in M_n$. Replacing Φ by the map $X \mapsto \Phi(X) - \Phi(0)$, we may assume that $O_\varepsilon(\Phi(A)) = O_\varepsilon(A)$ for all $A \in M_n$. Let $D = \text{diag}(w, w^2, \dots, w^{n-1}, 1)$ with $w = e^{i2\pi/n}$. Furthermore, let Γ be defined as in Lemma 2.1.2.

Assertion 1 There is a permutation matrix P such that one of the following holds for any $\mu \in \mathbf{C} \setminus \Gamma$.

(i) $P\Phi(E_{ij})P^t = u_{ij}E_{ij}$ and $P\Phi(\mu D + E_{ij})P^t = \mu D + v_{ij}E_{ij}$ with $u_{ij}, v_{ij} \in \mathbf{C}$ satisfying $|u_{ij}| = |v_{ij}| = 1$ whenever $1 \leq i, j \leq n$ and $i \neq j$.

(ii) $P\Phi(E_{ij})P^t = u_{ij}E_{ji}$ and $P\Phi(\mu D + E_{ij})P^t = \mu D + v_{ij}E_{ji}$ with $u_{ij}, v_{ij} \in \mathbf{C}$ satisfying $|u_{ij}| = |v_{ij}| = 1$ whenever $1 \leq i, j \leq n$ and $i \neq j$.

To prove the assertion, let $\mu \in \mathbf{C} \setminus \Gamma$. Suppose $\nu \in \mathbf{C}$ is such that $\nu, \nu\mu \notin \Gamma$. Since $O_\varepsilon(\Phi(\nu\mu D)) = O_\varepsilon(\nu\mu D) = \{\nu\mu w^j : 1 \leq j \leq n\}$, there is a permutation P (depending on μ and ν) such that $P\Phi(\nu\mu D)P^t = \nu\mu D + F$, where F has zero diagonal entries and $R_j(F)R_j(F^t) = 0$ for all $j = 1, \dots, n$. We will show that condition (i) or (ii) holds. Once this is done, we conclude that P is independent of the choice of μ and ν by examining $\Phi(E_{ij})$ for $1 \leq i, j \leq n$.

For simplicity, we assume that $P = I_n$. Otherwise, replace Φ by the map $X \mapsto P^t\Phi(X)P$. For pairs (i, j) with $i \neq j$ consider $V_{ij} = \Phi(\mu D + E_{ij})$. Since $O_\varepsilon(V_{ij}) = O_\varepsilon(\mu D + E_{ij})$, we see that V_{ij} has diagonal entries $\mu w, \dots, \mu w^{n-1}, \mu$. Since $\Phi(\nu\mu D)$ and $\nu\mu D$ have the same diagonal entries, and

$$O_\varepsilon(\Phi(\nu\mu D) - V_{ij}) = O_\varepsilon((\nu - 1)\mu D - E_{ij}) = \{(\nu - 1)\mu w^j : 1 \leq j \leq n\},$$

the vector of the diagonal of the matrix $(\nu\mu D - V_{ij})/\mu$ equals $\nu(w, w^2, \dots, w^n) - (w, w^2, \dots, w^n)R$ with $R \in \mathbf{P}_n$, and has entries in $\{(\nu - 1)w^j : 1 \leq j \leq n\}$. Since $\nu \in \mathbf{C} \setminus \Gamma$, it follows from Lemma 2.1.2 that $V_{ij} = \mu D + F_{ij}$ such that F_{ij} has zero diagonal and $R_k(F_{ij})R_k(F_{ij}^t) = 0$ for $k = 1, \dots, n$.

For pairs (i, j) with $i \neq j$ let $U_{ij} = \Phi(E_{ij})$. Since $O_\varepsilon(U_{ij}) = O_\varepsilon(E_{ij}) = \{0\}$ we see that for $k = 1, \dots, n$, $R_k(U_{ij})R_k(U_{ij}^t) = 0$ and U_{ij} has zero diagonal entries. Moreover, $O_\varepsilon(V_{ij} - U_{ji}) = O_\varepsilon(\mu D + E_{ij} - E_{ji})$ contains non-degenerate circular disks centered at μw^i and μw^j . Considering the disk with center μw^i , we see that either

- $R_i(U_{ij}) \neq 0 = R_i(U_{ij}^t)$ and $R_i(V_{ji}^t) \neq 0 = R_i(V_{ji})$ or
- $R_i(U_{ij}) = 0 \neq R_i(U_{ij}^t)$ and $R_i(V_{ji}^t) = 0 \neq R_i(V_{ji})$.

Suppose $R_1(U_{12}) \neq 0 = R_1(U_{12}^t)$. If U_{12} has a nonzero (k, j) entry with $j > 2$, then the j th row of U_{12} is zero as $R_j(U_{12})R_j(U_{12}^t) = 0$. Since $O_\varepsilon(V_{j1} - U_{1j}) =$

$O_\varepsilon(\mu D + E_{j1} - E_{1j})$ contains a unit disk centered at μw^j , either $R_j(V_{j1}^t)R_j(U_{1j}) \neq 0 = R_j(V_{j1}) = R_j(U_{1j}^t)$ or $R_j(V_{j1})R_j(U_{1j}^t) \neq 0 = R_j(V_{j1}^t) = R_j(U_{1j})$. In the former case, $O_\varepsilon(E_{1j} - E_{12}) = O_\varepsilon(U_{1j} - U_{12})$ contains a non-degenerate circular disk; in the latter case, $O_\varepsilon(\mu D + E_{j1} - E_{12}) = O_\varepsilon(V_{j1} - U_{12})$ contains a non-degenerate circular disk centered at μw^j . In both cases, we have a contradiction.

By the above paragraph, only the second column of U_{12} can be nonzero. Now, suppose U_{12} has a nonzero $(k, 2)$ entry with $k > 2$. Then $R_k(U_{12}) \neq 0 = R_k(U_{12}^t)$. Since $O_\varepsilon(V_{k1} - U_{1k}) = O_\varepsilon(\mu D + E_{k1} - E_{1k})$ contains a unit disk centered at μw^k , either $R_k(V_{k1}^t)R_k(U_{1k}) \neq 0 = R_k(V_{k1}) = R_k(U_{1k}^t)$ or $R_k(V_{k1})R_k(U_{1k}^t) \neq 0 = R_k(V_{k1}^t) = R_k(U_{1k})$. In the former case, $O_\varepsilon(\mu D + E_{k1} - E_{12}) = O_\varepsilon(V_{k1} - U_{12})$ contains a non-degenerate circular disk centered at μw^k ; in the latter case, $O_\varepsilon(E_{1k} - E_{12}) = O_\varepsilon(U_{1k} - U_{12})$ contains a non-degenerate circular disk. In both cases, we have a contradiction. Thus, we conclude that $U_{12} = u_{12}E_{12}$ for some nonzero u_{12} .

If $R_1(U_{12}) = 0 \neq R_1(U_{12}^t)$, we can use a similar argument to show that $U_{12} = u_{12}E_{21}$ for some nonzero u_{12} .

Applying the above argument to U_{ij} , we see that $U_{ij} = u_{ij}E_{ij}$ or $u_{ij}E_{ji}$ for pairs (i, j) with $i \neq j$.

Interchanging the roles of U_{ij} and V_{ij} in the above proof, we see that $V_{ij} = \mu D + v_{ij}E_{ij}$ or $\mu D + v_{ij}E_{ji}$ for pairs (i, j) with $i \neq j$.

Now, suppose $U_{12} = u_{12}E_{12}$. Since $O_\varepsilon(V_{j1} - U_{12}) = O_\varepsilon(\mu D + E_{j1} - E_{12})$, we see that $V_{j1} = \mu D + v_{j1}E_{j1}$ with $|u_{12}|^\varepsilon |v_{j1}|^{(1-\varepsilon)} = 1$ for all $j = 2, \dots, n$. Next, by the fact that $O_\varepsilon(V_{j1} - U_{kj}) = O_\varepsilon(\mu D + E_{j1} - E_{kj})$, we see that $U_{kj} = u_{kj}E_{jk}$ such that $|u_{kj}|^{(1-\varepsilon)} |v_{j1}|^\varepsilon = 1$ for all (j, k) with $j \neq k$, $j > 1$ and $k \in \{1, \dots, n\}$. In particular, there is u_j such that $|u_{kj}| = |u_j|$ for all $k \neq j$. Since $O_\varepsilon(U_{1j} - U_{j1}) = O_\varepsilon(E_{1j} - E_{j1})$, we see that $U_{j1} = u_{j1}E_{j1}$ with $\max\{|u_{1j}|^\varepsilon |u_{j1}|^{(1-\varepsilon)}, |u_{1j}|^{(1-\varepsilon)} |u_{j1}|^\varepsilon\} = 1$ for $j = 2, \dots, n$; since $O_\varepsilon(V_{ij} - U_{ji}) = O_\varepsilon(\mu D + E_{ij} - E_{ji})$, we see that $|v_{ij}|^\varepsilon |u_{ji}|^{(1-\varepsilon)} = 1 = |v_{ij}|^{(1-\varepsilon)} |u_{ji}|^\varepsilon$ for all pairs (i, j) with $i \neq j$. It follows that $|u_{ij}| = |v_{ij}| = 1$ for pairs (i, j) with $i \neq j$. Thus, condition (i) holds.

Suppose $U_{12} = u_{12}E_{21}$. We can use a similar argument to show that condition (ii) holds.

Assertion 2 There exist functions ψ_{ij} as described in the theorem such that the following hold.

(I) If conclusion (i) of Assertion 1 holds, then $P\Phi(\nu E_{ij})P^t = \psi_{ij}(\nu)E_{ij}$ for all $i \neq j$ and $\nu \in \mathbf{C}$.

(II) If conclusion (ii) of Assertion 1 holds, then $\varepsilon = 1/2$ and $P\Phi(\nu E_{ij})P^t = \psi_{ij}(\nu)E_{ji}$ for all $i \neq j$ and $\nu \in \mathbf{C}$.

To prove the assertion, let P satisfy the conclusion of Assertion 1. For simplicity, assume that $P = I_n$.

(I) Suppose condition (i) in Assertion 1 holds. Consider $\Phi(\nu E_{ij})$. Since

$$O_\varepsilon(\Phi(\nu E_{ij}) - \mu D - v_{rs} E_{rs}) = O_\varepsilon(\Phi(\nu E_{ij}) - \Phi(\mu D + E_{rs})) = O_\varepsilon(\nu E_{ij} - \mu D - E_{rs})$$

for any pairs (r, s) with $r \neq s$, we see that $\Phi(\nu E_{ij}) = \gamma E_{ij}$. Let $\psi_{ij} : \mathbf{C} \rightarrow \mathbf{C}$ be such that $\Phi(\nu E_{ij}) = \psi_{ij}(\nu) E_{ij}$.

We **claim** that $|\psi_{ij}(\nu_1) - \psi_{ij}(\nu_2)| = |\nu_1 - \nu_2|$ for any $\nu_1, \nu_2 \in \mathbf{C}$. Note that

$$\begin{aligned} O_\varepsilon(\Phi(\nu_1 E_{ij} + E_{ji}) - (\mu D + v_{rs} E_{rs})) &= O_\varepsilon(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\mu D + E_{rs})) \\ &= O_\varepsilon(\nu_1 E_{ij} + E_{ji} - \mu D - E_{rs}) \end{aligned}$$

for any pairs (r, s) with $r \neq s$. It follows that $\Phi(\nu_1 E_{ij} + E_{ji}) = \gamma E_{ij} + v_{ji} E_{ji}$. By the fact that $O_\varepsilon(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\nu_1 E_{ij})) = \{0\}$, we see that $\gamma = \psi_{ij}(\nu_1)$. Now,

$$\begin{aligned} O_\varepsilon(\psi_{ij}(\nu_1) E_{ij} + v_{ji} E_{ji} - \psi_{ij}(\nu_2) E_{ij}) &= O_\varepsilon(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\nu_2 E_{ij})) \\ &= O_\varepsilon((\nu_1 - \nu_2) E_{ij} + E_{ji}). \end{aligned}$$

This yields the desired conclusion. By Lemma 2.1.3, ψ_{ij} has the asserted form.

(II) Suppose condition (ii) in Assertion 1 holds. We can use a similar argument to that in the proof of (I) to conclude that $\Phi(\nu E_{ij}) = \psi_{ij}(\nu) E_{ji}$ for all (i, j) with $i \neq j$. Now,

$$O_\varepsilon(\psi_{12}(2) E_{21} - \psi_{31}(1) E_{13}) = O_\varepsilon(\Phi(2E_{12}) - \Phi(E_{31})) = O_\varepsilon(2E_{12} - E_{31}).$$

Thus, $|2|^{(1-\varepsilon)} = |2|^\varepsilon$ and hence $\varepsilon = 1/2$.

Assertion 3 The map Φ has the asserted form.

To prove the assertion, suppose condition (I) in Assertion 2 holds. For simplicity, we assume that $P = I_n$ and ψ_{ij} is the identity map for all pairs (i, j) . We will show that $\Phi(A) = A$ for all $A \in M_n$. Note that if $\mu \in \mathbf{C} \setminus \Gamma$ with sufficiently large magnitude, we will have

$$\begin{aligned} O_\varepsilon(\Phi(A) - \mu D - v_{rs} E_{rs}) &= O_\varepsilon(\Phi(A) - \Phi(\mu D + E_{rs})) \\ &= O_\varepsilon(A - \mu D - E_{rs}) = O_\varepsilon(\nu E_{ij} + E_{ki} - \mu D - E_{rs}). \end{aligned}$$

We consider five special cases before the general case.

Case 1 Suppose $A = \sum_{j=1}^n d_j E_{jj}$ is a diagonal matrix.

Assume $\Phi(A)$ has a nonzero entry at the (p, q) position for some $p \neq q$. Since $O_\varepsilon(\Phi(A)) = O_\varepsilon(A) = \{d_1, \dots, d_n\}$, we see that the p th column of $\Phi(A)$ is zero. But

then we can choose a suitable $\mu \in \mathbf{C} \setminus \Gamma$ so that $O_\varepsilon(\Phi(A) - \mu D - v_{qp}E_{qp})$ contains a non-degenerate disk centered at the p diagonal entry of $\Phi(A) - \mu D - v_{qp}E_{qp}$, where as $O_\varepsilon(A - \mu D - E_{qp}) = \{d_j - \mu w^j : 1 \leq j \leq n\}$, which is a contradiction. Thus, $\Phi(A)$ is a diagonal matrix. Since $O_\varepsilon(\phi(A) - \mu D - v_{12}E_{12}) = O_\varepsilon(A - \mu D - E_{12}) = \{d_j - \mu w^j : 1 \leq j \leq n\}$ for any $\mu \in \mathbf{C} \setminus \Gamma$, we see that $\Phi(A) = A$.

Case 2 Suppose $A = \nu E_{ij} + E_{ki}$ with $\nu \neq 0$ and $i \notin \{j, k\}$.

For notational simplicity, we assume that $(i, j) = (1, 2)$ so that $A = \nu E_{12} + E_{21}$ or $(i, j, k) = (1, 2, 3)$ so that $A = \nu E_{12} + E_{31}$. It is easy to adapt the arguments to the general case. Assume $\Phi(A)$ has a nonzero (p, q) entry with $p \neq q$ and $(p, q) \in \{1, \dots, n\} \times \{3, \dots, n\}$. Taking $(r, s) = (q, p)$ in the above, we see that $O_\varepsilon(\Phi(A) - \mu D - v_{qp}E_{qp}) = O_\varepsilon(A - \mu D - E_{qp})$ has μw^q as an isolated point. Since $\Phi(A) - \mu D - v_{qp}E_{qp}$ has nonzero (p, q) position, the q th column of $\Phi(A)$ has only one nonzero entry at the (q, p) position equal to v_{qp} . But then $O_\varepsilon(\mu D - \Phi(A))$ contains a non-degenerate circular disk centered at μw^q , whereas μw^q is an isolated point of $O_\varepsilon(\mu D - A)$, which is a contradiction. Similarly, assume that $\Phi(A)$ has a nonzero (p, q) entry with $(p, q) \in \{k, \dots, n\} \times \{1, 2\}$ with $k = 3$ or 4 depending on $A = \nu E_{12} + E_{21}$ or $A = \nu E_{12} + E_{31}$. Taking $(r, s) = (q, p)$ in the above, we see that $O_\varepsilon(\Phi(A) - \mu D - v_{qp}E_{qp}) = O_\varepsilon(A - \mu D - E_{qp})$ has μw^p as an isolated point. Thus, the p th row of $\Phi(A)$ has only one nonzero entry at the (q, p) position equal to v_{qp} . But then using $\Phi(\mu D) = \mu D$, we see that $O_\varepsilon(\mu D - \Phi(A))$ will contain a non-degenerate circular disk centered at μw^p , whereas $O_\varepsilon(\mu D - A)$ does not, which is a contradiction. Thus, $\Phi(A)$ only has a nonzero entry at the (p, q) positions with $(p, q) \in K$, where $K = \{(1, 2), (2, 1)\}$ or $K = \{(1, 2), (2, 1), (3, 1), (3, 2)\}$ depending on $A = \nu E_{12} + E_{21}$ or $A = \nu E_{13} + E_{31}$. If $A = \nu E_{12} + E_{21}$, then $O_\varepsilon(\Phi(A) - \nu E_{12}) = O_\varepsilon(A - \nu E_{12}) = \{0\}$ and $O_\varepsilon(\Phi(A) - E_{21}) = O_\varepsilon(A - E_{21}) = \{0\}$. We conclude that $\Phi(A) = \nu E_{12} + E_{21}$. Suppose $A = \nu E_{12} + E_{31}$. Since $O_\varepsilon(\Phi(A) - X) = O_\varepsilon(A - X)$ for $X \in \{0, \nu E_{12}, E_{23}, \mu D + E_{12}\}$ for a suitable $\mu \in \mathbf{C} \setminus \Gamma$, we see that $\Phi(A) = A$ as asserted.

Case 3 Suppose $A = \mu D + R$, where R has nonzero off diagonal entries in at most one row and μ satisfies $\mu|1 - w| > 2$ or $\mu = 0$.

For simplicity, assume that the nonzero off diagonal entries of $A = \mu D + \sum_{j=2}^n a_{1j}E_{1j}$ lie in the first row.

For $\mu|1 - w| > 2$ since (I) holds, we see that $\Phi(\mu D + E_{ij}) = \mu D + v_{ij}E_{ij}$ with $|v_{ij}| = 1$ for pairs (i, j) with $i \neq j$. Suppose $\Phi(A) = (y_{ij})$. Since $O_\varepsilon(\Phi(A) - \Phi(\mu D + E_{ij})) = O_\varepsilon(A - \mu D - E_{ij})$ for all pairs (i, j) with $i \neq j$, we see that $y_{ij} = 0$ for $i > 1$ and $i \neq j$. Moreover, since $O_\varepsilon(\Phi(A) - \Phi(\nu E_{1j} + E_{21})) = O_\varepsilon(A - \nu E_{1j} - E_{21})$ for all $\nu \in \mathbf{C}$ by Case 2, we see that $y_{1j} = a_{1j}$ for $j = 2, \dots, n$.

Now, suppose $\mu = 0$. Since $O_\varepsilon(\Phi(A) - \Phi(E_{ij})) = O_\varepsilon(A - E_{ij})$ for all pairs (i, j) with $i \neq j$, we see that $y_{ij} = 0$ for $i > 1$ and $i \neq j$. Moreover, since $O_\varepsilon(\Phi(A) - \Phi(\nu E_{1j} + E_{21})) = O_\varepsilon(A - \nu E_{1j} - E_{21})$ for all $\nu \in \mathbf{C}$ by Case 2, we see that $y_{1j} = a_{1j}$ for $j = 2, \dots, n$.

Case 4 Suppose $A = a_{ji}E_{ji} + \sum_{k \neq i} a_{ik}E_{ik}$ with $a_{ji} \neq 0$ and $R_i(A) \neq 0$.

We may assume without loss of generality that $(i, j) = (1, 2)$, $a_{21} \neq 0$, and $R_1(A) \neq 0$. Let $\Phi(A) = Y = (y_{rs})$.

There is a choice of μ for which $O_\varepsilon(Y - \mu D) = O_\varepsilon(A - \mu D)$ yields $y_{kk} = a_{kk} = 0$ for $k = 1, \dots, n$. We must have $y_{rs} = 0$ for $r > 2$ because

$$O_\varepsilon(Y - (\mu D + \nu E_{1r})) = O_\varepsilon(A - (\mu D + \nu E_{1r}))$$

which shows that the latter set will have a degenerate circular disk at $-\mu w^r$, whereas, if $y_{rs} \neq 0$, the former will have a non-degenerate circular disk at $-\mu w^r$ for $\nu \neq y_{1r}$. Now if $y_{21} \neq 0$ then

$$O_\varepsilon\left(Y - \left(\sum_{k \neq 1} a_{1k}E_{1k}\right)\right) = O_\varepsilon\left(A - \left(\sum_{k \neq 1} a_{1k}E_{1k}\right)\right) = O_\varepsilon(a_{21}E_{21}) = \{0\},$$

which implies $y_{1k} = a_{1k}$ for all $k \neq 1$. Similarly, if $y_{12} \neq 0$ then $y_{2k} = a_{2k}$ for all $k \neq 2$. Now, if $y_{21} = 0$ and $y_{12} \neq 0$ then $y_{21} = a_{21} \neq 0$ will be a contradiction. Thus $y_{21} = 0$ implies $y_{12} = 0$, which implies $O_\varepsilon(Y) = \{0\} \neq O_\varepsilon(\Phi(A))$, again a contradiction. Hence we have $\Phi(A) = A$.

Case 5 Suppose $A = (a_{ij})$ has exactly two indices i and j with $R_i(A) \neq 0$ and $R_j(A) \neq 0$.

For simplicity, assume that $i = 1$ and $j = 2$. Let $\Phi(A) = Y = (y_{ij})$. By an appropriate choice of μ and using $O_\varepsilon(Y - \mu D) = O_\varepsilon(A - \mu D)$, we see that $y_{kk} = a_{kk}$ for $k = 1, \dots, n$. As before, if $y_{rs} \neq 0$ for $r > 2$ and $r \neq s$, then $O_\varepsilon(Y - \mu D) = O_\varepsilon(A - \mu D)$ implies $R_r(Y^t) = 0$. Now

$$O_\varepsilon(Y - (\mu D + E_{1r})) = O_\varepsilon(A - (\mu D + E_{1r})).$$

The latter set has a degenerate circular disk at $a_{rr} - \mu w^r$, whereas the former set has a non-degenerate circular disk centered at that point. Thus we must have $y_{rs} = 0$ for $r > 2$ and $r \neq s$.

Now, using essentially the same arguments as in Case 4 yields $\Phi(A) = A$. Again, generalizing to any i and j with $i \neq j$, we have $\Phi(A) = A$ for all A with both $R_i(A) \neq 0$ and $R_j(A) \neq 0$.

General Case We complete the proof by now taking $A = (a_{ij})$ to be arbitrary. Let $\Phi(A) = Y = (y_{ij})$. As before, we can get $y_{kk} = a_{kk}$ by using μD with an appropriate choice of μ .

If $R_k(Y^t) \neq 0$, then using

$$O_\varepsilon \left(Y - \left(\mu D + \sum_{j \neq i} a_{kj} E_{kj} \right) \right) = O_\varepsilon \left(A - \left(\mu D + \sum_{j \neq i} a_{kj} E_{kj} \right) \right),$$

we get that the $y_{kj} = a_{kj}$ for all $j \neq k$ since the latter set has a degenerate circular disk centered at $a_{kk} - \mu w^k$.

If $R_k(Y^t) = 0$, then using

$$O_\varepsilon \left(Y - \left(\mu D + \nu E_{ik} + \sum_{j \neq i} a_{kj} E_{kj} \right) \right) = O_\varepsilon \left(A - \left(\mu D + \nu E_{ik} + \sum_{j \neq i} a_{kj} E_{kj} \right) \right),$$

where $i \neq j$ and $\nu \neq a_{ik}$, we conclude that $y_{kj} = a_{kj}$ for all $j \neq k$ since the latter set has a degenerate circular disk centered at $a_{kk} - \mu w^k$. \square

One can use a similar proof to obtain the structure of additive preservers of $O_\varepsilon(A)$, and then deduce the results on linear preservers.

Chapter 3

Product of Matrices

In this chapter, characterization is obtained for maps Φ on $n \times n$ matrices satisfying $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all matrices A and B , where $\mathcal{S}(X)$ is the Gershgorin region, the Brauer region, or the Ostrowski region.¹ Again our strategy is to show that Φ fixes all matrices after some modifications. An important step in our study is to extract information about the eigenvalues of $\Phi(A)$ using $\mathcal{S}(A)$ and $\mathcal{S}(A^2) = \mathcal{S}(\Phi(A)^2)$. To achieve this, we use the following basic result in matrix theory, (see [5, Theorem 3.2.4.2]).

Proposition 3.0.2 *Suppose $A \in M_n$ has n distinct eigenvalues and $B \in M_n$ satisfies $AB = BA$. Then there is a complex polynomial $p(z)$ of degree at most $n - 1$ such that $B = p(A)$.*

3.1 Gershgorin Sets

In this section we consider preservers of the Gershgorin set.

Theorem 3.1.1 *A mapping $\Phi : M_n \rightarrow M_n$ satisfies*

$$G(\Phi(A)\Phi(B)) = G(AB) \quad \text{for all } A, B \in M_n \quad (3.1)$$

if and only if there is $P \in \mathbf{P}_n$ and an invertible diagonal matrix D , where D is unitary such that Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}.$$

¹The material of this chapter is included in [8]

Let $X_{ij} = D + E_{ij}$, where $D = \mu \text{diag}(1, 2, \dots, n)$ for sufficiently large $\mu > 0$, say, $\mu > (2n + 1)$, so that $G(X_{ij}X_{rs})$ consists of n disjoint disks for any (i, j) and (r, s) pairs in the following.

Step 1 We can show that there exists a diagonal orthogonal matrix R and $P \in P_n$ such that $RP\Phi(D)P^t = D$ and $RP\Phi(X_{ij})P^t = D + \alpha_{ij}E_{ij}$ for $\alpha_{ij} \in \mathbf{C}$.

Since $G(\Phi(D)^2) = G(D^2) = \{\mu^2, (2\mu)^2, \dots, (n\mu)^2\}$, we see that $\Phi(D)^2$ is a diagonal matrix with diagonal entries $G(D^2)$. Note that $\Phi(D)$ commutes with $\Phi(D)^2$, so by Proposition 3.0.2, $\Phi(D)$ is a polynomial of $\Phi(D)^2$. Thus $\Phi(D)$ is a diagonal matrix with entries whose squares equal to $\mu^2, (2\mu)^2, \dots, (n\mu)^2$. Thus, $\Phi(D)$ has diagonal entries $\mu\mu, \pm 2\mu, \dots, \pm n\mu$. We can define a diagonal orthogonal matrix R and permutation matrix P such that $RP\Phi(D)P^t = D$.

As $G(\Phi(D)\Phi(X_{ij})) = G(DX_{ij})$, X_{ij} must have diagonal entries identical to those of D and a single row with nonzero off diagonal entries, namely, the i th row. Considering $G(\Phi(X_{ij})\Phi(X_{ji}))$ and $G(\Phi(X_{ji})\Phi(X_{ij}))$ we see that the (i, j) entry of X_{ij} must be nonzero. Suppose then that the (k, j) entry of X_{ij} is nonzero. Then $(k\mu)^2 \notin G(\Phi(X_{kj})\Phi(X_{ij}))$, a contradiction. Thus $RP\Phi(X_{ij})P^t = D + \alpha_{ij}E_{ij}$.

Step 2 Let R and P satisfy the conclusion in Step 1. We show that $RP\Phi(E_{ii})P^t = E_{ii}$ and $RP\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$ for $|\nu_{ij}| = 1$.

As $G(\Phi(E_{ii})\Phi(D)) = G(E_{ii}D) = \{i\mu, 0\}$, $P\Phi(E_{ii})P^t$ must have a single nonzero entry at the (i, i) position, and $P\Phi(E_{ii})P^t = \pm E_{ii}$. Replacing Φ by the map $A \mapsto \pm P\Phi(A)P^t$, we may assume $\Phi(E_{11}) = 1$ and $\Phi(E_{ii}) = \pm E_{ii}$.

Since $G(\Phi(D)\Phi(E_{ij})) = G(DE_{ij}) = D(0, |i\mu|)$, $\Phi(E_{ij})$ must be a matrix with zeroes on the diagonals and a single row, the i th row, of nonzero off diagonal entries. Suppose $\Phi(E_{ij})$ has a nonzero (i, k) entry. Then $\Phi(E_{ij})\Phi(X_{kj})$ will contain a disc centered at a location other than 0, a contradiction. Thus $\Phi(E_{ij}) = \nu_{ij}E_{ij}$, and as $G(\Phi(E_{ii})\Phi(E_{11})) = D(0, 1)$, $|\nu_{ij}| = 1$.

Step 3 We show that Φ has the asserted form.

By Step 1 and Step 2, if we replace Φ by a map $A \mapsto P\Phi(A)P^t$ for a suitable $P \in \mathbf{P}_n$, then $\Phi(E_{ij}) = \nu_{ij}E_{ij}$ with $\nu_{ij} \in \{\nu \in \mathbf{C} : |\nu| = 1\}$. Since $G(\Phi(E_{ij})\Phi(E_{ji})) = \{\nu_{ij}\nu_{ji}\} \cup \{0\} = G(E_{ij}E_{ji}) = \{1\} \cup \{0\}$, we have $\nu_{ji} = \frac{1}{\nu_{ij}}$. We can replace Φ by the map $A \mapsto D^{-1}\Phi(A)D$ where $D = \text{diag}(1, \frac{1}{\nu_{12}}, \frac{1}{\nu_{13}}, \dots, \frac{1}{\nu_{1n}})$ such that $\Phi(E_{1j}) = E_{1j}$, $\Phi(E_{j1}) = E_{j1}$, and $\Phi(E_{ij}) = \nu_{ij}E_{ij}$.

Let $A = E_{11} + E_{1s} + E_{s1} + E_{1t} + E_{t1} + E_{st} + E_{ts} + E_{ss} + E_{tt}$ for any $s, t \in (1, \dots, n)$, and $\Phi(A) = B = (b_{ij})$. Because, $G(\Phi(A)\Phi(X)) = D(b_{ij}, 1)$ for $X \in \{E_{1j}, E_{j1}\}$, we see that $b_{1s} = b_{1t} = b_{t1} = b_{s1} = 1$. Similarly, from $G(\Phi(A)\Phi(X)) = D(b_{ij}\nu_{ij}, 1) = D(1, 1)$ we see that $|b_{st}| = |b_{ts}| = 1$. Finally, $G(\Phi(A)\Phi(E_{kk})) = D(\pm b_{ij}, 1) =$

$D(1, 1)$ shows $b_{ss}, b_{tt} \in \{1, -1\}$. For simplicity, we assume $(s, t) = (2, 3)$. Thus $\Phi(A)^2$ has the following form:

$$B^2 = \begin{pmatrix} 3 & 1 + b_{ss} + b_{ts} & 1 + b_{st} + b_{tt} \\ 1 + b_{ss} + b_{st} & 3 & 1 + b_{ss}b_{st} + b_{tt}b_{st} \\ 1 + b_{ts} + b_{tt} & 1 + b_{st}b_{ss} + b_{st}b_{tt} & 3 \end{pmatrix} \oplus 0_{n-3}.$$

In order for the magnitudes of the off-diagonal entries to be such that $G(\Phi(A)^2) = G(A^2) = D(3, 6)$ we must have $b_{st} = b_{ts} = b_{ss} = b_{tt} = 1$. And since $G(\Phi(A)\Phi(X)) = D(1, 1)$ for $X \in \{E_{st}, E_{ts}, E_{tt}, E_{ss}\}$, we see that $\Phi(X) = X$. Varying s and t , we have $\Phi(E_{ij}) = E_{ij}$ for any i and j .

Now, consider any $A = (a_{ij}) \in M_n$ with $\Phi(A) = (b_{ij})$. Note $G(E_{ij}A) = D(a_{ji}, r) \cup \{0\}$, $r \in \mathbf{R}$ and $G(\Phi(E_{ij})\Phi(A)) = D(b_{ji}, r) \cup \{0\}$, so $a_{ji} = b_{ji}$, for all i, j . The assertion follows. \square

3.2 Brauer's Sets

In this section we consider preservers of Brauer's Set.

Theorem 3.2.1 *A mapping $\Phi : M_n \rightarrow M_n$ satisfies*

$$C(\Phi(A)\Phi(B)) = C(AB) \quad \text{for all } A, B \in M_n \quad (3.2)$$

if and only if there exist $P \in \mathbf{P}_n$ and an invertible diagonal matrix D , where D is unitary if $n \geq 3$, such that Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}.$$

Note that $C(X) = C(PXP^t)$ for any $P \in \mathbf{P}_n$, $C(X) = C(DXD^*)$ for any diagonal unitary matrix D , and $C(X) = C(DXD^{-1})$ for any invertible diagonal matrix D if $n = 2$. Hence, if $\Phi(A) = \pm(DP)A(DP)^{-1}$, then $\Phi(A)\Phi(B) = DP(AB)P^tD^{-1}$ and $C(AB) = C(\Phi(A)\Phi(B))$. We would like to prove the converse. In the following, we assume that $\Phi : M_n \rightarrow M_n$ satisfies (3.2).

We also make use of the following Lemma:

Lemma 3.2.2 *A matrix $A \in M_n$ is non-zero if and only if there is B such that $C(AB) \neq \{0\}$.*

Proof. Clearly, if $A = 0$ then $C(AB) = 0$ for any B . If $A \neq 0$, then for $B = A^*$, AB has nonzero trace and hence $C(AB) \neq \{0\}$.

Proof of Theorem 3.2.1

We first consider the case when $n = 2$.

Let $\Phi(E_{ij}) = F_{ij}$. Then $F_{ij} \neq 0$ by Lemma 3.2.2. Since $C(F_{jj}^2) = C(E_{jj}^2) = \{1, 0\}$, we see that F_{jj}^2 is in triangular form with eigenvalues 1, 0. Since F_{jj}^2 is a matrix with distinct eigenvalues, and F_{jj} commutes with it, by Proposition 3.0.2, F_{jj} is a polynomial of F_{jj}^2 . Therefore F_{jj} is a triangular matrix with diagonal entries $\{\mu_j, 0\}$ such that $\mu_j \in \{1, -1\}$. We may assume the (1, 1) entry of F_{11} is nonzero. Otherwise, replace Φ by the map $A \mapsto P\Phi(A)P^t$.

Consider the following two cases.

Case 1 $F_{11} = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$. Then one of the following holds.

(1.a) F_{22} is in upper triangular form. Since $C(F_{11}F_{22}) = \{0\}$, we have $F_{22} = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}$ with $b_2 \in \{1, -1\}$.

(1.b) F_{22} in lower triangular form. Since $C(F_{22}F_{11}) = \{0\}$, we have $F_{22} = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}$ with $b_1a_2 = 0$ and $b_2 \in \{1, -1\}$,

Case 2 Suppose $F_{11} = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}$. Then one of the following holds.

(2.a) F_{22} is in lower triangular form. Since $C(F_{11}F_{22}) = \{0\}$, we have $F_{22} = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}$ with $b_2 \in \{1, -1\}$.

(2.b) F_{22} in upper triangular form. Since $C(F_{22}F_{11}) = \{0\}$, we have $F_{22} = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}$ with $b_1a_2 = 0$ and $b_2 \in \{1, -1\}$.

The proofs of both cases follow similar steps. First, consider F_{12} , which by Lemma 3.2.2 we know to be nonzero. Examining $C(F_{jj}F_{12}) = \{0\} = C(F_{12}F_{jj})$, we can see that the diagonal entries of F_{12} must be zero. If F_{12} has two nonzero off diagonal entries then $C(F_{12}^2) \neq \{0\} = C(E_{12}^2)$ and we have a contradiction. We can see that (i) $F_{12} = \alpha E_{12}$ or (ii) $F_{12} = E_{21}/\alpha$ where α is some constant. A similar argument shows $F_{21} = \beta E_{21}$ or $F_{21} = E_{12}/\beta$. Since $C(F_{12}F_{21}) = \{1, 0\} = C(E_{12}E_{21})$, we see that in case (i), $F_{21} = \beta E_{21}$, in case (ii), $F_{21} = E_{21}/\beta$, and in both cases $\beta = 1/\alpha$. In both cases (i) and (ii), we can replace the map Φ by a map of the form $A \mapsto \pm D^{-1}\Phi(A)D$ with $D = \text{diag}(\alpha, 1)$. Since $C(F_{jj}X) = \{0\} = C(XF_{jj})$ for $X = E_{12}, E_{21}$, F_{jj} must have all zero off diagonal entries. We can then see $\Phi(E_{11}) = E_{11}$, and $\Phi(E_{22}) = \pm E_{22}$.

Now assume (i) holds, let $A = \sum_{i,j} E_{ij}$ be the matrix with all entries equal to one and $\Phi(A) = (b_{ij})$. Considering $C(F_{ij}B) = C(E_{ij}A) = \{1, 0\}$ we see that

$b_{11} = b_{12} = b_{21} = 1$ and $b_{22} = \pm 1$. Consider $\Phi(A)^2$:

$$\begin{pmatrix} 2 & 1 + b_{22} \\ 1 + b_{22} & 2 \end{pmatrix}.$$

Note $C(\Phi(A)^2) = C(A^2)$ does not consist of a single point, so $b_{22} = 1$. Since $C(\Phi(A)E_{22}) = \{\pm b_{22}, 0\} = C(AE_{22}) = \{1, 0\}$ we see that $\Phi(E_{22}) = E_{22}$. Because $\Phi(E_{ij}) = E_{ij}$ for all $i, j \in \{1, 2\}$ we conclude that $\Phi(A) = A$ for all $A \in M_2$ and the map has the asserted form. If (ii) holds, one can use similar steps to show that the modified map satisfies $\Phi(A) = A^t$. But then for $A = E_{11} + 10E_{22} + E_{12} = B^t$, we see that $C(AB) \neq C(A^tB^t) = C(\Phi(A)\Phi(B))$, which is a contradiction.

We next consider the case when $n > 2$.

Let $X_{ij} = D + E_{ij}$, where $D = \mu \text{diag}(1, 2, \dots, n)$ for sufficiently large $\mu > 0$ such that $C(X_{ij}X_{rs})$ consists of n disjoint connected regions in the following.

Step 1. There exist diagonal orthogonal matrix R and $P \in \mathbf{P}_n$ such that $RP\Phi(X_{ij})P^t = D + e_i v_{ij}^t$ so that the i th entry of v_{ij} is zero.

Since $C(\Phi(X_{ij})^2) = C(X_{ij}^2) = \{\mu^2, (2\mu)^2, \dots, (n\mu)^2\}$, we see that $\Phi(X_{ij})^2$ is a matrix with distinct eigenvalues and at most one nonzero row. Note that $\Phi(X_{ij})$ commutes with $\Phi(X_{ij})^2$. By Proposition 3.0.2, $\Phi(X_{ij})$ is a polynomial of $\Phi(X_{ij})^2$. Hence, it is a matrix with diagonal entries $\{\pm\mu, \pm 2\mu, \dots, \pm n\mu\}$ and at most one row with nonzero off diagonal entries. A similar argument holds for $\Phi(D)$, so $RP\Phi(D)P^t = D + x_k v^t$ for a diagonal unitary matrix R and $P \in \mathbf{P}_n$. Examining $C(\Phi(D)\Phi(X_{ij})) = \{\mu^2, (2\mu)^2, \dots, (n\mu)^2\}$ and $C(\Phi(X_{ij})\Phi(X_{ji})) \neq \{\pm\mu, \pm 2\mu, \dots, \pm n\mu\}$, we see that $RP\Phi(X_{ij})P^t = D + x_{\sigma(i,j)} v_{ij}^t$ for nonzero v_{ij} with $\sigma(i, j)$ entry being zero and R a diagonal matrix with entries in $\{1, -1\}$.

We next show that σ depends only on i and has a unique value for each i . Suppose

$$RP\Phi(X_{ij})P^t = D + x_p v_{ij}^t \quad \text{and} \quad RP\Phi(X_{rs})P^t = D + x_p v_{rs}^t$$

for $r \neq i$ with $x_p, v_{ij}, v_{rs} \in \mathbf{C}^n$. Since $RP\Phi(X_{ij})R\Phi(X_{rs})P^t = D^2 + Dx_p v_{rs}^t + x_p v_{ij}^t + x_p v_{ij}^t x_p v_{rs}^t$, we see that $C(\Phi(X_{ij})\Phi(X_{rs}))$ have no ovals of nonzero radius, a contradiction. Moreover, if $RP\Phi(X_{ij})P^t = D + x_p v_{ij}^t$ and $RP\Phi(X_{kj})P^t = D + x_q v_{kj}^t$, then $C(\Phi(X_{ij})\Phi(X_{kj}))$ would have an oval of nonzero radius, again a contradiction. The same argument holds for any i , we conclude that σ depends only on i and $\sigma(i) \neq \sigma(j)$ for $i \neq j$.

We now want to show that $\sigma(i) = i$. Since

$$C(\Phi(X_{ij})\Phi(X_{ji})) = \{1 + (i\mu)^2, 1 + (j\mu)^2, \dots, (n\mu)^2\},$$

we see that $\sigma(i)$ must be i or j . Suppose $\sigma(i) = j$. Then $\sigma(j) = i$. Hence $RP\Phi(X_{jk})P^t = D + x_i v_{jk}^t$ and $RP\Phi(X_{kj})P^t = D + x_k v_{jk}^t$ by a similar argument. But then $C(\Phi(X_{jk})\Phi(X_{kj}))$ will not contain the singleton at $1 + (j\mu)^2$, a contradiction. Thus $\sigma(i) = i$ and $RP\Phi(X_{ij})P^t = D + e_i v_{ij}^t$.

Step 2. We can show that $RP\Phi(X_{ij})P^t = D + \alpha_{ij}E_{ij}$.

Considering $C(\Phi(X_{ij})\Phi(X_{ji}))$ and $C(\Phi(X_{ji})\Phi(X_{ij}))$ we see that the (i, j) entry of X_{ij} must be nonzero so that $\{(i\mu)^2, (j\mu)^2\}$ is not in either of these sets. Suppose then that the (k, j) entry of X_{ij} is nonzero. Then $(k\mu)^2 \notin C(\Phi(X_{kj})\Phi(X_{ij}))$, a contradiction. Thus $RP\Phi(X_{ij})P^t = D + \alpha_{ij}E_{ij}$.

Step 3. We have $RP\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$ and $|\nu_{ij}| = 1$.

Considering $C(\Phi(D)\Phi(X_{ij}))$ for all i, j , we see that $RP\Phi(D)P^t = D$. Using

$$C(\Phi(D)\Phi(E_{ij})) = C(DE_{ij}) = \{0\},$$

we see E_{ij} must be a matrix with zeros on the diagonal and at most one row with nonzero off diagonal entries. Then since $C(\Phi(X_{ji})\Phi(E_{ij})) = \{1, 0\}$ has no discs of nonzero radius, $RP\Phi(E_{ij})P^t$ must have a single nonzero entry at (i, j) and $RP\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$.

The set $C(E_{ij}E_{ji}) = \{0, 1\}$ shows that $|\nu_{ij}||\nu_{ji}| = 1$. Consider $A = [a_{ij}]$ such that $a_{ij} \neq 0$ for every i, j and $\Phi(A) = [\alpha_{ij}]$. So we see that $C(E_{ji}A) = \{a_{ij}, 0\} = C(\Phi(E_{ji})\Phi(A)) = C(\nu_{ji}E_{rs}\Phi(A)) = \{\nu_{ji}\alpha_{sr}, 0\}$. So $a_{ij} = \nu_{ji}\alpha_{sr} = \frac{1}{\nu_{ij}}\alpha_{sr}$. It follows that every a_{ij} maps to an $\alpha_{sr} = \nu_{ij}a_{ij}$, and we can conclude that $|\nu_{ij}| = 1$.

Step 4. Replace Φ by the map $A \mapsto P^tAP$ where P satisfies the conclusion of Step 3. We show $\Phi(E_{ii}) = \pm E_{ii}$.

Let $\Phi(E_{ii}) = (f_{rs})$. For any $r \neq s$ such that $f_{rs} \neq 0$, $C(\Phi(E_{rs})\Phi(E_{ii})) \neq \{0\}$, so $\Phi(E_{ii})$ is a diagonal matrix. Consider $C(E_{ii}^2) = \{1, 0\} = C(\Phi(E_{ii})^2)$. From this we can see that $f_{ii} \in \{1, -1, 0\}$ for all i and there is at least 1 but not more than $n - 1$ diagonal entries for which $f_{ii} = 0$.

Suppose $\Phi(E_{ii}) = (f_{ij})$ has two nonzero diagonal entries $f_{ss} = \delta_s$ and $f_{kk} = \delta_k$, with $\delta \in \{1, -1\}$. Then $C(\Phi(E_{ii})\Phi(D))$ would have three distinct ovals of zero radius, and as $C(E_{ii}D) = \{i, 0\}$ this would be a contradiction and s or k must be equal to i . It follows that $\Phi(E_{ii}) = \pm E_{ii}$. Replacing Φ by the map $A \mapsto \pm\Phi(A)$, we can assume $\Phi(E_{11}) = E_{11}$ and we have $\Phi(E_{ii}) = \pm E_{ii}$.

Step 5. As $\Phi(E_{ii}) = \pm E_{ii}$, $\Phi(E_{11}) = E_{11}$, and $\Phi(E_{ij}) = \nu_{ij}E_{ij}$ we can show that $\Phi(E_{ii}) = E_{ii}$ and $\Phi(E_{ij}) = E_{ij}$.

As in the previous section, we may replace Φ by the map $A \mapsto D^{-1}\Phi(A)D$ where $D = \text{diag}(1, \frac{1}{v_{12}}, \frac{1}{v_{13}}, \dots, \frac{1}{v_{1j}})$ such that $\Phi(E_{1j}) = E_{1j}$ and $\Phi(E_{j1}) = E_{j1}$.

Note that this will not change the magnitude of (i, j) entry for $\Phi(E_{ij})$, so we may still assume that $\Phi(E_{ij}) = v_{ij}E_{ij}$. As with step 3 of the preceding section, we consider the matrix $A = E_{11} + E_{qs} + E_{s1} + E_{1t} + E_{t1} + E_{st} + E_{ts} + E_{ss} + E_{tt}$, where $\Phi(A) = B$ to show that $\Phi(E_{ij}) = E_{ij}$. As $C(\Phi(A)\Phi(X)) = \{b_{ij}\} \cup \{0\} = \{1\} \cup \{0\}$ for $X \in \{E_{1j}, E_{j1}\}$, we see that $b_{1s} = b_{1t} = b_{t1} = b_{s1} = 1$. Similarly, from $C(\Phi(A)\Phi(X)) = \{b_{ij}v_{ij}\} \cup \{0\} = \{1\} \cup \{0\}$ we see that $|b_{st}| = |b_{ts}| = 1$. And as $C(\Phi(A)\Phi(E_{kk})) = \{\pm b_{kk}\} \cup \{0\} = \{1\} \cup \{0\}$, we have $b_{ss}, b_{tt} \in \{1, -1\}$. For simplicity, we assume $(s, t) = (2, 3)$. Thus $\Phi(A)^2$ has the following form:

$$B^2 = \begin{pmatrix} 3 & 1 + b_{ss} + b_{ts} & 1 + b_{st} + b_{tt} \\ 1 + b_{ss} + b_{st} & 3 & 1 + b_{ss}b_{st} + b_{tt}b_{st} \\ 1 + b_{ts} + b_{tt} & 1 + b_{st}b_{ss} + b_{st}b_{tt} & 3 \end{pmatrix} \oplus 0_{n-3}.$$

In order for the magnitudes of the off diagonal entries to be such that $C(\Phi(A)^2) = C(A^2)$ we must have $b_{st} = b_{ts} = b_{ss} = b_{tt} = 1$. And as $C(\Phi(A)\Phi(X)) = \{1\} \cup \{0\}$ for $X \in \{E_{st}, E_{ts}, E_{tt}, E_{ss}\}$, $\Phi(X) = X$. As these results hold for any s and t , we have $\Phi(E_{ij}) = E_{ij}$.

Step 6 The map has the asserted form.

Replace Φ by $X \mapsto \pm P\Phi(X)P^t$ for a suitable $P \in \mathbf{P}_n$ and assume that $\Phi(E_{ij}) = E_{ij}$ for all (i, j) pair. Consider any $A = (a_{ij}) \in M_n$ with $\Phi(A) = (b_{ij})$. Since $C(E_{ij}B) = C(E_{ij}A)$, we see that $a_{ji} = b_{ji}$. Thus the map has the asserted form. \square

3.3 Inclusion Sets of Ostrowski

In this section we consider preservers of the Ostrowski Set.

Theorem 3.3.1 *Let $\varepsilon \in (0, 1)$. A mapping $\Phi : M_n \rightarrow M_n$ satisfies*

$$O_\varepsilon(\Phi(A)\Phi(B)) = O_\varepsilon(AB) \quad \text{for all } A, B \in M_n \quad (3.3)$$

if and only if there is $P \in \mathbf{P}_n$ and an invertible diagonal matrix D , where D is unitary unless $(n, \varepsilon) = (2, 1/2)$, such that Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}.$$

We will write $O(A)$ for $O_\varepsilon(A)$ for notational simplicity. For $X = (x_{ij}) \in M_n$, let $R_i(X) = \sum_{j \neq i} |x_{ij}|$ and $C_j(X) = \sum_{i \neq j} |x_{ij}|$ for $i, j = 1, \dots, n$. We will first consider the case when $\varepsilon \in (0, 1)$. In such a case, $R_j(X)^\varepsilon C_j(X)^{1-\varepsilon} = 0$ if and

only if $R_j(X)C_j(X) = 0$. We first show in Lemma 3.3.3 that there is $P \in \mathbf{P}_n$ and a diagonal orthogonal matrix R such that for X in a certain form, $RP\Phi(X)P^t$ and X have the same diagonal entries and triangular block pattern. Then we show in Lemma 3.3.4 that for the same matrices P and R , and a matrix Y with only one nonzero off-diagonal entry, $RP\Phi(Y)P^t$ and Y have the same diagonal entries and zero patterns. With this, we can show there is $P \in \mathbf{P}_n$ such that $P\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$ with $\nu_{ij} \neq 0$ for all $i, j \in \{1, \dots, n\}$ in Lemma 3.3.5. Then we construct the diagonal matrix such that the map Φ satisfies the stated theorem.

The following observations will be used in our proof.

Lemma 3.3.2 *Let $U \in M_n$.*

(a) *If U^2 consists of n disjoint disks including the degenerate disk centered at the (j, j) entry, then $R_j(U^2) = 0$ or $C_j(U^2) = 0$. If $R_j(U^2) = 0$ then $R_j(U) = 0$; if $C_j(U^2) = 0$ then $C_j(U) = 0$.*

(b) *If U^2 consists of n distinct points, then there is $Q \in \mathbf{P}_n$ such that $QUQ^t = \begin{pmatrix} D_1 & X \\ 0 & D_2 \end{pmatrix}$, where $D_1 \in M_k$ and $D_2 \in M_{n-k}$ are diagonal matrices. Moreover, if $QUQ^t = (u_{ij})$ and $V \in M_n$ with $QVQ^t = (v_{ij})$, then $u_{jj} = v_{jj}$ for $j = 1, \dots, k$ are centers of k of the disks of $O(VU)$ and $u_{jj} = v_{jj}$ for $j = k + 1, \dots, n$ are centers of $n - k$ of the disks of $O(UV)$.*

Proof. (a) Assume $U \in M_n$ satisfies the hypotheses. Then $R_j(U^2)C_j(U^2) = 0$. So, $R_j(U^2) = 0$ or $C_j(U^2) = 0$. Moreover, since U^2 has an eigenvalue in each of the disks in $O(U^2)$, U^2 has n distinct eigenvalues. Thus, any matrix commuting with U^2 is a polynomial of U^2 by Proposition 3.0.2. In particular, U is a polynomial of U^2 . Thus, if $C_j(U^2) = 0$, then U^2e_j is a multiple of e_j and so is Ue_j , and hence $C_j(U)$. Similarly, if $R_j(U^2) = 0$, then so is $R_j(U)$.

(b) If $O(U^2)$ consists of n distinct points, then $R_j(U^2)C_j(U^2) = 0$ for all $j = 1, \dots, n$. There exists $Q \in \mathbf{P}_n$ such that only the first k rows of QU^2Q has nonzero off-diagonal entries. Then $C_j(QU^2Q^t) = 0$ for $j = 1, \dots, k$. Now, QU^2Q^t has n distinct eigenvalues and QUQ^t is a polynomial of QU^2Q^t . Thus, QUQ^t has the asserted form.

The last assertion is easy to verify. □

Lemma 3.3.3 *Let $\mu > 0$ be such that $\mu > 5n$, $D = \mu \text{diag}(1, 2, \dots, n)$ and $N_k = E_{12} + E_{23} + \dots + E_{k,k+1}$ for $k = 1, \dots, n-1$. If $U \in \{D^m + N_k : k = 1, \dots, n-1, m = 1, 2, \dots\}$ and $V \in \{U^2, UU^t, U^tU\}$, then $O(V)$ is a union of n disjoint disks. Moreover, there is $P \in \mathbf{P}_n$ and $R = \text{diag}(r_1, \dots, r_n)$ with $r_1, \dots, r_n \in \{1, -1\}$ such that for any positive integer m , the following conditions hold.*

- (a) $RP\Phi(D^m)P^t = D^m$.
- (b) $RP\Phi(D^m + N_k)P^t$ has the form

$$\begin{pmatrix} * & u_k(m, \mu) \\ 0 & [\mu(k+1)]^m \end{pmatrix} \oplus \mu^m \text{diag}((k+2)^m, \dots, n^m)$$

for some nonzero vector $u_k(m, \mu) \in \mathbf{C}^k$ for $k = 1, \dots, n-1$. In particular, $RP\Phi(D^m + E_{12})P^t$ has the form $D^m + \nu_{12}E_{12}$ for some nonzero ν_{12} .

- (c) $RP\Phi(D^m + N_k^t)P^t$ has the form

$$\begin{pmatrix} * & 0 \\ v_k(m, \mu)^t & [\mu(k+1)]^m \end{pmatrix} \oplus \mu^m \text{diag}((k+2)^m, \dots, n^m)$$

for some nonzero vector $v_k(m, \mu) \in \mathbf{C}^k$ for $k = 1, \dots, n-1$.

Proof. Let μ, D, N_k, U and V satisfy the hypothesis.

Claim 1 *The matrix $O(V)$ satisfies the asserted condition.*

If $U = D^m + N_k$ and $V \in \{U^2, UU^t, U^tU\}$, it is easy to check that $R_j(V)$ and $C_j(V)$ are bounded above by $\mu^m(2n^m)$, and the distance between any two diagonal entries of V are bounded below by μ^{2m} . The radius of any disk in the Ostrowski set is at most $2\mu^m n^m$, which is less than half the distance between two centers. The result follows.

Claim 2 *There is $P \in \mathbf{P}_n$ and a diagonal orthogonal matrix R such that $RP^t\Phi(D^m)P$ has diagonal entries*

$$\mu^m, (\mu 2)^2, \dots, (\mu n)^m, \quad m = 1, 2, \dots$$

Moreover, for any positive integers m and k , if $Y = \Phi(D^m)$ and $Z = \Phi(D^k)$, then

$$R_j(Y)C_j(Z) = 0 = C_j(Y)R_j(Z) \quad \text{for all } j = 1, \dots, n.$$

To prove the claim, let $\Phi(D) = S$. Then $O(D^2) = O(S^2) = \{(\mu j)^2 : j = 1, \dots, n\}$. By Lemma 3.3.2 (b), there is $Q \in \mathbf{P}_n$ satisfying $Q^t S Q = \begin{pmatrix} S_1 & * \\ 0 & S_2 \end{pmatrix}$, where S_1 and S_2 are diagonal matrices such that $S_1^2 \oplus S_2^2$ has diagonal entries $\mu^2, (2\mu)^2, \dots, (n\mu)^2$ arranged in a certain order. Thus, there exists a diagonal

orthogonal matrix $R = \text{diag}(r_1, \dots, r_n)$ and $P \in \mathbf{P}_n$ such that $\tilde{S} = RP^tSP$ has diagonal entries $\mu, 2\mu, \dots, n\mu$ and $R_j(\tilde{S})C_j(\tilde{S}) = 0$ for $j = 1, \dots, n$.

For simplicity, we replace Φ by the map $X \mapsto RP^t\Phi(X)P$ and assume that $R = P = I_n$ in the following.

Under this assumption, we see that $\Phi(D) = X = (x_{ij})$ has diagonal entries $\mu, 2\mu, \dots, n\mu$, and $R_j(X)C_j(X) = 0$ for $j = 1, \dots, n$.

Suppose $m > 1$ and $\Phi(D^m) = Y = (y_{ij})$. Applying Lemma 3.3.2 (b) with $U = Y$, we see that Y has diagonal entries whose absolute values from the set $\{\mu^m, (2\mu)^m, \dots, (n\mu)^m\}$. Moreover, $R_j(Y)C_j(Y) = 0$ for $j = 1, \dots, n$.

Note that

$$O(XY) = O((D)(D^m)) = \{(j\mu)^{m+1} : j = 1, \dots, m\} = O((D^m)(D)) = O(YX).$$

If $R_j(X) \neq 0$, then $C_j(Y) = 0$. Otherwise, $C_j(X) = 0$ and $R_j(Y) = 0$, and hence $O(YX)$ contains a disk centered at $x_{jj}y_{jj}$ with nonzero radius, which is a contradiction. Thus,

$$O(XY) = \{(j\mu)^{m+1} : j = 1, \dots, n\} = \{x_{11}y_{11}, \dots, x_{nn}y_{nn}\}.$$

Since $x_{jj} = j\mu$, $\{|y_{jj}| : j = 1, \dots, n\} = \{(j\mu)^m : j = 1, \dots, n\}$, we conclude that $y_{jj} = (j\mu)^m$ for $j = 1, \dots, m$.

Now, if $Z = \Phi(D^k)$ for a positive integer k , then Z has diagonal entries

$$\{\mu^k, (2\mu)^k, \dots, (n\mu)^k\}$$

and satisfies $R_j(Z)C_j(Z) = 0$ for $j = 1, \dots, n$. Since

$$O(YZ) = O(ZY) = O(D^{m+k}) = \{(j\mu)^{m+k} : j = 1, \dots, n\},$$

we have $R_j(Y)C_j(Z) = 0 = C_j(Y)R_j(Z)$ for $j = 1, \dots, n$.

Claim 3 *Conditions (a) – (c) hold.*

For notational simplicity, we focus on the case when $m = 1$. One can check that the same proof holds for any positive integer m .

Let $J_1 \subseteq \{1, \dots, n\}$ contain all the indices j such that the j th row of $RP^t\Phi(D^\ell)P$ has nonzero off-diagonal entries for some positive integer ℓ , and $J_2 \subseteq \{1, \dots, n\}$ contains all the indices j such that the j th column of $RP^t(D^\ell)P$ has nonzero off-diagonal entries for some positive integer ℓ . By Claim 2, $J_1 \cup J_2 = \emptyset$.

Let $A = D + N_{n-1}$ and $\Phi(A) = B = (b_{ij})$. We are going to show that B has the form

$$\begin{pmatrix} * & u \\ 0 & \mu n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & 0 \\ v^t & \mu n \end{pmatrix}$$

for some nonzero $u, v \in \mathbf{C}^{n-1}$. Then for any positive integer ℓ , $O(\Phi(D^\ell)B) = O(D^\ell A)$ is a union of n disks with centers $\mu j^{\ell+1} : j = 1, \dots, n$. A similar conclusion holds for $O(B\Phi(D^\ell)) = O(AD^\ell)$. Note that

- (a) if $j \in J_1$, then $b_{jj}(j\mu)^\ell$ is a diagonal entry of $B\Phi(D^\ell)$ lying in $\{(k\mu)^{\ell+1} : k = 1, \dots, n\}$;
- (b) if $j \in J_2$, $b_{jj}(j\mu)^\ell$ is a diagonal entry of $\Phi(D^\ell)B$ lying in $\{(k\mu)^{\ell+1} : k = 1, \dots, n\}$;
- (c) if $j \notin J_1 \cup J_2$, then $b_{jj}(j\mu)^\ell$ is a diagonal entry of $B\Phi(D^\ell)$ as well as $\Phi(\mu D^\ell)B$ lying in $\{(k\mu)^{\ell+1} : k = 1, \dots, n\}$.

We see that for any $j \in \{1, \dots, n\}$,

$$b_{jj}(j\mu)^\ell \in \{(k\mu)^{\ell+1} : k = 1, \dots, n\}, \quad \ell = 1, 2, \dots$$

Consequently,

$$b_{jj} \in \{k\mu(k/j)^\ell : k = 1, \dots, n\}, \quad \ell = 1, 2, \dots\}.$$

It follows that

$$b_{jj} = \mu j \quad \text{for } j = 1, \dots, n.$$

Note that $O(B^2) = O(A^2)$ consists of n disks including the degenerate disk $\{(\mu n)^2\}$, we see that $R_n(B^2)C_n(B^2) = 0$. Applying Lemma 3.3.2(a) with $U = B$, we see that B has the form

$$(i) \begin{pmatrix} * & * \\ 0 & \mu n \end{pmatrix} \quad \text{or} \quad (ii) \begin{pmatrix} * & 0 \\ * & \mu n \end{pmatrix},$$

Applying the argument on B to $C = \Phi(A^t)$, we see that C has the form (i) or (ii) also.

Note that $O(CB) = O(A^t A)$ and $O(BC) = O(AA^t)$. First, B and C cannot be of the same form (i) or (ii). Otherwise, both $O(BC)$ and $O(CB)$ will have a disk centered at $\mu^2 n^2$, which is impossible. The fact that $O(BC) = O(AA^t)$ requires B has form (i) and C has form (ii). The nonzero radius of the disk centered at $(\mu n)^2$ in $O(BC)$ guarantees the $(1, 2)$ block of B and the $(2, 1)$ block of C are both nonzero.

Let $Z = \Phi(D)$. By $O(Z^2) = O(D^2)$, we see that Z has form (i) or (ii). Given $O(ZB) = O(DA)$, the $(2, 1)$ block of Z must be 0. Similarly, $O(CZ) = O(A^t D)$ requires the $(1, 2)$ block is 0. Therefore, $\Phi(D) = Z_2 \oplus [\mu n]$, where $Z_2 \in M_{n-1}$.

Now, set $A_2 = D + N_{n-2}$ and $B_2 = \Phi(A_2)$. Then, by a similar consideration of $O(D^\ell A_2) = O(\Phi(D^\ell)B_2)$, $O(A_2 D^\ell) = O(B_2 \Phi(D^\ell)B)$, $O(A_2) = O(B_2)$, we see

that B_2 has diagonal entries $\mu, 2\mu, \dots, n\mu$ such that $\Phi(A_2) = \hat{B}_2 \oplus [\mu n]$. Note that $O(\hat{B}_2^2)$ consists of $n - 1$ disks including the degenerate disk $\{\mu^2(n - 1)^2\}$. Thus, $\hat{B}_2 \in M_{n-1}$ has the form

$$\begin{pmatrix} * & * \\ 0 & \mu(n-1) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & 0 \\ * & \mu(n-1) \end{pmatrix}.$$

Similarly, $\Phi(A_2^t)$ has this form. Using $O(A_2A_2^t)$ and $O(A_2^t, A_2)$, we see that $\Phi(A_2)$ has upper triangular block form and $\Phi(A_2^t)$ has lower triangular block form. These two Ostrowski sets also show the $(1, 2)$ block of \hat{B}_2 and the $(2, 1)$ block of \hat{B}_2^t must be nonzero. Finally, $O(\Phi(D)\Phi(A_2)) = O(DA_2)$ and $O(\Phi(A_2^t)\Phi(D)) = O(A_2^tD)$ require that $\Phi(D) = Z_3 \oplus \text{diag}(\mu(n-1), \mu n)$.

Repeating the above argument for each disk, we can show that $\Phi(D + N_k)$ has the form

$$\begin{pmatrix} * & u_k(\mu) \\ 0 & \mu(k+1) \end{pmatrix} \oplus \text{diag}(\mu(k+2), \dots, \mu n)$$

and $\Phi(D + N_k^t)$ has the form

$$\begin{pmatrix} * & 0 \\ v_k(\mu)^t & \mu(k+1) \end{pmatrix} \oplus \text{diag}(\mu(k+2), \dots, \mu n)$$

for some nonzero vectors $u_k(\mu), v_k(\mu) \in \mathbf{C}^k$ for $k = 1, \dots, n - 1$. Furthermore, $\Phi(D) = D$.

Thus, we conclude (a) – (c) are valid. \square

Lemma 3.3.4 *Let μ, P , and R satisfy the hypothesis and conclusion of Lemma 3.3.3. Suppose $i \neq j$. Then there is a diagonal matrix \tilde{D} with diagonal entries $\mu, \dots, n\mu$, with the (i, i) entry equal to μ and (j, j) entry equal to 2μ , such that for any positive integer m , we have*

$$RP\Phi(\tilde{D}^m + E_{ij})P^t = \tilde{D}^m \tilde{M} + \nu_{ij}(m)E_{ij}$$

and

$$RP\Phi(\tilde{D}^m + E_{ji})P^t = \tilde{D}^m + \nu_{ji}(m)E_{ji}$$

for some nonzero numbers $\nu_{ij}(m), \nu_{ji}(m)$.

Proof. Let $Q \in \mathbf{P}_n$ be such that $QE_{ij}Q^t = E_{12}$ and $QE_{ji}Q^t = E_{21}$. Consider the map $\tilde{\Phi}$ defined by $X \mapsto \Phi(Q^tXQ)$. Then $O(\tilde{\Phi}(A)\tilde{\Phi}(B)) = O(AB)$ for all $A, B \in M_n$. By Lemma 3.3.3 (b) and (c), there exist $\hat{P} \in \mathbf{P}_n$ and a diagonal orthogonal matrix \hat{R} such that

$$\hat{R}\hat{P}\tilde{\Phi}(D^m + E_{12})\hat{P}^t = D^m + \nu_{ij}(m)E_{12}$$

and

$$\hat{R}\hat{P}\tilde{\Phi}(D^m + E_{21})\hat{P}^t = D^m + \nu_{ji}(m)E_{21}.$$

Set $\tilde{D} = Q^t P Q$, $\tilde{P} = Q^t \hat{P}$, $\tilde{R} = Q^t \hat{R} Q$. By the fact that $(Q^t E_{12} Q, Q^t E_{21} Q) = (E_{ij}, E_{ji})$, we have

$$\begin{aligned} \tilde{P}\Phi(\tilde{D}^m + E_{ij})\tilde{P}^t &= Q^t \hat{P}\Phi(\tilde{D}^m + E_{ij})\hat{P}^t Q = \tilde{D}^m(Q^t \hat{R} Q) + \nu_{ij}(m)E_{ij} \\ &= \tilde{D}^m \tilde{R} + \nu_{ij}(m)E_{ij} \end{aligned}$$

and

$$\begin{aligned} \tilde{P}\Phi(\tilde{D}^m + E_{ji})\tilde{P}^t &= Q^t \hat{P}\Phi(\tilde{D}^m + E_{ji})\hat{P}^t Q = \tilde{D}^m(Q^t \hat{R} Q) + \nu_{ij}(m)E_{ji} \\ &= \tilde{D}^m \tilde{R} + \nu_{ji}(m)E_{ji}. \end{aligned}$$

Using Lemma 3.3.3 (a), and comparing

$$O(\tilde{P}^m(\tilde{P} + E_{ij})) = O(\Phi(\tilde{P}^m)\Phi(\tilde{P} + E_{ij})), \quad m = 1, 2, \dots,$$

we see that $\tilde{P} = P$ and $\tilde{R} = R$. □

Lemma 3.3.5 *Let $R = \text{diag}(r_1, \dots, r_n)$ and P satisfy the conclusion of Lemma 3.3.3. Then there exist $\nu_{ij} \in \mathbf{C}$ such that $P\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$ for all (i, j) pairs, where*

$$(a) \nu_{jj} = r_j \text{ for } j = 1, \dots, n, \quad \text{and} \quad (b) \nu_{ij}\nu_{ji} = 1 \text{ if } i \neq j.$$

Consequently, $P\Phi(I)P^t = R$.

Proof. Let D, μ, R, P satisfy Lemma 3.3.3. For simplicity, assume $P = I$. Suppose $A = E_{jj}$ and $B = D^m$. Considering $O(\Phi(A)\Phi(B)) = O(AB)$ for each $m = 1, 2, \dots, n$, we see that the diagonal entries of $\Phi(A)$ are the same as that of $r_j E_{jj}$. By Lemma 3.3.4, for each $r \neq s$, there is a matrix $C = \tilde{D} + E_{sr}$ such that $R\Phi(C) = \tilde{D} + \nu_{sr}E_{sr}$, where $\nu_{sr} \in \mathbf{C}$ is nonzero, and \tilde{D} is a diagonal matrix with distinct diagonal entries. Now, considering $O(\Phi(A)\Phi(C)) = O(AC)$ for this matrix A , we see that the (r, s) entry of $\Phi(A)$ is zero. Since this is true for all $r \neq s$, we see that $\Phi(A) = r_j E_{jj}$ as asserted.

Now, suppose $i \neq j$, $A = E_{ij}$, and $B = D^m$. Because $O(\Phi(A)\Phi(B)) = O(AB)$ holds for each $m = 1, 2, \dots, n$, we see that all the diagonal entries of $\Phi(A)$ are zero. Furthermore, for any (r, s) pair with $r \neq s$ and $(r, s) \neq (i, j)$, there is a matrix $C = \tilde{D} + E_{sr}$ such that $R\Phi(C) = \tilde{D} + \nu_{sr}E_{sr}$, where $\nu_{sr} \in \mathbf{C}$ is nonzero, and \tilde{D} is a diagonal matrix with distinct diagonal entries. Once again using $O(\Phi(A)\Phi(C)) = O(AC)$, we see that the (r, s) entry of $\Phi(A)$ is zero. As a result, $\Phi(A)$ can only

has nonzero entry at the (i, j) position. Consider $O(E_{ij}E_{ji}) = O(\nu_{ij}E_{ij}\nu_{ji}E_{ji})$; we see that $\nu_{ij}\nu_{ji} = 1$.

Now, $O(IE_{ij}) = O(\Phi(I)\Phi(R))$ for all (i, j) pairs, so we get the conclusion on $\Phi(I)$. \square

We are now ready to present the

Proof of the necessity part of Theorem 3.3.1 when $\varepsilon \in (0, 1)$.

For simplicity, we assume Lemma 3.3.5 holds with $P = I$ and $r_1 = 1$. Otherwise, replace Φ by the map $X \mapsto r_1 P^{-1} \Phi(A) P$.

Case 1: $n = 2$.

Suppose $\varepsilon \neq 1/2$. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, and $\Phi(A) = B$. Since $O(AE_{ij}) = O(B\nu_{ij}\nu_{ij}E_{ij})$ by Lemma 3.3.5, we see that $B = \begin{pmatrix} 2 & \nu_{12} \\ 1/\nu_{12} & 0 \end{pmatrix}$. Since $O(A) = O(\Phi(I)\Phi(A)) = O(RB)$, we see that $|\nu_{12}|^\varepsilon |\nu_{21}|^{1-\varepsilon} = |\nu_{21}|^\varepsilon |\nu_{12}|^{1-\varepsilon}$. Thus, $|\nu_{12}| = |\nu_{21}| = 1$. Now, let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\Phi(A) = B$. Since $O(AE_{ij}) = O(B\nu_{ij}E_{ij})$ by Lemma 3.3.5, we see that $B = \begin{pmatrix} 1 & \nu_{12} \\ 1/\nu_{12} & r_2 \end{pmatrix}$. Since $O(B^2) = O(A^2)$, we see that $r_2 = 1$. The result follows.

If $\varepsilon = 1/2$, let

$$\Psi : A \mapsto r_1 \begin{pmatrix} 1/\nu_{12} & 0 \\ 0 & 1 \end{pmatrix} P^t \Phi(A) P \begin{pmatrix} \nu_{12} & 0 \\ 0 & 1 \end{pmatrix}.$$

From Lemma 3.3.5, we have $\Psi(E_{11}) = E_{11}$, $\Psi(E_{12}) = E_{12}$, $\Psi(E_{21}) = E_{21}$, and $\Psi(E_{22}) = r_2 E_{22}$ where $r_2 \in \{1, -1\}$.

Consider and let $\Psi(A) = B$. Using $O(AX) = O(B\Psi(X))$ for each $X = E_{ij}$, we see $B = \begin{pmatrix} 1 & 1 \\ 1 & r_2 \end{pmatrix}$. If $r_2 = -1$, then $O(A^2) \neq O(B^2)$ so $r_2 = 1$. Therefore, for any matrix C , $C = \Psi(C) = (D^{-1}P^t)\Phi(C)PD$ and $\Phi(C)$ is in the asserted form.

Case 2: $n \geq 3$

Assume $\varepsilon \neq 1/2$. Let $A = 3E_{11} + E_{1s} + E_{s1} + E_{1t} + E_{t1}$. From $O(AX) = O(B\Phi(X)) = \{b_{ij}\nu_{ij}, 0\}$, we determine that $B = 3E_{11} + \nu_{1s}E_{1s} + \nu_{s1}E_{s1} + \nu_{1t}E_{1t} + \nu_{t1}E_{t1}$. Using $O(A) = O(\Phi(I)\Phi(A)) = O(RB)$,

$$(|\nu_{1s}|)^\varepsilon \left(\frac{1}{|\nu_{1s}|} \right)^{1-\varepsilon} = 1.$$

Clearly, $|\nu_{1s}| = |\nu_{s1}| = 1$. Similarly, $|\nu_{1t}| = |\nu_{t1}| = 1$.

Assume $\varepsilon = 1/2$. Let $A = E_{11} + E_{1s} + E_{s1} + E_{1t} + E_{t1}$ and $\Phi(A) = B$. From $O(AX) = O(B\Phi(X)) = \{b_{ij}\nu_{ij}, 0\}$, we determine that $B = E_{11} + \nu_{1s}E_{1s} + \nu_{s1}E_{s1} +$

$\nu_{1t}E_{1t} + \nu_{t1}E_{t1}$. Consider $O(A^2) = O(B^2)$. Note $O(A^2)$ consists of 2 disks with centers 1 and 3 both with radius 2. So the disks of B^2 must have radius 2. From the disk centered at 3, we have,

$$(|\nu_{1s}| + |\nu_{1t}|)(|\nu_{s1}| + |\nu_{t1}|) = \left(\frac{1}{|\nu_{s1}|} + \frac{1}{|\nu_{t1}|} \right) (|\nu_{s1}| + |\nu_{t1}|) = 4.$$

Expanding, we see that $|\nu_{s1}|/|\nu_{t1}| = 1$, so $|\nu_{s1}| = |\nu_{t1}|$. Examining the radius of the disk centered at 1, we see that $(|\nu_{s1}| + 1)(1/|\nu_{s1}| + 1) = 4$. Therefore, $|\nu_{s1}| = 1$.

The following holds for all $\varepsilon \in (0, 1)$. Define $D = \text{diag}(1, \nu_{21}, \dots, \dots, \nu_{n1})$ and $\Psi : A \mapsto D\Phi(A)D^*$. Clearly, $\Psi(E_{11}) = E_{11}$. Let $A = E_{11} + E_{1s} + E_{s1} + E_{1t} + E_{t1} + E_{st} + E_{ts} + E_{ss} + E_{tt}$ for any $s, t \in (1, \dots, n)$. Let $\Psi(A) = B$. For simplicity, we assume $(s, t) = (2, 3)$. Using $O(AX) = O(B\Psi(X))$ for each $X = E_{ij}$, we see that

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & r_s & \nu_{st} \\ 1 & \nu_{ts} & r_t \end{pmatrix} \oplus 0_{n-3}.$$

$O(A^2)$ is a single disk centered at 3 with radius 6, which must be equal to $O(B^2)$. Therefore,

$$B^2 = \begin{pmatrix} 3 & 1 + r_s + \nu_{ts} & 1 + \nu_{st} + r_t \\ 1 + r_s + \nu_{st} & 3 & 1 + r_s\nu_{st} + r_t\nu_{st} \\ 1 + \nu_{ts} + r_t & 1 + \nu_{st}r_s + \nu_{st}r_t & 3 \end{pmatrix} \oplus 0_{n-3}.$$

If $r_s = -1$, then the maximum radius for all three disks is 4. This is also true if any of $r_t, \nu_{st}, \nu_{ts} = -1$. Note the maximum value of the off-diagonal entries is 3. Therefore, $r_s, r_t, \nu_{st}, \nu_{ts} = 1$. Since this holds for any s and t , we see $\Psi(C) = C$ for any matrix C . Since $A = \Psi(A) = D\Psi(A)D^{-1} = DRP\Phi(A)P^tD^{-1} = RDP\Phi(A)(DP)^{-1}$, Φ satisfies the conclusion of the theorem. \square

Chapter 4

Conclusion

In the previous chapters, we obtain characterization of maps satisfying

$$\mathcal{S}(\Phi(A) \circ \Phi(B)) = \mathcal{S}(A \circ B) \quad \text{for all } A, B \in M_n$$

for all $A, B \in M_n$ for the operation $A \circ B = A + B, A - B$ and AB , where $\mathcal{S}(X)$ is the Gershgorin region, Brauer region, and Ostrowski region. One may consider other binary operations on matrices such as the Jordan product: $A \circ B = AB + BA$, the Lie product: $A \circ B = AB - BA$, and the Joran triple product: $A \circ B = ABA$. Moreover, one may consider the $\mathcal{S}(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \mathcal{S}(A \circ \cdots \circ A_k)$ for all $A_1, \dots, A_k \in M_n$ for any associative binary operation \circ on matrices. One can also consider other eigenvalue inclusion regions $\mathcal{S}(A)$ of matrices. For instance, there has been some recent study on preservers of pseudospectrum of matrices.

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