

2016

## Majorana Positivity and the Fermion Sign Problem of Quantum Monte Carlo Simulations

Z. C. Wei

*Chinese Acad Sci, Inst Phys, POB 603, Beijing 100190, Peoples R China;*

T. Xiang

*Chinese Acad Sci, Inst Phys, POB 603, Beijing 100190, Peoples R China;*

Congjun Wu

*Univ Calif San Diego, Dept Phys, San Diego, CA 92093 USA;*

Shiwei Zhang

*College of William and Mary, Dept Phys, Williamsburg, VA 23187 USA*

Follow this and additional works at: <https://scholarworks.wm.edu/aspubs>

---

### Recommended Citation

Wei, Z. C., Wu, C., Li, Y., Zhang, S., & Xiang, T. (2016). Majorana positivity and the fermion sign problem of quantum Monte Carlo simulations. *Physical review letters*, 116(25), 250601.

This Article is brought to you for free and open access by the Arts and Sciences at W&M ScholarWorks. It has been accepted for inclusion in Arts & Sciences Articles by an authorized administrator of W&M ScholarWorks. For more information, please contact [scholarworks@wm.edu](mailto:scholarworks@wm.edu).

# Majorana Positivity and the Fermion Sign Problem of Quantum Monte Carlo Simulations

Z. C. Wei,<sup>1</sup> Congjun Wu,<sup>2,\*</sup> Yi Li,<sup>3</sup> Shiwei Zhang,<sup>4</sup> and T. Xiang<sup>1,5,†</sup>

<sup>1</sup>*Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing 100190, China*

<sup>2</sup>*Department of Physics, University of California, San Diego, California 92093, USA*

<sup>3</sup>*Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544*

<sup>4</sup>*Department of Physics, The College of William and Mary, Williamsburg, Virginia 23187*

<sup>5</sup>*Collaborative Innovation Center of Quantum Matter, Beijing 100190, China*

The sign problem is a major obstacle in quantum Monte Carlo simulations for many-body fermion systems. We examine this problem with a new perspective based on the Majorana reflection positivity and Majorana Kramers positivity. Two sufficient conditions are proven for the absence of the fermion sign problem. Our proof provides a unified description for all the interacting lattice fermion models previously known to be free of the sign problem based on the auxiliary field quantum Monte Carlo method. It also allows us to identify a number of new sign-problem-free interacting fermion models including, but not limited to, lattice fermion models with repulsive interactions but without particle-hole symmetry and interacting topological insulators with spin-flip terms.

PACS numbers: 02.70.Ss, 71.10.Fd, 71.27.+a

**Introduction:** A major difficulty in the study of strongly correlated systems is the exponentially large many-body Hilbert spaces which are usually difficult to handle by analytic methods. Unbiased numerical methods are therefore indispensable. Among various numerical approaches, the quantum Monte Carlo (QMC) method can yield accurate results by taking stochastic but importance sampling over very small but representative portions of the many-body Hilbert space. An advantage of QMC is that it is scalable with the system size if there is no sign problem. Unfortunately, the sign problem exists in most interacting fermion and frustrated quantum spin systems.

The origin and manifestation of the sign problem vary in different QMC algorithms. A frequently used algorithm for lattice fermions is the auxiliary field determinantal method [1, 2], in which the interaction terms are decoupled by the Hubbard-Stratonovich (HS) transformation into a superposition of quadratic fermion terms in the background of imaginary-time-dependent auxiliary fields. The fermion operators are then integrated out, yielding a fermion determinant which serves as the statistical weight for each HS field configuration. The sign problem emerges because this determinant is not always positive. In particular, the average value of the signs of these determinants often becomes exponentially small in the thermodynamic limit at low temperatures. This leads to uncontrollable statistical errors and ruins the calculation of QMC. Although a great deal of efforts have been made to solve, at least partially, this problem [3–12], a general solution is still lacking[13].

For certain classes of lattice fermion models, QMC simulations are proved to be sign-problem-free. Familiar examples include the positive- $U$  Hubbard model on a bipartite lattice at half-filling [2], the negative- $U$  Hubbard model [2, 14], and their  $SU(2N)$  generalizations [15–17].

The half-filled Kane-Mele-Hubbard model of interacting topological insulators is also sign-problem-free [18, 19]. For these models, after suitable HS decompositions with the Kramers time-reversal (TR) invariance [20, 21], each determinant is factorized into a product of two complex conjugate determinants defined in two subspaces with opposite spins. A number of non-factorizable models, such as the multi-component and multi-band Hubbard models [22–24], the negative- $U$  Hubbard models with spin-orbit coupling [25, 26], can also be shown to be free of the sign problem. In these systems, instead of the determinant itself being factorizable, the eigenvalues of the corresponding matrix are complex-conjugate paired, and real eigenvalues are doubly degenerate, thus the determinant is non-negative valued. Recently, the Majorana HS decomposition was introduced in QMC simulations [27, 28]. It is applied to spinless fermion models with repulsive interactions at the particle-hole symmetric point. For some of the above mentioned models, the positivity of fermion determinants can be understood from the algebraic structure of the orthogonal split group  $O(N, N)$  [29, 30].

The sign structures of the ground state wavefunctions of quantum lattice models are closely related to the reflection positivity of the Hamiltonian. The concept of reflection positivity was first introduced in the context of quantum field theory [31]. Its main application in condensed matter physics started from Lieb’s work on the spin reflection positivity in the Hubbard model [32, 33], and was recently applied to systems with Majorana reflection positivity [34–36]. For interacting fermion systems with these reflection positivities, it can be shown that their ground state wavefunctions are non-negative under certain suitably defined basis. However, these basis states are often non-local in real space, which is inconvenient for use in QMC simulations.

In this paper, we explore the QMC sign problem of lattice fermions from the perspective of Majorana reflection positivity and Majorana Kramers positivity defined below. We use the Majorana fermion representation, because it allows any fermionic systems, whether fermion number conserving or not, to be treated on equal footing. In the framework of the determinantal QMC algorithm, the statistical weight in the sampling is replaced by the trace of exponentials of fermion bilinears resulting from the HS decomposition, which is evaluated as a determinant. A HS decomposition is said to be a ‘‘positive decomposition’’ if all the generated determinants are positive semidefinite. Below we show that there are at least two kinds of positive decompositions which lead to QMC simulations free of the sign problem. We dub them *Majorana reflection positive decomposition* and *Majorana Kramers positive decomposition*, respectively. These do not exhaust all positive decompositions. They do, however, cover nearly all the interacting lattice fermion models that are previously known to be sign-problem-free. From these decompositions, we also identify a number of new models which are free of the sign problem.

Let us begin with the determinantal QMC algorithm for a general lattice model of Dirac fermions. The Hamiltonian  $H$  is a sum of a quadratic kinetic energy term  $H_0$  and an interaction term of 4-fermion operators  $H_I$  [1, 2, 37]. After the HS decomposition, the partition function  $Z$  is expressed as

$$Z = \text{Tre}^{-\beta H} = \lim_{M \rightarrow \infty} \sum_p \rho_p \quad (1)$$

$$\rho_p = \text{Tr} \prod_{k=1}^M e^{-\tau H_0} e^{-\tau H_I(\eta_k)}, \quad (2)$$

where  $\beta$  is the inverse temperature,  $\tau = \beta/M$  is the discrete time interval, and  $p = \{\eta_M(\{i\}), \dots, \eta_k(\{i\}), \dots, \eta_1(\{i\}), i = 1, \dots, N\}$  represents a time-sequence of the HS-field distributions, with  $N$  the lattice size. The decoupled interaction  $H_I(\eta_k)$  contains only two-fermion terms, and depends on the time-step size,  $\tau$ , and the spatial distribution of the HS fields,  $\eta_k(\{i\})$ . The value of  $\rho_p$  can be determined by tracing out the fermion degrees of freedom in  $H_0$  and  $H_I$ . The formula for determining  $\rho_p$  is given in Supplemental Material (SM)-I.

At each lattice site, a Dirac fermion can be represented using two Majorana fermions. Thus the original  $N$  Dirac fermions can be expressed in terms of  $2N$  Majorana fermions. We divide these  $2N$  Majorana fermions into two groups,  $\gamma_i^{(1)}$  and  $\gamma_i^{(2)}$  ( $1 \leq i \leq N$ ), and define their Clifford algebra operators as [36]

$$\Gamma_\alpha^+ = i^{[\frac{\alpha}{2}]} \gamma_{i_1}^{(1)} \dots \gamma_{i_m}^{(1)}, \quad \Gamma_\alpha^- = (-i)^{[\frac{\alpha}{2}]} \gamma_{i_1}^{(2)} \dots \gamma_{i_m}^{(2)}, \quad (3)$$

where  $\alpha$  represents a sequence  $\{i_1, i_2, \dots, i_m\}$  with  $1 \leq i_1 < \dots < i_m \leq N$ , and  $[x]$  equals the largest integer less

than or equal to  $x$ .  $\Gamma_\alpha^\pm$  is said to be even (odd) if  $m$  is even (odd). The reflection operation  $\theta$  is defined as an anti-linear automorphism map:  $\theta(i) = -i$ ,  $\theta(\gamma_i^{(1)}) = \gamma_i^{(2)}$  and  $\theta(\gamma_i^{(2)}) = \gamma_i^{(1)}$ . Clearly,  $\theta^2 = 1$  and  $\theta(\Gamma_\alpha^\pm) = \Gamma_\alpha^\mp$ . A bosonic operator  $O$  is Majorana reflection symmetric if  $\theta(O) = O$ , and is Majorana reflection positive if it further satisfies the condition [34, 35].

$$\text{Tr}[Q \circ \theta(Q)O] \geq 0, \quad (4)$$

where  $Q = \sum_\alpha c_\alpha \Gamma_\alpha^+$  is an arbitrary operator in the algebra spanned by the  $\Gamma^+$ -matrices with  $c_\alpha$ 's the complex coefficients, and  $Q \circ \theta(Q) = \sum_{\alpha\beta} c_\alpha c_\beta^* \Gamma_\alpha^+ \Gamma_\beta^-$ .

In the Majorana representation, the bilinear terms in the expression of  $\rho_p$ , including  $H_0$  and  $H_I(\tau_k)$ , each can be expressed as

$$H_{bl} = \gamma^T V \gamma, \quad (5)$$

where  $\gamma^T = (\gamma_i^{(1)}, \gamma_i^{(2)})^T$  and  $V$  is an  $2N \times 2N$  anti-symmetric matrix.  $V$  is the coefficient matrix of  $H_0$  or  $H_I(\tau_k)$  in the Majorana representation.

Majorana reflection positive decomposition:  $V$  is defined as a Majorana reflection positive kernel if it can be represented as

$$V = \begin{pmatrix} A & iB \\ -iB^T & A^* \end{pmatrix}, \quad (6)$$

where  $A$  and  $B$  are  $N \times N$  matrices.  $A$  is complex anti-symmetric satisfying  $A^T = -A$ .  $B = B^\dagger$  is a Hermitian matrix which is either positive semidefinite or negative semidefinite. ( $B$  can be either positive or negative semidefinite because, after a gauge transformation  $\gamma_j^{(2)} \rightarrow -\gamma_j^{(2)}$ ,  $B$  becomes  $-B$  and  $A$  remains unchanged.) A HS decomposition satisfying this condition will be called a Majorana reflection positive decomposition.

**Theorem 1**  $\rho_p$  is positive semi-definite if all the coefficient matrices of the bilinear fermion terms in Eq. (2) are Majorana reflection positive kernels.

This theorem can be proved in two steps. The first is to show that, if  $V$  is a Majorana reflection positive kernel, then  $\exp(-\tau \gamma^T V \gamma)$  is reflection positive. A proof on this was actually already given in Ref. [35]. This means that  $\rho_p$  is just the trace of a product of a series of reflection positive operators determined by the exponentials of the bilinear fermion operators  $H_0$  and  $H_I(\eta_k)$  in Eq. (2). The second step is to show that the product of a series of reflection positive operators is also reflection positive and its trace is non-negative. A proof of this, as a lemma, is given in SM-II. Combining the above results, we have  $\rho_p \geq 0$ . Thus the system is sign-problem-free in QMC simulations if all the kernels in Eq. (2) are Majorana reflection positive. The Majorana reflection can be regarded as a generalization of the PT transformation

discussed in Refs [38, 39]. However, this symmetry alone does not lead to the positivity of  $\rho_p$ . For example, if  $B$  is not positive semidefinite,  $\exp(-\tau\gamma^T V \gamma)$  remains Majorana reflection symmetric, but is no longer Majorana reflection positive. In this case,  $\rho_p$  is not always positive definite.

Despite its seeming simplicity, Theorem 1 covers all two and higher dimensional interacting fermion models previously known to be sign-problem-free in determinantal QMC simulations, without imposing explicitly the TR-invariance in the HS decomposition. These include the Hubbard model and its variations [2, 18, 19], the interacting spinless fermion model [27, 28], and other models whose coefficient matrices in the Dirac fermion representation have the orthogonal split  $O(N, N)$  group algebra structure [29]. Below we discuss two such examples, one for spinless fermions and the other for spin one-half systems, and prove they are sign problem free in new parameter regions unknown before.

The first example is an interacting spinless fermion model defined on a bipartite lattice. The model Hamiltonian is

$$H_0 = - \sum_{i,j \in A} c_i^\dagger B_{1,ij} c_j + \sum_{i,j \in B} c_i^\dagger B_{2,ij} c_j + \sum_{i \in A, j \in B} \left( c_i^\dagger F_{ij} c_j + h.c. \right), \quad (7)$$

$$H_I = \sum_{ij} V_{ij} \left( n_i - \frac{1}{2} \right) \left( n_j - \frac{1}{2} \right), \quad (8)$$

where  $n_i = c_i^\dagger c_i$ .  $B_1$  and  $B_2$  are real symmetric matrices, both of which are positive semidefinite (or, equivalently negative semidefinite).  $F$  is an arbitrary real matrix.  $V_{ij} \geq 0$  if  $i$  and  $j$  belong to different sublattices, and  $V_{ij} \leq 0$  otherwise.

$H_I$  can be decomposed into a bilinear form by taking the following HS transformation

$$e^{-\tau V_{ij} (n_i - \frac{1}{2})(n_j - \frac{1}{2})} = \frac{1}{2} e^{-\frac{\tau V_{ij}}{4}} \sum_{\eta=\pm} e^{\eta \lambda_{ij} (c_i^\dagger c_j + \nu c_j^\dagger c_i)}, \quad (9)$$

where  $\nu = -1$  if  $i$  and  $j$  belong to the same sublattices and  $\nu = +1$  otherwise. In Eq. (9),  $\eta$  is a discrete local HS field, and  $\lambda_{ij}$  is determined by the equation  $\sqrt{\nu} \lambda_{ij} = \cosh^{-1} \exp(\tau V_{ij}/2)$ .

It is simple to verify that both  $H_0$  and the decoupled interaction terms in the exponent of Eq. (9) can be cast into the form,

$$H'_{bl} = \sum_{i,j \in A} c_i^\dagger (C_{ij} - B_{1,ij}) c_j + \sum_{i,j \in B} c_i^\dagger (D_{ij} + B_{2,ij}) c_j + \sum_{i \in A, j \in B} \left( c_i^\dagger F_{ij} c_j + h.c. \right), \quad (10)$$

where  $C$  and  $D$  are real anti-symmetric matrices satisfying  $C = -C^T$  and  $D^T = -D$ . In SM-III. A, it is shown

that the matrix kernel of  $H'_{bl}$  is Majorana reflection positive. Therefore, the interacting spinless model  $H_0 + H_I$  is sign-problem-free, according to Theorem 1.

If both of  $B_1$  and  $B_2$  vanish, the above interacting fermion Hamiltonian defined on the honeycomb lattice is the model studied in Refs. [27–29]. It can be extended to include the on-site staggered chemical potential term, and remains sign-problem free [29, 40]. This is equivalent to only keeping the diagonal terms of  $B_{1,2}$ . Generally speaking, in the presence of  $B_{1,2}$ , Eq. (7) and Eq. (8) do not possess the particle-hole symmetry. For example, consider the case with  $B_{1,ij} = \mu \delta_{ij}$  if  $i \in A$  and  $j \in A$ , and  $B_{2,ij} = 0$ , which is equivalent to applying a uniform chemical potential  $\mu/2$  and a staggered on-site potential  $(-)^i \mu/2$  to the system. In the weak coupling limit, the single-particle spectrum splits into two bands and the band gap is approximately equal to  $\mu/2$ , which means that the chemical potential is located right at the bottom of the upper band. At zero temperature, the fermion density remains at half-filling. However, with increasing temperature, the fermion density begins to deviate from half-filling and the upper band is populated by fermions within an energy window of  $T$  starting from the bottom of that band. This implies that a spinless fermion model with repulsive interactions can be simulated without the sign problem away from half filling.

Now let us consider a second example, a spin-1/2 fermion model with Coulomb repulsion, spin-orbit coupling and spin-flip terms, again defined on a bipartite lattice. This is a generalized Kane-Mele-Hubbard model. The Hamiltonian  $H = H_0 + H_I$  is defined by

$$H_0 = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + i\lambda \sum_{\langle\langle ij \rangle\rangle \sigma} \sigma c_{i\sigma}^\dagger c_{j\sigma} + \sum_{ij} (-)^j h_{ij} c_{i\uparrow}^\dagger c_{j\downarrow} + h.c., \quad (11)$$

$$H_I = U \sum_i (n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2}), \quad (12)$$

where  $c_{i\sigma}$  ( $\sigma = \uparrow, \downarrow$ ) is the annihilation operator of fermion with spin  $\sigma$ .  $h_{ij}$  is a real symmetric positive (or, negative) semi-definite matrix. If we only keep the diagonal terms of  $h_{ij}$ , they reduce to an in-plane staggered magnetic field distribution. In the limit  $\lambda = 0$  and  $h_{ij} = 0$ , this Hamiltonian becomes the half-filled Hubbard model. For finite  $\lambda$  and  $h_{ij}$ , it breaks the SU(2) invariance. In the absence of  $h_{ij}$ , the  $z$ -component of total spin  $S_z$  remains conserved. In this case, Eq. (12) is known to be sign-problem-free [18, 19]. However, in the presence of  $h_{ij}$ , both the  $S_z$  conservation and the TR symmetry are broken. The previous proof for the absence of the sign problem is no longer valid [18, 19].

To show the above model is sign-problem free, let us first consider the following bilinear Hamiltonian of spin-

1/2 fermions,

$$H''_{bl} = \sum_{ij} \left( c_{i\uparrow}^\dagger M_{ij} c_{j\uparrow} + c_{i\downarrow}^\dagger M_{ij}^* c_{j\downarrow} \right) - \sum_{ij} h_{ij} \left( c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{j\downarrow} c_{i\uparrow} \right), \quad (13)$$

where  $M_{ij}$  is an arbitrary  $N \times N$  complex matrix. It can be shown that  $H''_{bl}$  is Majorana reflection positive. A proof of this is given in SM-III.B.

The Coulomb interaction  $H_I$  can be decomposed into a bilinear form by the following HS transformation,

$$e^{-\tau U(n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2})} = \frac{1}{2} e^{\frac{1}{4}\tau U} \sum_{\eta=\pm} e^{i\lambda'\eta(n_{i\uparrow} + n_{i\downarrow} - 1)}, \quad (14)$$

where  $\lambda' = \cos^{-1} \exp(-\tau U/2)$ . By taking a particle-hole transformation for the down-spin fermion operators  $c_{j\downarrow}^\dagger \rightarrow (-)^j c_{j\downarrow}$  and keeping the up-spin fermion operators unchanged, the bilinear exponent on the right hand side of Eq. (14) becomes  $i\lambda'\eta(n_{i\uparrow} - n_{i\downarrow})$ . Under the same transformation, the  $t$ - and  $\lambda$ -terms in  $H_0$  defined by Eq. (11) remain unchanged, but the  $h$ -term becomes the second term of Eq. (13). Thus  $H_0$  is also Majorana reflection positive, and the Kane-Mele-Hubbard model defined in Eq. (12) is free of the QMC sign problem according to Theorem 1.

The absence of the sign problem of the Hamiltonian defined by Eqs (11) and (12) is actually beyond the framework of TR-invariant decompositions in the Dirac fermion representation [20, 21]. It provides an opportunity to study the effect of TR-symmetry breaking in two-dimensional interacting topological insulators through QMC simulations [41, 42]. The  $h$ -terms, which flip electron spins, can arise from the scattering of magnetic impurities. The magnetic impurities on the edges of a two-dimensional topological insulator can destabilize the helical edge states by opening gaps. The interplay among interaction effects, band structure topology, and magnetic impurities, is an interesting topic that deserves further investigation.

Majorana Kramers positive decomposition: We next present a second theorem for the absence of sign problem based on the Kramers symmetry structure of Majorana fermions.

**Theorem 2**  $\rho_p$  is non-negative if there exist two transformation operators,  $S$  and  $P$ , such that

$$S^T V S = V^*, \quad (15)$$

$$P V P^{-1} = V, \quad (16)$$

where  $S$  is a real antisymmetric matrix satisfying  $S^2 = -I$  and  $S^T = -S$ ,  $P$  is a symmetric or antisymmetric Hermitian matrix satisfying  $P^2 = I$ , and  $P$  anti-commutes with  $S$ , i.e.,  $PS = -SP$ .

A proof of this theorem is given in SM-IV. The HS decomposition satisfying Eqs. (15) and (16) is termed as Majorana Kramers positive decomposition. It is a generalization of the Kramers TR-invariant decomposition used in the determinant QMC of Dirac fermions in Refs. [20, 21]. Here  $S$ , combined with complex conjugation  $C$ , defines an anti-unitary Kramers transformation operator  $T = SC$  satisfying  $T^2 = -1$ . A kernel  $V$  that satisfies Eqs. (15) and (16) is also invariant under the anti-linear transformation  $T' = PSC$ . If  $P$  is antisymmetric, then  $T'$  is also a Kramers operator, satisfying  $(T')^2 = -1$ , and  $V$  is not Majorana reflection symmetric. On the other hand, if  $P$  is symmetric, then  $(T')^2 = 1$ . In this case,  $V$  is both Majorana reflection and Kramers symmetric.

Eq. (15) ensures that the Majorana coefficient matrix  $V$  is symmetric under the TR transformation. But the symmetry alone does not assure  $\rho_p$  to be non-negative. This is because the eigenvalues of  $\prod_k \exp(-\tau V_k)$  always appear in pairs: If  $\Lambda_\alpha$  is an eigenvalue, so is  $\Lambda_\alpha^{-1}$ . Moreover, from Eq. (15), it can be shown that  $\Lambda_\alpha^*$  and  $(\Lambda_\alpha^*)^{-1}$  are also eigenvalues. If  $\Lambda_\alpha$  is modulus 1, then  $\Lambda_\alpha^* = \Lambda_\alpha^{-1}$ . In this case, these four eigenvalues reduce to two if  $\Lambda_\alpha$  is not doubly degenerate. According to the expression of  $\rho_p$  in terms of  $\Lambda_\alpha$ 's in SM-I,  $\rho_p$  may not be positive-definite. The condition defined by Eq. (16) adds an extra constraint to the Majorana coefficient matrix  $V$ . It enforces the double degeneracy of the eigenvalues of  $\prod_k \exp(-\tau V_k)$  when they are modulus one. This Kramers degeneracy assures  $\rho_p \geq 0$ .

Theorem 2 is valid independent of the specific representations of  $S$  and  $P$ . This implies that there is significant flexibility in choosing the HS decomposition scheme. Below we consider some simple realizations of  $S$  and  $P$  operators. Assuming the system contains  $N = 2L$  sites, we label the lattice sites by two indices  $(a, i)$  with  $a = 1, 2$  ( $a$  can be also regarded as the index for the orbital degrees of freedom) and  $i = 1, \dots, L$ , and the corresponding Majorana fermion operators by  $\gamma_{a,i}^{(\mu)}$  with  $\mu = 1, 2$  the index of the two Majorana fermions at each site. Operator  $S$  can then be taken as  $S = i\sigma_2 \otimes \tau_0 \otimes I$ , where  $I$  is the identity matrix in the sector of  $i$ , and  $\sigma_\alpha$  and  $\tau_\alpha$  denote the identity ( $\alpha = 0$ ) and Pauli ( $\alpha = 1, 2, 3$ ) matrices in the sectors of  $\mu$  and  $a$ , respectively. The role of  $S$  is to map  $\gamma_{a,i}^{(1)}$  to  $\gamma_{a,i}^{(2)}$  and  $\gamma_{a,i}^{(2)}$  to  $-\gamma_{a,i}^{(1)}$ .

$P$  can be either anti-symmetric or symmetric. An anti-symmetric  $P$  can be defined as  $\sigma_{1,3} \otimes \tau_2 \otimes I$ .  $S$  and  $P$  thus defined can be applied to the interacting fermion model investigated in Ref. [20, 21]. For a symmetric  $P$ , there are more choices, including any of the following,  $\sigma_1 \otimes \tau_\alpha \otimes I$  and  $\sigma_3 \otimes \tau_\alpha \otimes I$  ( $\alpha = 0, 1, 3$ ). For each of these, there exists a corresponding class of  $V$ 's satisfying Eqs. (15) and (16). For example for  $P = \sigma_1 \otimes \tau_0 \otimes I$ ,  $V$  can be generally expressed as

$$V = \sum_{\alpha} (\sigma_0 \otimes \tau_\alpha \otimes A_\alpha + i\sigma_1 \otimes \tau_\alpha \otimes B_\alpha), \quad (17)$$

where  $A_\alpha$  and  $B_\alpha$  with  $\alpha = 0, 1$ , and  $3$  are real anti-symmetric matrices, and  $A_2$  and  $B_2$  are imaginary symmetric matrices. The interacting fermion models that can be decomposed into the form of Eq. (17) would represent a new class of models without the QMC sign-problem. An example of sign-problem free Hamiltonian that satisfies Eq. (17) is given in SM. V.

**Summary:** We have shown that interacting fermion models are free of the sign problem in determinantal QMC simulations if the bilinear Hamiltonians obtained with the HS-decomposition possess the Majorana reflection positivity or the Majorana Kramers positivity. The two theorems we have proven cover all the sign-problem-free interacting lattice models that are previously known. It also allows us to identify a number of new interacting fermion models without the QMC sign problem.

**Acknowledgments** We thank L. Wang for helpful discussions. Z. C. W. and T. X. are supported by the National Natural Science Foundation of China (Grants No. 11190024 and No. 11474331) and by the National Basic Research Program of China (Grants No. 2011CB309703). C. W. is supported by the NSF DMR-1410375 and AFOSR FA9550-14-1-0168. C. W. acknowledges the CAS/SAFEA International Partnership Program for Creative Research Teams of China, and the President's Research Catalyst Awards CA-15-327861 from the University of California Office of the President. Y.L. thanks the Princeton Center for Theoretical Science at Princeton University for support. S.Z. is supported by the NSF (Grant no. DMR-1409510) and the Simons Foundation.

---

\* Electronic address: [wucj@physics.ucsd.edu](mailto:wucj@physics.ucsd.edu)

† Electronic address: [txiang@iphy.ac.cn](mailto:txiang@iphy.ac.cn)

- [1] R. Blankenbecler, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D **24**, 2278 (1981).
- [2] J. E. Hirsch, Phys. Rev. B **31**, 4403 (1985).
- [3] S. W. Zhang, eprint arXiv:cond-mat/9909090, in 'Quantum Monte Carlo Methods in Physics and Chemistry,' Ed. M.P. Nightingale and C.J. Umrigar (Kluwer Academic, Dordrecht, 1999 (1999)).
- [4] S. W. Zhang, Phys. Rev. Lett. **83**, 2777 (1999).
- [5] S. Chandrasekharan and U. J. Wiese, Phys. Rev. Lett. **83**, 3116 (1999).
- [6] S. E. Koonin, D. J. Dean, and K. Langanke, Phys. Rep. **278**, 1 (1997).
- [7] S. Rombouts, K. Heyde, and N. Jachowicz, Phys. Rev. C **58**, 3295 (1998).
- [8] M. Imada and T. Kashima, J. Phys. Soc. Jpn. **69**, 2723 (2000).
- [9] T. Kashima and M. Imada, J. Phys. Soc. Jpn. **70**, 2287 (2001).
- [10] S. Chandrasekharan and A. Li, Phys. Rev. D **85**, 091502 (2012).
- [11] M. C. Ogilvie and P. N. Meisinger, SIGMA **5**, 047 (2009).
- [12] M. C. Ogilvie, P. N. Meisinger, and T. D. Wisner, Int. J. Theo. Phys. **50**, 1042 (2010).
- [13] M. Troyer and U.-J. Wiese, Phys. Rev. Lett. **94**, 170201 (2005).
- [14] R. T. Scalettar *et al.*, Phys. Rev. Lett. **62**, 1407 (1989).
- [15] Z. Cai *et al.*, Phys. Rev. Lett. **110**, 220401 (2013).
- [16] T. C. Lang *et al.*, Phys. Rev. Lett. **111**, 066401 (2013).
- [17] D. Wang *et al.*, Phys. Rev. Lett. **112**, 156403 (2014).
- [18] D. Zheng, G.-M. Zhang, and C. Wu, Phys. Rev. B **84**, 205121 (2011).
- [19] M. Hohenadler, T. C. Lang, and F. F. Assaad, Phys. Rev. Lett. **106**, 100403 (2011).
- [20] S. Hands *et al.*, Eur. Phys. J. C **17**, 285 (2000).
- [21] C. Wu and S.-C. Zhang, Phys. Rev. B **71**, 155115 (2005).
- [22] C. Wu, J. P. Hu, and S. C. Zhang, Phys. Rev. Lett. **91**, 186402 (2003).
- [23] S. Capponi, C. Wu, and S. C. Zhang, Phys. Rev. B **70**, 220505(R) (2004).
- [24] E. Berg, M. A. Metlitski, and S. Sachdev, Science **338**, 1606 (2012).
- [25] H.-K. Tang, X. Yang, J. Sun, and H.-Q. Lin, EPL (Europhysics Letters) **107**, 40003 (2014).
- [26] H. Shi, P. Rosenberg, S. Chiesa, and S. W. Zhang, to be published (2016).
- [27] Z.-X. Li, Y.-F. Jiang, and H. Yao, Phys. Rev. B **91**, 241117 (2015).
- [28] Z.-X. Li, Y.-F. Jiang, and H. Yao, New Journal of Physics **17**, 085003 (2015).
- [29] L. Wang *et al.*, Phys. Rev. Lett. **115**, 250601 (2015).
- [30] Y.-H. Liu and L. Wang, Phys. Rev. B **92**, 235129 (2015).
- [31] K. Osterwalder and R. Schrader, Comm. Math. Phys. **31**, 83 (1973).
- [32] E. H. Lieb, Phys. Rev. Lett. **62**, 1201 (1989).
- [33] G. Tian, Journal of Statistical Physics **116**, 629 (2004).
- [34] A. Jaffe and F. L. Pedrocchi, Annales Henri Poincaré **16**, 189 (2015).
- [35] A. Jaffe and B. Janssens, Arxiv:1506.04197 .
- [36] Z.-C. Wei, X.-J. Han, Z.-Y. Xie, and T. Xiang, Phys. Rev. B **92**, 161105 (2015).
- [37] J. E. Hirsch, Phys. Rev. B **28**, 4059 (1983).
- [38] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
- [39] C. M. Bender and P. D. Mannheim, Physics Letters A **374**, 1616 (2010).
- [40] E. F. Huffman and S. Chandrasekharan, Phys. Rev. B **89**, 111101 (2014).
- [41] C. L. Kane and E. J. Mele, Physical Review Letters **95**, 146802 (2005).
- [42] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).