Mathematical and Numerical Analysis of Coupled Nonlinear Schrödinger Equations

Michael Christopher Essman

Follow this and additional works at: https://scholarworks.wm.edu/honorstheses

Part of the Mathematics Commons

Recommended Citation
https://scholarworks.wm.edu/honorstheses/714

This Honors Thesis is brought to you for free and open access by the Theses, Dissertations, & Master Projects at W&M ScholarWorks. It has been accepted for inclusion in Undergraduate Honors Theses by an authorized administrator of W&M ScholarWorks. For more information, please contact scholarworks@wm.edu.
Mathematical and Numerical Analysis of Coupled Nonlinear Schrödinger Equations

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelors of Science in Mathematics from The College of William and Mary

by

Michael Christopher Essman

Accepted for ____________________________________

Junping Shi, Director

Chi-Kwong Li

Robert Michael Lewis

Justin May

Williamsburg, VA
April 19, 2010
Mathematical and Numerical Analysis of Coupled Nonlinear Schrödinger Equations

Michael Essman

Email: mcessm@wm.edu

April 19, 2010
Abstract

Radially symmetric solutions of many important systems of partial differential equations can be reduced to systems of special ordinary differential equations. Using methods for determining explicit solutions given certain conditions and assumptions, we find and explore solutions to the one-dimensional Nonlinear Schrödinger problem. Specifically, we find a semi-trivial solution, then find explicit solutions with the methods derived from solving the semi-trivial solutions and from previous work in the field. We also developed a numerical solver for initial value problems for such systems based on Matlab, and we obtain numerical bifurcation diagrams. Various bifurcation diagrams of coupled Schrödinger equations from nonlinear physics are obtained, which suggests the uniqueness of the ground state solution.
# Contents

1 Introduction ...................................... 1  
   1.1 Background .................................. 1  
   1.2 Summary of results ............................ 4  

2 Explicit Solutions to the One-Dimensional System 5  
   2.1 Previous Results ................................ 5  
   2.2 Semi-Trivial Solutions .......................... 7  
   2.3 Explicit Solutions in One Dimension ............... 8  

3 Numerical Bifurcation Diagrams 11  
   3.1 Mathematical Setting ............................ 12  
   3.2 Numerical Methods .............................. 15  
   3.3 Numerical Bifurcation Diagrams ................. 17  
   3.4 Appendix: Program Code ....................... 22  

4 Conclusion .................................... 25  
   4.1 Conclusion .................................... 25  
   4.2 Acknowledgements .............................. 26
# List of Figures

2.1 Bright and Dark Solitons ............................... 7
2.2 Bright-Bright Solitons ............................... 10

3.1 Potential Solutions to the Initial Value NLS Problem 13
3.2 Solution Curves to the NLS Problem ................... 14
3.3 Bifurcation Diagrams of (3.2) .......................... 18
3.4 Bifurcation Diagrams of (3.2), Continued ............... 19
3.5 Ground State Solution of (3.13) ....................... 21
Chapter 1

Introduction

1.1 Background

The nonlinear Schrödinger (NLS) equation

\[ i\psi_t + \Delta \psi + \gamma |\psi|^2 \psi = 0, \]  

is a canonical and universal equation which is of major importance in continuum mechanics, plasma physics, nonlinear optics, and condensed matter (where it describes the behavior of a weakly interacting Bose gas and is known as the Gross-Pitaevskii (GP) equation). The coupled form of the above NLS equation has been receiving a lot of attention with recent experimental advances in multi-component Bose-Einstein condensates (BECs).

The Bose-Einstein condensate is a state of matter formed by a system of bosons confined in an external potential and cooled to temperatures approaching absolute zero. Under such supercooled conditions, a large fraction of the atoms collapse into the lowest quantum state of external potential. At that point, quantum effects become apparent on a macroscopic scale. The BEC has been an important issue in condensed material physics since physicists Cornell and Wieman were able to produce the condensate in 1995 \[4\] using gas composed of rubidium atoms cooled to 170 nanokelvin. Their achievement
earned them the 2001 Nobel Prize in Physics. It is well-known (see [18]) that NLS equations (or GP equations) provide a good description the behavior of the BEC’s and is the approach often applied to their theoretical analysis. Phase separation of different types of condenses has been one of recent interests from the experimental work of the Cornell and Wieman research group at NIST [27, 14]. It has also been suggested that multicomponent BECs offer the simplest tractable microscopic models in the proper universality class of cosmological systems and that solitary waves in multi-component BECs may have their analogs among cosmic strings. The two-component system is described by [16, 51, 32]:

\[
\begin{align*}
    i\hbar \frac{\partial \phi_1}{\partial t} & = \left( -\frac{\hbar^2}{2m_1} \Delta + V_1(x) + \lambda_1 |\phi_1|^2 \right) \phi_1 + \beta |\phi_2|^2 \phi_1, \\
    i\hbar \frac{\partial \phi_2}{\partial t} & = \left( -\frac{\hbar^2}{2m_2} \Delta + V_2(x) + \lambda_2 |\phi_2|^2 \right) \phi_2 + \beta |\phi_1|^2 \phi_2,
\end{align*}
\]

(1.2)

where \( x \in \mathbb{R}^n \) for \( n = 1, 2, 3 \), \( \phi_j \ (j = 1, 2) \) are the real-valued wave functions of two interacting condensates; \( V_j(x) \) are the trap potentials; and \( \lambda_j \) and \( \beta \) are the interaction strengths determined by the scattering lengths for binary collisions of like and unlike bosons. Another recent interest concerning coupled NLS equations is the propagation of soliton-like pulses in birefringent nonlinear fibers. Experiments have shown the existence of self-trapping incoherent beams in a nonlinear medium [25, 26]. Such findings are significant, because optical pulses propagating in a linear medium have a natural tendency to broaden in time (dispersion) and space (diffraction). Such broadening can be eliminated in a nonlinear medium that modifies its refractive index in the presence of light in such a way that dispersion or diffraction effects are counteracted by light-induced lensing. This can allow short pulses to propagate without changing their shape. Mathematically, propagation of solitons in nonlinear fiber couplers is described by the set of coupled nonlinear Schrödinger equations [1, 13, 24]:

\[
\begin{align*}
    i \frac{\partial \psi_j}{\partial z} & + \frac{1}{2} \frac{\partial^2 \psi_j}{\partial x^2} + \frac{1}{2} \frac{\partial^2 \psi_j}{\partial y^2} + \alpha \left( \sum_{j=1}^{K} |\psi_j|^2 \right) \psi_j = 0,
\end{align*}
\]

(1.3)

for \( j = 1, \cdots K \). Here, (complex-valued) \( \psi_j \) denotes the \( j \)-th component of the light beam, \( \alpha \) is a coefficient representing the strength of nonlinearity, \( (x, y) \) is the transverse
coordinate, $z$ is the coordinate along the direction of propagation, and $\sum |\psi_j|^2$ is the change in refractive index profile created by all the incoherent components in the light beam.

Only (1.2) is further considered, but (1.3) for $K = 2$ can also be treated similarly. Looking for pulse-like soliton solution of (1.2) of the form

$$\phi_j(t,x) = u_j(x) \exp(i\mu_j t/\hbar),$$  \hspace{1cm} (1.4)

(1.2) reduces to a system of elliptic PDEs:

$$\begin{cases}
\frac{\hbar^2}{2m_1} \Delta u_1 + V_1(x)u_1 + \lambda_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1 = \mu_1 u_1, \\
\frac{\hbar^2}{2m_2} \Delta u_2 + V_2(x)u_2 + \lambda_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2 = \mu_2 u_2.
\end{cases}$$  \hspace{1cm} (1.5)

Here, $\mu_j$ can be viewed as chemical potential. When $V_j \equiv 0$, the solutions of the homogeneous equation (1.5) are the canonical ground states. By renaming the parameters, the equation of the canonical ground states becomes

$$\begin{cases}
\Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, & x \in \mathbb{R}^n, \\
\Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, & x \in \mathbb{R}^n,
\end{cases}$$  \hspace{1cm} (1.6)

where $\lambda_i, \mu_i, \beta > 0$, and $n = 1, 2, 3$. Observe the positive solutions of (1.6) which decay to zero as $|x| \to \infty$. In this thesis we only consider (1.6).

Driven by the fascinating experiments of BECs and nonlinear optics, theoretical physicists have extensively investigated the coupled NLS equations as the main underlying theory in the last decade. Numerical simulations have produced results matching experimental data very well. Variations on the NLS equation have been observed, but exact soliton solutions are hard to obtain, especially for the higher spatial dimension case. Rigorous mathematical studies of the soliton solutions only began in recent years. It is known that such solutions are radially symmetric and decay exponentially [6]. Hence the system
to be considered is

\[
\begin{aligned}
\Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 &= 0, \quad x \in \mathbb{R}^n, \\
\Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 &= 0, \quad x \in \mathbb{R}^n, \\
u_1(x) > 0, \quad u_2(x) > 0, \quad x \in \mathbb{R}^n, \\
u_1(x) \to 0, \quad u_2(x) \to 0, \quad |x| \to \infty.
\end{aligned}
\]  

(1.7)

The existence of positive solutions of (1.7) has been considered recently by Amrosetti and Colorado [3], Bartsch, Wang and Wei [5], Dancer and Wei [11], de Figueiredo and Lopez [12], Lin and Wei [19, 20], Liu and Wang [21], Maia, Montefusco, and Pellacci [22], Sirakov [30], Wei and Weth [33, 34, 35] and many others. The methods involved in all these works are variational ones, as the solution \((u_1, u_2)\) of (1.6) is a critical point of the energy function

\[
E(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u_1|^2 + |\nabla u_2|^2 + \lambda_1 u_1^2 + \lambda_2 u_2^2) - \frac{1}{4} \int_{\mathbb{R}^n} (\mu_1 u_1^4 + 8\beta u_1^2 u_2^2 + \mu_2 u_2^4). 
\]  

(1.8)

### 1.2 Summary of results

For the case of \(n = 1\), the canonical ground states can be integrated for some values of \(\mu, \lambda, \) and \(\beta\) in (1.7). We first explore the solution to a semi-trivial (uncoupled) system. The solution to that system, along with work done previously in the field, provides a general form for the solution to the coupled system. It is found for (1.7) that a solution exists if \(\lambda_1 = \lambda_2\) and \(\beta < \min(\mu_1, \mu_2)\) or \(\beta > \max(\mu_1, \mu_2)\). In this case, a system of two bright solitons is found rather than a system of one bright soliton and one dark soliton. For most parameters and for \(n \geq 2\), however, an explicit form of the canonical ground states cannot be obtained. The numerical bifurcation diagrams generated confirm that fact. Furthermore, the numerical solver confirms the existence of ground state solutions for \(\beta < \min(\mu_1, \mu_2)\) and \(\beta > \max(\mu_1, \mu_2)\). Finally, with the numerical solver, we can approximate canonical ground states for the case \(n \geq 2\).
Chapter 2

Explicit Solutions to the One-Dimensional System

2.1 Previous Results

Explicit solutions to the general NLS system exist; however, certain assumptions must be placed on the system before such solutions are derived. We consider different coupled systems closely related to the NLS, and the results provide guidance on potential solutions for NLS systems.

Solutions for NLS equations of Kerr-type nonlinearity (in which the refractive index of a medium is dependent on the intensity of the optical pulse propagating through the medium) exist (see [1, 2]). Similar to (1.3), consider the system of equations

$$i \frac{\partial \phi_j}{\partial z} + \frac{1}{2} \frac{\partial^2 \phi_j}{\partial x^2} + \alpha \left( \sum_{j=1}^{K} |\phi_j|^2 \right) \phi_j = 0,$$

for $j = 1, \ldots, K$. Here, $\phi_j$ denotes the real-valued $j$-th component of the beam, $\alpha$ is a coefficient depicting the strength of the nonlinear effects, $x$ is the transverse coordinate, and $z$ is the coordinate along the direction of propagation.

Here we consider systems with nonzero boundary conditions, so the inverse scattering
technique will not work here. Stationary solutions to (2.1) are considered which are given by

\[ \psi_j(x, z) = \frac{1}{\sqrt{\alpha}} u_j(x) \exp(ik_j^2 z/2), \]  

(2.2)

with real valued functions for \( u_j(x) \). Now, the set of equations in (2.1) becomes a system of ordinary differential equations:

\[ \frac{d^2 u_j}{dx^2} + 2 \left[ \sum_{j=1}^{N} u_j^2 \right] u_j = k_j^2 u_j, \]  

(2.3)

which is integrable for a set of real \( \{k_j\} \). It is known from [15] that for the cases \( N = 2 \) and for \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \),

\[
\begin{align*}
    u_1''(x) + 2(u_1^2 + u_2^2)u_1 &= \lambda_1 u_1, \\
    u_2''(x) + 2(u_1^2 + u_2^2)u_2 &= \lambda_2 u_2,
\end{align*}
\]  

(2.4)

which, according to [1] yields solutions of bright and dark solitons. Here, \( u_1 \) depicts the behavior of the bright soliton and \( u_2 \) depicts the dark soliton. The solitons are on a finite background. Therefore, energy is conserved. Our solutions for \( u_1 \) and \( u_2 \) are the following:

\[ u_1(x) = \frac{\sqrt{a_2}}{\cosh(\sqrt{f} x)}, \quad u_2(x) = \frac{\sqrt{\lambda_2} \tanh(\sqrt{f} x)}{\sqrt{2}}. \]  

(2.5)

Here, \( f = \lambda_1 - \lambda_2 \) and \( a_2 = \frac{1}{2}(2\lambda_1 - \lambda_2) \) are both constants. An example is shown in Fig. 2.1. We notice the bright-dark soliton solution does not satisfy a zero boundary condition at \( \pm \infty \), and this solution only exists when \( \mu_1 = \mu_2 = \beta = 2 \) and \( \lambda_1 \neq \lambda_2 \) in (1.6). In this paper we look for solutions of (1.7), which are bright-bright solitons.
Figure 2.1: The bright and dark solitons. Observe that $u_1$, the bright soliton, is an even function while $u_2$, the dark soliton, is an odd function. Here, $\lambda_1 = 4$ and $\lambda_2 = 1$.

### 2.2 Semi-Trivial Solutions

Recall the canonic form of the NLS equation in one dimension:

$$
\begin{cases}
    u_1'' - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, & x > 0, \\
    u_2'' - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, & x > 0, \\
    u_1'(0) = 0, & u_1'(x) < 0, & u_1(x) > 0, & \lim_{|x|\to\infty} u_1(x) = 0, \\
    u_2'(0) = 0, & u_2'(x) < 0, & u_2(x) > 0, & \lim_{|x|\to\infty} u_2(x) = 0.
\end{cases}
$$

(2.6)

We find a new explicit solution to the equation. We commence investigation with the semi-trivial solution where exactly one of $u_1$ or $u_2$ is 0. This generates two possible systems:

$$
U_1 = \begin{cases}
    u_1'' - \lambda_1 u_1 + \mu_1 u_1^3 = 0, \\
    u_1'(0) = 0, & u_1'(x) < 0, \\
    u_1(x) > 0, & \lim_{|x|\to\infty} u_1(x) = 0.
\end{cases}
$$

(2.7)
\[ U_2 = \begin{cases} 
 u''_2 - \lambda_2 u_2 + \mu_2 u_2^3 = 0, \\
 u'_2(0) = 0, \quad u'_2(x) < 0, \\
 u_2(x) > 0, \quad \lim_{|x| \to \infty} u_2(x) = 0. 
\end{cases} \] (2.8)

Both (2.7) and (2.8) reduce to a scalar equation:
\[ u'' - \lambda u + \mu u^3 = 0, \quad u(x) > 0, \quad u'(x) \leq 0, \quad x > 0, \quad \lim_{|x| \to \infty} u(x) = 0. \] (2.9)

It is known from [2] that (2.9) has an explicit solution:
\[ u(x) = \sqrt{\frac{2\lambda}{\mu}} \frac{1}{\cosh(\sqrt{\lambda}x)}. \] (2.10)

This is the similar bright soliton in (2.5).

Hence, if we define \( u_j(x; \lambda_j, \mu_j) = \sqrt{\frac{2\lambda_j}{\mu_j}} (\cosh(\sqrt{\lambda_j}x))^{-1} \) for each \( j = 1, 2 \), then \( U_1 = (u_1, 0) \) and \( U_2 = (0, u_2) \) are solved. Therefore, when the two equations in the system are decoupled, an explicit solution with a zero-component is found, and we call such solutions semi-trivial solutions.

### 2.3 Explicit Solutions in One Dimension

Observe, however, a non-trivial solution may exist when certain assumptions are imposed on the system. Recall the solutions \( u_1 \) and \( u_2 \) to (2.4) takes the respective forms found in (2.5). For (2.4), the parameters \( \mu_1, \mu_2, \) and \( \beta \) are all equal to 2 in (1.6). Some more general approaches are taken in [15]. We notice that the form of \( u_1 \) in (2.5) involves a hyperbolic trigonometric function in the form of \( 1/\cosh \). A more general form of \( u_1 \) inspired by the solution presented in (2.5) is considered as a possible solution to the system (2.6).

Suppose \( u_1(x) = a_1[\cosh(bx)]^{-1} \) and \( u_2(x) = a_2[\cosh(bx)]^{-1} \). These satisfy all of the initial conditions imposed in (2.6). The solution to both equations is found if the following
equations are satisfied:

\[
\begin{align*}
\frac{a_1(b^2\cosh(bx))^2 - 2b^2 - \lambda_1 \cosh(bx)^2 + \mu_1 a_1^2 + \beta a_2^2)}{\cosh(bx)^3} &= 0, \\
\frac{a_2(b^2\cosh(bx))^2 - 2b^2 - \lambda_2 \cosh(bx)^2 + \mu_2 a_2^2 + \beta a_1^2)}{\cosh(bx)^3} &= 0.
\end{align*}
\] (2.11)

Since \(u_i(x) > 0\) for \(i = 1, 2, \beta > 0\), and \(a_1\) and \(a_2\) are not equal to zero (else the system becomes semi-trivial as above), the following equations yield a solution:

\[
\begin{align*}
b^2 \cosh(bx)^2 - 2b^2 - \lambda_1 \cosh(bx)^2 + \mu_1 a_1^2 + \beta a_2^2 &= 0, \\
b^2 \cosh(bx)^2 - 2b^2 - \lambda_2 \cosh(bx)^2 + \mu_2 a_2^2 + \beta a_1^2 &= 0.
\end{align*}
\] (2.12)

This generates a system of four equations which must be satisfied for the solution:

\[
\begin{align*}
b^2 - \lambda_1 &= 0, \\
b^2 - \lambda_2 &= 0, \\
-2b^2 + \mu_1 a_1^2 + \beta a_2^2 &= 0, \\
-2b^2 + \mu_2 a_2^2 + \beta a_1^2 &= 0.
\end{align*}
\] (2.13)

In this case, the system is satisfied if \(2b^2 = 2\lambda_1 = 2\lambda_2 = \mu_1 a_1^2 + \beta a_2^2 = \mu_2 a_2^2 + \beta a_1^2\). Assume \(\mu_1\) and \(\mu_2\) are fixed. Furthermore, assume \(\lambda_1 = \lambda_2\) (henceforth referred to simply as \(\lambda\)). Without loss of generality, it is assumed that \(\mu_1 \geq \mu_2\). Also, \(b = \sqrt{\lambda}\). Using simple elimination, it followes that \(a_1 = \sqrt{\frac{2\lambda(\beta - \mu_1)}{\beta^2 - \mu_1 \mu_2}}\) and \(a_2 = \sqrt{\frac{2\lambda(\beta - \mu_2)}{\beta^2 - \mu_1 \mu_2}}\). Since neither \(a_1\) nor \(a_2\) can be a negative value (and \(a_1 = 0\) or \(a_2 = 0\) yields the semi-trivial solution explored previously), it follows that \(\beta\) must satisfy \(\beta > 0\) and \(\beta < \min\{\mu_1, \mu_2\}\) or \(\beta > \max\{\mu_1, \mu_2\}\). Else, a positive solution to (2.6) is not found when \(\beta\) falls within the range \((\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\})\) since either \(a_1\) or \(a_2\) becomes complex. Given those constraints, we find a positive solution to (2.6) of the form

\[
\begin{align*}
u_1(x) &= \frac{\sqrt{2\lambda(\beta - \mu_1)}}{\sqrt{\lambda x}} ,
\nu_2(x) &= \frac{\sqrt{2\lambda(\beta - \mu_2)}}{\sqrt{\lambda x}}.
\end{align*}
\] (2.14)
Figure 2.2: Two bright solitons. Observe that again, both solitons are even functions. Here, $\lambda = 4$, $\mu_1 = 2$, and $\mu_2 = 4$. For the figure on the left, $\beta = 1$, while for the figure on the right, $\beta = 10$.

See Figure 2.2 for graphs of two examples of bright-bright solitons. Note that both $u_1$ and $u_2$ take the form $1/\cosh$, are maximized at $x = 0$. When $\beta < \min\{u_1, u_2\}$, the height of $u_1$ is larger than that of $u_2$, and the order is reversed when $\beta > \max\{u_1, u_2\}$. 
Chapter 3

Numerical Bifurcation Diagrams

The explicit solutions in Chapter 2 cannot be found in dimension $n \geq 2$. In this chapter, we use numerical methods to study the ground state solutions of the coupled NLS. Since the solutions of (1.7) are radially symmetric, they satisfy

$$\begin{cases} 
  u''_1 + \frac{n-1}{r} u'_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, & r > 0, \\
  u''_2 + \frac{n-1}{r} u'_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, & r > 0, \\
  u'_1(0) = 0, & u'_1(r) < 0, & \lim_{r \to \infty} u_1(r) = 0, \\
  u'_2(0) = 0, & u'_2(r) < 0, & \lim_{r \to \infty} u_2(r) = 0.
\end{cases} \tag{3.1}$$

In particular, the solution satisfies the initial value problem:

$$\begin{cases} 
  u''_1 + \frac{n-1}{r} u'_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, & r > 0, \\
  u''_2 + \frac{n-1}{r} u'_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, & r > 0, \\
  u_1(0) = A > 0, & u'_1(0) = 0, \\
  u_2(0) = B > 0, & u'_2(0) = 0.
\end{cases} \tag{3.2}$$

The initial value problem (3.2) is considered here. The basic mathematical setting is given in Section 3.1; the numerical method is introduced in Section 3.2; and numerical bifurcation diagrams for (3.2) are presented along with further observations in Section 11.
3.3. The \texttt{Matlab} program for generating the bifurcation diagrams is included in Section 3.4. The results indicate that for all possible parameters in (3.1), the positive solution of (3.1) is \textit{unique}. This has not been proved for general coupled Schrödinger equations.

### 3.1 Mathematical Setting

The initial value problem (3.2) is considered. The local existence and uniqueness of the solution to (3.2) can be proved via a standard application of the contraction mapping principle (see, for example, [28], Lemma 2.1). The solution of (3.2) is denoted by \((u_1(r; A, B), u_2(r; A, B))\), or simply \((u_1(r), u_2(r))\) when there is no confusion. The solution \((u_1(r), u_2(r))\) can be extended to a maximal interval \((0, R)\) such that \(u_1(r) > 0\) and \(u_2(r) > 0\) in \((0, R)\). Note that this includes the case where \((u_1(r), u_2(r))\) can be extended to \(t = R\) and \(u_1(R)u_2(R) = 0\).

Two types of solutions are important here. If

\[
\begin{align*}
  u(r) > 0, & \quad v(r) > 0, \quad u'(r) < 0, \quad v'(r) < 0, \quad 0 < r < \infty, \\
  u(R) = v(R) = 0,
\end{align*}
\]

then \((u(r), v(r))\) is a \textit{ground state solution}; if

\[
\begin{align*}
  u(r) > 0, & \quad v(r) > 0, \quad u'(r) < 0, \quad v'(r) < 0, \quad 0 < r < R, \\
  u(R) = v(R) = 0,
\end{align*}
\]

then \((u(r), v(r))\) is a \textit{crossing solution}. From the result of [6], any solution of (1.7) is radially symmetric thus a solution of (3.2) satisfying (3.3), and any solution on a ball \(B_R\) is also radially symmetric.

Define

\[
\begin{align*}
  f(u_1, u_2) \equiv F_1(u_1, u_2) &= -\lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1^2 u_2, \\
  g(u_1, u_2) \equiv F_2(u_1, u_2) &= -\lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1 u_2^2.
\end{align*}
\]

The set \(\{f(u_1, u_2) = 0\}\) consists of the line \(\{u_1 = 0\}\) and the ellipse \(E_1 = \{\mu_1 u_1^3 + \beta u_1^2 = 0\}\), and the set \(\{g(u_1, u_2) = 0\}\) consists of the line \(\{u_2 = 0\}\) and the ellipse
\[ E_2 = \{ \beta u_1^2 + \mu_2 u_2^2 = \lambda_2 \}. \]

Let

\[ \beta_1 = \min \left\{ \frac{\lambda_2}{\lambda_1} \mu_1, \frac{\lambda_1}{\lambda_2} \mu_2 \right\}, \quad \text{and} \quad \beta_2 = \max \left\{ \frac{\lambda_2}{\lambda_1} \mu_1, \frac{\lambda_1}{\lambda_2} \mu_2 \right\}. \]  

(3.6)

Then it is easy to show that when \( 0 < \beta < \beta_1 \) and \( \beta > \beta_2 \), \( E_1 \) and \( E_2 \) intersect exactly once in the first quadrant, and when \( \beta_1 < \beta < \beta_2 \), \( E_1 \) and \( E_2 \) do not intersect; hence, one ellipse is inside the other one (see Figure 3.1). In the first case, the unique intersection point of \( f = 0 \) and \( g = 0 \) is the global minimizer of the potential function \( F(u_1, u_2) \) in the first quadrant. It is assumed that \( \min F(u_1, u_2) = -\gamma_1 < 0 \). Define \( F_\gamma = \{(u_1, u_2) \in \mathbb{R}_+^2 : F(u_1, u_2) \leq \gamma \} \); then, there exists a \( 0 < \gamma_2 < \gamma_1 \) such that when \( -\gamma_1 < \gamma < -\gamma_2 \), \( F_\gamma \) is a connected closed subset.

Define the following regions in \( \mathbb{R}_+^2 \):

\[ I = \{(u_1, u_2) \in \mathbb{R}_+^2 : f(u_1, u_2) > 0, g(u_1, u_2) > 0 \}, \]

\[ II = \{(u_1, u_2) \in \mathbb{R}_+^2 : f(u_1, u_2) < 0, g(u_1, u_2) < 0 \}, \]

\[ III = \{(u_1, u_2) \in \mathbb{R}_+^2 : f(u_1, u_2) < 0, g(u_1, u_2) > 0 \}, \]

\[ IV = \{(u_1, u_2) \in \mathbb{R}_+^2 : f(u_1, u_2) > 0, g(u_1, u_2) < 0 \}. \]

(3.7)

Figure 3.1: The regions of possible initial values (A, B): solid lines are \( f(u_1, u_2) = 0 \) and \( g(u_1, u_2) = 0 \) respectively; and the dashed line is \( F(u_1, u_2) = 0 \). (left): \( 0 < \beta < \beta_1 \) and \( \beta > \beta_2 \); (right) \( \beta_1 < \beta < \beta_2 \).
For \((A, B) \in II \cup III \cup IV\), \(u' > 0\) or \(v' > 0\), hence \((u, v)\) cannot be a ground state or a crossing solution. For \((A, B) \in I\), \(u' < 0\) and \(v' < 0\); thus

\[ T = T(A, B) = \sup\{r > 0 : u_1(r) > 0, u_2(r) > 0, u_1'(r) < 0, u_2'(r) < 0, \ r \in (0, T)\} \]

exists. \(I\) is partitioned into the following classes:

\[ B = \{(A, B) \in I : T < \infty, u_1(T) = 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) < 0\}, \]
\[ G = \{(A, B) \in I : T < \infty, u_1(T) > 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) < 0\}, \]
\[ R = \{(A, B) \in I : T < \infty, u_1(T) > 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) < 0\}, \]
\[ Y = \{(A, B) \in I : T < \infty, u_1(T) > 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) < 0\}, \]
\[ Q = \{(A, B) \in I : T = \infty, \lim_{r \to \infty} u_1(r) = \lim_{r \to \infty} u_2(r) = 0\}, \]
\[ P = I \setminus (B \cup G \cup R \cup Y \cup Q). \]

Then each of \(B, G, R, Y\) is an open subset of \(\mathbb{R}^2_+\) if it is non-empty. The boundary between \(B\) and \(G\) represents the initial values for crossing solutions, and the crossing time \(T\) satisfies \(u_1(T) = u_2(T) = 0\), and each element in \(Q\) is a ground state solution.

![Figure 3.2: Solution curves of \((3.2)\) when \(n = 3, \mu_1 = \lambda_1 = \lambda_2 = 1, \mu_2 = 2, \beta = 0.01\). Initial values: \(u(0) = 8\) in all three; (left) \(v(0) = 9\) \((u(R) = 0\) and \(v(R) > 0\)); (middle) \(v(0) = 11.4\) \((u(R) = v(R) = 0,\) crossing solution\); (right) \(v(0) = 13\) \((u(R) > 0\) and \(v(R) = 0)\).]
3.2 Numerical Methods

Computational methods also solve initial value problems like (3.2). Indeed, a more general problem is considered:

\[
\begin{align*}
  u_1'' + \frac{n-1}{r} u_1' + f(u_1, u_2) &= 0, \quad r > 0, \\
  u_2'' + \frac{n-1}{r} u_2' + g(u_1, u_2) &= 0, \quad r > 0, \\
  u_1(0) &= A > 0, \quad u_1'(0) = 0, \\
  u_2(0) &= B > 0, \quad u_2'(0) = 0.
\end{align*}
\] (3.9)

The system (3.9) is expanded from two second-order differential equations into a system of four first-order differential equations

\[
\begin{align*}
  u_1' &= v_1, \\
  v_1' &= -\frac{n-1}{r} u_1' - f(u_1, u_2), \\
  u_2' &= v_2, \\
  v_2' &= -\frac{n-1}{r} u_2' - g(u_1, u_2), \\
  u_1(0) &= A > 0, \quad v_1(0) = 0, \\
  u_2(0) &= B > 0, \quad v_2(0) = 0.
\end{align*}
\] (3.10)

The space of initial values \{\(A, B\) : \(A_b \leq A \leq A_e, B_b \leq B \leq B_e\}\) is then discretized to a two-dimensional data structure:

\[
\{(A_i, B_j) : 0 \leq i \leq n, \ 0 \leq j \leq m\},
\]

where \(A_i = A_b + (i/n)(A_e - A_b)\) and \(B_j = B_b + (j/n)(B_e - B_b)\). Then for each initial value \((A_i, B_j)\), (3.10) is solved by using an ODE solver in Matlab, until the solution reaches a stopping time which is defined as

\[
T = \sup \{r > 0 : u_1(r)u_1'(r)u_2(r)u_2'(r) \neq 0\}.
\] (3.11)
In fact, the stopping time is only detected if the initial value \((A, B)\) satisfies

\[ f(A, B) > 0, \quad \text{and} \quad g(A, B) > 0; \]

(3.12)

that is, if \((A, B)\) belongs to region I defined in (3.7). If \((A, B) \in II \cup III \cup IV\), then initially \(u'(r) > 0\) or \(v'(r) > 0\) for small \(r > 0\), and the solution cannot satisfy (3.8) or (3.9). On the bifurcation graph, the color cyan represents the data point \((A_i, B_j)\) if \((A_i, B_j) \in II \cup III \cup IV\).

On the other hand, if the initial value \((A_i, B_j) \in IV\), then for some \(\varepsilon > 0\), \(u_1(r), u_2(r) > 0\) and \(u_1'(r), u_2'(r) < 0\) for \(r \in (0, \varepsilon)\), hence \(T\) is well-defined. As the solution reaches \(T\), the data point is colored according to the classification in (3.8): blue for \(u_1(T) = 0\); green for \(u_1'(T) = 0\); red for \(u_2(T) = 0\); and yellow for \(u_2'(T) = 0\). Notice that it is certainly possible to have both \(u_1\) and \(u_2\) equaling zero simultaneously, but in general such initial values \((A, B)\) only form boundary curves on \(\mathbb{R}_+^2\) between the open subsets \(B, G, R, Y\) and the cyan region \(C = II \cup III \cup IV\).

In the bifurcation diagrams (see for example, Fig. 3.3), cyan areas are bordered by the highlighted curves \(f(u, v) = 0\) and \(g(u, v) = 0\). Furthermore, the boundary curve between the red and blue regions gives all initial values for the crossing solution for which \(u_1(T) = u_2(T) = 0\); the boundary curve between the yellow and green regions gives all initial values for which \(u_1'(T) = u_2'(T) = 0\), which indeed give radially symmetric solutions satisfying the Neumann boundary condition of (3.9). In both diagrams in Fig 3.3, there is only one point common to the closure of all four (red, blue, green, yellow) regions, and that point is exactly the one corresponding to the ground state solution. Note that (3.9) cannot have a solution with \(u_1(T) = u_2(T) = u_1'(T) = u_2'(T) = 0\) for finite \(T\) from the uniqueness of the solution to ODE.
3.3 Numerical Bifurcation Diagrams

By using the numerical scheme described in Section 3, the distribution of the qualitative behavior of solutions to the shooting problem (3.9) can be investigated.

From the discussion of the nonlinearities \( f(u, v) \) and \( g(u, v) \) in the coupled Schrödinger equations in Section 2, two possible bifurcation points arise:

\[
\beta_1 = \min \left\{ \frac{\lambda_2}{\lambda_1} \mu_1, \frac{\lambda_1}{\lambda_2} \mu_2 \right\}, \quad \text{and} \quad \beta_2 = \max \left\{ \frac{\lambda_2}{\lambda_1} \mu_1, \frac{\lambda_1}{\lambda_2} \mu_2 \right\}.
\]

In \([3, 5, 12]\), two other possible bifurcation points are identified. Let \( \phi_a \) be the unique positive radially symmetric solution of

\[
\begin{cases}
\Delta \phi - a \phi + \phi^3 = 0, & x \in \mathbb{R}^n, \\
\phi(x) \to 0, & |x| \to \infty.
\end{cases}
\] (3.13)

For \( \eta > 0 \) define

\[
- \nu_1(\eta) = \text{principal eigenvalue of the operator } M_0(k) = -\Delta k - \eta \phi_0 k.
\] (3.14)

For the existence and uniqueness of \( \phi_a \), refer to \([6]\). It is also known \((6)\) that \( M_0 \) has a unique positive eigenvalue \( \nu_1(\eta) \), and the property of \( \nu_1(\eta) \) can be found in \([12]\). Then Theorem 1.1 of \([12]\) shows that when \( n = 1, 2, 3 \), there exist \( 0 < \beta_1^* < \beta_2^* < \infty \) such that when \( 0 < \beta < \beta_1^* \) and \( \beta > \beta_2^* \), (1.7) has a solution. Here, if \( \lambda_1 = 1 \), then the \( \beta_i^* \) satisfy

\[
\beta_1^* = \min \{ \beta_a, \beta_b \}, \quad \beta_2^* = \max \{ \beta_a, \beta_b \},
\]

where \( \nu_1(\frac{\beta_a}{\mu_1}) = \lambda_2, \quad \nu_1(\frac{\beta_a}{\mu_2}) = \frac{1}{\lambda_2}. \) (3.15)

One can show that (see \([12]\) Theorems 1.2 and 1.3)

\[
0 < \beta_1 < \beta_1^* < \beta_2^* < \beta_2.
\]
Figure 3.3: Bifurcation diagrams of (3.2). The coordinates are \((A, B)\), the initial values in (3.9). Here \(0 \leq A, B \leq 10, 300 \times 300\) points in \((A, B) \in [0, 10]^2\) are sampled, \(n = 3, \mu_1 = \mu_2 = 1, \lambda_1 = 1, \lambda_2 = 2\).
Figure 3.4: Bifurcation diagrams of (3.2). The coordinates are \((A, B)\), the initial values in (3.9). Here (except (a)) \(0 \leq A, B \leq 5\), 300 \times 300 points in \((A, B) \in [0, 5]^2\) are sampled, \(n = 3, \mu_1 = \mu_2 = 1, \lambda_1 = 1, \lambda_2 = 2\). In (a), \(0 \leq A, B \leq 10\).
This existence result can be seen in the numerical bifurcation diagrams of the shooting problem (3.9). In the numerical experiment, \((\lambda_1, \lambda_2, \mu_2, \mu_2) = (1, 2, 1, 1)\) is fixed, and \(\beta\) is a free parameter. Hence \(\beta_1 = 0.5\) and \(\beta_2 = 2\). In Fig. 3.3 one can see that \(\beta_1^* \approx 0.85\). As \(\beta \to \beta_1^*\), the green region (for which \(u_1'(T) = 0\)) shrinks to empty near \((A, B) = (0, 6)\). This indicates a convergence of the ground states of the system to the semi-trivial state \((u_1(r), u_2(r)) = (0, \phi_2)\), where \(\phi_2\) is the unique solution of (3.13) with the same \(\lambda_2\). From Fig. 3.3 \(\phi_2(0) \approx 6.13\). This also confirms the bifurcation diagram suggested in [3, 5].

In Fig. 3.3 observe that the structure of the bifurcation diagrams undergoes several topological changes as \(\beta\) increases from \(\beta = 0\) to \(\beta = 1\). When \(0 < \beta < \beta_1\), the cyan region borders the green region along the curve \(-1 + u_1^2 + \beta u_2^2 = 0\), and borders the yellow region by the curve \(-2 + u_2^2 + \beta u_1^2 = 0\). Hence the boundary curve between red-green region and blue-yellow region connects with the intersection point of \(-1 + u_1^2 + \beta u_2^2 = 0\) and \(-2 + u_2^2 + \beta u_1^2 = 0\) (see Fig. 3.3 (a), (b)). For \(\beta_1 < \beta < \beta_1^*\), only the yellow region encircles the non-admissible region in the lower-left corner, and the yellow region also shares a boundary with \(u_1 = 0\) while the green region shrinks (see Fig. 3.3 (c), (d)). For \(\beta > \beta_1^*\), the blue region reaches the vertical boundary \(u_1 = 0\), and it separates the yellow and red regions (see Fig. 3.3 (e), (f)).

For \(\beta_1^* (\approx 0.85) < \beta_1 < \beta_2^* (\approx 1.2)\), it has been conjectured that (1.7) has no ground state solution. Fig. 3.3 (f) appears to support that claim as the green region (where \(u_1'(T) = 0\)) is empty for this parameter range. At \(\beta = 1\), the red region is also absent on the diagram (even with an expanded plotting region). Hence the bifurcation diagram consists solely of blue and yellow regions (see Fig. 3.4 (a)).

As \(\beta\) increases from \(\beta = 1\), a similar sequence of bifurcations occurs; see Fig. 3.4 (b)-(f). Here, the plotting window \((A, B) \in [0, 5] \times [0, 5]\) grants a better view of the systems’ behavior. The red region reappears as \(\beta\) increases from \(\beta = 1\) but from the right lower corner. At \(\beta = \beta_2^* \approx 1.2\), the blue region branches off the \(u_1\)-axis. This blue region represents the bifurcation from the semitrivial solution \((u_1(r), u_2(r)) = (\phi_1, 0)\). In Fig.
\[ \phi_2(0) \approx 4.32, \] that is, exactly the last point of the blue region in contact with \( u_1 \)-axis (Fig. 3.4 (b)). As \( \beta \) crosses \( \beta_2^* \), a green region emerges from the \( u_1 \)-axis, and a ground state bifurcates from the semitrivial solution. Again, the ground state is indicated by the unique point to the closure of all four regions (Fig. 3.4 (c)). At \( \beta = \beta_2 = 2 \) (Fig. 3.4 (d)), the green region reaches to the cyan region of non-admissible initial values, and for \( \beta > \beta_2 \), the yellow and green regions encircle the non-admissible region in the lower-left corner (Fig. 3.4 (e)). But one can see that when \( \beta \) is large, the boundaries of the yellow and green regions do not intersect the \( u_1 \) and \( u_2 \) axes, which is different from the small \( \beta \) case (\( \beta = 0.01 \) and \( \beta = 0.2 \) in Fig. 3.3 (a), (b)).

![Figure 3.5: Ground state of (3.13) when \( n = 3 \). (left) \( a = 2 \), ground state \( \phi(0) \approx 6.13; \) (right) \( a = 1 \), ground state \( \phi(0) \approx 4.32. \)](image)

Observations of numerical bifurcation diagrams are summarized as follows along with a few conjectures:

1. The shooting problem (3.9), in general, possesses four types of solutions with stopping condition \( u_i(T) = 0 \) or \( u_i'(T) = 0 \) for \( i = 1, 2 \), and the set of solutions satisfying these conditions is an open subset of \( \mathbb{R}^2_+ \).

2. The absence of at least one type of region implies the nonexistence of a ground state solution of the coupled Schrödinger equation (1.7). This occurs when \( \beta \in (\beta_1^*, \beta_2^*) \). Such a non-existence result has not yet been proved rigorously for any non-trivial case, to the best of our knowledge.
3. The common boundary point of the four regions is a ground state solution of (1.7), and in all of the numerical experiments here, the ground state solution appears to be *unique*. In general, the uniqueness of the ground state of (1.7) is not known except for some special cases (see [9, 17, 23]), and the nondegeneracy of the ground state is studied in [11].

4. When the ground state exists, a monotone increasing curve in $\mathbb{R}^2_+$ separates the blue-yellow and green-red regions. This curve contains all solutions satisfying $u_1(T) = u_2(T) = 0$ or $u'_1(T) = u'_2(T) = 0$. The monotonicity of such a curve has been proved in some similar problems [9, 10].

### 3.4 Appendix: Program Code

Here is the *Matlab* code for generating the bifurcation diagrams:

```matlab
function CSUMSprog4

b= ; %this is the first initial vb:c:values tested for A and B (the user must input these)
c= ; %this is the number of steps (the user must input these)
d= ; %this is the final initial value tested for A and B (the user must input these)

eps = .0000001;
Q = linspace(b,d,c); %this maps a vector of initial values
\text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}
S = repmat(Q,[c 1]); %this remaps the vector to make a new matrix
\text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}
T = transpose(S); % This flips S
\text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}
A(:,:,1)=S; %This stores A's values into a matrix
\text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}
A(:,:,2)=T; %this stores B's values into a matrix
\text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}

\text{tstart} = 0.0001;
\text{tfinal} = 20;

\text{hold on} %this is so it graphs all the points instead of erasing them

for q=1:1:c;
    for r=1:1:c;
        y0=[A(q,r,1),0,A(q,r,2),0];
        opts_events = odeset('Events',@events,'OutputSel',1); %sets the events used in ode113
        \text{user\_entry} = \text{input('dummy value: will not be saved')} \text{'}

    end
end
```

22
[t,y,te,ye,ie] = ode113(@CSUMS1,[tstart,tfinal],y0,opts_events); %runs ode

user_entry = input('dummy value: will not be saved ')

if size(ie)==[0,0]
    A(q,r,3)=0;
elseif size(ye)==[0,0]
    A(q,r,3)=0;
elseif size(ye)==[2,4]
    if abs(min(ye(2,1))-min(ye(2,3)))<eps
        A(q,r,3)=ye(1,1);
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(2,1))-min(ye(2,4)))<eps
        A(q,r,3)=ye(1,1);
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(2,2))-min(ye(2,3)))<eps
        A(q,r,3)=ye(1,3);
        plot(A(q,r,1),A(q,r,2),'.b');
    elseif abs(min(ye(2,2))-min(ye(2,4)))<eps
        A(q,r,3)=0;
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(2,1))) < eps %if y(1) reaches 0, it stores y(1) and plots a blue point
        A(q,r,3) = ye(1,1); %this records value y1
        plot(A(q,r,1),A(q,r,2),'.b');
    elseif abs(min(ye(2,2))) < eps %if y(2) reaches 0, it stores y(2) and plots a green point
        A(q,r,3) = ye(1,1); %this records value y2
        plot(A(q,r,1),A(q,r,2),'.g');
    elseif abs(min(ye(2,3))) < eps %if y(3) or y(4) reaches 0, it stores y(3) and plots a red point
        A(q,r,3) = ye(1,3);
        plot(A(q,r,1),A(q,r,2),'.r');
    else %if everything else worked correctly, this will plot a yellow point if the slope of the second line reaches 0
        A(q,r,3) = ye(1,3);
        plot(A(q,r,1),A(q,r,2),'.y');
    end
else
    if abs(min(ye(:,1))-min(ye(:,3)))<eps
        A(q,r,3)=ye(1,1);
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(:,1))-min(ye(:,4)))<eps
        A(q,r,3)=ye(1,1);
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(:,2))-min(ye(:,3)))<eps
        A(q,r,3)=ye(1,3);
        plot(A(q,r,1),A(q,r,2),'.b');
    elseif abs(min(ye(:,2))-min(ye(:,4)))<eps
        A(q,r,3)=0;
        plot(A(q,r,1),A(q,r,2),'.c');
    elseif abs(min(ye(:,3))) < eps %if y(1) reaches 0, it stores y(1) and plots a blue point
        A(q,r,3) = ye(1,1); %this records value y1
        plot(A(q,r,1),A(q,r,2),'.b');
    elseif abs(min(ye(:,2))) < eps %if y(2) reaches 0, it stores y(2) and plots a green point
        A(q,r,3) = ye(1,1); %this records value y2
        plot(A(q,r,1),A(q,r,2),'.g');
    elseif abs(min(ye(:,3))) < eps %if y(3) or y(4) reaches 0, it stores y(3) and plots a red point
        A(q,r,3) = ye(1,3);
        plot(A(q,r,1),A(q,r,2),'.r');
    end
end
else %nothing goes here;
  A(q,r,3) = ye(1,3);
  plot(A(q,r,1),A(q,r,2),'y.'),
end
end
end

hold off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dy = CSUMS1 (t,y)
  n = ; %the user must input these
  mu1 = ; %the user must input these
  mu2 = ; %the user must input these
  delta1 = ; %the user must input these
  delta2 = ; %the user must input these
  beta = ; %the user must input these
  p = ; %the user must input these
  q = ; %the user must input these

dy = zeros(4,1);
  dy(1) = y(2);
  dy(2) = -(n-1)*y(2)/t + delta1*y(1) - mu1*y(1)^(p+1) - beta*y(1)*(y(3)^2);
  dy(3) = y(4);
  dy(4) = -(n-1)*y(4)/t + delta2*y(3) - mu2*y(3)^(q+1) - beta*y(3)*(y(1)^2);

f=(-delta1+mu1*y(1)^(p+1)*y(3)^2);
g=(-delta2+mu2*y(1)^(q+1)*y(3)^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [value,isterminal,direction] = events(t,y)
  value = [y(1),y(2),y(3),y(4)]; %the event is recorded at y(1),y(2),y(3),y(4)
  isterminal = [1,1,1,1]; %terminates function once a zero is reached by either function
  direction = [-1,1,-1,1]; %y(1),y(3) value is initially positive, crossing zero to negative;
  %y(2),y(4) value goes from negative to positive as
  %y(1),y(3) are minimized
Chapter 4

Conclusion

4.1 Conclusion

We studied soliton solutions of coupled Schrödinger equations, which is an important model in nonlinear optics and quantum physics. An explicit solution exists to the semi-trivial (uncoupled) systems (2.7) and (2.8) in one dimension. Specifically, the solution takes the form of hyperbolic trigonometric functions (namely $u_j = \sqrt{2\lambda_j \mu_j} (\cosh(\sqrt{\lambda_j} x))^{-1}$).

Furthermore, through the work of Ahkmediev, Kroólikowski and Snyder in [2], the solution to the coupled system found in (2.6) is known for $\mu_1 = \mu_2 = \beta = 2$ when $\lambda_1$ and $\lambda_2$ are both free parameters. This gives a system of one bright soliton (represented by $u_1$) and one dark soliton (represented by $u_2$). When $\lambda_1 = \lambda_2$, though, the system yields two bright solitons satisfying the ground state conditions. Solutions when $\lambda_1 = \lambda_2$ do not necessarily exist, however. It must be that either $\beta < \min(\mu_1, \mu_2)$ or $\beta > \max(\mu_1, \mu_2)$ in that case.

We developed a numerical solver using Matlab that confirms this result for higher dimensional case $n \geq 2$, for which explicit solutions cannot be found. We base the numerical method on solving the initial value problem of the system of ODEs in (3.10), and the solver detects the point where one component of the solution loses its positivity.
or monotonicity. The program then plots the numerical bifurcation diagram using the information how the solution is terminated, and the common boundary point of the different regions of initial values (colored according to their solution behavior) is the one for the ground state solution (bright-bright soliton). The solver further bolsters the conjecture that positive ground state solutions are unique, although they do not necessarily exist as evident in Figure 3.4.

The new explicit solutions found here present much future work. Certainly, we can directly test an explicit solution with an approximate solution found by the numerical bifurcation method. Also, the new explicit solutions need to be tested for energy conservation; otherwise, they have no physical analog.

4.2 Acknowledgements

Thanks to the National Science Foundation (NSF) for providing support for this work under the CSUMS grant.

Furthermore, thanks to Professor Shi for his guidance and encouragement through this entire process. Also, thanks to Professors Lewis and Phillips who helped develop the Matlab program central to this paper.

Finally, thanks to my parents, Brian and Ellen. I could not have been able to achieve what I have without their constant love and support.


[9] Chern, Jann-Long; Lin, Chang-Shou; Shi, Junping, Uniqueness of solution to a

[10] Chern, Jann-Long; Chen, Zhi-You; Tang, Yong-Li; Lin, Chang-Shou; Shi, Jun-
ping, On the Uniqueness and Structure of Solutions to a Coupled Elliptic System.

equations with attractive interaction. Trans. Amer. Math. Soc. 361 (2009), no. 3,
1189–1208.

[12] de Figueiredo, Djairo G.; Lopes, Orlando, Solitary waves for some nonlinear
1, 149–161.


[14] Hall, D. S.; Matthews, M. R.; Ensher, J. R.; Wieman, C. E.; Cornell, E. A., Dy-
namics of component separation in a binary mixture of Bose-Einstein condensates.


[16] Ho Tin-Lun; Shenoy, V. B., Binary mixtures of Bose condensates of Alkali atoms.

[17] Li, Congming; Ma, Li, Uniqueness of positive bound states to Schrödinger systems

[18] Lieb, Elliott H.; Seiringer, Robert, Derivation of the Gross-Pitaevskii equation for

28


