2016

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Dedicated to Professor Hans Schneider.

Abstract

Let A be an irreducible (entrywise) nonnegative $n \times n$ matrix with eigenvalues

$$\rho, \lambda_2 = b + ic, \lambda_3 = b - ic, \lambda_4, \cdots, \lambda_n,$$

where ρ is the Perron eigenvalue. It is shown that for any $t \in [0, \infty)$ there is a nonnegative matrix with eigenvalues

$$\rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n,$$

whenever $\tilde{t} \geqslant \gamma_n t$ with $\gamma_3 = 1, \gamma_4 = 2, \gamma_5 = \sqrt{5}$ and $\gamma_n = 2.25$ for $n \geqslant 6$. The result improves that of Guo et al. Our proof depends on an auxiliary result in geometry asserting that the area of an n-sided convex polygon is bounded by γ_n times the maximum area of a triangle lying inside the polygon.

2000 Mathematics Subject Classification. 15A48, 15A18.

Key words and phrases. Non-negative matrices, Perron eigenvalue, perturbation.

1 Introduction

The *nonnegative inverse eigenvalue problem* concerns the study of necessary and sufficient conditions for a given set of complex numbers $\lambda_1, \ldots, \lambda_n$ to be the eigenvalues of an (entrywise) nonnegative matrix. This problem has attracted the attention of many authors, and is still open; for example, see [4] and its references. In connection to this study, researchers study the change of the Perron eigenvalue under the perturbation of the other real or complex eigenvalues of a given nonnegative matrix. Here are several results in this direction.

(1) In [6], the author proved the following:

Suppose $\rho, \lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of an $n \times n$ nonnegative matrix A such that ρ is the Perron eigenvalue, and λ_2 is real. Then for any $0 \leqslant t \leqslant \tilde{t}$, there is a nonnegative matrix with eigenvalues $\rho + \tilde{t}, \lambda_2 \pm t, \lambda_3, \cdots, \lambda_n$.

(2) Laffey [9] and Guo et al. [5] obtained the following independently:

Suppose $\rho, \lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of an $n \times n$ nonnegative matrix A such that ρ is the Perron eigenvalue, and $(\lambda_2, \lambda_3) = (b + ic, b - ic)$ is a (non-real) complex conjugate pair. Then for any $\tilde{t}, t \in [0, \infty)$ with $2t \leqslant \tilde{t}$, there is a nonnegative matrix with eigenvalues $\rho + \tilde{t}, \lambda_2 - t, \lambda_3 - t, \lambda_4 \cdots, \lambda_n$.

(3) In [5, Proposition 3.1], Guo and Guo showed that:

Suppose $\rho, \lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of an $n \times n$ nonnegative matrix A such that ρ is the Perron eigenvalue, and $(\lambda_2, \lambda_3) = (b + ic, b - ic)$ is a (non-real) complex conjugate pair. Then for any $\tilde{t}, t \in [0, \infty)$ with $4t \leqslant \tilde{t}$, there is a nonnegative matrix with eigenvalues $\rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n$.

The authors also pose the problem of finding the smallest constant c for which the above result holds with $4t$ replaced by ct. In [3] Cronin and Laffey show that $c = 1$ for $n = 3$, $c = 2$ for $n = 4$ and $c \geqslant 2$ for $n \geqslant 5$. They further show that for $c > 2$, the result holds for sufficiently small t but the question about arbitrary t is left open.

The results in (1) and (2) above were shown to be optimal in the sense that the conclusion may fail if $\tilde{t} < t$ in (1) and $\tilde{t} < 2t$ in (2). However, the result in (3) may be strengthened. In this paper, we improve the third result, and prove the following.

Theorem 1.1. *Suppose $\rho, \lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of an $n \times n$ nonnegative matrix A such that ρ is the Perron eigenvalue, and $\lambda_2 = b + ic$ and $\lambda_3 = b - ic$ are (non-real) complex conjugate pairs. Then for any $t \in [0, \infty)$ there is a nonnegative matrix with eigenvalues*

$$\rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n,$$

whenever $\tilde{t} \geqslant \gamma_n t$ with $\gamma_3 = 1, \gamma_4 = 2, \gamma_5 = \sqrt{5}$ and $\gamma_n = 2.25$ for $n \geqslant 6$.

Our proof depends on the following geometrical result, which is of independent interest [7].

Proposition 1.2. *Suppose $n \in \{3, 4, 5, 6\}$. The area of an n-sided convex hexagon $\mathcal{P} \subseteq \mathbb{R}^2$ is bounded by γ_n times the maximum area of the triangles lying inside $\mathcal{P}$, where*

$$\gamma_3 = 1, \ \gamma_4 = 2, \ \gamma_5 = \sqrt{5}, \ \gamma_6 = 2.25,$$

and these bounds are best possible.

One easily sees that the maximum area of the triangles lying inside a convex polygon is attained at a triangle formed by 3 of the vertices of the polygon.

The proof of Theorem 1.1 is given in Section 2, and the technical proof of Proposition 1.2 and some remarks are given in Section 3.

2 Proof of Theorem 1.1

We begin with two lemmas. The first one can be found in [8].

Lemma 2.1. *Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a nonnegative matrix. Then there is a nonnegative matrix with constant row sums with eigenvalues $\lambda_1, \ldots, \lambda_n$.*

The next lemma concerns the change of r eigenvalues, $\lambda_1, \ldots, \lambda_r$ with $r < n$, and leaving invariant the other eigenvalues of an $n \times n$ matrix A by a rank-r perturbation. It can be viewed as an extension of the result in [10]; see also [2, Theorems 27 and 33].

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $X = [x_1|x_2|\cdots|x_r] \in \mathbb{C}^{n\times r}$ be such that $\operatorname{rank}(X) = r$ and $AX = XD$, where $D \in \mathbb{C}^{r\times r}$ with eigenvalues $\lambda_1, \cdots, \lambda_r$. Then for any $r \times n$ matrix C, the matrix $A + XC$ has eigenvalues $\mu_1, \cdots, \mu_r, \lambda_{r+1}, \cdots, \lambda_n$, where $\mu_1, \cdots, \mu_r$ are eigenvalues of the matrix $D + CX$.*

Proof. Let $S = [X|Y]$ be a nonsingular matrix with $S^{-1} = \begin{bmatrix} U \\ V \end{bmatrix}$, with $U \in \mathbb{C}^{r\times n}$. Then $UX = I_r, VY = I_{n-r}$ and $(VX)^t = UY = O_{r\times(n-r)}$. Because $AX = XD$, we have

$$S^{-1}AS = \begin{bmatrix} U \\ V \end{bmatrix} A[X, Y] = \begin{bmatrix} D & UAY \\ 0 & VAY \end{bmatrix} \tag{2.1}$$

and

$$S^{-1}XCS = \begin{bmatrix} I_r \\ 0 \end{bmatrix} CS = \begin{bmatrix} C \\ 0 \end{bmatrix} [X|Y] = \begin{bmatrix} CX & CY \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$S^{-1}(A + XC)S = S^{-1}AS + S^{-1}XCS = \begin{bmatrix} D + CX & UAY + CY \\ 0 & VAY \end{bmatrix}.$$

Now, from (2.1) we have $\sigma(VAY) = \{\lambda_{r+1}, \cdots, \lambda_n\}$ and therefore

$$\sigma(A + XC) = \sigma(D + CX) \cup \{\lambda_{r+1}, \cdots, \lambda_n\}.$$

□

We are now ready to present the proof of Theorem 1.1.

Let $A \in \Omega_\rho$ be an $n \times n$ non-negative real matrix with eigenvalues $\rho, b+ic, b-ic, \lambda_4, \cdots, \lambda_n$, and let $u\pm iv$ be eigenvectors of A corresponding to the eigenvalues $b\pm ic$, where $u = (u_1, u_2, \cdots, u_n)^T, v = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n$. Then we have the following equality for $n \times 2$ matrices:

$$A[u|v] = [u|v] \begin{bmatrix} b & c \\ -c & b \end{bmatrix}. \tag{2.2}$$

We adopt an idea in [5] and let

$$M = \begin{bmatrix} 1 & \cdots & 1 \\ u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{bmatrix}.$$

Denote by $P = P(u, v)$ a point in $\mathbb{R}^2$ with co-ordinate (u, v). By Analytic Geometry, suppose

$$\det(i, j, k) = \det\left(\begin{bmatrix} 1 & 1 & 1 \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{bmatrix}\right), \qquad 1 \leqslant i,\ j,\ k \leqslant n.$$

Then $|\det(i, j, k)|$ is 2 times the area of the triangle with vertices $P_i(u_i, v_i)$, $P_j(u_j, v_j)$ and $P_k(u_k, v_k)$. Moreover, $\det(i, j, k) > 0$ if and only if the points $P_i \to P_j \to P_k \to P_i$ are not collinear and appear in counterclockwise direction in $\mathbb{R}^2$.

Replacing (A, u, v) by (QAQ^T, Qu, Qv) for a suitable permutation matrix Q, we may assume that

$$\varDelta = \det(1, 2, 3) = \max_{1 \leqslant i,j,k \leqslant n} \det(i, j, k). \tag{2.3}$$

Recall that $e = (1, \ldots, 1)^T$. Since $e, u + iv, u - iv$ are the eigenvectors of the distinct eigenvalues $\rho, \lambda_2, \lambda_3$, so e, u, v are linearly independent over $\mathbb{R}$. It follows that $\varDelta = \det(1, 2, 3) > 0$. Let

$$x = (x_1, x_2, x_3, 0, \cdots, 0)^T \quad \text{and} \quad y = (y_1, y_2, y_3, 0, \cdots, 0)^T$$

satisfy

$$x^T e = 0,\ x^T u = 1,\ x^T v = 0;\ y^T e = 0,\ y^T u = 0,\ y^T v = 1, \tag{2.4}$$

that is,

$$\begin{bmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} x_1 = \frac{1}{\varDelta}(v_2 - v_3),\ x_2 = \frac{1}{\varDelta}(v_3 - v_1),\ x_3 = \frac{1}{\varDelta}(v_1 - v_2), \\ y_1 = \frac{1}{\varDelta}(u_3 - u_2),\ y_2 = \frac{1}{\varDelta}(u_1 - u_3),\ y_3 = \frac{1}{\varDelta}(u_2 - u_1). \end{aligned} \tag{2.5}$$

and

$$[x, y]^T [u, v] = I_2.$$

Suppose

$$[u|v][x|y]^T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & \cdots & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & 0 & \cdots & 0 \end{bmatrix}. \tag{2.6}$$

Then for $i = 1, \ldots, n$,

$$\alpha_{i1} = u_i x_1 + v_i y_1 = \frac{1}{\varDelta}\det(i, 2, 3) - \frac{1}{\varDelta}(u_2 v_3 - u_3 v_2),$$

$$\alpha_{i2} = u_i x_2 + v_i y_2 = \frac{1}{\Delta}\det(1,i,3) - \frac{1}{\Delta}(u_3 v_1 - u_1 v_3),$$

$$\alpha_{i3} = u_i x_3 + v_i y_3 = \frac{1}{\Delta}\det(1,2,i) - \frac{1}{\Delta}(u_1 v_2 - u_2 v_1).$$

If

$$c_{i1} = \alpha_{i1} - \alpha_{21} = \frac{1}{\Delta}\det(i,2,3),\ c_{i2} = \alpha_{i2} - \alpha_{32} = \frac{1}{\Delta}\det(1,i,3),\ c_{i3} = \alpha_{i3} - \alpha_{23} = \frac{1}{\Delta}\det(1,2,i),$$

then

$$c_{11} \geqslant c_{i1}, \quad c_{22} \geqslant c_{i2}, \quad c_{33} \geqslant c_{i3}, \tag{2.7}$$

because $\Delta = \det(1,2,3) \geqslant \det(i,j,k)$ for all $1 \leqslant i,j,k \leqslant n$. Let

$$c_{i1} = \min_{l=1,2\ldots,n} c_{l1}, \quad c_{j2} = \min_{l=1,2\ldots,n} c_{l2}, \quad c_{k3} = \min_{l=1,2\ldots,n} c_{l3}.$$

Then $c_{i1} \leqslant c_{l1},\ c_{j2} \leqslant c_{l2}$ and $c_{k3} \leqslant c_{l3}$ for all $l = 1,2\ldots,n$. Therefore, we have

$$\alpha_{i1} \leqslant \alpha_{l1},\ \alpha_{j2} \leqslant \alpha_{l2} \text{ and } \alpha_{k3} \leqslant \alpha_{l3} \text{ for all } l = 1,2\ldots,n. \tag{2.8}$$

Assume that $n \geqslant 6$, and that $1,2,3,i,j,k$ are distinct, and focus on

$$\tilde{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ u_1 & u_2 & u_3 & u_i & u_j & u_k \\ v_1 & v_2 & v_3 & v_i & v_j & v_k \end{bmatrix}. \tag{2.9}$$

Note that for the following points in $\mathbb{R}^2$,

$$P_1(u_1,v_1),\ P_2(u_2,v_2),\ P_3(u_3,v_3),\ P_i(u_i,v_i),\ P_j(u_j,v_j),\ P_k(u_k,v_k),$$

- the area of a triangle formed by any three of these points is not larger than $\dfrac{\det(1,2,3)}{2}$, which is the area of the triangle with vertices P_1, P_2, P_3;
- $c_{i1} \leqslant \frac{\det(2,2,3)}{\Delta} = \frac{\det(2,3,3)}{\Delta} = 0,\ \ c_{j2} \leqslant \frac{\det(1,1,3)}{\Delta} = \frac{\det(1,3,3)}{\Delta} = 0,\ \ c_{k3} \leqslant \frac{\det(1,2,1)}{\Delta} = \frac{\det(1,2,2)}{\Delta} = 0.$

Thus, $\det(i,2,3),\ \det(1,j,3),\ \det(1,2,k) \in (-\infty,0]$. Note that $\det(r,s,t) \leqslant 0$ if and only if $P_r,\ P_s,\ P_t$ are collinear or they are in clockwise direction. Let ℓ_1 (respectively, $\ell_2,\ \ell_3$) be the line through P_1 (respectively, $P_2,\ P_3$) parallel to $\overline{P_2P_3}$, (respectively, $\overline{P_1P_3},\ \overline{P_1P_2}$). Suppose ℓ_2 and ℓ_3 (respectively, ℓ_1 and ℓ_3, ℓ_1 and ℓ_2) intersect at Q_1 (respectively, Q_2 and Q_3). Since $\det(i,2,3) \leqslant 0$ and $|\det(1,2,i)|,\ |\det(1,3,i)| \leqslant \det(1,2,3)$, P_i lies in the triangle $Q_1P_3P_2$. Similarly, P_j and P_k lie in the triangles $P_1P_3Q_2$ and $P_1Q_3P_2$ respectively. Thus $P_1P_kP_2P_iP_3P_j$ is a convex hexagon (including the degenerate cases, when it is a triangle, quadrilateral or pentagon). Moreover, the vertices $P_1,\ P_j,\ P_3,\ P_i,\ P_2,\ P_k,\ P_1$ are in clockwise direction. By Proposition 1.2,

$$\frac{5}{4} \geqslant \frac{1}{\Delta}\left(|\det(i,2,3)| + |\det(1,j,3)| + |\det(1,2,k)|\right) = -(c_{i1} + c_{j2} + c_{k3}) \geqslant 0.$$

It follows that

$$-1 \geqslant \alpha_{i1} + \alpha_{j2} + \alpha_{k3} = c_{i1} + \alpha_{21} + c_{j2} + \alpha_{32} + c_{k3} + \alpha_{23} \geqslant -\frac{5}{4} - 1 = -2.25. \tag{2.10}$$

Suppose $\tilde{t} \geqslant 2.25t \geqslant 0$. Let

$$\delta = \frac{\tilde{t} + t(\alpha_{i1} + \alpha_{j2} + \alpha_{k3})}{3} \geqslant \frac{\tilde{t} - 2.25t}{3} \geqslant 0$$

Set

$$z = (-t\alpha_{i1} + \delta,\ -t\alpha_{j2} + \delta,\ -t\alpha_{k3} + \delta, 0, \cdots, 0)^T \quad \text{and } \tilde{A} = A + [e|u|v][z|tx|ty]^T.$$

By direct computation, we have

$$[z|tx|ty]^T[e|u|v] = \begin{bmatrix} \tilde{t} & * & * \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}.$$

By Lemma 2.2, the eigenvalues of $\tilde{A}$ are $\rho + \tilde{t},\ \sigma_2,\ \sigma_3,\ \lambda_4, \cdots,\ \lambda_n$, where $\sigma_2,\ \sigma_3$ are the eigenvalues of $\begin{bmatrix} b & c \\ -c & b \end{bmatrix} + tI_2$, that is, $\sigma_2 = b + t + ic,\ \sigma_3 = b + t - ic$.

Let

$$[e|u|v][z|tx|ty]^T = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & 0 & \cdots & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & 0 & \cdots & 0 \end{bmatrix}$$

By (2.8), we have

$$\begin{aligned} \beta_{l1} = &\quad t(\alpha_{l1} - \alpha_{i1}) + \delta \geqslant 0 \\ \beta_{l2} = &\quad t(\alpha_{l2} - \alpha_{j2}) + \delta \geqslant 0 \\ \beta_{l3} = &\quad t(\alpha_{l3} - \alpha_{k3}) + \delta \geqslant 0. \end{aligned}$$

Thus, $\tilde{A}$ also has nonnegative entries. Hence, $\tilde{A}$ is the desired matrix.

Suppose $n = 5,\ 4,\ 3$. Then the matrix $\tilde{M}$ in (2.9) has at most n columns. Nevertheless, we can apply a similar argument and use the corresponding result in Proposition 1.2 to construct the desired matrix $\tilde{A}$. We omit the details. □

3 Proof of Proposition 1.2

The purpose of this section is to prove the Proposition 1.2. The results for $n = 3$ is trivial.

We will assume that $P_1, \ldots, P_n$ are vertices of the convex polygon arranged in counterclockwise direction. The following two facts are useful in our discussion.

(a) One can apply an affine transformation $v \mapsto Tv + v_0$ for some invertible 2×2 matrix T and $v_0 \in \mathbb{R}^2$ to the points $P_1, \ldots, P_n$ without affecting the hypothesis and conclusion of the result.

(b) One can always find an affine map to send any 3 vertices of the polygon to any 3 non-collinear points.

Suppose $n = 4$. One may apply an affine transformation and assume that $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$ are the vertices of the triangle of largest area. Since all the triangles inside the quadrilateral have area at most $1/2$, the fourth vertex is in the triangle with vertices $(0,0), (1,1), (0,1)$. The conclusion of Proposition 1.2 follows readily.

Suppose $n = 5$ and $P_1, \ldots, P_5$ are vertices of a convex pentagon arranged in counterclockwise direction. Let T be a triangle of largest area.

Case 1. T has two sides in common with the pentagon. We may assume that $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$ are the vertices of T. Then P_4 and P_5 have to lie in the triangle with vertices $(1,0)$, $(1,1)$, $(0,1)$ and the conclusion of Proposition 1.2 follows readily.

Case 2. T has only one side in common with the pentagon. We may assume that $P_1 = (0,0)$, $P_2 = (1,0)$, $P_4 = (0,1)$ are the vertices of T. Then we have

(a) $P_3 = (u_3, v_3)$ lies in the triangle with vertices $(1,0)$, $(1,1)$, $(0,1)$, and

(b) $P_5 = (-u_5, v_5)$ lies in the triangle with vertices $(0,0)$, $(0,1)$, $(-1,1)$.

By applying the affine transformation $(x,\ y) \mapsto (1-(x+y),\ y)$, if necessary, we may assume that $v_3 \geqslant v_5$. For the convenience of calculation, we will use $\varDelta(i,j,k)$ to denote twice the area of the triangle with vertices P_i, P_j, P_k. We will show that subject to the constraints (a), (b) and $\varDelta(2,3,5) \leqslant 1$, we have $\varDelta(1,2,4) + \varDelta(2,3,4) + \varDelta(1,4,5) \leqslant \sqrt{5}$, where the equality holds at $(u_3, v_3) = (2, \sqrt{5}-1)/2$ and $(-u_5, v_5) = (1-\sqrt{5}, \sqrt{5}-1)/2$.

By direct calculation, we have

$$\varDelta(2,3,5) = v_3(1+u_5) - (1-u_3)v_5 \quad \text{ and}$$

$$\varDelta(1,2,4) + \varDelta(2,3,4) + \varDelta(1,4,5) = u_3 + u_5 + v_3\,.$$

So we need to show that subject to the constraints

$$u_3 \leqslant 1 \leqslant u_3 + v_3,\ 0 \leqslant u_5 \leqslant v_5 \leqslant v_3 \leqslant 1,\ v_3(1+u_5) - (1-u_3)v_5 \leqslant 1\,, \tag{3.1}$$

the maximum value of $u_3 + u_5 + v_3$ is $\sqrt{5}$.

We can replace v_5 by v_3 without changing $u_3 + u_5 + v_3$ or violating the constraints. So we will assume that $v_5 = v_3$. Then the constraints in (3.1) becomes

$$u_3 \leqslant 1 \leqslant u_3 + v_3,\ 0 \leqslant u_5 \leqslant v_3 \leqslant 1,\ (u_3 + u_5)v_3 \leqslant 1$$

So we have $u_3 + u_5 \leqslant 1 + v_3$ and $\dfrac{1}{v_3}$. Therefore, for fixed $0 \leqslant v_3 \leqslant 1$, the maximum of $u_3 + u_5 + v_3$ is equal to $1 + 2v_3$, if $1 + v_3 \leqslant \dfrac{1}{v_3} \Leftrightarrow v_3 \leqslant \dfrac{\sqrt{5}-1}{2}$, and $v_3 + \dfrac{1}{v_3}$ if $1 + v_3 \leqslant \dfrac{1}{v_3} \Leftrightarrow v_3 \geqslant \dfrac{\sqrt{5}-1}{2}$. Maximizing over v_3 in both cases, we have the maximum value $\sqrt{5}$ attained at $v_3 = \dfrac{\sqrt{5}-1}{2}$. Thus the maximum of $u_3 + u_5 + v_3$ is attained at $u_3 = 1,\ u_5 = v_3 = v_5 = \dfrac{\sqrt{5}-1}{2}$. We note that for these values of $u_3,\ u_5,\ v_3,\ v_5$, we actually have $\Delta(i,j,k) \leqslant 1$ for all $1 \leqslant i < j < k \leqslant 5$.

Finally, we consider the intricate case when $n = 6$. Suppose a (non-degenerate) convex hexagon has vertices $P_1(x_1, y_1), \dots, P_6(x_6, y_6)$ arranged in counterclockwise direction. We will prove that

$$\frac{\text{Area of the hexagon with vertices } P_1,\ P_2, \dots, P_6}{\max\{\text{Area of triangle with vertices } P_i,\ P_j,\ P_k : 1 \leqslant i < j < k \leqslant 6\}} \leqslant \frac{9}{4}, \tag{3.2}$$

where the inequality becomes an equality for the hexagon $\mathcal{H}_0$ with vertices

$$(0,0),\ (1,0),\ (\frac{5}{6}, \frac{2}{3}),\ (0,1),\ (-\frac{1}{4}, 1),\ (-\frac{2}{3}, \frac{2}{3}).$$

Note that a direct calculation shows that the area of the triangle with vertices $(0,0),\ (1,0),\ (0,1)$ is $\frac{1}{2}$, which is maximum among all triangles with vertices from $\mathcal{H}_0$.

Lemma 3.1. *Suppose the maximum of the left hand side of (3.2) is attained at some hexagon $\mathcal{H}$ with vertices $P_1, \dots, P_6$. Then*

$$\max\{\textit{Area of triangle with vertices } P_i,\ P_j,\ P_k : 1 \leqslant i < j < k \leqslant 6\}$$

is attained at some triangle with at least one side in common with the boundary of $\mathcal{H}$.

Proof. Let M be the maximum of the left hand side of (3.2) over all (non-degenerate) convex hexagon. Clearly, M exists and $\frac{9}{4} \leqslant M \leqslant 4$.

Suppose the maximum of the left hand side of (3.2) is attained at some hexagon $\mathcal{H}$ with vertices $P_1, \dots, P_6$, labeled in counterclockwise direction. We are going to prove the result by contradiction.

Suppose the maximum of the area of triangles with vertices $P_i,\ P_j,\ P_k,\ 1 \leqslant i < j < k \leqslant 6$ can only be attained at triangles with no side in common with the hexagon $\mathcal{H}$. Without loss of generality, we may assume that the maximum is attained at the triangle with vertices $P_1,\ P_3,\ P_5$. Using an affine transformation, we may assume that $P_1 = (0,0)$, $P_3 = (1,0)$ and $P_5 = (0,1)$. For the convenience of notation and computation, let

$$\Delta(i,j,k) = 2 \times (\text{area of triangle with vertices } P_i,\ P_j,\ P_k)$$

for $1 \leqslant i < j < k \leqslant 6$. By our assumption, we have

$$\Delta(1,3,5) = 1,\ \Delta(2,4,6) \leqslant 1 \ \text{ and } \Delta(i,j,k) < 1 \ \text{ for all } (i,j,k) \neq (1,3,5),\ (2,4,6). \tag{3.3}$$

We will prove that under the conditions in (3.3), the area of the hexagon $\mathcal{H}$ is less than or equal to 1, which contradicts the fact that $M \geqslant \frac{9}{4}$ as shown by our example before Lemma 3.1.

In the following, we will prove that under the conditions in (3.3), we have

$$\Delta_0 = \Delta(1,2,3) + \Delta(3,4,5) + \Delta(1,5,6) \leqslant 1 \tag{3.4}$$

Suppose $P_2 = (u_1, -v_1)$, $P_4 = (u_2, v_2)$ and $P_6 = (-u_3, v_3)$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & u_1 & 1 & u_2 & 0 & -u_3 \\ 0 & -v_1 & 0 & v_2 & 1 & v_3 \end{bmatrix}.$$

Then $|\Delta(i,j,k)|$ is equal to the determinant of the submatrix of A lying in columns i, j, k. By (3.3), we have

$$\begin{aligned} \Delta(1,3,5) &= 1 \text{ is the maximum, among all } \Delta(i,j,k) \\ \Delta(2,4,6) &= (u_2 - u_1)(v_1 + v_3) + (u_1 + u_3)(v_1 + v_2) \leqslant 1, \quad \text{and} \end{aligned} \tag{3.5}$$

$$0 \leqslant v_1 < u_1 < 1, \qquad u_2 < 1,\ v_2 < 1, \quad u_2 + v_2 \geqslant 1, \quad 0 \leqslant u_3 < v_3 < 1\,.$$

By direct computation, we have

$$\Delta_0 = u_2 + u_3 + v_1 + v_2 - 1.$$

Note that the area of the triangle with vertices P_i, P_j, P_k will not change if we replace P_i by $P_i + d(P_j - P_k)$ for any $d \in \mathbb{R}$. Thus, $\Delta(1,3,5)$ will not be affected and $\Delta(2,4,6)$ will not change under the following transformations:

1. $(u_1, v_1, u_2, v_2, u_3, v_3) \to (u_1 + (u_2 + u_3)d, v_1 + (v_3 - v_2)d, u_2, v_2, u_3, v_3)$,
2. $(u_1, v_1, u_2, v_2, u_3, v_3) \to (u_1, v_1, u_2 + (u_1 + u_3)d, v_2 - (v_1 + v_3)d, u_3, v_3)$,
3. $(u_1, v_1, u_2, v_2, u_3, v_3) \to (u_1, v_1, u_2, v_2, u_3 + (u_1 - u_2)d, v_3 + (v_1 + v_2)d)$

For $(i,j,k) \neq (1,3,5)$ and $(2,4,6)$, $\Delta(i,j,k) < 1$ will hold for sufficiently small $d > 0$, whereas Δ_0 will change to

1. $\Delta_0 + (v_3 - v_2)d$,
2. $\Delta_0 + (u_1 + u_3 - v_1 - v_3)d$,
3. $\Delta_0 + (u_1 - u_2)d$,

respectively. By the maximality of Δ_0, we must have

$$v_2 - v_3 = (u_1 + u_3 - v_1 - v_3) = (u_1 - u_2) = 0\,,$$

which gives

$$u_1 = u_2,\ v_1 = u_2 + u_3 - v_3,\ v_2 = v_3\,.$$

Substituting into $\Delta(2,4,6)$, we have

$$\Delta(2,4,6) = (u_2 + u_3)^2 \leqslant 1 \ \Rightarrow\ (u_2 + u_3) \leqslant 1\,.$$

Substituting into Δ_0, we have

$$\Delta_0 = 2u_2 + 2u_3 - 1 \leqslant 1,$$

which is the desired contradiction. □

By Lemma 3.1, we can assume that the largest triangle Δ in the hexagon $\mathcal{H}$ has at least one side in common with $\mathcal{H}$. We consider two cases.

Case 1 Δ has two sides in common with $\mathcal{H}$. Then we may assume that Δ is the triangle with vertices $P_1,\ P_2,\ P_3$. Using an affine transformation, we may assume that $P_1 = (0,0)$, $P_2 = (1,0)$ and $P_3 = (0,1)$. Then P_4, P_5 and P_6 have to lie inside the triangle with vertices, $(0,0),\ (1,1)$ and $(0,\ 1)$. Therefore, $\mathcal{H}$ has area less than or equal to 1, a contradiction.

Case 2 Δ has one side in common with $\mathcal{H}$. Then we may assume that Δ is the triangle with vertices $P_1,\ P_2,\ P_4$.

Using an affine transformation, we may assume that $P_1 = (0,0)$, $P_2 = (1,0)$ and $P_4 = (0,1)$. Let $P_3 = (u_1, v_1)$, $P_5 = (-u_2, v_2)$ and $P_6 = (-u_3, v_3)$, where $u_1, u_2, u_3, v_1, v_2, v_3 \geqslant 0$. So, we have a hexagon with vertices $(0,0),\ (1,0),\ (u_1,v_1),\ (0,1),\ (-u_2,v_2),\ (-u_3,v_3)$. Since the hexagon is convex, we have

$$u_1 + v_1 \geqslant 1,\ v_2 \geqslant v_3,\ u_3v_2 \geqslant u_2v_3, \ \text{ and } u_3v_2 - u_2v_3 \geqslant u_3 - u_2 \tag{3.6}$$

Let

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & u_1 & 0 & -u_2 & -u_3 \\ 0 & 0 & v_1 & 1 & v_2 & v_3 \end{bmatrix}.$$

Then $|\tilde{\Delta}(i,j,k)|$ is the determinant of the submatrix of $\tilde{A}$ lying in columns i, j, k, and assume that

$$\tilde{\Delta}(1,2,4) = 1, \quad \text{ and } \quad \tilde{\Delta}(i,j,k) \leqslant 1 \ \text{ for all } 1 \leqslant i < j < k \leqslant 6 \tag{3.7}$$

It follows from (3.7) that

(a) (u_1, v_1) lies in the triangle with vertices $(1,0),\ (1,1),\ (0,1)$. Equivalently, $0 \leqslant 1 - u_1 \leqslant v_1 \leqslant 1$.

(b) $(-u_2, v_2)$ and $(-u_3, v_3)$ lie in the triangle with vertices $(0,0)$, $(0,1)$, $(-1,1)$. Equivalently,

$$0 \leqslant u_2 \leqslant v_2 \leqslant 1 \quad \text{and} \quad 0 \leqslant u_3 \leqslant v_3 \leqslant 1.$$

Let

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = \tilde{\Delta}(2,3,4) + \tilde{\Delta}(1,4,5) + \tilde{\Delta}(1,5,6) = u_1 + u_2 + v_1 + u_3 v_2 - u_2 v_3 - 1\,.$$

Suppose g attains a maximum M at $(u_1, v_1, u_2, v_2, u_3, v_3)$ subject to the constraints (3.6) and (3.7). We are going to show that

$$M \leqslant \frac{5}{4} \tag{3.8}$$

Lemma 3.2. *Suppose $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfy (a) and (b) such that $g(u_1, v_1, u_2, v_2, u_3, v_3) \geqslant \frac{5}{4}$. Then*

$$u_1 + v_1 \geqslant \frac{5}{4}, \quad v_2 \geqslant \frac{1}{4}.$$

Proof. Suppose at some $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfying (a) and (b), $g(u_1, v_1, u_2, v_2, u_3, v_3) \geqslant \frac{5}{4}$. Then

$$\begin{aligned} \frac{5}{4} &\leqslant u_1 + u_2 + v_1 + u_3 v_2 - u_2 v_3 - 1 \\ &= u_1 + v_1 - 1 + u_2(1 - v_2 + u_3) + (v_2 - u_2)u_3 \\ &\leqslant (u_1 + v_1 - 1) + u_2 + (v_2 - u_2) \\ &= (u_1 + v_1 - 1) + v_2. \end{aligned}$$

Since $(u_1 + v_1 - 1), v_2 \leqslant 1$, the result follows. □

Let us focus on the following constraints.

(c) $\tilde{\Delta}(1,3,5) = u_2 v_1 + u_1 v_2 \leqslant 1$,

(d) $\tilde{\Delta}(1,3,6) = u_3 v_1 + u_1 v_3 \leqslant 1$,

(e) $\tilde{\Delta}(2,3,5) = v_1 - v_2 + u_2 v_1 + u_1 v_2 \leqslant 1$,

(f) $\tilde{\Delta}(2,3,6) = v_1 - v_3 + u_3 v_1 + u_1 v_3 \leqslant 1$,

Consider the maximization problems under the following constraints:

1. M_1 = maximum of g under the constraints $v_1 \leqslant v_3$, (a), (b), (c), (d) and (3.6).
2. M_2 = maximum of g under the constraints $v_3 \leqslant v_1 \leqslant v_2$, (a), (b), (c) and (f).
3. M_3 = maximum of g under the constraints $v_2 \leqslant v_1$, (a), (b), (f) and (3.6).

Because $v_3 \leqslant v_2$, we have $M \leqslant \max\{M_1,\ M_2,\ M_3\}$. So (3.8) will follow from the following.

Proposition 3.3. $M_1,\ M_3 \leqslant M_2 \leqslant \frac{5}{4}$.

Proof. First we show that $M_1, M_3 \leqslant \max\left\{M_2,\ \frac{5}{4}\right\}$. Let

$$\begin{aligned}
g_1(u_1,v_1,u_2,v_2,u_3,v_3) &= \tilde{\Delta}(1,3,5) = u_2v_1 + u_1v_2 \\
g_2(u_1,v_1,u_2,v_2,u_3,v_3) &= \tilde{\Delta}(1,3,6) = u_3v_1 + u_1v_3 \\
g_3(u_1,v_1,u_2,v_2,u_3,v_3) &= \tilde{\Delta}(2,3,5) = v_1 - v_2 + u_2v_1 + u_1v_2 \\
g_4(u_1,v_1,u_2,v_2,u_3,v_3) &= \tilde{\Delta}(2,3,6) = v_1 - v_3 + u_3v_1 + u_1v_3\,.
\end{aligned}$$

Suppose M_1 is attained at $P = (u_1,v_1,u_2,v_2,u_3,v_3)$ satisfying the constraints $v_1 \leqslant v_3$, (a), (b), (c), (d) and (3.6). Note that

$$\begin{aligned}
g_1(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) &= g_1(u_1,v_1,u_2,v_2,u_3,v_3) - (u_3v_2 - u_2v_3)d \\
&\leqslant g_1(u_1,v_1,u_2,v_2,u_3,v_3)\,, \\
g_2(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) &= g_2(u_1,v_1,u_2,v_2,u_3,v_3)\,, \\
g(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) &= g(u_1,v_1,u_2,v_2,u_3,v_3) + (v_3 - u_3)d \\
&\geqslant g(u_1,v_1,u_2,v_2,u_3,v_3)\,.
\end{aligned}$$

If $v_1 < v_3$, then we may let $d = (v_3 - v_1)/v_3$ and replace (u_1, v_1) by $(u_1 - u_3d, v_1 + v_3d) = (\tilde{u}_1, v_3)$ with $\tilde{u}_1 = u_1 - u_3(v_3 - v_1)/v_3$. Then by the fact that $0 \leqslant u_3 \leqslant v_3 \leqslant 1$,

$$\begin{aligned}
\tilde{u}_1 &\geqslant u_1 - (v_3 - v_1) = u_1 + v_1 - v_3 \geqslant 1 - v_3 \geqslant 0 \\
\tilde{u}_1 + v_3 &\geqslant u_1 + v_1 \geqslant 1\,.
\end{aligned}$$

Thus, this replacement will neither decrease M_1 nor violate the constraints (a), (b), (c), (d), (3.6). In that case, P also satisfies (f). Therefore, $M_1 \leqslant M_2$.

Suppose M_3 is attained at $P = (u_1,v_1,u_2,v_2,u_3,v_3)$ satisfying the constraints $v_2 \leqslant v_1$, (a), (b), (e) and (f). We may assume that $M_3 \geqslant \frac{5}{4}$. Then, by Lemma 3.2, $v_2 \geqslant \frac{1}{4}$. Note that

$$\begin{aligned}
g_3(u_1 + (1+u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) &= g_3(u_1,v_1,u_2,v_2,u_3,v_3)\,, \\
g_4(u_1 + (1+u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) &= g_4(u_1,v_1,u_2,v_2,u_3,v_3) - (v_2 - v_3 + u_3v_2 - u_2v_3)d \\
&\leqslant g_4(u_1,v_1,u_2,v_2,u_3,v_3)\,, \\
g(u_1 + (1+u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) &= g(u_1,v_1,u_2,v_2,u_3,v_3) + (1 + u_2 - v_2)d \\
&\geqslant g(u_1,v_1,u_2,v_2,u_3,v_3)\,.
\end{aligned}$$

If $v_1 > v_2$, we may let $d = (v_1 - v_2)/v_2$ and replace (u_1, v_1) by $(u_1 + (1 + u_2)d, v_1 - v_2 d) = (\hat{u}_1, v_2)$ so that $\hat{u}_1 = u_1 + (1 + u_2)d$. Then

$$\begin{aligned}
\hat{u}_1 &\geqslant u_1 \geqslant 0, \\
\hat{u}_1 + v_2 &= u_1 + \frac{(1 + u_2)(v_1 - v_2)}{v_2} + v_2 \\
&= u_1 + v_1 + \frac{(1 + u_2 - v_2)(v_1 - v_2)}{v_2} \geqslant u_1 + v_1 \geqslant 1.
\end{aligned}$$

Such a replacement will neither decrease M_3 nor violate the constraints (a), (b), (f), and (3.6). In that case, P also satisfies (c). Therefore, $M_3 \leqslant M_2$.

It remains to prove $M_2 \leqslant \frac{5}{4}$. Note that we have relaxed the constraint (3.6) in the definition of M_2 to simplify the arguments in the following. On the other hand, we cannot use the assumption that $P_1, \ldots, P_6$ are the vertices of a convex polygon anymore. To establish our result, We need one more lemma.

Lemma 3.4. *M_2 is attained at some $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfying one of the following conditions:*

1. $v_1 = v_2 = v_3$.
2. $\tilde{\Delta}(1, 3, 5) = 1$, $v_3 = u_3$, $\tilde{\Delta}(2, 3, 6) < 1$ *and* $v_3 = v_1$.
3. $\tilde{\Delta}(1, 3, 5) = 1$, $v_3 = u_3$ *and* $\tilde{\Delta}(2, 3, 6) = 1$.

Proof. Suppose M_2 is attained at some $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfying $v_3 \leqslant v_1 \leqslant v_2$, (a), (b), (c) and (f). If $v_2 = v_3$, then $v_1 = v_2 = v_3$.

Suppose $v_2 > v_3$. We first show that $\tilde{\Delta}(1, 3, 5) = 1$. Assume that $\tilde{\Delta}(1, 3, 5) < 1$. Note that

$$\begin{aligned}
g_1(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) &= g_1(u_1, v_1, u_2, v_2, u_3, v_3) + v_1 d + u_1 e, \\
g_4(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) &= g_4(u_1, v_1, u_2, v_2, u_3, v_3)\,, \\
g(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) &= g(u_1, v_1, u_2, v_2, u_3, v_3) + (1 - v_3)d + u_3 e\,.
\end{aligned}$$

Then we can do the following to increase g to derive a contradiction. (1) If $v_2 < 1$, then take a suitable $d = e > 0$. (2) If $v_2 = 1$, then $\Delta(1, 3, 5) = u_1 v_2 + u_2 v_1 < 1$ implies that $u_2 < 1$ as $u_1 + v_1 \geqslant 1$. We may let $d > 0 = e$.

Next, we show that we may assume that $v_3 = u_3$. Note that

$$\begin{aligned}
g_1(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1 d) &= g_1(u_1, v_1, u_2, v_2, u_3, v_3)\,, \\
g_4(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1 d) &= g_4(u_1, v_1, u_2, v_2, u_3, v_3)\,, \\
g(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1 d) &= g(u_1, v_1, u_2, v_2, u_3, v_3) + (1 - v_2)d\,.
\end{aligned}$$

Since $u_1 + v_1 > 1$, we may decrease $v_3 - u_3$ without decreasing g. Hence, we may assume that $v_3 = u_3$.

We further claim that $v_2 > u_2$. If it is not true and $v_2 = u_2$, then $\tilde{\Delta}(1,3,5) = (v_1 + u_1)u_2 = 1$, and $1 + u_2 = 1 + v_2 \geqslant u_1 + v_1 = 1/u_2$ so that $1 + u_2 \geqslant 1/u_2 \geqslant 0$. Hence $u_2 \in [(\sqrt{5}-1)/2, 1]$, and

$$g(u_1, \ldots, v_3) = 1/u_2 + u_2 - 1 < 5/4 \qquad \text{for } u_2 \in [(\sqrt{5}-1)/2, 1],$$

which is a contradiction.

Now, we can show that $\tilde{\Delta}(2,3,6) = 1$ or $v_3 = v_1$. Note that

$$\begin{aligned} g_1(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) &= g_1(u_1, v_1, u_2, v_2, u_3, v_3)\,, \\ g_4(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) &= g_4(u_1, v_1, u_2, v_2, u_3, v_3) + (u_1 + v_1 - 1)d, \\ g(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) &= g(u_1, v_1, u_2, v_2, u_3, v_3) + (v_2 - u_2)d\,. \end{aligned}$$

Suppose $\tilde{\Delta}(2,3,6) < 1$. If $v_3 < v_1$, then we can increase g by choosing $d > 0$, a contradiction. So we have $v_3 = v_1$. □

Now we can finish the proof of Proposition 3.3.

Suppose $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfies $v_3 \leqslant v_1 \leqslant v_2$, (a), (b), (c), (f) and one of the conditions in Lemma 3.4, we will show that $g(u_1, v_1, u_2, v_2, u_3, v_3) \leqslant \dfrac{5}{4}$ according to the three conditions.

Case 2.1 Suppose $v_1 = v_2 = v_3 = v$. Then we have

$$\tilde{\Delta}(1,3,5) = (u_1 + u_2)v\,, \qquad \tilde{\Delta}(2,4,6) = (u_1 + u_3)v\,,$$

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = u_1 + u_2(1 - v) + v + u_3 v - 1.$$

We need to maximize $g(u_1, v_1, u_2, v_2, u_3, v_3)$ subject to the constraints:

$$(u_1 + u_2)v \leqslant 1 \quad \Leftrightarrow \quad u_2 \leqslant \frac{1 - u_1}{v},$$

$$(u_1 + u_3)v \leqslant 1 \quad \Leftrightarrow \quad u_3 \leqslant \frac{1 - u_1}{v},$$

and

$$\frac{5}{4} \leqslant u_1 + v_1 \leqslant 2,\ 0 \leqslant u_2,\ u_3 \leqslant v \leqslant 1.$$

Because $\left(v - \dfrac{1}{2}\right)^2 \geqslant 0$, it follows that $v^2 \geqslant v - \dfrac{1}{4} \geqslant 1 - u_1$, and hence $1 \geqslant \dfrac{1 - u_1}{v^2}$. Therefore, the maximum of $g(u_1, v_1, u_2, v_2, u_3, v_3)$ occurs at $u_2 = u_3 = \dfrac{1 - u_1}{v}$. Then

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = u_1 + v + \frac{1 - u_1}{v} - 1 = h(u_1, v).$$

Since $\dfrac{\partial h}{\partial v} = 1 - \dfrac{1-u_1}{v^2} \geqslant 0$, the maximum of h occurs at $v = 1$, which gives $h(u_1, 1) = 1 < \dfrac{5}{4}$.

Case 2.2 Suppose $\tilde{\Delta}(1,3,5) = 1$, $v_3 = u_3 = v_1 = v$. Then we have

$$\tilde{\Delta}(1,3,5) = u_2 v + u_1 v_2 = 1 \Rightarrow u_2 = \frac{(1-u_1v_2)}{v}$$

and

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = (u_1 + v)(1 + v_2) + \frac{1-u_1v_2}{v} - 2 = k(u_1, v_2, v).$$

So we want to maximize $k(u_1, v_2, v)$ subject to

$$\tilde{\Delta}(2,3,6) = v(u_1+v) \leqslant 1, \quad \frac{1}{4} \leqslant \frac{5}{4} - u_1 \leqslant v \leqslant v_2 \leqslant 1, \quad \frac{1-u_1v_2}{v} \leqslant v_2.$$

Equivalently,

$$\frac{1}{4} \leqslant \frac{5}{4} - u_1 \leqslant v \leqslant \frac{1}{v+u_1} \leqslant v_2 \leqslant 1.$$

Note that $\dfrac{\partial k}{\partial v_2} = v - u_1\left(\dfrac{1}{v} - 1\right)$.

Suppose $\dfrac{\partial k}{\partial v_2} \geqslant 0$, i.e., $u_1 \leqslant \dfrac{v^2}{1-v}$. Then the maximum of k occurs at $v_2 = 1$ so that

$$k(u_1, 1, v) = 2u_1 + \frac{(1-u_1)}{v} + 2v - 2.$$

Elementary calculus shows that the maximum of $2u_1 + \dfrac{(1-u_1)}{v} + 2v - 2$ with

$$\frac{1}{4} \leqslant \frac{5}{4} - u_1 \leqslant v \leqslant \frac{1}{v+u_1} \leqslant 1, \quad u_1 \leqslant \frac{v^2}{1-v}$$

occurs at $v = \dfrac{2}{3}$, $u_1 = \dfrac{5}{6}$ and $k(\dfrac{5}{6}, 1, \dfrac{2}{3}) = \dfrac{5}{4}$.

Suppose $\dfrac{\partial k}{\partial v_2} < 0$, i.e., $u_1 < \dfrac{v^2}{1-v}$. Then the maximum of k occurs at $v_2 = \dfrac{1}{(u_1+v)}$ so that

$$k(u_1, \frac{1}{(u_1+v)}, v) = u_1 + v + \frac{1}{(u_1+v)} - 1.$$

Direct calculation shows that the maximum of $u_1 + v + \dfrac{1}{(u_1+v)} - 1$ in

$$\frac{1}{4} \leqslant \frac{5}{4} - u_1 \leqslant v \leqslant \frac{1}{v+u_1} \leqslant 1, \; u_1 \geqslant \frac{v^2}{1-v}$$

occurs at $u_1 = 1$, $v = \dfrac{\sqrt{5}-1}{2}$, which gives $v_2 = \dfrac{\sqrt{5}-1}{2}$ and $k(1, \dfrac{\sqrt{5}-1}{2}, \dfrac{\sqrt{5}-1}{2}) = \sqrt{5} - 1 < \dfrac{5}{4}$.

Case 2.3 $\tilde{\Delta}(1,3,5)=1$, $\tilde{\Delta}(2,3,6)=1$ and $v_3=u_3$. Then we have

$$u_1=\frac{(1-v_1+v_3-v_1v_3)}{v_3},\ u_2=\frac{v_1v_2+v_3+v_1v_2v_3-v_2-v_2v_3}{v_1v_3},$$

and

$$g(u_1,v_1,u_2,v_2,u_3,v_3)=\frac{(1-v_1)(v_1-v_2)+v_3+(v_2-1)v_3^2}{v_1v_3}=\ell(v_1,v_2,v_3).$$

So we want to maximize $\ell(v_1,v_2,v_3)$ subject to

$$\frac{1}{4}\leqslant\frac{5}{4}-\frac{(1-v_1+v_3-v_1v_3)}{v_3}\leqslant v_1\leqslant v_2\leqslant 1,\quad \frac{v_1v_2+v_3+v_1v_2v_3-v_2-v_2v_3}{v_1v_3}\leqslant v_2\leqslant 1.$$

Equivalently,

$$v_1\leqslant v_2\leqslant 1,\quad \frac{1}{1+v_3}\leqslant v_1\leqslant\frac{4-v_3}{4}.$$

Note that $\dfrac{\partial\ell}{\partial v_2}=\dfrac{v_1+v_3^2-1}{v_1v_3}$.

Suppose $v_1+v_3^2\geqslant 1$. The maximum of ℓ occurs at $v_2=1$ so that $\ell(v_1,1,v_3)=\dfrac{v_3-(1-v_1)^2}{v_1v_3}$. Direct calculation shows that the maximum of $\dfrac{v_3-(1-v_1)^2}{v_1v_3}$ with

$$v_1\leqslant v_2\leqslant 1,\quad \frac{1}{1+v_3}\leqslant v_1\leqslant\frac{4-v_3}{4},\quad v_1+v_3^2\geqslant 1$$

occurs at $v_1=v_3=\dfrac{2}{3}$ and $\ell(\dfrac{2}{3},1,\dfrac{2}{3})=\dfrac{5}{4}$.

Suppose $v_1+v_3^2<1$. The maximum of k occurs at $v_2=v_1$ so that $\ell(v_1,v_1,v_3)=\dfrac{1-(1-v_1)v_3}{v_1}$. Direct calculation shows that the maximum of $\dfrac{1-(1-v_1)v_3}{v_1}$ in

$$v_1\leqslant v_2\leqslant 1,\ \frac{1}{1+v_3}\leqslant v_1\leqslant\frac{4-v_3}{4},\ v_1+v_3^2\leqslant 1$$

occurs at $v_1=\dfrac{2}{3}$, $v_3=\dfrac{1}{2}$ and $\ell(\dfrac{2}{3},\dfrac{2}{3},\dfrac{1}{2})=\dfrac{5}{4}$. $\square$

Remarks Several comments related to Proposition 1.2 are in order.

1. The proof of Proposition 1.2 is direct but quite lengthy. A shorter proof is desirable.

2. One might expect that a symmetry argument can be used to show that the solution of Proposition 1.2 is attained at a regular hexagon by a suitable affine transform when $n=6$, but it is not the case as shown by our result.

3. Following Proposition 1.2, a natural problem to study is to determine the optimal bound of the ratio between the area of an n-sided convex polygon $\mathcal{P}_n$ and the maximal area of an m-sided polygon $\mathcal{P}_m \subset \mathcal{P}_n$ for $m < n$.

Acknowledgment

We would like to thank the referee for providing the references [3] and [7]. The research of Li and Poon was supported by USA NSF, and HK RGC. Li was an honorary professor of the Shanghai University, and an honorary professor of the University of Hong Kong. The research of Wang was done while he was visiting the College of William and Mary during the academic year 2013-14 under the support of China Scholarship Council.

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