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Perturbing eigenvalues of non-negative matrices

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Dedicated to Professor Hans Schneider.

Abstract
Let $A$ be an irreducible (entrywise) nonnegative $n \times n$ matrix with eigenvalues
\[ \rho, \lambda_2 = b + ic, \lambda_3 = b - ic, \lambda_4, \cdots, \lambda_n, \]
where $\rho$ is the Perron eigenvalue. It is shown that for any $t \in [0, \infty)$ there is a nonnegative
matrix with eigenvalues
\[ \rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n, \]
whenever $\tilde{t} \geq \gamma_n t$ with $\gamma_3 = 1, \gamma_4 = 2, \gamma_5 = \sqrt{5}$ and $\gamma_n = 2.25$ for $n \geq 6$. The result improves
that of Guo et al. Our proof depends on an auxiliary result in geometry asserting that the area
of an $n$-sided convex polygon is bounded by $\gamma_n$ times the maximum area of a triangle lying
inside the polygon.

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1 Introduction
The nonnegative inverse eigenvalue problem concerns the study of necessary and sufficient con-
ditions for a given set of complex numbers $\lambda_1, \ldots, \lambda_n$ to be the eigenvalues of an (entrywise)
nonnegative matrix. This problem has attracted the attention of many authors, and is still open;
for example, see [4] and its references. In connection to this study, researchers study the change of
the Perron eigenvalue under the perturbation of the other real or complex eigenvalues of a given
nonnegative matrix. Here are several results in this direction:

(1) In [6], the author proved the following:

Suppose $\rho, \lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of an $n \times n$ nonnegative matrix $A$ such that $\rho$ is
the Perron eigenvalue, and $\lambda_2$ is real. Then for any $0 \leq t \leq \tilde{t}$, there is a nonnegative matrix
with eigenvalues $\rho + \tilde{t}, \lambda_2 \pm t, \lambda_3, \cdots, \lambda_n$. 

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(2) Laffey [9] and Guo et al. [5] obtained the following independently:

Suppose \( \rho, \lambda_2, \lambda_3, \ldots, \lambda_n \) are the eigenvalues of an \( n \times n \) nonnegative matrix \( A \) such that \( \rho \) is the Perron eigenvalue, and \( (\lambda_2, \lambda_3) = (b + ic, b - ic) \) is a (non-real) complex conjugate pair. Then for any \( \tilde{t}, t \in [0, \infty) \) with \( 2t \leq \tilde{t} \), there is a nonnegative matrix with eigenvalues \\
\( \rho + \tilde{t}, \lambda_2 - t, \lambda_3 - t, \lambda_4 \cdots, \lambda_n \).

(3) In [5, Proposition 3.1], Guo and Guo showed that:

Suppose \( \rho, \lambda_2, \lambda_3, \ldots, \lambda_n \) are the eigenvalues of an \( n \times n \) nonnegative matrix \( A \) such that \( \rho \) is the Perron eigenvalue, and \( (\lambda_2, \lambda_3) = (b + ic, b - ic) \) is a (non-real) complex conjugate pair. Then for any \( \tilde{t}, t \in [0, \infty) \) with \( 4t \leq \tilde{t} \), there is a nonnegative matrix with eigenvalues \\
\( \rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n \).

The authors also pose the problem of finding the smallest constant \( c \) for which the above result holds with \( 4t \) replaced by \( ct \). In [3] Cronin and Laffey show that \( c = 1 \) for \( n = 3 \), \( c = 2 \) for \( n = 4 \) and \( c \geq 2 \) for \( n \geq 5 \). They further show that for \( c > 2 \), the result holds for sufficiently small \( t \) but the question about arbitrary \( t \) is left open.

The results in (1) and (2) above were shown to be optimal in the sense that the conclusion may fail if \( \tilde{t} < t \) in (1) and \( \tilde{t} < 2t \) in (2). However, the result in (3) may be strengthened. In this paper, we improve the third result, and prove the following.

**Theorem 1.1.** Suppose \( \rho, \lambda_2, \lambda_3, \ldots, \lambda_n \) are the eigenvalues of an \( n \times n \) nonnegative matrix \( A \) such that \( \rho \) is the Perron eigenvalue, and \( \lambda_2 = b + ic \) and \( \lambda_3 = b - ic \) are (non-real) complex conjugate pairs. Then for any \( t \in [0, \infty) \) there is a nonnegative matrix with eigenvalues \\
\( \rho + \tilde{t}, \lambda_2 + t, \lambda_3 + t, \lambda_4 \cdots, \lambda_n \),

whenever \( \tilde{t} \geq \gamma_n t \) with \( \gamma_3 = 1, \gamma_4 = 2, \gamma_5 = \sqrt{5} \) and \( \gamma_n = 2.25 \) for \( n \geq 6 \).

Our proof depends on the following geometrical result, which is of independent interest [7].

**Proposition 1.2.** Suppose \( n \in \{3, 4, 5, 6\} \). The area of an \( n \)-sided convex hexagon \( P \subseteq \mathbb{R}^2 \) is bounded by \( \gamma_n \) times the maximum area of the triangles lying inside \( P \), where

\( \gamma_3 = 1, \gamma_4 = 2, \gamma_5 = \sqrt{5}, \gamma_6 = 2.25 \),

and these bounds are best possible.

One easily sees that the maximum area of the triangles lying inside a convex polygon is attained at a triangle formed by 3 of the vertices of the polygon.

The proof of Theorem 1.1 is given in Section 2, and the technical proof of Proposition 1.2 and some remarks are given in Section 3.
2 Proof of Theorem 1.1

We begin with two lemmas. The first one can be found in [8].

Lemma 2.1. Suppose \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of a nonnegative matrix. Then there is a nonnegative matrix with constant row sums with eigenvalues \( \lambda_1, \ldots, \lambda_n \).

The next lemma concerns the change of \( r \) eigenvalues, \( \lambda_1, \ldots, \lambda_r \) with \( r < n \), and leaving invariant the other eigenvalues of an \( n \times n \) matrix \( A \) by a rank-\( r \) perturbation. It can be viewed as an extension of the result in [10]; see also [2, Theorems 27 and 33].

Lemma 2.2. Let \( A \in \mathbb{C}^{n \times n} \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Let \( X = [x_1|x_2| \cdots |x_r] \in \mathbb{C}^{n \times r} \) be such that \( \text{rank}(X) = r \) and \( AX = XD \), where \( D \in \mathbb{C}^{r \times r} \) with eigenvalues \( \lambda_1, \ldots, \lambda_r \). Then for any \( r \times n \) matrix \( C \), the matrix \( A + XC \) has eigenvalues \( \mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n \), where \( \mu_1, \ldots, \mu_r \) are eigenvalues of the matrix \( D + CX \).

Proof. Let \( S = [X|Y] \) be a nonsingular matrix with \( S^{-1} = \begin{bmatrix} U & V \end{bmatrix} \), with \( U \in \mathbb{C}^{r \times n} \). Then \( UX = I_r, VY = I_{n-r} \) and \( (VX)^t = UY = O_{r \times (n-r)} \). Because \( AX = XD \), we have

\[
S^{-1}AS = \begin{bmatrix} U & V \end{bmatrix} A[X,Y] = \begin{bmatrix} D & UAY \\ 0 & VAY \end{bmatrix}
\] (2.1)

and

\[
S^{-1}XCS = \begin{bmatrix} I_r & 0 \end{bmatrix} CS = \begin{bmatrix} C & 0 \end{bmatrix} [X|Y] = \begin{bmatrix} CX & CY \\ 0 & 0 \end{bmatrix}.
\]

Thus,

\[
S^{-1}(A + XC)S = S^{-1}AS + S^{-1}XCS = \begin{bmatrix} D + CX & UAY + CY \\ 0 & VAY \end{bmatrix}.
\]

Now, from (2.1) we have \( \sigma(VAY) = \{\lambda_{r+1}, \ldots, \lambda_n\} \) and therefore

\[
\sigma(A + XC) = \sigma(D + CX) \cup \{\lambda_{r+1}, \ldots, \lambda_n\}.
\]

We are now ready to present the proof of Theorem 1.1.

Let \( A \in \Omega_{\rho} \) be an \( n \times n \) non-negative real matrix with eigenvalues \( \rho, b+ic, b-ic, \lambda_4, \ldots, \lambda_n \), and let \( u \pm iv \) be eigenvectors of \( A \) corresponding to the eigenvalues \( b \pm ic \), where \( u = (u_1, u_2, \ldots, u_n)^T \), \( v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n \). Then we have the following equality for \( n \times 2 \) matrices:

\[
A[u|v] = [u|v] \begin{bmatrix} b & c \\ -c & b \end{bmatrix}.
\] (2.2)

We adopt an idea in [5] and let

\[
M = \begin{bmatrix} 1 & \cdots & 1 \\ u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{bmatrix}.
\]
Denote by \( P = P(u, v) \) a point in \( \mathbb{R}^2 \) with co-ordinate \((u, v)\). By Analytic Geometry, suppose

\[
\det(i, j, k) = \det \begin{pmatrix} 1 & 1 & 1 \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{pmatrix}, \quad 1 \leq i, j, k \leq n.
\]

Then \( |\det(i, j, k)| \) is 2 times the area of the triangle with vertices \( P_i(u_i, v_i), P_j(u_j, v_j) \) and \( P_k(u_k, v_k) \). Moreover, \( \det(i, j, k) > 0 \) if and only if the points \( P_i \rightarrow P_j \rightarrow P_k \rightarrow P_i \) are not collinear and appear in counterclockwise direction in \( \mathbb{R}^2 \).

Replacing \((A, u, v)\) by \((QAQ^T, Qu, Qv)\) for a suitable permutation matrix \(Q\), we may assume that

\[
\Delta = \det(1, 2, 3) = \max_{1 \leq i, j, k \leq n} \det(i, j, k).
\]

Recall that \( e = (1, \ldots, 1)^T \). Since \( e, u + iv, u - iv \) are the eigenvectors of the distinct eigenvalues \( \rho, \lambda_2, \lambda_3 \), so \( e, u, v \) are linearly independent over \( \mathbb{R} \). It follows that \( \Delta = \det(1, 2, 3) > 0 \). Let

\[
x = (x_1, x_2, x_3, 0, \ldots, 0)^T \quad \text{and} \quad y = (y_1, y_2, y_3, 0, \ldots, 0)^T
\]

satisfy

\[
x^T e = 0, \ x^T u = 1, \ x^T v = 0; \ y^T e = 0, \ y^T u = 0, \ y^T v = 1,
\]

that is,

\[
\begin{bmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Then

\[
x_1 = \frac{1}{\Delta} (v_2 - v_3), \ x_2 = \frac{1}{\Delta} (v_3 - v_1), \ x_3 = \frac{1}{\Delta} (v_1 - v_2),
\]

\[
y_1 = \frac{1}{\Delta} (u_3 - u_2), \ y_2 = \frac{1}{\Delta} (u_2 - u_3), \ y_3 = \frac{1}{\Delta} (u_1 - u_2).
\]

and

\[
\]

Suppose

\[
[u|v][x|y]^T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & \cdots & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & 0 & \cdots & 0 \end{bmatrix}. \quad (2.6)
\]

Then for \( i = 1, \ldots, n \),

\[
\alpha_{i1} = u_i x_1 + v_i y_1 = \frac{1}{\Delta} \det(i, 2, 3) - \frac{1}{\Delta} (u_2 v_3 - u_3 v_2),
\]
\[
\alpha_{i2} = u_i x_2 + v_i y_2 = \frac{1}{\Delta} \det(1, i, 3) - \frac{1}{\Delta} (u_3 v_1 - u_1 v_3),
\]
\[
\alpha_{i3} = u_i x_3 + v_i y_3 = \frac{1}{\Delta} \det(1, 2, i) - \frac{1}{\Delta} (u_1 v_2 - u_2 v_1).
\]

If
\[
c_{i1} = \alpha_{i1} - \alpha_{21} = \frac{1}{\Delta} \det(i, 2, 3),
\]
\[
c_{i2} = \alpha_{i2} - \alpha_{32} = \frac{1}{\Delta} \det(1, i, 3),
\]
\[
c_{i3} = \alpha_{i3} - \alpha_{23} = \frac{1}{\Delta} \det(1, 2, i),
\]
then
\[
c_{11} \geq c_{i1}, \quad c_{22} \geq c_{i2}, \quad c_{33} \geq c_{i3},
\]
(2.7)
because \(\Delta = \det(1, 2, 3) \geq \det(i, j, k)\) for all \(1 \leq i, j, k \leq n\). Let
\[
c_{i1} = \min_{l=1,2,\ldots,n} c_{l1}, \quad c_{i2} = \min_{l=1,2,\ldots,n} c_{l2}, \quad c_{i3} = \min_{l=1,2,\ldots,n} c_{l3}.
\]
Then \(c_{i1} \leq c_{i2} \leq c_{i2}\) and \(c_{i3} \leq c_{i3}\) for all \(l = 1, 2, \ldots, n\). Therefore, we have
\[
\alpha_{i1} \leq c_{i1}, \quad \alpha_{i2} \leq c_{i2} \quad \text{and} \quad \alpha_{i3} \leq c_{i3} \quad \text{for all} \quad l = 1, 2, \ldots, n.
\]
(2.8)
Assume that \(n \geq 6\), and that \(1, 2, 3, i, j, k\) are distinct, and focus on
\[
\hat{M} = \left[\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & u_2 & u_3 & u_i & u_j & u_k \\
u_2 & v_3 & v_i & v_j & v_k
\end{array}\right].
\]
(2.9)
Note that for the following points in \(\mathbb{R}^2\),
\[
P_1(u_1, v_1), \quad P_2(u_2, v_2), \quad P_3(u_3, v_3), \quad P_1(u_i, v_i), \quad P_j(u_j, v_j), \quad P_k(u_k, v_k),
\]
- the area of a triangle formed by any three of these points is not larger than \(\frac{\det(1, 2, 3)}{2}\), which is the area of the triangle with vertices \(P_1, P_2, P_3\);
- \(c_{i1} \leq \frac{\det(1, 2, 3)}{\Delta} = \frac{\det(2, 3, 3)}{\Delta} = 0, \quad c_{i2} \leq \frac{\det(1, 1, 3)}{\Delta} = \frac{\det(1, 3, 3)}{\Delta} = 0, \quad c_{i3} \leq \frac{\det(1, 2, 1)}{\Delta} = \frac{\det(1, 2, 2)}{\Delta} = 0.
\]
Thus, \(\det(i, 2, 3), \det(1, j, 3), \det(1, 2, k) \in (-\infty, 0]\). Note that \(\det(r, s, t) \leq 0\) if and only if \(P_r, P_s, P_t\) are collinear or they are in clockwise direction. Let \(\ell_1\) (respectively, \(\ell_2, \ell_3\)) be the line through \(P_1\) (respectively, \(P_2, P_3\)) parallel to \(P_2P_3\) (respectively, \(P_1P_3, P_1P_2\)). Suppose \(\ell_2\) and \(\ell_3\) (respectively, \(\ell_1\) and \(\ell_1, \ell_2\) and \(\ell_2\)) intersect at \(Q_1\) (respectively, \(Q_2\) and \(Q_3\)). Since \(\det(i, 2, 3) \leq 0\) and \(|\det(1, 2, i)|, |\det(1, 3, i)| \leq \det(1, 2, 3), P_i\) lies in the triangle \(Q_1P_3P_2\). Similarly, \(P_j\) and \(P_k\) lie in the triangles \(P_1P_3Q_2\) and \(P_3Q_2P_j\) respectively. Thus \(P_1P_2P_3P_1P_2\) is a convex hexagon (including the degenerate cases, when it is a triangle, quadrilateral or pentagon). Moreover, the vertices \(P_1, P_j, P_3, P_1, P_2, P_k, P_1\) are in clockwise direction. By Proposition 1.2,
\[
\frac{5}{4} \geq \frac{1}{\Delta} (|\det(i, 2, 3)| + |\det(1, j, 3)| + |\det(1, 2, k)|) = -(c_{i1} + c_{i2} + c_{i3}) \geq 0.
\]
It follows that
\[-1 \geq \alpha_{i1} + \alpha_{j2} + \alpha_{k3} = c_{i1} + \alpha_{21} + c_{j2} + \alpha_{32} + c_{k3} + \alpha_{23} \geq \frac{5}{4} - 1 = -2.25. \tag{2.10}\]

Suppose \(\tilde{t} \geq 2.25t \geq 0\). Let
\[
\delta = \frac{\tilde{t} + t(\alpha_{i1} + \alpha_{j2} + \alpha_{k3})}{3} \geq \frac{\tilde{t} - 2.25t}{3} \geq 0
\]
Set
\[
z = (-t\alpha_{i1} + \delta, -t\alpha_{j2} + \delta, -t\alpha_{k3} + \delta, 0, \cdots, 0)^T \quad \text{and} \quad \tilde{A} = A + [e|u|v][z|tx|ty]^T.
\]

By direct computation, we have
\[
[z|tx|ty]^T[e|u|v] = \begin{bmatrix}
\tilde{t} & * & *
0 & t & 0
0 & 0 & t
\end{bmatrix}.
\]

By Lemma 2.2, the eigenvalues of \(\tilde{A}\) are \(\rho + \tilde{t}, \sigma_2, \sigma_3, \lambda_4, \cdots, \lambda_n\), where \(\sigma_2, \sigma_3\) are the eigenvalues of
\[
\begin{bmatrix}
b & c
-c & b
\end{bmatrix} + tI_2, \quad \text{that is,} \quad \sigma_2 = b + t + ic, \quad \sigma_3 = b + t - ic.
\]

Let
\[
[e|u|v][z|tx|ty]^T = \begin{bmatrix}
\beta_{i1} & \beta_{i2} & \beta_{i3} & 0 & \cdots & 0 \\
\beta_{j1} & \beta_{j2} & \beta_{j3} & 0 & \cdots & 0 \\
\beta_{k1} & \beta_{k2} & \beta_{k3} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n1} & \beta_{n2} & \beta_{n3} & 0 & \cdots & 0
\end{bmatrix}
\]

By (2.8), we have
\[
\beta_{i1} = t(\alpha_{i1} - \alpha_{i1}) + \delta \geq 0
\]
\[
\beta_{i2} = t(\alpha_{i2} - \alpha_{j2}) + \delta \geq 0
\]
\[
\beta_{i3} = t(\alpha_{i3} - \alpha_{k3}) + \delta \geq 0.
\]

Thus, \(\tilde{A}\) also has nonnegative entries. Hence, \(\tilde{A}\) is the desired matrix.

Suppose \(n = 5, 4, 3\). Then the matrix \(\tilde{M}\) in (2.9) has at most \(n\) columns. Nevertheless, we can apply a similar argument and use the corresponding result in Proposition 1.2 to construct the desired matrix \(\tilde{A}\). We omit the details.

\[\square\]

\section{3 Proof of Proposition 1.2}

The purpose of this section is to prove the Proposition 1.2. The results for \(n = 3\) is trivial.

We will assume that \(P_1, \ldots, P_n\) are vertices of the convex polygon arranged in counterclockwise direction. The following two facts are useful in our discussion.
(a) One can apply an affine transformation \( v \mapsto Tv + v_0 \) for some invertible \( 2 \times 2 \) matrix \( T \) and \( v_0 \in \mathbb{R}^2 \) to the points \( P_1, \ldots, P_n \) without affecting the hypothesis and conclusion of the result.

(b) One can always find an affine map to send any 3 vertices of the polygon to any 3 non-collinear points.

Suppose \( n = 4 \). One may apply an affine transformation and assume that \( P_1 = (0, 0) \), \( P_2 = (1, 0) \), \( P_3 = (1, 1) \) are the vertices of the triangle of largest area. Since all the triangles inside the quadrilateral have area at most \( \frac{1}{2} \), the fourth vertex is in the triangle with vertices \( (0, 0), (1, 1), (0, 1) \). The conclusion of Proposition 1.2 follows readily.

Suppose \( n = 5 \) and \( P_1, \ldots, P_3 \) are vertices of a convex pentagon arranged in counterclockwise direction. Let \( T \) be a triangle of largest area.

**Case 1.** \( T \) has two sides in common with the pentagon. We may assume that \( P_1 = (0, 0) \), \( P_2 = (1, 0) \), \( P_3 = (1, 1) \) are the vertices of \( T \). Then \( P_4 \) and \( P_5 \) have to lie in the triangle with vertices \( (0, 0), (1, 1), (0, 1) \) and the conclusion of Proposition 1.2 follows readily.

**Case 2.** \( T \) has only one side in common with the pentagon. We may assume that \( P_1 = (0, 0) \), \( P_2 = (1, 0) \), \( P_4 = (0, 1) \) are the vertices of \( T \). Then we have

(a) \( P_3 = (u_3, v_3) \) lies in the triangle with vertices \( (1, 0), (1, 1), (0, 1) \), and

(b) \( P_5 = (-u_5, v_5) \) lies in the triangle with vertices \( (0, 0), (0, 1), (-1, 1) \).

By applying the affine transformation \((x, y) \mapsto (1 - (x + y), y)\), if necessary, we may assume that \( v_3 \geq v_5 \). For the convenience of calculation, we will use \( \Delta(i, j, k) \) to denote twice the area of the triangle with vertices \( P_i, P_j, P_k \). We will show that subject to the constraints (a), (b) and \( \Delta(2, 3, 5) \leq 1 \), we have \( \Delta(1, 2, 4) + \Delta(2, 3, 4) + \Delta(1, 4, 5) \leq \sqrt{5} \), where the equality holds at \((u_3, v_3) = (2, \sqrt{5} - 1)/2 \) and \((-u_5, v_5) = (1 - \sqrt{5}, \sqrt{5} - 1)/2 \).

By direct calculation, we have

\[
\Delta(2, 3, 5) = v_3(1 + u_5) - (1 - u_3)v_5 \quad \text{and} \\
\Delta(1, 2, 4) + \Delta(2, 3, 4) + \Delta(1, 4, 5) = u_3 + u_5 + v_3.
\]

So we need to show that subject to the constraints

\[
u_3 \leq 1 \leq u_3 + v_3, \quad 0 \leq u_5 \leq v_5 \leq v_3 \leq 1, \quad v_3(1 + u_5) - (1 - u_3)v_5 \leq 1, \quad (3.1)
\]

the maximum value of \( u_3 + u_5 + v_3 \) is \( \sqrt{5} \).

We can replace \( v_5 \) by \( v_3 \) without changing \( u_3 + u_5 + v_3 \) or violating the constraints. So we will assume that \( v_5 = v_3 \). Then the constraints in (3.1) becomes

\[
u_3 \leq 1 \leq u_3 + v_3, \quad 0 \leq u_5 \leq v_3 \leq 1, \quad (u_3 + u_5)v_3 \leq 1.
\]
So we have $u_3 + u_5 \leq 1 + v_3$ and $\frac{1}{v_3}$. Therefore, for fixed $0 \leq v_3 \leq 1$, the maximum of $u_3 + u_5 + v_3$ is equal to $1 + 2v_3$, if $1 + v_3 \leq \frac{1}{v_3} \iff v_3 \leq \frac{\sqrt{5} - 1}{2}$, and $v_3 + \frac{1}{v_3}$ if $1 + v_3 \leq \frac{1}{v_3} \iff v_3 \geq \frac{\sqrt{5} - 1}{2}$.

Maximizing over $v_3$ in both cases, we have the maximum value $\sqrt{5}$ attained at $v_3 = \frac{\sqrt{5} - 1}{2}$. Thus the maximum of $u_3 + u_5 + v_3$ is attained at $u_3 = 1$, $u_5 = v_3 = v_5 = \frac{\sqrt{5} - 1}{2}$. We note that for these values of $u_3$, $u_5$, $v_3$, $v_5$, we actually have $\Delta(i, j, k) \leq 1$ for all $1 \leq i < j < k \leq 5$.

Finally, we consider the intricate case when $n = 6$. Suppose a (non-degenerate) convex hexagon has vertices $P_1(x_1, y_1), \ldots, P_6(x_6, y_6)$ arranged in counterclockwise direction. We will prove that

\[
\frac{\text{Area of the hexagon with vertices } P_1, P_2, \ldots, P_6}{\max\{\text{Area of triangle with vertices } P_i, P_j, P_k : 1 \leq i < j < k \leq 6\}} \leq \frac{9}{4},
\]

(3.2)

where the inequality becomes an equality for the hexagon $H_0$ with vertices

$$(0, 0), (1, 0), \left(\frac{5}{6}, \frac{2}{3}\right), (0, 1), \left(-\frac{1}{4}, 1\right), \left(-\frac{2}{3}, \frac{2}{3}\right).$$

Note that a direct calculation shows that the area of the triangle with vertices $(0, 0), (1, 0), (0, 1)$ is $\frac{1}{2}$, which is maximum among all triangles with vertices from $H_0$.

**Lemma 3.1.** Suppose the maximum of the left hand side of (3.2) is attained at some hexagon $H$ with vertices $P_1, \ldots, P_6$. Then

\[
\max\{\text{Area of triangle with vertices } P_i, P_j, P_k : 1 \leq i < j < k \leq 6\}
\]

is attained at some triangle with at least one side in common with the boundary of $H$.

**Proof.** Let $M$ be the maximum of the left hand side of (3.2) over all (non-degenerate) convex hexagon. Clearly, $M$ exists and $\frac{9}{4} \leq M \leq 4$.

Suppose the maximum of the left hand side of (3.2) is attained at some hexagon $H$ with vertices $P_1, \ldots, P_6$, labeled in counterclockwise direction. We are going to prove the result by contradiction.

Suppose the maximum of the area of triangles with vertices $P_i, P_j, P_k, 1 \leq i < j < k \leq 6$ can only be attained at triangles with no side in common with the hexagon $H$. Without loss of generality, we may assume that the maximum is attained at the triangle with vertices $P_1, P_3, P_5$. Using an affine transformation, we may assume that $P_1 = (0, 0), P_3 = (1, 0)$ and $P_5 = (0, 1)$. For the convenience of notation and computation, let

\[
\Delta(i, j, k) = 2 \times (\text{area of triangle with vertices } P_i, P_j, P_k)
\]

for $1 \leq i < j < k \leq 6$. By our assumption, we have

\[
\Delta(1, 3, 5) = 1, \Delta(2, 4, 6) \leq 1 \ \text{and} \ \Delta(i, j, k) < 1 \ \text{for all} \ (i, j, k) \neq (1, 3, 5), (2, 4, 6).
\]

(3.3)
We will prove that under the conditions in (3.3), the area of the hexagon $H$ is less than or equal to 1, which contradicts the fact that $M \geq \frac{9}{4}$ as shown by our example before Lemma 3.1.

In the following, we will prove that under the conditions in (3.3), we have

\[
\Delta_0 = \Delta(1, 2, 3) + \Delta(3, 4, 5) + \Delta(1, 5, 6) \leq 1 \quad (3.4)
\]

Suppose $P_2 = (u_1, -v_1)$, $P_3 = (u_2, v_2)$ and $P_6 = (-u_3, v_3)$. Let

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & u_1 & 1 & u_2 & 0 & -u_3 \\
0 & -v_1 & 0 & v_2 & 1 & v_3
\end{bmatrix}.
\]

Then $|\Delta(i, j, k)|$ is equal to the determinant of the submatrix of $A$ lying in columns $i, j, k$. By (3.3), we have

\[
\Delta(1, 3, 5) = 1 \text{ is the maximum, among all } \Delta(i, j, k)
\]

\[
\Delta(2, 4, 6) = (u_2 - u_1)(v_1 + v_3) + (u_1 + u_3)(v_1 + v_2) \leq 1, \quad \text{and} \quad (3.5)
\]

\[
0 \leq v_1 < u_1 < 1, \quad u_2 < 1, \quad v_2 < 1, \quad u_2 + v_2 \geq 1, \quad 0 \leq u_3 < v_3 < 1.
\]

By direct computation, we have

\[
\Delta_0 = u_2 + u_3 + v_1 + v_2 - 1.
\]

Note that the area of the triangle with vertices $P_i, P_j, P_k$ will not change if we replace $P_i$ by $P_i + d(P_j - P_k)$ for any $d \in \mathbb{R}$. Thus, $\Delta(1, 3, 5)$ will not be affected and $\Delta(2, 4, 6)$ will not change under the following transformations:

1. $(u_1, v_1, u_2, v_2, u_3, v_3) \rightarrow (u_1 + (u_2 + u_3)d, v_1 + (v_3 - v_2)d, u_2, v_2, u_3, v_3),$

2. $(u_1, v_1, u_2, v_2, u_3, v_3) \rightarrow (u_1, v_1, u_2 + (u_1 + u_3)d, v_2 - (v_1 + v_3)d, u_3, v_3),$

3. $(u_1, v_1, u_2, v_2, u_3, v_3) \rightarrow (u_1, v_1, u_2, v_2, u_3 + (u_1 - u_2)d, v_3 + (v_1 + v_2)d)$

For $(i, j, k) \neq (1, 3, 5)$ and $(2, 4, 6)$, $\Delta(i, j, k) < 1$ will hold for sufficiently small $d > 0$, whereas $\Delta_0$ will change to

1. $\Delta_0 + (v_3 - v_2)d,$

2. $\Delta_0 + (u_1 + u_3 - v_1 - v_3)d,$

3. $\Delta_0 + (u_1 - u_2)d,$

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respectively. By the maximality of $\Delta_0$, we must have
\[
v_2 - v_3 = (u_1 + u_3 - v_1 - v_3) = (u_1 - u_2) = 0,
\]
which gives
\[
  u_1 = u_2, \quad v_1 = u_2 + u_3 - v_3, \quad v_2 = v_3.
\]
Substituting into $\Delta(2, 4, 6)$, we have
\[
  \Delta(2, 4, 6) = (u_2 + u_3)^2 \leq 1 \implies (u_2 + u_3) \leq 1.
\]
Substituting into $\Delta_0$, we have
\[
  \Delta_0 = 2u_2 + 2u_3 - 1 \leq 1,
\]
which is the desired contradiction.

By Lemma 3.1, we can assume that the largest triangle $\Delta$ in the hexagon $H$ has at least one side in common with $H$. We consider two cases.

**Case 1** $\Delta$ has two sides in common with $H$. Then we may assume that $\Delta$ is the triangle with vertices $P_1$, $P_2$, $P_3$. Using an affine transformation, we may assume that $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_3 = (0, 1)$. Then $P_1$, $P_3$ and $P_6$ have to lie inside the triangle with vertices, $(0, 0)$, $(1, 1)$ and $(0, 1)$. Therefore, $H$ has area less than or equal to 1, a contradiction.

**Case 2** $\Delta$ has one side in common with $H$. Then we may assume that $\Delta$ is the triangle with vertices $P_1$, $P_2$, $P_4$.

Using an affine transformation, we may assume that $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_4 = (0, 1)$. Let $P_3 = (u_1, v_1)$, $P_5 = (-u_2, v_2)$ and $P_6 = (-u_3, v_3)$, where $u_1, u_2, u_3, v_1, v_2, v_3 \geq 0$. So, we have a hexagon with vertices $(0, 0)$, $(1, 0)$, $(u_1, v_1)$, $(0, 1)$, $(-u_2, v_2)$, $(-u_3, v_3)$. Since the hexagon is convex, we have
\[
u_1 + v_1 \geq 1, \quad v_2 \geq v_3, \quad u_3v_2 \geq u_2v_3, \quad \text{and} \quad u_3v_2 - u_2v_3 \geq u_3 - u_2 \quad \quad (3.6)
\]
Let
\[
\tilde{A} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & u_1 & -u_2 & -u_3 \\
0 & 0 & v_1 & v_2 & v_3
\end{bmatrix}.
\]
Then $|\tilde{\Delta}(i, j, k)|$ is the determinant of the submatrix of $\tilde{A}$ lying in columns $i, j, k$, and assume that
\[
\tilde{\Delta}(1, 2, 4) = 1, \quad \text{and} \quad |\tilde{\Delta}(i, j, k)| \leq 1 \quad \text{for all} \quad 1 \leq i < j < k \leq 6 \quad \quad (3.7)
\]
It follows from (3.7) that
\[
(a) \quad (u_1, v_1) \text{ lies in the triangle with vertices } (1, 0), (1, 1), (0, 1). \quad \text{Equivalently, } 0 \leq 1 - u_1 \leq v_1 \leq 1.
\]
(b) \((-u_2, v_2\) and \((-u_3, v_3\) lie in the triangle with vertices \((0, 0), (0, 1), (-1, 1)\). Equivalently,

\[
0 \leq u_2 \leq v_2 \leq 1 \quad \text{and} \quad 0 \leq u_3 \leq v_3 \leq 1.
\]

Let

\[
g(u_1, v_1, u_2, v_2, u_3, v_3) = \tilde{\Delta}(2, 3, 4) + \tilde{\Delta}(1, 4, 5) + \tilde{\Delta}(1, 5, 6) = u_1 + u_2 + v_1 + u_3v_2 - u_2v_3 - 1.
\]

Suppose \(g\) attains a maximum \(M\) at \((u_1, v_1, u_2, v_2, u_3, v_3)\) subject to the constraints (3.6) and (3.7).

We are going to show that

\[
M \leq \frac{5}{4} \quad (3.8)
\]

**Lemma 3.2.** Suppose \((u_1, v_1, u_2, v_2, u_3, v_3)\) satisfy (a) and (b) such that \(g(u_1, v_1, u_2, v_2, u_3, v_3) \geq \frac{5}{4}\).

Then

\[
u_1 + v_1 \geq \frac{5}{4}, \quad v_2 \geq \frac{1}{4}.
\]

**Proof.** Suppose at some \((u_1, v_1, u_2, v_2, u_3, v_3)\) satisfying (a) and (b), \(g(u_1, v_1, u_2, v_2, u_3, v_3) \geq \frac{5}{4}\)

Then

\[
\frac{5}{4} \leq u_1 + u_2 + v_1 + u_3v_2 - u_2v_3 - 1
\]

\[
= u_1 + v_1 - 1 + u_2(1 - v_2 + u_3) + (v_2 - u_2)u_3
\]

\[
\leq (u_1 + v_1 - 1) + u_2 + (v_2 - u_2)
\]

\[
= (u_1 + v_1 - 1) + v_2.
\]

Since \((u_1 + v_1 - 1), v_2 \leq 1\), the result follows. \(\square \)

Let us focus on the following constraints.

(c) \(\tilde{\Delta}(1, 3, 5) = u_2v_1 + u_1v_2 \leq 1\),

(d) \(\tilde{\Delta}(1, 3, 6) = u_3v_1 + u_1v_3 \leq 1\),

(e) \(\tilde{\Delta}(2, 3, 5) = v_1 - v_2 + u_2v_1 + u_1v_2 \leq 1\),

(f) \(\tilde{\Delta}(2, 3, 6) = v_1 - v_3 + u_2v_1 + u_1v_3 \leq 1\),

Consider the maximization problems under the following constraints:

1. \(M_1 = \text{maximum of } g\) under the constraints \(v_1 \leq v_3, (a), (b), (c), (d)\) and (3.6).

2. \(M_2 = \text{maximum of } g\) under the constraints \(v_3 \leq v_1 \leq v_2, (a), (b)\) and (f).

3. \(M_3 = \text{maximum of } g\) under the constraints \(v_2 \leq v_1, (a), (b), (f)\) and (3.6).

Because \(v_3 \leq v_2\), we have \(M \leq \max\{M_1, M_2, M_3\}\). So (3.8) will follow from the following.
Proposition 3.3. \( M_1, M_3 \leq M_2 \leq \frac{5}{4} \).

**Proof.** First we show that \( M_1, M_3 \leq \max \left\{ M_2, \frac{5}{4} \right\} \). Let

\[
g_1(u_1, v_1, u_2, v_2, u_3, v_3) = \Delta(1, 3, 5) = u_2v_1 + u_1v_2
\]

\[
g_2(u_1, v_1, u_2, v_2, u_3, v_3) = \Delta(1, 3, 6) = u_3v_1 + u_1v_3
\]

\[
g_3(u_1, v_1, u_2, v_2, u_3, v_3) = \Delta(2, 3, 5) = v_1 - v_2 + u_2v_1 + u_1v_2
\]

\[
g_4(u_1, v_1, u_2, v_2, u_3, v_3) = \Delta(2, 3, 6) = v_1 - v_3 + u_3v_1 + u_1v_3.
\]

Suppose \( M_1 \) is attained at \( P = (u_1, v_1, u_2, v_2, u_3, v_3) \) satisfying the constraints \( v_1 \leq v_3, \) (a), (b), (c), (d) and (3.6). Note that

\[
g_1(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) = g_1(u_1, v_1, u_2, v_2, u_3, v_3) - (u_3v_2 - u_2v_3)d
\]

\[
\leq g_1(u_1, v_1, u_2, v_2, u_3, v_3),
\]

\[
g_2(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) = g_2(u_1, v_1, u_2, v_2, u_3, v_3),
\]

\[
g(u_1 - u_3d, v_1 + v_3d, u_2, v_2, u_3, v_3) = g(u_1, u_2, u_3, v_3) + (v_3 - u_3)d
\]

\[
\geq g(u_1, v_1, u_2, v_2, u_3, v_3).
\]

If \( v_1 < v_3 \), then we may let \( d = (v_3 - v_1)/v_3 \) and replace \( (u_1, v_1) \) by \( (u_1 - u_3d, v_1 + v_3d) = (\tilde{u}_1, v_3) \)

with \( \tilde{u}_1 = u_1 - u_3(v_3 - v_1)/v_3 \). Then by the fact that \( 0 \leq u_3 \leq v_3 \leq 1 \),

\[
\tilde{u}_1 \geq u_1 - (v_3 - v_1) = u_1 + v_1 - v_3 \geq 1 - v_3 \geq 0
\]

\[
\tilde{u}_1 + v_3 \geq u_1 + v_1 \geq 1.
\]

Thus, this replacement will neither decrease \( M_1 \) nor violate the constraints (a), (b), (c), (d), (3.6).

In that case, \( P \) also satisfies (f). Therefore, \( M_1 \leq M_2 \).

Suppose \( M_3 \) is attained at \( P = (u_1, v_1, u_2, v_2, u_3, v_3) \) satisfying the constraints \( v_2 \leq v_1, \) (a), (b), (e) and (f). We may assume that \( M_3 \geq \frac{5}{4} \). Then, by Lemma 3.2, \( v_2 \geq \frac{1}{4} \). Note that

\[
g_3(u_1 + (1 + u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) = g_3(u_1, v_1, u_2, v_2, u_3, v_3),
\]

\[
g_4(u_1 + (1 + u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) = g_4(u_1, v_1, u_2, v_2, u_3, v_3) - (v_2 - v_3 + u_3v_2 - u_2v_3)d
\]

\[
\leq g_4(u_1, v_1, u_2, v_2, u_3, v_3),
\]

\[
g(u_1 + (1 + u_2)d, v_1 - v_2d, u_2, v_2, u_3, v_3) = g(u_1, v_1, u_2, v_2, u_3, v_3) + (1 + u_2 - v_2)d
\]

\[
\geq g(u_1, v_1, u_2, v_2, u_3, v_3).
\]

\[\text{12}\]
If $v_1 > v_2$, we may let $d = (v_1 - v_2)/v_2$ and replace $(u_1, v_1)$ by $(u_1 + (1 + u_2)d, v_1 - v_2d) = (\hat{u}_1, v_2)$ so that $\hat{u}_1 = u_1 + (1 + u_2)d$. Then

$$\hat{u}_1 \geq u_1 \geq 0,$$

$$\hat{u}_1 + v_2 = u_1 + \frac{(1 + u_2)(v_1 - v_2)}{v_2} + v_2$$

$$= u_1 + v_1 + \frac{(1 + u_2 - v_2)(v_1 - v_2)}{v_2} \geq u_1 + v_1 \geq 1.$$

Such a replacement will neither decrease $M_3$ nor violate the constraints (a), (b), (f), and (3.6). In that case, $P$ also satisfies (c). Therefore, $M_3 \leq M_2$.

It remains to prove $M_2 \leq \frac{5}{4}$. Note that we have relaxed the constraint (3.6) in the definition of $M_2$ to simplify the arguments in the following. On the other hand, we cannot use the assumption that $P_1, \ldots, P_6$ are the vertices of a convex polygon anymore. To establish our result, we need one more lemma.

**Lemma 3.4.** $M_2$ is attained at some $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfying one of the following conditions:

1. $v_1 = v_2 = v_3$.
2. $\hat{\Delta}(1, 3, 5) = 1$, $v_3 = u_3$, $\hat{\Delta}(2, 3, 6) < 1$ and $v_3 = v_1$.
3. $\hat{\Delta}(1, 3, 5) = 1$, $v_3 = u_3$ and $\hat{\Delta}(2, 3, 6) = 1$.

**Proof.** Suppose $M_2$ is attained at some $(u_1, v_1, u_2, v_2, u_3, v_3)$ satisfying $v_3 \leq v_1 \leq v_2$, (a), (b), (c) and (f). If $v_2 = v_3$, then $v_1 = v_2 = v_3$.

Suppose $v_2 > v_3$. We first show that $\hat{\Delta}(1, 3, 5) = 1$. Assume that $\hat{\Delta}(1, 3, 5) < 1$. Note that

$$g_1(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) = g_1(u_1, v_1, u_2, v_2, u_3, v_3) + v_1d + u_1e,$$

$$g_4(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) = g_4(u_1, v_1, u_2, v_2, u_3, v_3),$$

$$g(u_1, v_1, u_2 + d, v_2 + e, u_3, v_3) = g(u_1, v_1, u_2, v_2, u_3, v_3) + (1 - v_3)d + u_3e.$$

Then we can do the following to increase $g$ to derive a contradiction. (1) If $v_2 < 1$, then take a suitable $d = e > 0$. (2) If $v_2 = 1$, then $\Delta(1, 3, 5) = u_1v_2 + u_2v_1 < 1$ implies that $u_2 < 1$ as $u_1 + v_1 \geq 1$. We may let $d > 0 = e$.

Next, we show that we may assume that $v_3 = u_3$. Note that

$$g_1(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1d) = g_1(u_1, v_1, u_2, v_2, u_3, v_3),$$

$$g_4(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1d) = g_4(u_1, v_1, u_2, v_2, u_3, v_3),$$

$$g(u_1, v_1, u_2, v_2, u_3 - (1 - u_1)d, v_3 - v_1d) = g(u_1, v_1, u_2, v_2, u_3, v_3) + (1 - v_2)d.$$
Since \( u_1 + v_1 > 1 \), we may decrease \( v_3 - u_3 \) without decreasing \( g \). Hence, we may assume that \( v_3 = u_3 \).

We further claim that \( v_2 > u_2 \). If it is not true and \( v_2 = u_2 \), then \( \bar{\Delta}(1,3,5) = (v_1 + u_1)u_2 = 1 \), and \( 1 + u_2 = 1 + v_2 \geq u_1 + v_1 = 1/u_2 \) so that \( 1 + u_2 \geq 1/u_2 \geq 0 \). Hence \( u_2 \in [(\sqrt{5} - 1)/2, 1] \), and

\[
g(u_1, \ldots, v_3) = 1/u_2 + u_2 - 1 < 5/4 \quad \text{for } u_2 \in [(\sqrt{5} - 1)/2, 1],
\]

which is a contradiction.

Now, we can show that \( \bar{\Delta}(2,3,6) = 1 \) or \( v_3 = v_1 \). Note that

\[
g_1(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) = g_1(u_1, v_1, u_2, v_2, u_3, v_3),
\]

\[
g_4(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) = g_4(u_1, v_1, u_2, v_2, u_3, v_3) + (u_1 + v_1 - 1)d,
\]

\[
g(u_1, v_1, u_2, v_2, u_3 + d, v_3 + d) = g(u_1, v_1, u_2, v_2, u_3, v_3) + (v_2 - u_2)d.
\]

Suppose \( \bar{\Delta}(2,3,6) < 1 \). If \( v_3 < v_1 \), then we can increase \( g \) by choosing \( d > 0 \), a contradiction. So we have \( v_3 = v_1 \).

**Now we can finish the proof of Proposition 3.3.**

Suppose \( (u_1, v_1, u_2, v_2, u_3, v_3) \) satisfies \( v_3 \leq v_1 \leq v_2 \), (a), (b), (c), (f) and one of the conditions in Lemma 3.4, we will show that \( g(u_1, v_1, u_2, v_2, u_3, v_3) \leq \frac{5}{4} \) according to the three conditions.

**Case 2.1** Suppose \( v_1 = v_2 = v_3 = v \). Then we have

\[
\bar{\Delta}(1,3,5) = (u_1 + u_2)v, \quad \bar{\Delta}(2,4,6) = (u_1 + u_3)v,
\]

\[
g(u_1, v_1, u_2, v_2, u_3, v_3) = u_1 + u_2(1 - v) + v + u_3v - 1.
\]

We need to maximize \( g(u_1, v_1, u_2, v_2, u_3, v_3) \) subject to the constraints:

\[
(u_1 + u_2)v \leq 1 \iff u_2 \leq \frac{1 - u_1}{v},
\]

\[
(u_1 + u_3)v \leq 1 \iff u_3 \leq \frac{1 - u_1}{v},
\]

and

\[
\frac{5}{4} \leq u_1 + v_1 \leq 2, \quad 0 \leq u_2, \quad u_3 \leq v \leq 1.
\]

Because \( \left(v - \frac{1}{2}\right)^2 \geq 0 \), it follows that \( v^2 \geq v - \frac{1}{4} \geq 1 - u_1 \), and hence \( 1 \geq \frac{1 - u_1}{v^2} \). Therefore, the maximum of \( g(u_1, v_1, u_2, v_2, u_3, v_3) \) occurs at \( u_2 = u_3 = \frac{1 - u_1}{v} \). Then

\[
g(u_1, v_1, u_2, v_2, u_3, v_3) = u_1 + v + \frac{1 - u_1}{v} - 1 = h(u_1, v).
\]
Since $\frac{\partial h}{\partial v} = 1 - \frac{1 - u_1}{v^2} \geq 0$, the maximum of $h$ occurs at $v = 1$, which gives $h(u_1, 1) = \frac{1}{4} < \frac{5}{4}$.

**Case 2.2** Suppose $\Delta(1, 3, 5) = 1$, $v_3 = u_3 = v_1 = v$. Then we have

$$\Delta(1, 3, 5) = u_2 v + u_1 v_2 = 1 \implies u_2 = \frac{(1 - u_1 v_2)}{v}$$

and

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = (u_1 + v)(1 + v_2) + \frac{1 - u_1 v_2}{v} - 2 = k(u_1, v_2, v).$$

So we want to maximize $k(u_1, v_2, v)$ subject to

$$\Delta(2, 3, 6) = v(u_1 + v) \leq 1, \quad \frac{1}{4} \leq \frac{5}{4} - u_1 \leq v \leq v_2 \leq 1, \quad \frac{1 - u_1 v_2}{v} \leq v_2.$$

Equivalently,

$$\frac{1}{4} \leq \frac{5}{4} - u_1 \leq v \leq \frac{1}{v + u_1} \leq v_2 \leq 1.$$

Note that $\frac{\partial k}{\partial v_2} = v = u_1 \left(\frac{1}{v} - 1\right)$.

Suppose $\frac{\partial k}{\partial v_2} \geq 0$, i.e., $u_1 \leq \frac{v^2}{1 - v}$. Then the maximum of $k$ occurs at $v_2 = 1$ so that

$$k(u_1, 1, v) = 2u_1 + \frac{(1 - u_1)}{v} + 2v - 2.$$

Elementary calculus shows that the maximum of $2u_1 + \frac{(1 - u_1)}{v} + 2v - 2$ with

$$\frac{1}{4} \leq \frac{5}{4} - u_1 \leq v \leq \frac{1}{v + u_1} \leq 1, \quad u_1 \leq \frac{v^2}{1 - v}$$

occurs at $v = \frac{2}{3}$, $u_1 = \frac{5}{6}$ and $k(\frac{5}{6}, 1, \frac{2}{3}) = \frac{5}{4}$.

Suppose $\frac{\partial k}{\partial v_2} < 0$, i.e., $u_1 < \frac{v^2}{1 - v}$. Then the maximum of $k$ occurs at $v_2 = \frac{1}{(u_1 + v)}$ so that

$$k(u_1, \frac{1}{(u_1 + v)}, v) = u_1 + v + \frac{1}{(u_1 + v)} - 1.$$

Direct calculation shows that the maximum of $u_1 + v + \frac{1}{(u_1 + v)} - 1$ in

$$\frac{1}{4} \leq \frac{5}{4} - u_1 \leq \frac{1}{v + u_1} \leq 1, \quad u_1 \geq \frac{v^2}{1 - v}$$

occurs at $u_1 = 1$, $v = \frac{\sqrt{5} - 1}{2}$, which gives $v_2 = \frac{\sqrt{5} - 1}{2}$ and $k(1, \frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} - 1}{2}) = \sqrt{5} - 1 < \frac{5}{4}$.  

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**Case 2.3** \(\bar{\Delta}(1,3,5) = 1, \bar{\Delta}(2,3,6) = 1\) and \(v_3 = u_3\). Then we have

\[
\begin{align*}
u_1 &= \frac{(1-v_1 + v_3 - v_1 v_3)}{v_3}, \quad u_2 = \frac{v_1 v_2 + v_3 + v_1 v_2 v_3 - v_2 - v_2 v_3}{v_1 v_3},
\end{align*}
\]

and

\[
g(u_1, v_1, u_2, v_2, u_3, v_3) = \frac{(1-v_1)(v_1 - v_2) + v_3 + (v_2 - 1)v_3^2}{v_1 v_3} = \ell(v_1, v_2, v_3).
\]

So we want to maximize \(\ell(v_1, v_2, v_3)\) subject to

\[
\frac{1}{4} \leq \frac{5}{4} - \frac{(1-v_1 + v_3 - v_1 v_3)}{v_3} \leq v_1 \leq v_2 \leq 1, \quad \frac{v_1 v_2 + v_3 + v_1 v_2 v_3 - v_2 - v_2 v_3}{v_1 v_3} \leq v_2 \leq 1.
\]

Equivalently,

\[
v_1 \leq v_2 \leq 1, \quad \frac{1}{1 + v_3} \leq v_1 \leq \frac{4 - v_3}{4}.
\]

Note that \(\frac{\partial \ell}{\partial v_2} = \frac{v_1 + v_3^2 - 1}{v_1 v_3}\).

Suppose \(v_1 + v_3^2 \geq 1\). The maximum of \(\ell\) occurs at \(v_2 = 1\) so that \(\ell(v_1, 1, v_3) = \frac{v_3 - (1 - v_1)^2}{v_1 v_3}\).

Direct calculation shows that the maximum of \(\frac{v_3 - (1 - v_1)^2}{v_1 v_3}\) with

\[
v_1 \leq v_2 \leq 1, \quad \frac{1}{1 + v_3} \leq v_1 \leq \frac{4 - v_3}{4}, \quad v_1 + v_3^2 \geq 1
\]

occurs at \(v_1 = v_3 = \frac{2}{3}\) and \(\ell\left(\frac{2}{3}, 1, \frac{2}{3}\right) = \frac{5}{4}\).

Suppose \(v_1 + v_3^2 < 1\). The maximum of \(\ell\) occurs at \(v_2 = v_1\) so that \(\ell(v_1, v_1, v_3) = \frac{1 - (1 - v_1)v_3}{v_1}\).

Direct calculation shows that the maximum of \(\frac{1 - (1 - v_1)v_3}{v_1}\) in

\[
v_1 \leq v_2 \leq 1, \quad \frac{1}{1 + v_3} \leq v_1 \leq \frac{4 - v_3}{4}, \quad v_1 + v_3^2 \leq 1
\]

occurs at \(v_1 = \frac{2}{3}, v_3 = \frac{1}{2}\) and \(\ell\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{2}\right) = \frac{5}{4}\).

**Remarks** Several comments related to Proposition 1.2 are in order.

1. The proof of Proposition 1.2 is direct but quite lengthy. A shorter proof is desirable.

2. One might expect that a symmetry argument can be used to show that the solution of Proposition 1.2 is attained at a regular hexagon by a suitable affine transform when \(n = 6\), but it is not the case as shown by our result.
3. Following Proposition 1.2, a natural problem to study is to determine the optimal bound of the ratio between the area of an \( n \)-sided convex polygon \( P_n \) and the maximal area of an \( m \)-sided polygon \( P_m \subset P_n \) for \( m < n \).

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