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Factoring a Quadratic Operator as a Product of Two Positive Contractions

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Factoring a Quadratic Operator as a Product of Two Positive Contractions

Chi-Kwong Li and Ming-Cheng Tsai

Abstract. Let T be a quadratic operator on a complex Hilbert space H. We show that T can be written as a product of two positive contractions if and only if T is of the form

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some $a, b \in [0,1]$ and strictly positive operator P with $\|P\| \le |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}$. Also, we give a necessary condition for a bounded linear operator T with operator matrix $\left(\begin{smallmatrix} T_1 & T_3 \\ 0 & T_2 \end{smallmatrix}\right)$ on $H \oplus K$ that can be written as a product of two positive contractions.

1 Introduction

There has been considerable interest in studying the factorization of bounded linear operators (see [2–5, 15]). For example, a 2×2 matrix C can be written as a product of two orthogonal projections if and only if C is the identity operator or C is unitarily similar to $\left(\begin{smallmatrix} a & \sqrt{a(1-a)} \\ 0 & 0 \end{smallmatrix}\right)$ for some $a \in [0,1]$. For more results about products of orthogonal projections, one may consult [1, 7, 8, 11]. Note that one can write an $n \times n$ matrix C as a product of two positive (semi-definite) operators exactly when C is similar to a positive operator (see [14, Theorem 2.2]). However, in the infinite-dimensional case, the product of two positive operators may not be similar to a positive operator (see [12], [15, Example 2.11]). For more development in this direction, one may consult [12, 14, 15].

In this paper, we study the problem when a bounded linear operator T on a complex Hilbert space H can be written as a product of two positive contractions. In this case, T must be a contraction, and we have that

$$-I/8 \le \operatorname{Re} T \quad \text{and} \quad -I/4 \le \operatorname{Im} T \le I/4$$

(see [10, Theorem 1.1 and Corollary 4.3]). In Proposition 2.4, we give a necessary condition for this problem when T has operator matrix

$$\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \quad \text{on} \quad H \oplus K.$$

In such a case, T_1 and T_2 must also be products of two positive contractions. This is an extension of the result of Wu in [14, Corollary 2.3] concerning the finite dimensional

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case. However, even for a 2×2 matrix C, it is not easy to determine when it is the product of two positive contractions. For example, consider

$$C = \frac{1}{25}\begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}.$$

The diagonalizable contraction C is similar to a positive operator. Thus, it is a product of two positive operators. Moreover, C satisfies $-I/8 \leq \operatorname{Re} C$ and $-I/4 \leq \operatorname{Im} C \leq I/4$. However, we will see that C cannot be written as a product of two positive contractions by Lemma 2.1.

Let $B(H)$ be the algebra of bounded linear operators acting on a complex Hilbert space H. We identify $B(H)$ with M_n, the algebra of $n \times n$ complex matrices, if H has finite dimension n. Recall that a bounded linear operator $T \in B(H)$ is positive (resp., strictly positive) if $\langle Th, h \rangle \geq 0$ (resp., $\langle Th, h \rangle > 0$) for every $h \neq 0$ in H. As usual, we write $T \geq 0$ (resp., $T > 0$) when T is positive (resp., strictly positive).

We call $T \in B(H)$ a quadratic operator if $(T - aI)(T - bI) = 0$ for some scalars a, $b \in \mathbb{C}$. Every quadratic operator $T \in B(H)$ is unitarily similar to

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some $a, b \in \mathbb{C}$, $P > 0$ (see [13]). In this paper, we prove the following theorem.

Theorem 1.1 *A quadratic operator $T \in B(H)$ with operator matrix*

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \textit{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some $a, b \in \mathbb{C}$ and $P > 0$, can be written as a product of two positive contractions if and only if $a, b \in [0,1]$ and

$$\|P\| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}.$$

2 Proof

First we consider the 2×2 case so that we can identify $B(H) = M_2$ and $H = \mathbb{C}^2$.

Lemma 2.1 *Suppose $C = \left(\begin{smallmatrix} a & z \\ 0 & b \end{smallmatrix}\right)$ with $z \geq 0$. Then C is a product of two positive contractions if and only if a, $b \in [0,1]$ and*

$$z \in S = \left\{ c : 0 \leq c \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)} \right\}.$$

If the above equivalent conditions hold, then there are continuous maps $a_{ij}(z), b_{ij}(z)$ for $1 \leq i, j \leq 2$ with

$$\begin{aligned} &0 \leq a_{ii}(z), \quad a_{12}(z) = a_{21}(z) \geq 0, \quad 0 \leq (a_{ij}(z)) \leq I, \\ &b_{ii}(z) \leq 1, \quad b_{12}(z) = b_{21}(z) \leq 0, \quad 0 \leq (b_{ij}(z)) \leq I. \end{aligned} \tag{2.1}$$

such that

$$(a_{ij}(z))(b_{ij}(z)) = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}, \quad z \in S. \tag{2.2}$$

Proof We first prove the sufficiency. Without loss of generality, we can assume that $0 \le a \le b \le 1$. If $a = b$ or $b = 1$, then $z = 0$ and $C = \operatorname{diag}(a, 1)\operatorname{diag}(1, b)$. In the sequel, we may assume $0 \le a < b < 1$, and consider two cases.

Case 1. $0 = a < b < 1$. For $z \in S$, we have that $z^2 \le b(1-b)$, and hence $(z^2/b) + b \le (1-b) + b = 1$. Consider

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} z^2/b & z \\ z & b \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

Then A is rank 1 with eigenvalue $(z^2/b) + b$, and $C = AB$. Evidently, $a_{ij}(z)$ and $b_{ij}(z)$ are continuous maps for $1 \le i, j \le 2$ and satisfy (2.1) and (2.2).

Case 2. $0 < a < b < 1$. For $z \in S$, we have

$$a + b - \frac{z^2}{(1-a)(1-b)} \ge a + b - (\sqrt{a} - \sqrt{b})^2 = 2\sqrt{ab}.$$

Let $\lambda_1(z) \ge \lambda_2(z)$ be roots of the equation

$$\lambda^2 - \left(a + b - \frac{z^2}{(1-a)(1-b)}\right)\lambda + ab = 0.$$

Then $a \le \lambda_2(z) \le \lambda_1(z) \le b$ and $\lambda_1(z), \lambda_2(z)$ are continuous maps on $z \in S$. Note that

$$\lambda_1(z)\lambda_2(z) = ab, \quad \lambda_1(z) + \lambda_2(z) = a + b - \frac{z^2}{(1-a)(1-b)}.$$

We have

$$z = \sqrt{\frac{(1-a)(1-b)(\lambda_j - a)(b - \lambda_j)}{\lambda_j}}, \quad j = 1, 2. \tag{2.3}$$

We will construct

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \gamma \begin{pmatrix} a_3 & -a_2 \\ -a_2 & a_4 \end{pmatrix}$$

such that A has eigenvalues $1, \lambda_1$, B has eigenvalues $1, \lambda_2$, and $C = AB$. First, we set

$$\gamma = \frac{\lambda_2}{b} = \frac{a}{\lambda_1} < 1. \tag{2.4}$$

Because $1 - b - \gamma + b\gamma = (1-b)(1-\gamma) > 0$, we can let

$$a_3 = \frac{b - a}{1 + b\gamma - \gamma - a} < \frac{b-a}{b-a} = 1$$

so that by (2.4),

$$\begin{aligned} a_3 - \lambda_1 &= \frac{(b-a)}{(1 + b\gamma - \gamma - a)} - \frac{a}{\gamma} = \frac{\gamma b - \gamma a - a - \gamma ab + \gamma a + a^2}{\gamma(1 + b\gamma - \gamma - a)} \\ &= \frac{\frac{1}{\gamma}(\gamma b - a)(1-a)}{(1 + b\gamma - \gamma - a)} = \frac{(b - \lambda_1)(1-a)}{(1 + b\gamma - \gamma - a)} \ge 0. \end{aligned}$$

Then we can let $a_1 = 1 + \lambda_1 - a_3 > 0$ so that $a_1 + a_3 = 1 + \lambda_1$, and

$$a_2 = \sqrt{a_1 a_3 - \lambda_1} = \sqrt{(1 + \lambda_1 - a_3)a_3 - \lambda_1} = \sqrt{(1 - a_3)(a_3 - \lambda_1)}$$

so that $a_1a_3 - a_2^2 = \lambda_1$. As a result, $a_1 + a_3 = 1 + \lambda_1$, $\det(A) = \lambda_1$, and hence A has eigenvalues $1, \lambda_1$. Further, let

$$a_4 = \frac{1}{a_3}\left(\frac{\lambda_2}{\gamma^2} + a_2^2\right),$$

so that $\gamma^2(a_3a_4 - a_2^2) = \lambda_2$. Then by (2.4),

$$\begin{aligned}
\gamma(a_3 + a_4) &= \gamma a_3 + \frac{\gamma}{a_3}\left(\frac{\lambda_2}{\gamma^2} + a_2^2\right) = \frac{\gamma}{a_3}\left(\frac{\lambda_2}{\gamma^2} + (a_3 - \lambda_1 + \lambda_1 a_3)\right) \\
&= \frac{\gamma}{a_3}\left(\frac{\lambda_2}{\gamma^2} - \lambda_1\right) + \gamma(1 + \lambda_1) = \frac{\gamma}{a_3}\frac{(b-a)}{\gamma} + \gamma(1+\lambda_1) \\
&= 1 + b\gamma - \gamma - a + \gamma + \gamma\lambda_1 = 1 + \lambda_2.
\end{aligned}$$

As a result, $\operatorname{tr} B = 1 + \lambda_2$ and $\det(B) = \lambda_2$. Therefore, B has eigenvalues $1, \lambda_2$. Denote by $(AB)_{ij}$ the (i, j) entry of AB. By (2.4),

$$(AB)_{11} = \gamma(a_1a_3 - a_2^2) = \gamma\lambda_1 = a, \quad (AB)_{22} = \gamma(a_3a_4 - a_2^2) = \gamma(\lambda_2/\gamma^2) = b.$$

Clearly, $(AB)_{21} = \gamma(a_2a_3 - a_3a_2) = 0$. By (2.4) and (2.3),

$$\begin{aligned}
(AB)_{12} &= \gamma a_2(a_4 - a_1) = \gamma\sqrt{(1-a_3)(a_3-\lambda_1)}\big((a_3+a_4) - (a_3+a_1)\big) \\
&= \frac{\gamma\sqrt{(1-b-\gamma+b\gamma)(b-\lambda_1)(1-a)}}{(1+b\gamma-\gamma-a)}\left(\frac{(1+\lambda_2)}{\gamma} - (1+\lambda_1)\right) \\
&= \frac{\sqrt{(1-b)(1-\gamma)(1-a)(b-\lambda_1)}}{(1+b\gamma-\gamma-\gamma\lambda_1)}(1+\lambda_2-\gamma-\gamma\lambda_1) \\
&= \sqrt{(1-b)(1-a)(1-\gamma)(b-\lambda_1)} = \sqrt{\frac{(1-b)(1-a)(\lambda_1-a)(b-\lambda_1)}{\lambda_1}} = z.
\end{aligned}$$

For the converse, since A and B are positive contractions with $\sigma(C) = \sigma(AB) = \sigma(B^{1/2}AB^{1/2}) \subseteq [0, \infty)$, we have $0 \le a, b \le 1$. Without loss of generality, we may assume $a \le b$. First, consider $\|A\| = \|B\| = 1$. Then the assumption $C = AB$ implies C is unitarily similar to

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_2 & b_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 b_1 & \alpha_1 b_2 \\ b_2 & b_4 \end{pmatrix},$$

where $\left(\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_4 \end{smallmatrix}\right)$ is unitarily similar to $\left(\begin{smallmatrix} \alpha_2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $0 \le \alpha_1, \alpha_2 \le 1$, $\alpha_2 \le b_1, b_4 \le 1$ and $b_2 \ge 0$. Thus, we have $1+\alpha_2 = b_1+b_4$, $a+b = \alpha_1b_1+b_4$, $ab = \alpha_1\alpha_2 = \alpha_1(b_1b_4-b_2^2)$, and $a^2 + b^2 + z^2 = \alpha_1^2(b_1^2 + b_2^2) + b_2^2 + b_4^2$. These imply that

$$z^2 = \left[\alpha_1^2(b_1^2 + b_2^2) + b_2^2 + b_4^2\right] - \left[(\alpha_1b_1 + b_4)^2 - 2\alpha_1\alpha_2\right] = (1-\alpha_1)^2 b_2^2.$$

Hence we may assume $\alpha_1 < 1$. In addition, we also obtain that

$$a + b = \alpha_1b_1 + b_4 = \alpha_1b_1 + 1 + \alpha_2 - b_1 = 1 + \alpha_2 - (1-\alpha_1)b_1$$

and hence

$$\begin{aligned} b_1 &= \frac{1}{1-\alpha_1}(1+\alpha_2-a-b) = \frac{1}{1-\alpha_1}\big[(1-a)(1-b)-ab+\alpha_2\big] \\ &= \frac{1}{1-\alpha_1}\big[\alpha_2(1-\alpha_1)+(1-a)(1-b)\big], \end{aligned}$$

where the last equality follows from $ab = \alpha_1\alpha_2$. Let $c = (1-a)(1-b)/(1-\alpha_1)$. Then $b_1 = \alpha_2 + c$ and $b_4 = 1 - c$. By a direct computation, we see that

$$\begin{aligned} z^2 &= (1-\alpha_1)^2 b_2^2 = (1-\alpha_1)^2(b_1b_4-\alpha_2) \\ &= (1-\alpha_1)^2\big[(\alpha_2+c)(1-c)-\alpha_2\big] \qquad (\text{because } \alpha_2 = b_1b_4 - b_2^2) \\ &= c(1-\alpha_1)\big[(1-\alpha_1)(1-\alpha_2)-c(1-\alpha_1)\big] \\ &= (1-a)(1-b)\big[(a+b)-(\alpha_1+\alpha_2)\big], \end{aligned}$$

where the last equality follows from $c = (1-a)(1-b)/(1-\alpha_1)$ and $ab = \alpha_1\alpha_2$. Since $ab = \alpha_1\alpha_2$, we have $\alpha_1 + \alpha_2 \ge 2\sqrt{\alpha_1\alpha_2} = 2\sqrt{ab}$. This implies that

$$z \le |\sqrt{a}-\sqrt{b}|\sqrt{(1-a)(1-b)}.$$

In general, since $C = \alpha\begin{pmatrix} a/\alpha & z/\alpha \\ 0 & b/\alpha \end{pmatrix} = \alpha(\frac{A}{\|A\|})(\frac{B}{\|B\|})$, where $0 < \alpha = \|A\|\|B\| \le 1$, the scalars a, b, z in the above can be replaced by $a/\alpha, b/\alpha, z/\alpha$, respectively, to get $0 \le a/\alpha, b/\alpha \le 1$ and

$$\frac{z}{\alpha} \le \sqrt{\left(1-\tfrac{a}{\alpha}\right)\left(1-\tfrac{b}{\alpha}\right)}\ \left|\sqrt{\tfrac{a}{\alpha}}-\sqrt{\tfrac{b}{\alpha}}\right|.$$

This shows that $0 \le a, b \le \alpha \le 1$ and

$$z \le |\sqrt{a}-\sqrt{b}|\sqrt{(\alpha-a)\left(1-\tfrac{b}{\alpha}\right)} \le |\sqrt{a}-\sqrt{b}|\sqrt{(1-a)(1-b)}.$$

This proves the necessity. ∎

In order to prove Theorem 1.1, we need the following fact; see, for example, [9, p. 547].

Lemma 2.2 *Let A be a bounded linear operator of the form*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \text{ on } H \oplus K,$$

where H and K are Hilbert spaces. Then A is positive if and only if A_{11} and A_{22} are both positive and there exists a contraction D mapping K into H satisfying $A_{12} = A_{11}^{1/2} D A_{22}^{1/2}$.

Lemma 2.3 *Suppose $a_{11}(z), a_{22}(z), a_{12}(z) = a_{21}(z)$ are continuous real-valued functions defined on $S \subseteq [0,\infty)$ such that*

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \ge 0$$

for all $z \in S$. Then

$$\begin{pmatrix} a_{11}(P) & a_{12}(P) \\ a_{21}(P) & a_{22}(P) \end{pmatrix} \ge 0$$

on $H \oplus H$ for all positive operators $P \in B(H)$ with spectrum in S.

Proof Since $A \geq 0$, we have $a_{11}(z), a_{22}(z) \geq 0$, and

$$0 \leq a_{12}(z)a_{21}(z) \leq a_{11}(z)a_{22}(z), \qquad z \in S.$$

Define $h(z)$ by

$$h(z) := \begin{cases} \frac{a_{12}(z)}{a_{11}^{1/2}(z)a_{22}^{1/2}(z)} & \text{if } |a_{12}(z)| > 0, \\ 0 & \text{if } a_{12}(z) = 0. \end{cases}$$

Then $h(z)$ is a bounded Borel function on S with $|h(z)| \leq 1$ that satisfies

$$a_{12}(z) = a_{11}^{1/2}(z)h(z)a_{22}^{1/2}(z).$$

By the spectral theorem, for all positive operators $P \in B(H)$ with spectrum in S, we have $a_{11}(P) \geq 0$, $a_{22}(P) \geq 0$, $a_{12}(P) = a_{21}(P) \geq 0$, and

$$a_{12}(P) = a_{11}^{1/2}(P)h(P)a_{22}^{1/2}(P)$$

for the contraction $h(P) \in B(H)$. Our assertion follows from Lemma 2.2. ■

In the finite dimensional case, Wu [14, Corollary 2.3] has shown that if $C = \left(\begin{smallmatrix} C_1 & C_3 \\ 0 & C_2 \end{smallmatrix}\right)$ is a product of two positive operators, then so are C_1 and C_2. Proposition 2.4 gives another proof, which holds for both finite and infinite dimensional Hilbert spaces. In fact, it is also true that positive operators are replaced by positive contractions.

Proposition 2.4 *Let T be a bounded linear operator of the form*

$$\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \text{ on } H \oplus K,$$

where H and K are both Hilbert spaces. If T is a product of two positive contractions, then so are T_1 and T_2.

Proof By our assumption and Lemma 2.2, we may assume that $T = AB$, where A and B are of the form

$$\begin{pmatrix} A_1 & A_1^{1/2}D_1A_2^{1/2} \\ A_2^{1/2}D_1^*A_1^{1/2} & A_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 & B_1^{1/2}D_2B_2^{1/2} \\ B_2^{1/2}D_2^*B_1^{1/2} & B_2 \end{pmatrix} \quad \text{on } H \oplus K,$$

respectively, such that $0 \leq A_1 \leq I_H$, $0 \leq A_2 \leq I_K$, $0 \leq B_1 \leq I_H$, $0 \leq B_2 \leq I_K$, D_1 and D_2 are contractions from K into H. From $T = AB$, we obtain that

$$T_1 = A_1B_1 + A_1^{1/2}D_1(A_2^{1/2}B_2^{1/2}D_2^*B_1^{1/2}), \tag{2.5}$$

$$A_2^{1/2}(D_1^*A_1^{1/2}B_1^{1/2})B_1^{1/2} = -A_2^{1/2}(A_2^{1/2}B_2^{1/2}D_2^*)B_1^{1/2}, \tag{2.6}$$

$$T_2 = (A_2^{1/2}D_1^*A_1^{1/2}B_1^{1/2})D_2B_2^{1/2} + A_2B_2.$$

Let E_1 be the restriction of $A_2^{1/2}$ to $(\ker A_2^{1/2})^\perp$; then E_1 is injective. Since $0 \leq A_2^{1/2} \leq I_K$, so we can consider the (possibly unbounded) inverse

$$E := E_1^{-1}\colon \operatorname{ran} A_2^{1/2} \longrightarrow (\ker A_2^{1/2})^\perp$$

such that $EA_2^{1/2} = P_0$, where P_0 is the orthogonal projection from K onto $\overline{\operatorname{ran} A_2^{1/2}}$. Hence by (2.6), we derive that

$$A_2^{1/2}B_2^{1/2}D_2^*B_1^{1/2} = P_0(A_2^{1/2}B_2^{1/2}D_2^*B_1^{1/2}) = -P_0(D_1^*A_1^{1/2}B_1).$$

Moreover, substitute this into (2.5) to get

$$\begin{aligned} T_1 &= A_1B_1 - A_1^{1/2}D_1\big(P_0(D_1^*A_1^{1/2}B_1)\big) = \big[A_1^{1/2}(I_H - D_1P_0D_1^*)A_1^{1/2}\big]B_1 \\ &= \big[A_1^{1/2}(I_H - (P_0D_1^*)^*(P_0D_1^*))A_1^{1/2}\big]B_1. \end{aligned}$$

Note that $\|P_0D_1^*\| \le 1$ implies that

$$0 \le \big(I_H - (P_0D_1^*)^*(P_0D_1^*)\big) \le I_H.$$

Therefore, $T_1 = [(A_1^{1/2}P_1^*)P_1A_1^{1/2}]B_1$, where $P_1^*P_1 = I_H - (P_0D_1^*)^*(P_0D_1^*)$ for some positive contraction P_1 on H. This shows that T_1 is a product of two positive contractions. Similarly, we can show that T_2^* is a product of two positive contractions, and hence so is T_2. This completes our proof. ■

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 We first prove the necessity. By assumption, we can focus on the part

$$\begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \in B(H_3 \oplus H_3)$$

for some $P > 0$. Now, consider a 2×2 matrix $\left(\begin{smallmatrix} a & z \\ 0 & b \end{smallmatrix}\right)$ with $a, b \in [0,1]$ and

$$z \in S := \left\{c : 0 \le c \le |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}\right\}.$$

Then by Lemma 2.1, there are continuous maps $a_{ij}(z), b_{ij}(z)$ for $1 \le i, j \le 2$ with $a_{12}(z) = a_{21}(z) \ge 0$, $b_{12}(z) = b_{21}(z) \le 0$ and satisfying

$$0 \le (a_{ij}(z)) \le I_2,\ \ 0 \le (b_{ij}(z)) \le I_2, \qquad (a_{ij}(z))(b_{ij}(z)) = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}, \quad z \in S.$$

By Lemma 2.3,

$$0 \le (a_{ij}(P)) \le I \quad \text{ and } \quad 0 \le (b_{ij}(P)) \le I.$$

By the spectral theorem on positive operators,

$$(a_{ij}(P))(b_{ij}(P)) = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}.$$

To prove the converse, suppose there is a factorization of the quadratic operator $T \in B(H)$ with operator matrix $aI \oplus bI \oplus \left(\begin{smallmatrix} aI & P \\ 0 & bI \end{smallmatrix}\right)$ for some $P \ge 0$ as the product of two positive contractions. By Proposition 2.4, we know that

$$T_1 = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} = AB \qquad \text{for some} \quad 0 \le A, B \le I,\ A, B \in B(H_3 \oplus H_3).$$

We may use the Berberian construction (see [6]) to embed H_3 into a larger Hilbert space K_3, $B(H_3)$ into $B(K_3)$. Suppose

$$A = (A_{ij})_{1\le i,j\le 2}, \quad B = (B_{ij})_{1\le i,j\le 2} \in B(H_3 \oplus H_3).$$

Then P, A, and B are extended to $\widetilde{P} \in B(K_3)$, $\widetilde{A} = (\widetilde{A}_{ij})_{1\le i,j\le 2} \in B(K_3 \oplus K_3)$, and $\widetilde{B} = (\widetilde{B}_{ij})_{1\le i,j\le 2} \in B(K_3 \oplus K_3)$, respectively, such that the following conditions hold:

(a) $\widetilde{P} \ge 0$ with $\|P\| = \|\widetilde{P}\|$ such that all the elements in $\sigma(\widetilde{P})$ are eigenvalues of $\widetilde{P}$.
(b) $0 \le \widetilde{A}, \widetilde{B} \le I$ such that $\widetilde{T}_1 = \begin{pmatrix} aI & \widetilde{P} \\ 0 & bI \end{pmatrix} = \widetilde{A}\widetilde{B}$.

Since $\widetilde{P} \ge 0$ and $\sigma(\widetilde{P})$ are eigenvalues of $\widetilde{P}$, the quadratic operator $\widetilde{T}_1$ is unitarily similar to $\begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix} \oplus T_2$ admitting a factorization as the product of two positive contractions. By Proposition 2.4, we see that $\begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix}$ is a product of two positive contractions. Thus,

$$\|P\| \le |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}. \qquad \blacksquare$$

Remark 2.5 Inspired by a comment of the referee, we see that if one considers the set of operators of the form $\begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}$ with respect to a fixed orthonormal basis, then our proof of Theorem 1.1 shows that the decomposition depends continuously on P, and therefore continuously on T.

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