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# Factoring a Quadratic Operator as a Product of Two Positive Contractions

Chi-Kwong Li and Ming-Cheng Tsai

*Abstract.* Let  $T$  be a quadratic operator on a complex Hilbert space  $H$ . We show that  $T$  can be written as a product of two positive contractions if and only if  $T$  is of the form

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some  $a, b \in [0, 1]$  and strictly positive operator  $P$  with  $\|P\| \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}$ . Also, we give a necessary condition for a bounded linear operator  $T$  with operator matrix  $\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$  on  $H \oplus K$  that can be written as a product of two positive contractions.

## 1 Introduction

There has been considerable interest in studying the factorization of bounded linear operators (see [2–5, 15]). For example, a  $2 \times 2$  matrix  $C$  can be written as a product of two orthogonal projections if and only if  $C$  is the identity operator or  $C$  is unitarily similar to  $\begin{pmatrix} a & \sqrt{a(1-a)} \\ 0 & 0 \end{pmatrix}$  for some  $a \in [0, 1]$ . For more results about products of orthogonal projections, one may consult [1, 7, 8, 11]. Note that one can write an  $n \times n$  matrix  $C$  as a product of two positive (semi-definite) operators exactly when  $C$  is similar to a positive operator (see [14, Theorem 2.2]). However, in the infinite-dimensional case, the product of two positive operators may not be similar to a positive operator (see [12], [15, Example 2.11]). For more development in this direction, one may consult [12, 14, 15].

In this paper, we study the problem when a bounded linear operator  $T$  on a complex Hilbert space  $H$  can be written as a product of two positive contractions. In this case,  $T$  must be a contraction, and we have that

$$-I/8 \leq \operatorname{Re} T \quad \text{and} \quad -I/4 \leq \operatorname{Im} T \leq I/4$$

(see [10, Theorem 1.1 and Corollary 4.3]). In Proposition 2.4, we give a necessary condition for this problem when  $T$  has operator matrix

$$\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \quad \text{on} \quad H \oplus K.$$

In such a case,  $T_1$  and  $T_2$  must also be products of two positive contractions. This is an extension of the result of Wu in [14, Corollary 2.3] concerning the finite dimensional

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case. However, even for a  $2 \times 2$  matrix  $C$ , it is not easy to determine when it is the product of two positive contractions. For example, consider

$$C = \frac{1}{25} \begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}.$$

The diagonalizable contraction  $C$  is similar to a positive operator. Thus, it is a product of two positive operators. Moreover,  $C$  satisfies  $-I/8 \leq \operatorname{Re} C$  and  $-I/4 \leq \operatorname{Im} C \leq I/4$ . However, we will see that  $C$  cannot be written as a product of two positive contractions by Lemma 2.1.

Let  $B(H)$  be the algebra of bounded linear operators acting on a complex Hilbert space  $H$ . We identify  $B(H)$  with  $M_n$ , the algebra of  $n \times n$  complex matrices, if  $H$  has finite dimension  $n$ . Recall that a bounded linear operator  $T \in B(H)$  is positive (resp., strictly positive) if  $\langle Th, h \rangle \geq 0$  (resp.,  $\langle Th, h \rangle > 0$ ) for every  $h \neq 0$  in  $H$ . As usual, we write  $T \geq 0$  (resp.,  $T > 0$ ) when  $T$  is positive (resp., strictly positive).

We call  $T \in B(H)$  a quadratic operator if  $(T - aI)(T - bI) = 0$  for some scalars  $a, b \in \mathbb{C}$ . Every quadratic operator  $T \in B(H)$  is unitarily similar to

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some  $a, b \in \mathbb{C}, P > 0$  (see [13]). In this paper, we prove the following theorem.

**Theorem 1.1** *A quadratic operator  $T \in B(H)$  with operator matrix*

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

*for some  $a, b \in \mathbb{C}$  and  $P > 0$ , can be written as a product of two positive contractions if and only if  $a, b \in [0, 1]$  and*

$$\|P\| \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}.$$

## 2 Proof

First we consider the  $2 \times 2$  case so that we can identify  $B(H) = M_2$  and  $H = \mathbb{C}^2$ .

**Lemma 2.1** *Suppose  $C = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}$  with  $z \geq 0$ . Then  $C$  is a product of two positive contractions if and only if  $a, b \in [0, 1]$  and*

$$z \in S = \{c : 0 \leq c \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}\}.$$

*If the above equivalent conditions hold, then there are continuous maps  $a_{ij}(z), b_{ij}(z)$  for  $1 \leq i, j \leq 2$  with*

$$(2.1) \quad \begin{aligned} 0 \leq a_{ii}(z), \quad a_{12}(z) = a_{21}(z) \geq 0, \quad 0 \leq (a_{ij}(z)) \leq I, \\ b_{ii}(z) \leq 1, \quad b_{12}(z) = b_{21}(z) \leq 0, \quad 0 \leq (b_{ij}(z)) \leq I. \end{aligned}$$

*such that*

$$(2.2) \quad (a_{ij}(z))(b_{ij}(z)) = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}, \quad z \in S.$$

**Proof** We first prove the sufficiency. Without loss of generality, we can assume that  $0 \leq a \leq b \leq 1$ . If  $a = b$  or  $b = 1$ , then  $z = 0$  and  $C = \text{diag}(a, 1) \text{diag}(1, b)$ . In the sequel, we may assume  $0 \leq a < b < 1$ , and consider two cases.

**Case 1.**  $0 = a < b < 1$ . For  $z \in S$ , we have that  $z^2 \leq b(1-b)$ , and hence  $(z^2/b) + b \leq (1-b) + b = 1$ . Consider

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} z^2/b & z \\ z & b \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then  $A$  is rank 1 with eigenvalue  $(z^2/b) + b$ , and  $C = AB$ . Evidently,  $a_{ij}(z)$  and  $b_{ij}(z)$  are continuous maps for  $1 \leq i, j \leq 2$  and satisfy (2.1) and (2.2).

**Case 2.**  $0 < a < b < 1$ . For  $z \in S$ , we have

$$a + b - \frac{z^2}{(1-a)(1-b)} \geq a + b - (\sqrt{a} - \sqrt{b})^2 = 2\sqrt{ab}.$$

Let  $\lambda_1(z) \geq \lambda_2(z)$  be roots of the equation

$$\lambda^2 - \left( a + b - \frac{z^2}{(1-a)(1-b)} \right) \lambda + ab = 0.$$

Then  $a \leq \lambda_2(z) \leq \lambda_1(z) \leq b$  and  $\lambda_1(z), \lambda_2(z)$  are continuous maps on  $z \in S$ . Note that

$$\lambda_1(z)\lambda_2(z) = ab, \quad \lambda_1(z) + \lambda_2(z) = a + b - \frac{z^2}{(1-a)(1-b)}.$$

We have

$$(2.3) \quad z = \sqrt{\frac{(1-a)(1-b)(\lambda_j - a)(b - \lambda_j)}{\lambda_j}}, \quad j = 1, 2.$$

We will construct

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \gamma \begin{pmatrix} a_3 & -a_2 \\ -a_2 & a_4 \end{pmatrix}$$

such that  $A$  has eigenvalues 1,  $\lambda_1$ ,  $B$  has eigenvalues 1,  $\lambda_2$ , and  $C = AB$ . First, we set

$$(2.4) \quad \gamma = \frac{\lambda_2}{b} = \frac{a}{\lambda_1} < 1.$$

Because  $1 - b - \gamma + b\gamma = (1-b)(1-\gamma) > 0$ , we can let

$$a_3 = \frac{b-a}{1+b\gamma-\gamma-a} < \frac{b-a}{b-a} = 1$$

so that by (2.4),

$$\begin{aligned} a_3 - \lambda_1 &= \frac{(b-a)}{(1+b\gamma-\gamma-a)} - \frac{a}{\gamma} = \frac{\gamma b - \gamma a - a - \gamma ab + \gamma a + a^2}{\gamma(1+b\gamma-\gamma-a)} \\ &= \frac{\frac{1}{\gamma}(\gamma b - a)(1-a)}{(1+b\gamma-\gamma-a)} = \frac{(b-\lambda_1)(1-a)}{(1+b\gamma-\gamma-a)} \geq 0. \end{aligned}$$

Then we can let  $a_1 = 1 + \lambda_1 - a_3 > 0$  so that  $a_1 + a_3 = 1 + \lambda_1$ , and

$$a_2 = \sqrt{a_1 a_3 - \lambda_1} = \sqrt{(1 + \lambda_1 - a_3) a_3 - \lambda_1} = \sqrt{(1 - a_3)(a_3 - \lambda_1)}$$

so that  $a_1 a_3 - a_2^2 = \lambda_1$ . As a result,  $a_1 + a_3 = 1 + \lambda_1$ ,  $\det(A) = \lambda_1$ , and hence  $A$  has eigenvalues  $1, \lambda_1$ . Further, let

$$a_4 = \frac{1}{a_3} \left( \frac{\lambda_2}{\gamma^2} + a_2^2 \right),$$

so that  $\gamma^2(a_3 a_4 - a_2^2) = \lambda_2$ . Then by (2.4),

$$\begin{aligned} \gamma(a_3 + a_4) &= \gamma a_3 + \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} + a_2^2 \right) = \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} + (a_3 - \lambda_1 + \lambda_1 a_3) \right) \\ &= \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} - \lambda_1 \right) + \gamma(1 + \lambda_1) = \frac{\gamma}{a_3} \frac{(b-a)}{\gamma} + \gamma(1 + \lambda_1) \\ &= 1 + b\gamma - \gamma - a + \gamma + \gamma\lambda_1 = 1 + \lambda_2. \end{aligned}$$

As a result,  $\text{tr } B = 1 + \lambda_2$  and  $\det(B) = \lambda_2$ . Therefore,  $B$  has eigenvalues  $1, \lambda_2$ . Denote by  $(AB)_{ij}$  the  $(i, j)$  entry of  $AB$ . By (2.4),

$$(AB)_{11} = \gamma(a_1 a_3 - a_2^2) = \gamma\lambda_1 = a, \quad (AB)_{22} = \gamma(a_3 a_4 - a_2^2) = \gamma(\lambda_2/\gamma^2) = b.$$

Clearly,  $(AB)_{21} = \gamma(a_2 a_3 - a_3 a_2) = 0$ . By (2.4) and (2.3),

$$\begin{aligned} (AB)_{12} &= \gamma a_2(a_4 - a_1) = \gamma \sqrt{(1-a_3)(a_3-\lambda_1)} \left( (a_3 + a_4) - (a_3 + a_1) \right) \\ &= \frac{\gamma \sqrt{(1-b-\gamma+b\gamma)(b-\lambda_1)(1-a)}}{(1+b\gamma-\gamma-a)} \left( \frac{(1+\lambda_2)}{\gamma} - (1+\lambda_1) \right) \\ &= \frac{\sqrt{(1-b)(1-\gamma)(1-a)(b-\lambda_1)}}{(1+b\gamma-\gamma-\gamma\lambda_1)} (1+\lambda_2-\gamma-\gamma\lambda_1) \\ &= \sqrt{(1-b)(1-a)(1-\gamma)(b-\lambda_1)} = \sqrt{\frac{(1-b)(1-a)(\lambda_1-a)(b-\lambda_1)}{\lambda_1}} = z. \end{aligned}$$

For the converse, since  $A$  and  $B$  are positive contractions with  $\sigma(C) = \sigma(AB) = \sigma(B^{1/2}AB^{1/2}) \subseteq [0, \infty)$ , we have  $0 \leq a, b \leq 1$ . Without loss of generality, we may assume  $a \leq b$ . First, consider  $\|A\| = \|B\| = 1$ . Then the assumption  $C = AB$  implies  $C$  is unitarily similar to

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_2 & b_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 b_1 & \alpha_1 b_2 \\ b_2 & b_4 \end{pmatrix},$$

where  $\begin{pmatrix} b_1 & b_2 \\ b_2 & b_4 \end{pmatrix}$  is unitarily similar to  $\begin{pmatrix} \alpha_2 & 0 \\ 0 & 1 \end{pmatrix}$  for some  $0 \leq \alpha_1, \alpha_2 \leq 1$ ,  $\alpha_2 \leq b_1, b_4 \leq 1$  and  $b_2 \geq 0$ . Thus, we have  $1 + \alpha_2 = b_1 + b_4$ ,  $a + b = \alpha_1 b_1 + b_4$ ,  $ab = \alpha_1 \alpha_2 = \alpha_1 (b_1 b_4 - b_2^2)$ , and  $a^2 + b^2 + z^2 = \alpha_1^2 (b_1^2 + b_2^2) + b_2^2 + b_4^2$ . These imply that

$$z^2 = [\alpha_1^2 (b_1^2 + b_2^2) + b_2^2 + b_4^2] - [(\alpha_1 b_1 + b_4)^2 - 2\alpha_1 \alpha_2] = (1 - \alpha_1)^2 b_2^2.$$

Hence we may assume  $\alpha_1 < 1$ . In addition, we also obtain that

$$a + b = \alpha_1 b_1 + b_4 = \alpha_1 b_1 + 1 + \alpha_2 - b_1 = 1 + \alpha_2 - (1 - \alpha_1) b_1$$

and hence

$$\begin{aligned} b_1 &= \frac{1}{1-\alpha_1}(1+\alpha_2-a-b) = \frac{1}{1-\alpha_1}[(1-a)(1-b)-ab+\alpha_2] \\ &= \frac{1}{1-\alpha_1}[\alpha_2(1-\alpha_1)+(1-a)(1-b)], \end{aligned}$$

where the last equality follows from  $ab = \alpha_1\alpha_2$ . Let  $c = (1-a)(1-b)/(1-\alpha_1)$ . Then  $b_1 = \alpha_2 + c$  and  $b_4 = 1 - c$ . By a direct computation, we see that

$$\begin{aligned} z^2 &= (1-\alpha_1)^2 b_2^2 = (1-\alpha_1)^2 (b_1 b_4 - \alpha_2) \\ &= (1-\alpha_1)^2 [(\alpha_2 + c)(1-c) - \alpha_2] \quad (\text{because } \alpha_2 = b_1 b_4 - b_2^2) \\ &= c(1-\alpha_1)[(1-\alpha_1)(1-\alpha_2) - c(1-\alpha_1)] \\ &= (1-a)(1-b)[(a+b) - (\alpha_1 + \alpha_2)], \end{aligned}$$

where the last equality follows from  $c = (1-a)(1-b)/(1-\alpha_1)$  and  $ab = \alpha_1\alpha_2$ . Since  $ab = \alpha_1\alpha_2$ , we have  $\alpha_1 + \alpha_2 \geq 2\sqrt{\alpha_1\alpha_2} = 2\sqrt{ab}$ . This implies that

$$z \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}.$$

In general, since  $C = \alpha \begin{pmatrix} a/\alpha & z/\alpha \\ 0 & b/\alpha \end{pmatrix} = \alpha \left( \frac{A}{\|A\|} \right) \left( \frac{B}{\|B\|} \right)$ , where  $0 < \alpha = \|A\| \|B\| \leq 1$ , the scalars  $a, b, z$  in the above can be replaced by  $a/\alpha, b/\alpha, z/\alpha$ , respectively, to get  $0 \leq a/\alpha, b/\alpha \leq 1$  and

$$\frac{z}{\alpha} \leq \sqrt{\left(1 - \frac{a}{\alpha}\right)\left(1 - \frac{b}{\alpha}\right)} \quad \left| \sqrt{\frac{a}{\alpha}} - \sqrt{\frac{b}{\alpha}} \right|.$$

This shows that  $0 \leq a, b \leq \alpha \leq 1$  and

$$z \leq |\sqrt{a} - \sqrt{b}| \sqrt{(\alpha - a)\left(1 - \frac{b}{\alpha}\right)} \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}.$$

This proves the necessity. ■

In order to prove Theorem 1.1, we need the following fact; see, for example, [9, p. 547].

**Lemma 2.2** *Let  $A$  be a bounded linear operator of the form*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \text{ on } H \oplus K,$$

where  $H$  and  $K$  are Hilbert spaces. Then  $A$  is positive if and only if  $A_{11}$  and  $A_{22}$  are both positive and there exists a contraction  $D$  mapping  $K$  into  $H$  satisfying  $A_{12} = A_{11}^{1/2} D A_{22}^{1/2}$ .

**Lemma 2.3** *Suppose  $a_{11}(z), a_{22}(z), a_{12}(z) = a_{21}(z)$  are continuous real-valued functions defined on  $S \subseteq [0, \infty)$  such that*

$$A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \geq 0$$

for all  $z \in S$ . Then

$$\begin{pmatrix} a_{11}(P) & a_{12}(P) \\ a_{21}(P) & a_{22}(P) \end{pmatrix} \geq 0$$

on  $H \oplus H$  for all positive operators  $P \in B(H)$  with spectrum in  $S$ .

**Proof** Since  $A \geq 0$ , we have  $a_{11}(z), a_{22}(z) \geq 0$ , and

$$0 \leq a_{12}(z)a_{21}(z) \leq a_{11}(z)a_{22}(z), \quad z \in S.$$

Define  $h(z)$  by

$$h(z) := \begin{cases} \frac{a_{12}(z)}{a_{11}^{1/2}(z)a_{22}^{1/2}(z)} & \text{if } |a_{12}(z)| > 0, \\ 0 & \text{if } a_{12}(z) = 0. \end{cases}$$

Then  $h(z)$  is a bounded Borel function on  $S$  with  $|h(z)| \leq 1$  that satisfies

$$a_{12}(z) = a_{11}^{1/2}(z)h(z)a_{22}^{1/2}(z).$$

By the spectral theorem, for all positive operators  $P \in B(H)$  with spectrum in  $S$ , we have  $a_{11}(P) \geq 0$ ,  $a_{22}(P) \geq 0$ ,  $a_{12}(P) = a_{21}(P) \geq 0$ , and

$$a_{12}(P) = a_{11}^{1/2}(P)h(P)a_{22}^{1/2}(P)$$

for the contraction  $h(P) \in B(H)$ . Our assertion follows from Lemma 2.2.  $\blacksquare$

In the finite dimensional case, Wu [14, Corollary 2.3] has shown that if  $C = \begin{pmatrix} C_1 & C_3 \\ 0 & C_2 \end{pmatrix}$  is a product of two positive operators, then so are  $C_1$  and  $C_2$ . Proposition 2.4 gives another proof, which holds for both finite and infinite dimensional Hilbert spaces. In fact, it is also true that positive operators are replaced by positive contractions.

**Proposition 2.4** *Let  $T$  be a bounded linear operator of the form*

$$\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \text{ on } H \oplus K,$$

where  $H$  and  $K$  are both Hilbert spaces. If  $T$  is a product of two positive contractions, then so are  $T_1$  and  $T_2$ .

**Proof** By our assumption and Lemma 2.2, we may assume that  $T = AB$ , where  $A$  and  $B$  are of the form

$$\begin{pmatrix} A_1 & A_1^{1/2}D_1A_2^{1/2} \\ A_2^{1/2}D_1^*A_1^{1/2} & A_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 & B_1^{1/2}D_2B_2^{1/2} \\ B_2^{1/2}D_2^*B_1^{1/2} & B_2 \end{pmatrix} \quad \text{on } H \oplus K,$$

respectively, such that  $0 \leq A_1 \leq I_H$ ,  $0 \leq A_2 \leq I_K$ ,  $0 \leq B_1 \leq I_H$ ,  $0 \leq B_2 \leq I_K$ ,  $D_1$  and  $D_2$  are contractions from  $K$  into  $H$ . From  $T = AB$ , we obtain that

$$(2.5) \quad T_1 = A_1B_1 + A_1^{1/2}D_1(A_2^{1/2}B_2^{1/2}D_2^*B_1^{1/2}),$$

$$(2.6) \quad A_2^{1/2}(D_1^*A_1^{1/2}B_1^{1/2})B_1^{1/2} = -A_2^{1/2}(A_2^{1/2}B_2^{1/2}D_2^*)B_1^{1/2},$$

$$T_2 = (A_2^{1/2}D_1^*A_1^{1/2}B_1^{1/2})D_2B_2^{1/2} + A_2B_2.$$

Let  $E_1$  be the restriction of  $A_2^{1/2}$  to  $(\ker A_2^{1/2})^\perp$ ; then  $E_1$  is injective. Since  $0 \leq A_2^{1/2} \leq I_K$ , so we can consider the (possibly unbounded) inverse

$$E := E_1^{-1}: \text{ran } A_2^{1/2} \longrightarrow (\ker A_2^{1/2})^\perp$$

such that  $EA_2^{1/2} = P_0$ , where  $P_0$  is the orthogonal projection from  $K$  onto  $\overline{\text{ran } A_2^{1/2}}$ . Hence by (2.6), we derive that

$$A_2^{1/2} B_2^{1/2} D_2^* B_1^{1/2} = P_0(A_2^{1/2} B_2^{1/2} D_2^* B_1^{1/2}) = -P_0(D_1^* A_1^{1/2} B_1).$$

Moreover, substitute this into (2.5) to get

$$\begin{aligned} T_1 &= A_1 B_1 - A_1^{1/2} D_1 (P_0(D_1^* A_1^{1/2} B_1)) = [A_1^{1/2} (I_H - D_1 P_0 D_1^*) A_1^{1/2}] B_1 \\ &= [A_1^{1/2} (I_H - (P_0 D_1^*)^* (P_0 D_1^*)) A_1^{1/2}] B_1. \end{aligned}$$

Note that  $\|P_0 D_1^*\| \leq 1$  implies that

$$0 \leq (I_H - (P_0 D_1^*)^* (P_0 D_1^*)) \leq I_H.$$

Therefore,  $T_1 = [(A_1^{1/2} P_1^*) P_1 A_1^{1/2}] B_1$ , where  $P_1^* P_1 = I_H - (P_0 D_1^*)^* (P_0 D_1^*)$  for some positive contraction  $P_1$  on  $H$ . This shows that  $T_1$  is a product of two positive contractions. Similarly, we can show that  $T_2^*$  is a product of two positive contractions, and hence so is  $T_2$ . This completes our proof. ■

Now we are ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1** We first prove the necessity. By assumption, we can focus on the part

$$\begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \in B(H_3 \oplus H_3)$$

for some  $P > 0$ . Now, consider a  $2 \times 2$  matrix  $\begin{pmatrix} a & z \\ 0 & b \end{pmatrix}$  with  $a, b \in [0, 1]$  and

$$z \in S := \{c : 0 \leq c \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}\}.$$

Then by Lemma 2.1, there are continuous maps  $a_{ij}(z), b_{ij}(z)$  for  $1 \leq i, j \leq 2$  with  $a_{12}(z) = a_{21}(z) \geq 0, b_{12}(z) = b_{21}(z) \leq 0$  and satisfying

$$0 \leq (a_{ij}(z)) \leq I_2, \quad 0 \leq (b_{ij}(z)) \leq I_2, \quad (a_{ij}(z))(b_{ij}(z)) = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}, \quad z \in S.$$

By Lemma 2.3,

$$0 \leq (a_{ij}(P)) \leq I \quad \text{and} \quad 0 \leq (b_{ij}(P)) \leq I.$$

By the spectral theorem on positive operators,

$$(a_{ij}(P))(b_{ij}(P)) = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}.$$

To prove the converse, suppose there is a factorization of the quadratic operator  $T \in B(H)$  with operator matrix  $aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}$  for some  $P \geq 0$  as the product of two positive contractions. By Proposition 2.4, we know that

$$T_1 = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} = AB \quad \text{for some } 0 \leq A, B \leq I, A, B \in B(H_3 \oplus H_3).$$

We may use the Berberian construction (see [6]) to embed  $H_3$  into a larger Hilbert space  $K_3, B(H_3)$  into  $B(K_3)$ . Suppose

$$A = (A_{ij})_{1 \leq i, j \leq 2}, \quad B = (B_{ij})_{1 \leq i, j \leq 2} \in B(H_3 \oplus H_3).$$



Then  $P$ ,  $A$ , and  $B$  are extended to  $\tilde{P} \in B(K_3)$ ,  $\tilde{A} = (\tilde{A}_{ij})_{1 \leq i, j \leq 2} \in B(K_3 \oplus K_3)$ , and  $\tilde{B} = (\tilde{B}_{ij})_{1 \leq i, j \leq 2} \in B(K_3 \oplus K_3)$ , respectively, such that the following conditions hold:

- (a)  $\tilde{P} \geq 0$  with  $\|P\| = \|\tilde{P}\|$  such that all the elements in  $\sigma(\tilde{P})$  are eigenvalues of  $\tilde{P}$ .
- (b)  $0 \leq \tilde{A}, \tilde{B} \leq I$  such that  $\tilde{T}_1 = \begin{pmatrix} aI & \tilde{P} \\ 0 & bI \end{pmatrix} = \tilde{A}\tilde{B}$ .

Since  $\tilde{P} \geq 0$  and  $\sigma(\tilde{P})$  are eigenvalues of  $\tilde{P}$ , the quadratic operator  $\tilde{T}_1$  is unitarily similar to  $\begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix} \oplus T_2$  admitting a factorization as the product of two positive contractions. By Proposition 2.4, we see that  $\begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix}$  is a product of two positive contractions. Thus,

$$\|P\| \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}. \quad \blacksquare$$

**Remark 2.5** Inspired by a comment of the referee, we see that if one considers the set of operators of the form  $\begin{pmatrix} a & P \\ 0 & bI \end{pmatrix}$  with respect to a fixed orthonormal basis, then our proof of Theorem 1.1 shows that the decomposition depends continuously on  $P$ , and therefore continuously on  $T$ .

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