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Inverse spectral problems for collections of leading principal submatrices of tridiagonal matrices

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\section*{1. Introduction}

An $n$-by-$n$ matrix $A = (a_{ij})$ is called tridiagonal if $|i - j| > 1$ implies $a_{ij} = 0$. Such a matrix is irreducible if and only if $|i - j| = 1$ implies $a_{ij} \neq 0$. We are interested here in real symmetric (equivalently, complex hermitian) tridiagonal matrices, which, of
course, have distinct real eigenvalues if they are irreducible [5]. Principal submatrices of tridiagonal matrices are tridiagonal, and leading principal submatrices (based upon consecutive indices beginning with 1) of irreducible tridiagonal matrices are irreducible, as well. We let $A_k$ denote the $k$-by-$k$ leading principal submatrix of $A$, and denote its eigenvalues by

$$\lambda_{k,1} < \lambda_{k,2} < \cdots < \lambda_{k,k}.$$  

The classical interlacing inequalities [5] apply, so that

$$\lambda_{k,i} \leq \lambda_{k-h,i} \leq \lambda_{k,i+h}$$

for $h = 1, \ldots, n - k$, with strict inequalities for $h = 1$. The strict inequalities are special to irreducible tridiagonal matrices. It is a well known result that, given the $2n - 1$ real numbers

$$\lambda_{n,1} < \lambda_{n-1,1} < \lambda_{n,2} < \lambda_{n-1,2} < \cdots < \lambda_{n-1,n-1} < \lambda_{n,n},$$

there is an irreducible, real symmetric tridiagonal matrix $A$ such that

$$\sigma(A_n) = \{\lambda_{n,1}, \ldots, \lambda_{n,n}\}$$

and

$$\sigma(A_{n-1}) = \{\lambda_{n-1,1}, \ldots, \lambda_{n-1,n-1}\}.$$  

Moreover, $A$ is unique up to diagonal unitary similarity [3]. This is intuitively plausible as $A$ is described by the $2n - 1$ independent real parameters $a_{11}, \ldots, a_{nn}, a_{n1}, \ldots, a_{n-1,n}$ (because of symmetry) and there are $2n - 1$ real targets, meeting all known necessary conditions. We call this the classical case, though it seems first to have been proven in [3] (uniqueness) and [4] (existence). It has been of interest, both numerically and otherwise in a number of applications.

2. The reasonable conjecture

Our interest here lies in a question whose answer would greatly generalize the classical result about eigenvalues of $A_n$ and $A_{n-1}$. We are motivated, in part, by the need for such assignment results in the multiplicity list problem for trees. See [8], [6], and [1] for specifics, as well as for general background. This also has natural interest as a theoretical problem, and one that has physical applications [2]. Given $2n - 1$ real numbers, when are we able to partition the numbers into sets $S_1, \ldots, S_n$ so that $S_i \subset \sigma(A_i)$ for $i = 1, \ldots, n$, for some real symmetric $n$-by-$n$ irreducible tridiagonal matrix $A$? Of course, we allow some $S_i$’s to be empty, as in the classical case.
Of course, the partition of the $2n - 1$ numbers must be consistent with interlacing generally, and strict interlacing between $A_i$ and $A_{i-1}$, in particular. This is assumed in the classical case. We also assume (as follows there) that our $2n - 1$ numbers are distinct. The sizes of the sets $S_i$, $i = 1, \ldots, n$ must not cause an upper left corner of $A$ to be oversubscribed. We should not expect to assign more than $2i - 1$ eigenvalues to $A_i$ and its principal submatrices, as $A_i$ has only $2i - 1$ independent, nonzero entries. This is automatic in the classical case, but not in the generalization. For example, if $n = 4$ and only one eigenvalue is assigned to $A_4$, then 6 are assigned to $A_1$, $A_2$ and $A_3$, which is one too many.

However, given reasonable sizes of $n$ sets $S_1, \ldots, S_n$, we expect that we can partition any $2n - 1$ real numbers into the sets of these sizes so that a matrix exists with $S_i \subseteq \sigma(A_i)$ for each $i = 1, \ldots, n$. The formal statement of this conjecture is as follows.

**Conjecture 1.** Given a list $L$ of $2n - 1$ distinct real numbers, $l_1 < l_2 < \cdots < l_{2n-1}$, and $n$ non-negative integers, $z_1, \ldots, z_n$, satisfying

$$z_i \leq i,$$

$$\sum_{i=1}^{k} z_i \leq 2k - 1,$$

and

$$\sum_{i=1}^{n} z_i = 2n - 1,$$

there exists an $n$-by-$n$ real symmetric tridiagonal matrix $A$ with

$$\sigma(A_i) \supset S_i,$$

in which $S_1, \ldots, S_n$ is some partition of $L$ satisfying $|S_i| = z_i$ for each $i$.

3. Review of known cases

There are several inverse eigenvalue problems of the type we consider in our main conjecture for which the existence of a solution is guaranteed by previous theorems. One such case is the inverse eigenvalue problem that asks that the $2n - 1$ values in a given list be assigned to the $2n - 1$ eigenvalues of $A$ and $A_{n-1}$. The following classical result ensures the existence of such a matrix $A$.

**Theorem 1.** (See [3,4].) Let $\lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,n}$ and $\lambda_{n-1,1} < \lambda_{n-1,2} < \cdots < \lambda_{n-1,n-1}$ be $2n - 1$ real numbers such that

$$\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1},$$
for all $i = 1, \ldots, n - 1$. Then there exists a real $n$-by-$n$ symmetric tridiagonal matrix $A$ such that the eigenvalues of $A$ are $\lambda_{n,1}, \ldots, \lambda_{n,n}$ and the eigenvalues of $A_{n-1}$ are $\lambda_{n-1,1}, \ldots, \lambda_{n-1,n-1}$. Furthermore, this matrix is unique up to the signs of the super-diagonal entries.

Another case of the main conjecture is the inverse eigenvalue problem that asks that the $2n - 1$ values in a given list be assigned to the set composed of the maximum and minimum eigenvalues of each of the $n$ leading principal submatrices $A_i$, while $A_1$ has only one eigenvalue. The existence of a solution is ensured by the following result.

**Theorem 2.** (See [7].) Let

$$\lambda_{n,1} < \lambda_{n-1,1} < \lambda_{n-2,1} < \cdots < \lambda_{2,1} < \lambda_{1,1} < \lambda_{2,2} < \cdots < \lambda_{n-2,n-2} < \lambda_{n-1,n-1} < \lambda_{n,n}$$

be $2n - 1$ real numbers. Then there exists a real $n$-by-$n$ symmetric tridiagonal matrix $A$ such that $\lambda_{k,1}$ and $\lambda_{k,k}$ are the smallest and largest eigenvalues of $A_k$, for $k = 1, \ldots, n$. Furthermore, this matrix is unique up to the signs of the super-diagonal entries.

The proof of this theorem cleverly uses the recursive characteristic polynomial relationship for symmetric tridiagonal matrices to show that such a matrix $A$ may be constructed by solving several 2-by-2 systems of linear equations.

We may use these two results and a small amount of further explanation to discover that the main conjecture is true in the 2-by-2 and 3-by-3 cases. For the $n = 2$ case, the matrix $A$ only has 3 eigenvalues of leading principal submatrices. Thus, there is only one way we can choose a set of 3 eigenvalues to target, and the result of Theorem 1 guarantees the existence of a matrix $A$ with 3 numbers from a list assigned to these 3 eigenvalues. We consider the $n = 3$ case of the problem with help from the following lemma.

**Lemma 1.** Let $A$ be a real symmetric tridiagonal matrix such that $\{\lambda_{n,i}\}_{i=1}^n = \sigma(A)$, $\{\lambda_{n-1,i}\}_{i=1}^{n-1} = \sigma(A_{n-1})$, and $\{\lambda_{n-2,i}\}_{i=1}^{n-2} = \sigma(A_{n-2})$. Then

$$E_{k-2}(\{\lambda_{n-2,i}\}) = E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))$$

for $k = 2, \ldots, n$, where $E_j(\{\lambda_{k,i}\})$ denotes the $j$th symmetric sum of the eigenvalues of $A_k$. Here we use the convention,

$$E_0(\{\lambda_{n,i}\}) = 1, \forall n$$

and

$$E_k(\{\lambda_{n,i}\}) = 0, \forall n < k.$$
Proof. Let $P_k$ denote the characteristic polynomial of $A_k$ and let the $x_i$ denote $a_{ii}$ for $i = 1, \ldots, n$ and $y_i$ denote $a_{i,i+1}$ for $i = 1, \ldots, n-1$. The well known recursive relation for characteristic polynomials of symmetric tridiagonal matrices gives:

$$P_{n-2}(\lambda) = \frac{P_n(\lambda) - P_{n-1}(\lambda)(\lambda - x_n)}{-y_{n-1}^2}.$$  

If we rewrite $x_n$ and $-y_{n-1}^2$ in terms of the symmetric sums of the eigenvalues of $A$ and $A_{n-1}$:

$$x_n = E_1(\{\lambda_{n,i}\}) - E_1(\{\lambda_{n-1,i}\}),$$
$$-y_{n-1}^2 = E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_2(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))$$

and then equate coefficients of the polynomials on the left and right sides of the equation, we find that

$$E_{k-2}(\{\lambda_{n-2,i}\}) = \frac{E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))},$$

for $k = 2, \ldots, n$. □

Note that the formula given in the above lemma is a formula (up to a sign change) of the coefficients of the characteristic polynomial of $A_{n-2}$ in terms of the spectra of $A$ and $A_{n-1}$.

Proof of 3-by-3 case of Main Conjecture. There are only 3 suitable ways to specify a set of 5 leading principal eigenvalues of $A$: 1) the 3 eigenvalues of $A$ and the 2 eigenvalues of $A_2$; 2) 2 eigenvalues of $A$, 2 eigenvalues of $A_2$, and the eigenvalue of $A_1$; 3) the 3 eigenvalues of $A$, one eigenvalue of $A_2$ and the eigenvalue of $A_1$. The first two cases are covered by Theorems 1 and 2, respectively. Thus, we have only to prove the third case. This case was originally proven by Nuckols in [8] but we present the argument here.

The result of Theorem 1 allows us to be certain of the existence of a tridiagonal matrix $A_3$ if we choose the spectra of $A_3$ and $A_2$ so that the spectrum of $A_2$ interlaces that of $A_3$. The strategy here is to show that once $\lambda_{3,1} < \lambda_{3,2} < \lambda_{2,2} < \lambda_{3,3}$ are fixed, we can find a $\lambda_{2,1} \in (\lambda_{3,1}, \lambda_{3,2})$ so that the matrix that realizes the leading principal eigenvalues $\{\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,3}, \lambda_{2,1}, \lambda_{2,2}\}$ also realizes $\lambda_{1,1} \in (\lambda_{3,1}, \lambda_{3,2})$ as the eigenvalue of $A_1$. The characteristic polynomial of $A_1$ is a linear polynomial. So we find that

$$\lambda_{1,1} = \frac{E_3(\{\lambda_{3,i}\}) + E_2(\{\lambda_{2,i}\})(E_1(\{\lambda_{2,i}\}) - E_1(\{\lambda_{3,i}\}))}{E_2(\{\lambda_{3,i}\}) - E_2(\{\lambda_{2,i}\}) + E_1(\{\lambda_{2,i}\})(E_1(\{\lambda_{2,i}\}) - E_1(\{\lambda_{3,i}\}))} = \frac{\lambda_{3,1}\lambda_{3,2}\lambda_{3,3} + (\lambda_{2,1}\lambda_{2,2})(\lambda_{2,1} + \lambda_{2,2} - \lambda_{3,1} - \lambda_{3,2} - \lambda_{3,3})}{\lambda_{3,1}\lambda_{3,2} + \lambda_{3,1}\lambda_{3,3} + \lambda_{3,2}\lambda_{3,3} - \lambda_{2,1}\lambda_{2,2} + (\lambda_{2,1} + \lambda_{2,2})(\lambda_{2,1} + \lambda_{2,2} - \lambda_{3,1} - \lambda_{3,2} - \lambda_{3,3})}. \quad \Box$$
We can think of the above expression for $\lambda_{1,1}$ as a function of the variable $\lambda_{2,1}$ whose domain is $(\lambda_{3,1}, \lambda_{3,2})$. Note that both the numerator and denominator of the function are continuous functions of $\lambda_{2,1}$ and that Theorem 1 guarantees the existence of a matrix $A_3$ with spectrum $\{\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,3}\}$ and with $\{\lambda_{2,1}, \lambda_{2,2}\} = \sigma(A_2)$. Thus, $a_{23} \neq 0$ and the denominator of the expression is never 0.

If we take the limit of the function as $\lambda_{2,1}$ approaches $\lambda_{3,1}$ we find that

$$
\lim_{\lambda_{2,1} \to \lambda_{3,1}} \lambda_{1,1} = \lambda_{3,1} \frac{\lambda_{2,2}(\lambda_{2,2} - \lambda_{3,2} - \lambda_{3,3}) + \lambda_{3,2} \lambda_{3,3}}{\lambda_{2,2}(\lambda_{2,2} - \lambda_{3,2} - \lambda_{3,3}) + \lambda_{3,2} \lambda_{3,3}} = \lambda_{3,1}.
$$

If we take the limit of the function as $\lambda_{2,1}$ approaches $\lambda_{3,2}$ we find that

$$
\lim_{\lambda_{2,1} \to \lambda_{3,2}} \lambda_{1,1} = \lambda_{3,2} \frac{\lambda_{2,2}(\lambda_{2,2} - \lambda_{3,1} - \lambda_{3,3}) + \lambda_{3,1} \lambda_{3,3}}{\lambda_{2,2}(\lambda_{2,2} - \lambda_{3,1} - \lambda_{3,3}) + \lambda_{3,1} \lambda_{3,3}} = \lambda_{3,2}.
$$

We see that the function is onto the entire interval $(\lambda_{3,1}, \lambda_{3,2})$, by the Intermediate Value Theorem. This proves the lemma for the ordering $\lambda_{3,1} < \lambda_{1,1} < \lambda_{3,2} < \lambda_{2,2} < \lambda_{3,3}$. The proof of the lemma for the ordering $\lambda_{3,1} < \lambda_{2,1} < \lambda_{3,2} < \lambda_{1,1} < \lambda_{3,3}$ uses an analogous argument. \qed

4. New results

An inverse eigenvalue problem similar to the ones resolved by known results is the problem that asks that the $2n - 1$ values from a list be assigned to the set composed of the $n$ eigenvalues of $A$, any $n - 2$ eigenvalues of $A_{n-1}$, and any 1 eigenvalue of $A_{n-2}$. This problem is more accessible when we choose the 1 eigenvalue of $A_{n-2}$ to be either the largest or smallest eigenvalue of $A_{n-2}$. The problem is then resolved by the following result.

**Lemma 2.** Let $\{\lambda_{n,i}\}_{i=1}^{n}$, $\{\lambda_{n-1,i}\}_{i=2}^{n-1}$, and $\lambda_{n-2,1}$ be $2n - 1$ numbers such that

$$
\lambda_{n,1} < \lambda_{n-2,1} < \lambda_{n,2}
$$

and

$$
\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1}, \forall i = 2, \ldots, n - 1.
$$

Then there exists an $n$-by-$n$ real symmetric tridiagonal matrix $A$ such that $\sigma(A) = \{\lambda_{i,n}\}_{i=1,\ldots,n}$, $\{\lambda_{n-1,i}\}_{i=2,\ldots,n-1} \subset \sigma(A_{n-1})$, and $\lambda_{n-2,1} \in \sigma(A_{n-2})$.

**Proof.** The classical result allows us to choose the sets $\{\lambda_{n,i}\}$ and $\{\lambda_{n-1,i}\}$ so that the members of the latter strictly interlace the members of the former and be certain that
a matrix $A$ exists with $\{\lambda_{n,i}\} = \sigma(A)$ and $\{\lambda_{n-1,i}\} = \sigma(A_{n-1})$. Given the numbers assigned to us in the statement of Theorem 1, all we must show is that there exists a $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$ so that the matrix $A$ realizes $\lambda_{n-1,2} \in (\lambda_{n,1}, \lambda_{n,2})$ as an eigenvalue of $A_{n-2}$.

Without loss of generality, we may assume that $\lambda_{n,1} = 0$, for if we prove the existence of a matrix for this case, then we have proven it for all $\lambda_{n,1}$ by a shift of the matrix by a scalar multiple of the identity matrix. Lemma 1 gives:

$$E_{k-2}\{\lambda_{n-2,i}\} = \frac{E_k\{\lambda_{n,i}\} - E_k\{\lambda_{n-1,i}\} + E_{k-1}\{\lambda_{n-1,i}\}(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\})}{E_2\{\lambda_{n,i}\} - E_2\{\lambda_{n-1,i}\} + E_1\{\lambda_{n-1,i}\}(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\}))$$

for $k = 2, \ldots, n$. When $k = n$, we have that $E_n\{\lambda_{n,i}\} = 0$, since $\lambda_{n,1} = 0$; and $E_n\{\lambda_{n-1,i}\} = 0$, by convention. Thus,

$$E_{n-2}\{\lambda_{n-2,i}\} = \frac{E_{n-1}\{\lambda_{n-1,i}\}(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\})}{E_2\{\lambda_{n,i}\} - E_2\{\lambda_{n-1,i}\} + E_1\{\lambda_{n-1,i}\}(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\}))}$$

The denominator of the right hand side of the above equation never vanishes because it is equal to $-a_{n-1,n}^2$, the negative square of the bottom right super-diagonal entry of $A$. We note that $\{E_{k-2}\{\lambda_{n-2,i}\}\}$ are the coefficients (plus or minus) of $P_{n-2}(\lambda)$, the characteristic polynomial of $A_{n-2}$. These vary continuously as a function of $\lambda_{n-1,1}$ when the rest of the spectra of $A_n$ and $A_{n-1}$ is fixed. Since the coefficients of the polynomial vary continuously, the roots of the polynomial vary continuously. In particular, $\lambda_{n-2,1}$ varies continuously as a function of $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$. We see that

$$\lim_{\lambda_{n-1,1} \to \lambda_{n,1}} \prod_{i=1}^{n-2} (\lambda_{n-2,i}) = \lim_{\lambda_{n-1,1} \to 0} \prod_{i=1}^{n-2} (\lambda_{n-2,i})$$

$$= \frac{0(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\})}{E_2\{\lambda_{n,i}\} - E_2\{\lambda_{n-1,i}\} + E_1\{\lambda_{n-1,i}\}(E_1\{\lambda_{n-1,i}\) - E_1\{\lambda_{n,i}\}))} = 0$$

Since $\prod_{i=1}^{n-2} (\lambda_{n-2,i})$ approaches 0, at least one member of $\sigma(A_{n-2})$ approaches 0.
However, for $i = 2, \ldots, n - 2$, the interlacing inequalities imply

$$0 < \lambda_{n,2} < \lambda_{n-2,i}.$$ 

Thus, $\lambda_{n-2,1}$ is the only eigenvalue of $A_{n-2}$ which may approach 0 and

$$\lim_{\lambda_{n-1,1} \to \lambda_{n,1}} (\lambda_{n-2,1}) = 0.$$ 

Thus, 0 is the greatest lower bound of $\lambda_{n-2,1}$.

Now, the interlacing inequalities guarantee that $\lambda_{n-1,1} < \lambda_{n-2,1}$, so as $\lambda_{n-1,1}$ approaches $\lambda_{n,2}$, we still have that $\lambda_{n-1,1} < \lambda_{n-2,1}$. This gives us the result that the least upper bound for $\lambda_{n-2,1}$ is at least $\lambda_{n,2}$. So we have shown that $(\lambda_{n,1}, \lambda_{n,2})$ is contained within the interval between the greatest lower bound and least upper bound of $\lambda_{n-2,1}$. Thus, by the intermediate value theorem, for every $\lambda_{n-2,1} \in (\lambda_{n,1}, \lambda_{n,2})$, there exists some $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$ so that the matrix $A$ realizes $\lambda_{n-2,1}$ as an eigenvalue of $A_{n-2}$. □

Instead of targeting the smallest eigenvalue of $A_{n-2}$ we could have targeted the largest by fixing $\sigma(A_n), \{\lambda_{n-1,i}\}_{i=1}^{n-2} \subset \sigma(A_{n-1})$ and choosing $\lambda_{n-2,n-2} \in (\lambda_{n,n-1}, \lambda_{n,n})$. For that proof, we would assume $\lambda_{n,n} = 0$ and use steps analogous to those used in the above proof.

We can further generalize the result of Lemma 2 to show that we can use an eigenvalue of $A_{n-1}$ to target an eigenvalue for $A_k$ for any $k \leq n - 2$.

**Lemma 3.** For $k \leq n - 2$, let $\{\lambda_{n,i}\}_{i=1}^{n}$, $\{\lambda_{n-1,i}\}_{i=2}^{n-1}$, and $\lambda_{k,1}$ be $2n - 1$ numbers such that

$$\lambda_{n,1} < \lambda_{k,1} < \lambda_{n,2}$$

and

$$\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1} \forall i = 2, \ldots, n - 1.$$

Then there exists an $n \times n$ real symmetric tridiagonal matrix $A$ such that $\sigma(A) = \{\lambda_{n,i}\}_{i=1}^{n}, \{\lambda_{n-1,i}\}_{i=2}^{n-1} \subset \sigma(A_{n-1})$, and $\{\lambda_{k,1}\} \subset \sigma(A_k)$.

**Proof.** Without loss of generality, we will assume that $\lambda_{n,1} = 0$ and we will first show by induction that

$$\lim_{\lambda_{n-1,1} \to \lambda_{n,1}} (\lambda_{k,1}) = \lambda_{n,1} \forall k < n.$$ 

The case $k = n - 1$ holds trivially and $k = n - 2$ is the case covered by the previous lemma. We now assume that it holds for $k = m$ and show that this implies its validity for $k = m - 1$. The following formula holds:
\[ E_{m-1}(\{\lambda_{m-1,i}\}) \]
\[ = \frac{E_m(\{\lambda_{m,i}\})(E_1(\{\lambda_{m,i}\}) - E_1(\{\lambda_{m+1,i}\}))}{E_2(\{\lambda_{m+1,i}\}) - E_2(\{\lambda_{m,i}\}) + E_1(\{\lambda_{m,i}\})(E_1(\{\lambda_{m,i}\}) - E_1(\{\lambda_{m+1,i}\}))}. \]

The inductive assumption gives
\[ \lim_{\lambda_{n-1,i} \to \lambda_{n,1}} (\lambda_{m,1}) = 0. \]
Thus, \[ \lim_{\lambda_{n-1,i} \to \lambda_{n,1}} (E_m(\{\lambda_{m,i}\})) = 0 \] and thus \[ \lim_{\lambda_{n-1,i} \to \lambda_{n,1}} (E_{m-1}(\{\lambda_{m-1,i}\})) = 0. \]
Thus we see that \[ \lim_{\lambda_{n-1,i} \to \lambda_{n,1}} (\lambda_{m-1,1}) = 0. \]

Now each \( \lambda_{k,1} \) is a continuous function of \( \lambda_{n-1,1} \) when the rest of the spectra of \( A \) and \( A_{n-1} \) are fixed. We’ve shown that \( \lambda_{n,1} \) is the greatest lower bound of \( \lambda_{k,1} \). Also, it is obvious that the least upper bound for \( \lambda_{k,1} \) is at least \( \lambda_{2,n} \). Thus, by the intermediate value theorem, for each \( \lambda_{k,1} \in (\lambda_{n,1}, \lambda_{n,2}) \), there exists some \( \lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2}) \) so that the matrix \( A \) guaranteed to exist by the classical result also realizes \( \lambda_{k,1} \) as an eigenvalue of \( A_k \). \( \square \)

We would like to show that we can use the eigenvalues of \( A_{n-1} \) to simultaneously target multiple eigenvalues of \( A_{n-2} \). In order to do this, we first have to analyze the behavior of the eigenvalues of \( A_{n-2} \) as a function of the eigenvalues of \( A_{n-1} \). We would like to explore the partial derivatives of the expression for the coefficients of \( P_{n-2} \) as a function of \( \lambda_{n-1,j} \), as \( \lambda_{n-1,j} \) ranges from \( \lambda_{n,j} \) to \( \lambda_{n,j+1} \). The expression for the coefficients is:

\[ E_{k-2}(\{\lambda_{n-2,i}\}) \]
\[ = \frac{E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}. \]

We will use some notation to help save space and emphasize the role of \( \lambda_{n-1,j} \). Let
\[ x = \lambda_{n-1,j} \]
\[ R = \{\lambda_{n-2,i}\}_{i=1,\ldots,n-2} \]
\[ S = \{\lambda_{n-1,i}\}_{i=1,\ldots,n-1} \]
\[ T = \{\lambda_{n,i}\}_{i=1,\ldots,n} \]
\[ S - x = \{\lambda_{n-1,i}\} \setminus \{x\} \]
\[ N_k(x) = E_k(T) - E_k(S) + E_{k-1}(S)(E_1(S) - E_1(T)) \]
\[ D(x) = E_2(T) - E_2(S) + E_1(S)(E_1(S) - E_1(T)) \]
and note that \( E_{k-2}(R)(x) = \frac{N_k(x)}{D(x)} \). The following facts may be quickly checked.
\[ E_k(S)(x) = xE_{k-1}(S - x) + E_k(S - x). \]
\[ \frac{d}{dx} E_k(S) = E_{k-1}(S - x). \]
\[
\begin{align*}
\frac{d}{dx} N(x) &= -E_{k-1}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T)) + E_{k-1}(S), \\
\frac{d}{dx} D(x) &= -E_1(S-x) + 2E_1(S) - E_1(T) = 2x + E_1(S-x) - E_1(T).
\end{align*}
\]

**Lemma 4.** For each \( k = 3, \ldots, n \), \( E_{k-2}(R)(x) \) has a critical point if and only if

\[\frac{1}{2}(-E_1(S-x) + E_1(T)) \in (\lambda_{n,j}, \lambda_{n,j+1}).\]

**Proof.**

\[
\frac{d}{dx} E_{k-2}(R)(x) = \frac{D(x)N'(x) - N(x)D'(x)}{(D(x))^2}.
\]

By using the above properties, we find that

\[
\begin{align*}
N'(x) &= -E_{k-1}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T)) + E_{k-1}(S) \\
&= xE_{k-2}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T)) \\
&= E_{k-2}(S-x)(x + E_1(S) - E_1(T)) \\
&= E_{k-2}(S-x)(2x + E_1(S-x) - E_1(T)) \\
&= E_{k-2}(S-x)D'(x).
\end{align*}
\]

Thus,

\[
D(x)N'(x) - N(x)D'(x) = D'(x)(E_{k-2}(S-x)D(x) - N(x)).
\]

\( D'(x) = 0 \) if and only if \( x = \frac{1}{2}(-E_1(S-x) + E_1(T)) \). So we have shown that \( E_{k-2}(R)(x) \) has a critical point if \( \frac{1}{2}(-E_1(S-x) + E_1(T)) \in (\lambda_{n,j}, \lambda_{n,j+1}) \). To show that this is the only possible critical point, we will show that the expression \( E_{k-2}(S-x)D(x) - N(x) \) is constant with respect to \( x \).

We will first consider the expression \( E_{k-2}(S-x)D(x) \) and collect from it only the terms non-constant with respect to \( x \).

\[
E_{k-2}(S-x)D(x) = E_{k-2}(S-x)(E_2(T) - E_2(S) + E_1(S)(E_1(S) - E_1(T))) \\
= E_{k-2}(S-x)(c_1 - E_2(S) + (E_1(S))^2 - E_1(S)E_1(T)) \\
= E_{k-2}(S-x)(c_1 - (xE_1(S-x) + E_2(S-x)) \\
+ (x + E_1(S-x))^2 - (x + E_1(S-x))E_1(T))
\]

If we eliminate all constant terms from this last expression, we obtain:

\[
E_{k-2}(S-x)(x^2 + x(E_1(S-x) - E_1(T))).
\]

\[V.\ Higgins, C. Johnson / Linear Algebra and its Applications 489 (2016) 104–122\]

113
Next, consider the expression $N(x)$ and collect from it only the terms non-constant with respect to $x$.

$$N(x) = E_k(T) - E_k(S) + E_{k-1}(S)(E_1(S) - E_1(T))$$

$$= c_2 - E_k(S) + E_{k-1}(S)E_1(S) - E_{k-1}(S)E_1(T)$$

$$= c_2 - (xE_{k-1}(S - x) + E_k(S - x))$$

$$+ (xE_{k-2}(S - x) + E_{k-1}(S - x))(x + E_1(S - x))$$

$$- (xE_{k-2}(S - x) + E_{k-1}(S - x))E_1(T)$$

If we eliminate all constant terms from this last expression, we obtain:

$$E_{k-2}(S - x)x^2 + xE_{k-2}(S - x)E_1(S - x) - xE_1(T)E_{k-2}(S - x)$$

$$= E_{k-2}(S - x)(x^2 + x(E_1(S - x) - E_1(T)))$$

Thus, $E_{k-2}(S - x)D(x) - N(x)$ is constant with respect to $x$.  □

**Lemma 5.**

$$\frac{d}{dx} E_{k-2}(S)(x) = \sum_{i=1}^{n-2} \frac{d}{dx} \lambda_{n-2,i} E_{k-3}(R - \lambda_{n-2,i}).$$

**Proof.** This formula follows from the product rule.  □

**Lemma 6.** For any $k \in \{3, 4, \ldots, n\}$, if $x_0$ is a critical point of $E_{k-2}(R)(x)$, it is also a critical point of $\lambda_{n-2,i}(x)$ for each $i = 1, \ldots, n - 2$.

**Proof.** Let $x_0$ be a critical point of $E_{k-2}(R)(x)$ for some $k \in \{3, \ldots, n\}$. Then by **Lemma 5**, $x_0$ is a critical point of each $E_{k-2}$. We obtain the following system of equations:

$$\begin{align*}
\lambda'_{n-2,1} + \lambda'_{n-2,2} + \cdots + \lambda'_{n-2,n-2} &= 0 \\
\lambda'_{n-2,1}(\lambda_{n-2,2} + \lambda_{n-2,3} + \cdots + \lambda_{n-2,n-2}) + \cdots + \lambda'_{n-2,n-2}(\lambda_{n-2,1} + \lambda_{n-2,2} + \cdots + \lambda_{n-2,n-1}) &= 0 \\
&\vdots \\
\lambda'_{n-2,1}(\lambda_{n-2,2}\lambda_{n-2,3}\cdots\lambda_{n-2,n-2}) + \cdots + \lambda'_{n-2,n-2}(\lambda_{n-2,1}\lambda_{n-2,2}\cdots\lambda_{n-2,n-1}) &= 0
\end{align*}$$

We may regard this as a system of linear equations in the variables $\{\lambda'_{n-2,i}\}$. The matrix of coefficients for this system is:

$$\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
E_1(R \setminus \lambda_{n-2,1}) & E_1(R \setminus \lambda_{n-2,2}) & E_1(R \setminus \lambda_{n-2,3}) & \cdots & E_1(R \setminus \lambda_{n-2,n-2}) \\
E_2(R \setminus \lambda_{n-2,1}) & E_2(R \setminus \lambda_{n-2,2}) & E_2(R \setminus \lambda_{n-2,3}) & \cdots & E_2(R \setminus \lambda_{n-2,n-2}) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
E_{n-3}(R \setminus \lambda_{n-2,1}) & E_{n-3}(R \setminus \lambda_{n-2,2}) & E_{n-3}(R \setminus \lambda_{n-2,3}) & \cdots & E_{n-3}(R \setminus \lambda_{n-2,n-2})
\end{bmatrix}$$
The determinant of this matrix has the same magnitude as the Vandermonde determinant:

\[
\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t}).
\]

This determinant is nonzero by virtue of the distinct \(\{\lambda_{n-2,i}\}\). Thus, our system of equations has only the trivial solution. We have shown that when \(x_0\) is a critical point of some \(E_{k-2}(R)\), it is a critical point of each member of \(R\):

\[
\lambda'_{1,n-2}(x_0) = \lambda'_{2,n-2}(x_0) = \cdots = \lambda'_{n-2,n-2}(x_0) = 0. \tag*{\square}
\]

**Lemma 7.** If \(x_0\) is a critical point of \(\lambda_{n-2,i}(x)\), \(x_0\) is also a critical point of \(E_{k-2}(R)(x)\) for some \(k = 3, \ldots, n\).

**Proof.** Suppose \(x_0\) is a critical point of \(\lambda_{n-2,i}(x)\) but that \(x_0\) is not a critical point of some \(E_{k-2}(R)\). Then by Lemma 4, \(x_0\) is not a critical point of any of the \(\{E_{k-2}(R)\}\).

As in the proof of Lemma 6, we obtain a set of linear equations that may be represented by the following equation:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
E_1(R \setminus \lambda_{n-2,1}) & E_1(R \setminus \lambda_{n-2,2}) & \cdots & E_1(R \setminus \lambda_{n-2,n-2}) \\
E_2(R \setminus \lambda_{n-2,1}) & E_2(R \setminus \lambda_{n-2,2}) & \cdots & E_2(R \setminus \lambda_{n-2,n-2}) \\
\vdots & \vdots & \ddots & \vdots \\
E_{n-3}(R \setminus \lambda_{n-2,1}) & E_{n-3}(R \setminus \lambda_{n-2,2}) & \cdots & E_{n-3}(R \setminus \lambda_{n-2,n-2})
\end{bmatrix}
\begin{bmatrix}
\lambda'_{n-2,1} \\
\lambda'_{n-2,2} \\
\lambda'_{n-2,3} \\
\vdots \\
\lambda'_{n-2,n-2}
\end{bmatrix}
= 
\begin{bmatrix}
E_1(R)' \\
E_2(R)' \\
E_3(R)' \\
\vdots \\
E_{n-2}(R)'
\end{bmatrix}
\]

We can find a formula for the magnitude of the \(\lambda'_{n-2,i}\) by applying Cramer’s Rule:

\[
|\lambda'_{n-2,i}| = \left| \frac{E_1(R)'\lambda_{n-2,i}^{n-3} - E_2(R)'\lambda_{n-2,i}^{n-4} + E_3(R)'\lambda_{n-2,i}^{n-5} - \cdots + (-1)^{n-3}E_{n-2}(R)'} {\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})} \right|
\]

If we evaluate this relationship at a point \(x_0\) for which \(\lambda'_{n-2,i}(x_0) = 0\) but \(E_k(R)(x_0) \neq 0, \forall k = 1, \ldots, n-2\), we find that

\[
E_1(R)'\lambda_{n-2,i}^{n-3} - E_2(R)'\lambda_{n-2,i}^{n-4} + E_3(R)'\lambda_{n-2,i}^{n-5} - \cdots + (-1)^{n-3}E_{n-2}(R)' = 0,
\]

and

\[
\frac{\lambda_{n-2,i}^{n-3}}{E_1(R)'} - \frac{E_2(R)'}{E_1(R)'}\lambda_{n-2,i}^{n-4} + \frac{E_3(R)'}{E_1(R)'}\lambda_{n-2,i}^{n-5} - \cdots + (-1)^{n-3}\frac{E_{n-2}(R)'}{E_1(R)'} = 0.
\]
We wish to show that the roots of this polynomial are values that lie outside the range of $\lambda_{n-2,i}$. We will begin by examining the coefficients of the polynomial. Recall that we showed in the proof of Lemma 4 that

$$E_{k-2}(R)'(x) = \frac{D'(x)(E_{k-2}(S - x)D(x) - N_k(x))}{(D(x))^2}$$

Since $x_0$ is not a critical point of $E_{k-2}(R)(x)$, $D'(x_0) \neq 0$ and we find that

$$\frac{E_{k-2}(R)'(x_0)}{E_1(R)'(x_0)} = \frac{E_{k-2}(S - x)D(x_0) - N_k(x_0)}{E_1(S - x)D(x_0) - N_3(x_0)}$$

Since $D$ is a non-vanishing function, we may divide both the numerator and the denominator of the right hand side of the above equation by $D$ to find that

$$\frac{E_{k-2}(R)'(x_0)}{E_1(R)'(x_0)} = \frac{E_{k-2}(S - x) - \frac{N_k}{D}(x_0)}{E_1(S - x) - \frac{N_3}{D}(x_0)} = \frac{E_{k-2}(S - x) - E_{k-2}(R)}{E_1(S - x) - E_1(R)}$$

After multiplying by a common denominator of these coefficients, our polynomial takes the following form:

$$(E_1(S - x) - E_1(R))\lambda_{n-2}^{n-3} - (E_2(S - x) - E_2(R))\lambda_{n-2,i}^{n-4} + \ldots$$

$$+ (-1)^{n-3}(E_{n-2}(S - x) - E_{n-2}(R)) = 0$$

We make the substitution

$$E_k(R) = \lambda_{n-2,i}E_{k-1}(R \setminus \lambda_{n-2,i}) + E_k(R \setminus \lambda_{n-2,i})$$

and note that

$$(E_k(S - x) - \lambda_{n-2,i}E_{k-1}(R \setminus \lambda_{n-2,i}) - E_k(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^m$$

$$- (E_{k+1}(S - x) - \lambda_{n-2,i}E_k(R \setminus \lambda_{n-2,i}) - E_{k+1}(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m-1}$$

$$= -E_{k-1}(R \setminus \lambda_{n-2,i})\lambda_{n-2,i}^{m+1} + (E_k(S - x) + E_k(R \setminus \lambda_{n-2,i}) - E_k(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^m$$

$$- (E_{k+1}(S - x) + E_{k+1}(R \setminus \lambda_{n-2,i}) - E_{k+1}(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m-1}$$

$$= -E_{k-1}(R \setminus \lambda_{n-2,i})\lambda_{n-2,i}^{m+1} + E_k(S - x)\lambda_{n-2,i}^m - E_{k+1}(S - x)\lambda_{n-2,i}^{m-1}.$$
It is now clear that the set of roots of this polynomial is the set \((S - x) = \{\lambda_{n-1,i}\} \setminus \lambda_{n-1,j}\). However, this contradicts the bounds for \(\lambda_{n-2,i}\) set by the strict interlacing inequalities. Therefore, there exists no point which is a critical point of some \(\lambda_{n-2,i}\) but not a critical point of an \(E_{k-2}(R)\). □

We would now like to consider the behavior of the function

\[
F_k : (\lambda_{n,1}, \lambda_{n,2}) \times (\lambda_{n,2}, \lambda_{n,3}) \times \ldots \times (\lambda_{n,k}, \lambda_{n,k+1}) \to (\lambda_{n,1}, \lambda_{n,3})
\]

\[
\times (\lambda_{n,2}, \lambda_{n,4}) \times \ldots \times (\lambda_{n,k}, \lambda_{n,k+2})
\]

\[
(\lambda_{n-1,1}, \lambda_{n-1,2}, \ldots, \lambda_{n-1,k}) \mapsto (\lambda_{n-2,1}, \lambda_{n-2,2}, \ldots, \lambda_{n-2,k})
\]

which, for \(k \in \{1, \ldots, n - 2\}\), maps the \(k\) smallest eigenvalues of \(A_{n-1}\) to the \(k\) smallest eigenvalues of \(A_{n-2}\) when the rest of the spectra of \(A\) and \(A_{n-1}\) are assumed to be fixed. Our goal is to show that the image of this function includes its domain. Our next task is to examine the Jacobian of the function \(F_k\). In the proof of Lemma 7, we used Cramer’s rule to find a formula for the \(j\)th partial derivative of \(\lambda_{n-2,i}\). We discovered that

\[
\frac{\partial}{\partial \lambda_{n-1,j}} \lambda_{n-2,i} = (-1)^{i+1} \frac{E_1(R)_{\lambda_{n-1,j}} \lambda_{n-2,i}^{n-3} - E_2(R)_{\lambda_{n-1,j}} \lambda_{n-2,i}^{n-4} + \ldots + (-1)^{n-3} E_{n-2}(R)_{\lambda_{n-1,j}}}{\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})}
\]

\[
= (-1)^{i+1} \frac{E_1(R)_{\lambda_{n-1,j}} \sum_{k=0}^{n-2} ((-1)^k E_k(\{\lambda_{n-1,i}\}_{i \neq j}) \lambda_{n-2,i}^{n-2-k}}{\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})}
\]

\[
= E_1(R)_{\lambda_{n-1,j}} (-1)^{i+1} \frac{\prod_{r \neq j} (\lambda_{n-2,i} - \lambda_{n-1,r})}{\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})},
\]

assuming \(E_1(R)_{\lambda_{n-1,j}} \neq 0\). In the above expressions, \(E_1(R)_{\lambda_{n-1,j}}\) denotes \(\frac{\partial}{\partial \lambda_{n-1,j}} E_1(R) \times (\lambda_{n-1,j})\). We now consider the magnitude of the determinant of the Jacobian, denoted \(|J_k|\). After factoring common terms from rows and columns, we find that

\[
|J_k| = \frac{\prod_{j \leq k} (E_1(R)_{\lambda_{n-1,j}}) \prod_{1 \leq i < k, r > k} (\lambda_{n-2,i} - \lambda_{n-1,r})}{\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})} \left| \left\{ \prod_{j \neq r \leq k} (\lambda_{n-2,i} - \lambda_{n-1,r}) \right\}_{ij} \right|
\]
\[
\prod_{j \leq k} (E_1(R)_{\lambda_{n-1,j}}) \prod_{1 \leq i \leq k} (\prod_{r > k} (\lambda_{n-2,i} - \lambda_{n-1,r}))
\prod_{1 \leq s < t \leq n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})
\prod_{1 \leq u < v \leq k} (\lambda_{n-2,u} - \lambda_{n-2,v}) \prod_{1 \leq w < x \leq k} (\lambda_{n-1,w} - \lambda_{n-1,x}).
\]

Thus, the Jacobian is singular only when a partial derivative is zero for the set \{\lambda_{n-2,i}\} or when interlacing bounds are violated. Since the partial derivatives are continuous, by the inverse function theorem we have that the function \(F_k\) is locally invertible at all points at which \(J_k\) is nonsingular. We have now accumulated enough information about the function to prove the following.

**Theorem 3.** Let \(\lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,n}, \lambda_{n-2,1} < \lambda_{n-2,2} < \cdots < \lambda_{n-2,k}\), and \(\lambda_{n-1,k+1} < \lambda_{n-1,k+2} < \cdots < \lambda_{n-1,n-1}\) be \(2n - 1\) real numbers such that

\[
\lambda_{n,i} < \lambda_{n-2,i} < \lambda_{n,i+1},
\]

for all \(i = 1, \ldots, k\) and

\[
\lambda_{n,j} < \lambda_{n-1,j} < \lambda_{n,j+1}
\]

for all \(j = k + 1, \ldots, n - 1\). Then for \(k = 1, \ldots, n - 2\), there exists an \(n\)-by-\(n\) real symmetric tridiagonal matrix \(A\) such that the eigenvalues of \(A\) are \(\lambda_{n,1}, \ldots, \lambda_{n,n}\), the \(k\) smallest eigenvalues of \(A_{n-2}\) are \(\lambda_{n-2,1}, \ldots, \lambda_{n-2,k}\), and the \(n - 1 - k\) largest eigenvalues of \(A_{n-1}\) are \(\lambda_{n-1,k+1}, \ldots, \lambda_{n-1,n-1}\).

**Proof.** By the result of Theorem 1 we are able to ensure \(A\) realizes all the desired eigenvalues of \(A\) and \(A_{n-1}\) as long as those of \(A_{n-1}\) interlace those of \(A_n\). We fix \(\lambda_{n-1,k+1}, \ldots, \lambda_{n-1,n-1}\) as eigenvalues of \(A_{n-1}\). We now need to show that there exist real numbers \(x_1 < x_2 < \cdots < x_k\) interlacing the smallest \(k+1\) eigenvalues of \(A\) such that

\[
F_k(x_1, \ldots, x_k) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,k}).
\]

We will show this by induction on \(k\). The inductive assumption is that there exist real numbers \(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}\) such that \(F_{m-1}(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1})\) and that, furthermore, \(x_i^{(m-1)}\) lies in the interval \((\lambda_{n,i}, c_i)\), where \(c_i\) is the unique zero (if it exists) of \(\frac{\partial}{\partial \lambda_{n-1,i}} \lambda_{n-2,i}\). \(c_i\) is a continuous function of the spectra of \(A\) and \(A_{n-1}\), excluding the eigenvalue \(\lambda_{n-1,i}\). Thus, \(c_i\) is a constant whenever the rest of the spectra is held fixed. We use the convention that \(c_i = \lambda_{n,i+1}\) in case the zero is not contained within the interlacing boundaries. We know that, apart from the interlacing bounds, the only points where the Jacobian of \(F_k\) is
singular are the zeroes of the partial derivatives of the function. The mention of \( c_i \) in our inductive assumption is motivated by the suspicion that we can restrict our function \( F_k \) to a subrectangle of its domain such that the function is invertible on this subrectangle.

For the \( k = 1 \) case, note that \( F_1(\lambda_{n-1,1}) \) has at most one critical point, which we call \( c_1 \), using the convention that \( c_1 = \lambda_{n,2} \) if there is no critical point. Since the function approaches \( \lambda_{n,1} \) and \( \lambda_{n,2} \) at its boundaries, the function is monotone increasing on \( (\lambda_{n,1}, c_1) \) and monotone decreasing on \( (c_1, \lambda_{n,2}) \). Thus, for any \( \lambda_{n-2,1} \in (\lambda_{n,1}, \lambda_{n,2}) \) in the image of \( F_1 \), there exists a unique preimage \( x_1(1) \in (\lambda_{n,1}, c_1) \). Thus, our claim holds for the case \( k = 1 \).

Now we assume that our proposition holds for \( k = m-1 \). We seek values \( x_1^{(m)}, \ldots, x_m^{(m)} \) such that

\[
F_m(x_1^{(m)}, \ldots, x_m^{(m)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m}).
\]

By our inductive assumption, when we have fixed \( x_m^{(m-1)} \in (\lambda_{n,m}, \lambda_{n,m+1}) \), we are able to find values \( x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)} \) such that

\[
F_m-1(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1}).
\]

Thus,

\[
F_m(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}, x_m^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1}, y),
\]

for some \( y \in (\lambda_{n,m}, \lambda_{n,m+1}) \).

We have assumed that the first \( m-1 \) components of this preimage lie on a subrectangle on which the function is devoid of critical points. Thus, the function is locally invertible in the first \( m-1 \) components. This allows us to move \( \lambda_{n-1,m} \) continuously along the interval of \( (\lambda_{n,m}, \lambda_{n,m+1}) \) while the smallest \( m-1 \) eigenvalues of \( A_{n-2} \) are kept fixed as \( \lambda_{n-1,1}, \ldots, \lambda_{n-1,m-1} \). We know from the argument used in the proof of Lemma 3 that

\[
\lim_{\lambda_{n-1,m} \to \lambda_{n,m}} (\lambda_{n-2,m}) = \lambda_{n,m}
\]

and that

\[
\lim_{\lambda_{n-1,m} \to \lambda_{n,m+1}} (\lambda_{n-2,m}) = \lambda_{n,m+1}.
\]

Thus, by the intermediate value theorem, we may conclude that we can realize any desired \( \lambda_{n-2,m} \in (\lambda_{n,m}, \lambda_{n,m+1}) \).

We still must show that the \( m \)th component of the preimage lies in the interval \( (\lambda_{n,m}, c_m) \). We can show this by contradiction. Assume that the \( m \)th component of the preimage lies in the interval \( [c_m, \lambda_{n,m+1}) \). Then we may fix the other components of the preimage and think of \( \lambda_{n-2,m} \) as a function of \( \lambda_{n-1,m} \). On this interval, the function is
monotone decreasing, but has a lower bound of $\lambda_{n,m+1}$. Thus, it takes on only values higher than $\lambda_{n,m+1}$ on this interval, which excludes the value $\lambda_{n-2,m}$. This is a contradiction. Hence, the $m$th component of the preimage lies in the interval $(\lambda_{n,m}, c_m)$. □

We hope that further analysis of the behavior of eigenvalues of leading principal submatrices as a function of the eigenvalues of $A$ and $A_{n-1}$ will provide us with tools for proving more classes of cases of the general conjecture.

5. 4-by-4 cases

In Section 3 we showed that the conjecture is true for the 2-by-2 and 3-by-3 cases. Each case of the main conjecture can be identified with a sequence $z_1, \ldots, z_n$ majorized by $2k - 1$. There are 14 such sequences in the case $n = 4$. We are able to show that the conjecture holds for 8 of these cases, and we list them here.

(0,0,3,4): This is covered by Theorem 1.
(0,1,2,4) and (0,2,1,4): These are covered by Theorem 3.
(1,0,2,4): This is covered by Lemma 3.
(1,1,1,4): Given $L = l_1 < \cdots < l_7$, we choose $l_1, l_3, l_5, l_7$ to be the eigenvalues of $A$ and $l_6$ to be the largest eigenvalue of $A_3$. By Theorem 3, we choose the middle eigenvalue of $A_3$ between $l_5$ and $l_7$ so that it fixes $l_4$ as the largest eigenvalue of $A_2$ while we vary the smallest eigenvalue of $A_3$ between $l_4$ and $l_3$. By the argument used in Lemma 3, we see that as $\lambda_{3,1}$ approaches $l_1$, $\lambda_{1,1}$ approaches $l_1$ and that as $\lambda_{3,1}$ approaches $l_3$, $\lambda_{1,1}$ approaches $l_3$. So by the Intermediate Value Theorem, there is a value for $\lambda_{3,1}$ between $l_1$ and $l_3$ so that $\lambda_{1,1} = l_2$. Thus, there exists a matrix for which $l_1, l_3, l_5, l_7 \in \sigma(A)$, $l_6 \in \sigma(A_3)$, $l_4 \in \sigma(A_2)$, and $l_2 \in \sigma(A_1)$.

(1,2,2,2), (0,2,3,2), and (1,1,3,2): Note that the sequences (1,2,2), (0,2,3), and (1,1,3) are each resolved 3-by-3 cases of the conjecture. Thus, if $L$ consists of the numbers $l_1 < l_2 < \cdots < l_7$, in each case we may choose as $A_3$ the matrix realizing $l_2, \ldots, l_6$ as the desired eigenvalues of upper left submatrices of $A_3$ and then use the method given in [7] for completing the matrix so that it realizes $l_1$ and $l_7$ as the largest and smallest eigenvalues of $A_4$.

6. Other problems

Instead of focusing on $2n - 1$ eigenvalues of leading principal submatrices only, we may widen our scope to include inverse eigenvalue problems dealing with $2n - 1$ eigenvalues of any of the principal submatrices. We do not currently have a conjecture about which of the many possible problems will always have a solution since, while leading principal matrices are nested, matrices in the more general problem may be nested, have overlapping entries, or be disjoint. We present two quick examples of these types of problems.
Problem 1. Let $A_{i+1}^{i+1}$ denote the 2-by-2 submatrix of $A$ obtained by crossing out all rows and columns except the $i$th and $(i+1)$st. Submatrices of consecutive indices, $A_i^{i+1}$ and $A_{i+1}^{i+2}$ overlap in one entry. Given a list of $2n-1$ distinct real numbers, there exists an $n$-by-$n$ real symmetric tridiagonal matrix $A$ which realizes the $2n-1$ numbers as: the single eigenvalue of $A_1$ and the $2$ eigenvalues of each of the $A_{i+1}^{i+1}$, $i = 1, \ldots, n - 1$.

Proof. Suppose $a_{11}$ has been chosen. Then we may choose the eigenvalues, $\lambda_1^{1,2}$ and $\lambda_2^{1,2}$ of $A_1^{2}$ to be any values such that

$$\lambda_1^{1,2} < a_{11} < \lambda_2^{1,2}.$$

This choice determines the matrix $A_1^{2}$, and we thus obtain the entry $a_{22}$, which is also the upper left entry of the matrix $A_2^{3}$. In this recursive manner, we may construct the matrix as long as we have chosen the eigenvalues so that the inequality

$$\lambda_1^{i,i+1} < a_{i,i} < \lambda_2^{i,i+1}$$

is always satisfied. Depending on the given list, there may be many ways to make this choice, but we note that the choice satisfying

$$\lambda_1^{i+1,i+2} < \lambda_1^{i,i+1} < \lambda_2^{i,i+1} < \lambda_2^{i+1,i+2}$$

for all $i = 1, \ldots, n - 2$ will always satisfy the required inequalities. \[\square\]

Problem 2. Let $B^{n-k}$ denote the bottom right $(n-k)$-by-$(n-k)$ submatrix of $A$. Given a list $l_1 < \cdots < l_{2n-1}$ of $2n-1$ distinct real numbers, there exists an $n$-by-$n$ real symmetric tridiagonal matrix $A$ which realizes the $2n-1$ numbers as: All $k$ eigenvalues of $A_k$, all $(k-1)$ eigenvalues of $A_{k-1}$, all $(n-k)$ eigenvalues of $B^{n-k}$, all $(n-k-1)$ eigenvalues of $B^{n-k-1}$, and $1$ eigenvalue of the entire matrix $A$.

Proof. We note that $A_k$ and $B^{n-k}$ are disjoint. By the classical result, we will be able to choose the eigenvalues of $A_k$ and $A_{k-1}$ independently of those of $B^{n-k}$ and $B^{n-k-1}$. There is a single independent entry of the matrix, $a_{k,k+1}$ not contained in either $A_k$ or $B^{n-k}$. All we must determine is when we may choose this entry so that the matrix realizes a desired eigenvalue $\lambda_0$. If we let $P_k$ and $Q_{n-k}$ denote the characteristic polynomials of $A_k$ and $B^{n-k}$, respectively, we may compute the characteristic polynomial of $A$ by expanding along the $k$th row:

$$P_n(\lambda) = P_k(\lambda)Q_{n-k}(\lambda) - a_{k,k+1}^2 P_{k-1}(\lambda)Q_{n-k-1}(\lambda).$$

$$a_{k,k+1}^2 = \frac{P_kQ_{n-k}}{P_{k-1}Q_{n-k-1}}(\lambda_0).$$

Thus, $a_{k,k+1}$ is real if and only if $\lambda_0$ is less than an odd number of eigenvalues in the union of the spectra of $A_k$, $A_{k-1}$, $B^{n-k}$, and $B^{n-k-1}$. Thus, there exists a matrix realizing
the desired eigenvalues as long as we choose the single desired eigenvalue of $A$ to be some even-indexed $l_{2j}$ from the list. All other eigenvalues may be chosen freely, as long as they are chosen so that the spectra of $A_{k-1}$ and $B^{n-k-1}$ interlace the spectra of $A_k$ and $B^{n-k}$. □

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References