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Inverse spectral problems for collections of leading principal submatrices of tridiagonal matrices

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ABSTRACT

Which assignments from $2n - 1$ arbitrary, distinct real numbers as eigenvalues of designated leading principal submatrices permit a real symmetric tridiagonal matrix? We raise this question, motivated both by known results and recent work on multiplicities and interlacing equalities in symmetric matrices whose graph is a given tree. Known results are reviewed, a general conjecture is given, and several new partial results are proved.

1. Introduction

An n-by-n matrix $A = (a_{ij})$ is called tridiagonal if $|i - j| > 1$ implies $a_{ij} = 0$. Such a matrix is irreducible if and only if $|i - j| = 1$ implies $a_{ij} \neq 0$. We are interested here in real symmetric (equivalently, complex hermitian) tridiagonal matrices, which, of

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course, have distinct real eigenvalues if they are irreducible [5]. Principal submatrices of tridiagonal matrices are tridiagonal, and leading principal submatrices (based upon consecutive indices beginning with 1) of irreducible tridiagonal matrices are irreducible, as well. We let A_k denote the k-by-k leading principal submatrix of A, and denote its eigenvalues by

$$\lambda_{k,1} < \lambda_{k,2} < \cdots < \lambda_{k,k}.$$

The classical interlacing inequalities [5] apply, so that

$$\lambda_{k,i} \leq \lambda_{k-h,i} \leq \lambda_{k,i+h}$$

for $h = 1, \ldots, n-k$, with strict inequalities for $h = 1$. The strict inequalities are special to irreducible tridiagonal matrices. It is a well known result that, given the $2n-1$ real numbers

$$\lambda_{n,1} < \lambda_{n-1,1} < \lambda_{n,2} < \lambda_{n-1,2} < \cdots < \lambda_{n-1,n-1} < \lambda_{n,n},$$

there is an irreducible, real symmetric tridiagonal matrix A such that

$$\sigma(A_n) = \{\lambda_{n,1}, \ldots, \lambda_{n,n}\}$$

and

$$\sigma(A_{n-1}) = \{\lambda_{n-1,1}, \ldots, \lambda_{n-1,n-1}\}.$$

Moreover, A is unique up to diagonal unitary similarity [3]. This is intuitively plausible as A is described by the $2n-1$ independent real parameters $a_{11}, \ldots, a_{nn}, a_{12}, \ldots, a_{n-1,n}$ (because of symmetry) and there are $2n-1$ real targets, meeting all known necessary conditions. We call this the classical case, though it seems first to have been proven in [3] (uniqueness) and [4] (existence). It has been of interest, both numerically and otherwise in a number of applications.

2. The reasonable conjecture

Our interest here lies in a question whose answer would greatly generalize the classical result about eigenvalues of A_n and A_{n-1}. We are motivated, in part, by the need for such assignment results in the multiplicity list problem for trees. See [8], [6], and [1] for specifics, as well as for general background. This also has natural interest as a theoretical problem, and one that has physical applications [2]. Given $2n-1$ real numbers, when are we able to partition the numbers into sets $S_1, \ldots, S_n$ so that $S_i \subset \sigma(A_i)$ for $i = 1, .., n$, for some real symmetric n-by-n irreducible tridiagonal matrix A? Of course, we allow some S_i's to be empty, as in the classical case.

Of course, the partition of the $2n-1$ numbers must be consistent with interlacing generally, and strict interlacing between A_i and A_{i-1}, in particular. This is assumed in the classical case. We also assume (as follows there) that our $2n-1$ numbers are distinct. The sizes of the sets S_i, $i=1,\dots,n$ must not cause an upper left corner of A to be oversubscribed. We should not expect to assign more than $2i-1$ eigenvalues to A_i and its principal submatrices, as A_i has only $2i-1$ independent, nonzero entries. This is automatic in the classical case, but not in the generalization. For example, if $n=4$ and only one eigenvalue is assigned to A_4, then 6 are assigned to A_1, A_2 and A_3, which is one too many.

However, given reasonable sizes of n sets $S_1,\dots,S_n$, we expect that we can partition any $2n-1$ real numbers into the sets of these sizes so that a matrix exists with $S_i \subset \sigma(A_i)$ for each $i=1,\dots,n$. The formal statement of this conjecture is as follows.

Conjecture 1. *Given a list L of $2n-1$ distinct real numbers, $l_1 < l_2 < \cdots < l_{2n-1}$, and n non-negative integers, $z_1,\dots,z_n$, satisfying*

$$z_i \le i,$$

$$\sum_{i=1}^{k} z_i \le 2k-1,$$

and

$$\sum_{i=1}^{n} z_i = 2n-1,$$

there exists an n-by-n real symmetric tridiagonal matrix A with

$$\sigma(A_i) \supset S_i,$$

in which $S_1,\dots,S_n$ is some partition of L satisfying $|S_i| = z_i$ for each i.

3. Review of known cases

There are several inverse eigenvalue problems of the type we consider in our main conjecture for which the existence of a solution is guaranteed by previous theorems. One such case is the inverse eigenvalue problem that asks that the $2n-1$ values in a given list be assigned to the $2n-1$ eigenvalues of A and A_{n-1}. The following classical result ensures the existence of such a matrix A.

Theorem 1. *(See [3,4].) Let $\lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,n}$ and $\lambda_{n-1,1} < \lambda_{n-1,2} < \cdots < \lambda_{n-1,n-1}$ be $2n-1$ real numbers such that*

$$\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1},$$

for all $i = 1, \dots, n-1$. Then there exists a real n-by-n symmetric tridiagonal matrix A such that the eigenvalues of A are $\lambda_{n,1}, \dots, \lambda_{n,n}$ and the eigenvalues of A_{n-1} are $\lambda_{n-1,1}, \dots, \lambda_{n-1,n-1}$. Furthermore, this matrix is unique up to the signs of the super-diagonal entries.

Another case of the main conjecture is the inverse eigenvalue problem that asks that the $2n-1$ values in a given list be assigned to the set composed of the maximum and minimum eigenvalues of each of the n leading principal submatrices A_i, while A_1 has only one eigenvalue. The existence of a solution is ensured by the following result.

Theorem 2. *(See [7].) Let*

$$\lambda_{n,1} < \lambda_{n-1,1} < \lambda_{n-2,1} < \cdots < \lambda_{2,1} < \lambda_{1,1} < \lambda_{2,2} < \cdots < \lambda_{n-2,n-2} < \lambda_{n-1,n-1} < \lambda_{n,n}$$

be $2n-1$ real numbers. Then there exists a real n-by-n symmetric tridiagonal matrix A such that $\lambda_{k,1}$ and $\lambda_{k,k}$ are the smallest and largest eigenvalues of A_k, for $k = 1, \dots, n$. Furthermore, this matrix is unique up to the signs of the super-diagonal entries.

The proof of this theorem cleverly uses the recursive characteristic polynomial relationship for symmetric tridiagonal matrices to show that such a matrix A may be constructed by solving several 2-by-2 systems of linear equations.

We may use these two results and a small amount of further explanation to discover that the main conjecture is true in the 2-by-2 and 3-by-3 cases. For the $n = 2$ case, the matrix A only has 3 eigenvalues of leading principal submatrices. Thus, there is only one way we can choose a set of 3 eigenvalues to target, and the result of Theorem 1 guarantees the existence of a matrix A with 3 numbers from a list assigned to these 3 eigenvalues. We consider the $n = 3$ case of the problem with help from the following lemma.

Lemma 1. *Let A be a real symmetric tridiagonal matrix such that $\{\lambda_{n,i}\}_{i=1}^{n} = \sigma(A)$, $\{\lambda_{n-1,i}\}_{i=1}^{n-1} = \sigma(A_{n-1})$, and $\{\lambda_{n-2,i}\}_{i=1}^{n-2} = \sigma(A_{n-2})$. Then*

$$\begin{aligned} &E_{k-2}(\{\lambda_{n-2,i}\}) \\ &= \frac{E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}, \end{aligned}$$

for $k = 2, \dots, n$, where $E_j\{\lambda_{k,i}\}$ denotes the jth symmetric sum of the eigenvalues of A_k. Here we use the convention,

$$E_0(\{\lambda_{n,i}\}) = 1, \forall n$$

and

$$E_k(\{\lambda_{n,i}\}) = 0, \forall n < k.$$

Proof. Let P_k denote the characteristic polynomial of A_k and let the x_i denote a_{ii} for $i = 1, .., n$ and y_i denote $a_{i,i+1}$ for $i = 1, \dots, n-1$. The well known recursive relation for characteristic polynomials of symmetric tridiagonal matrices gives:

$$P_{n-2}(\lambda) = \frac{P_n(\lambda) - P_{n-1}(\lambda)(\lambda - x_n)}{-y_{n-1}^2}.$$

If we rewrite x_n and $-y_{n-1}^2$ in terms of the symmetric sums of the eigenvalues of A and A_{n-1}:

$$\begin{aligned} x_n &= E_1(\{\lambda_{n,i}\}) - E_1(\{\lambda_{n-1,i}\}), \\ -y_{n-1}^2 &= E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})) \end{aligned}$$

and then equate coefficients of the polynomials on the left and right sides of the equation, we find that

$$\begin{aligned} &E_{k-2}(\{\lambda_{n-2,i}\}) \\ &= \frac{E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}, \end{aligned}$$

for $k = 2, \dots, n$. □

Note that the formula given in the above lemma is a formula (up to a sign change) of the coefficients of the characteristic polynomial of A_{n-2} in terms of the spectra of A and A_{n-1}.

Proof of 3-by-3 case of Main Conjecture. There are only 3 suitable ways to specify a set of 5 leading principal eigenvalues of A: 1) the 3 eigenvalues of A and the 2 eigenvalues of A_2; 2) 2 eigenvalues of A, 2 eigenvalues of A_2, and the eigenvalue of A_1; 3) the 3 eigenvalues of A, one eigenvalue of A_2 and the eigenvalue of A_1. The first two cases are covered by Theorems 1 and 2, respectively. Thus, we have only to prove the third case. This case was originally proven by Nuckols in [8] but we present the argument here.

The result of Theorem 1 allows us to be certain of the existence of a tridiagonal matrix A_3 if we choose the spectra of A_3 and A_2 so that the spectrum of A_2 interlaces that of A_3. The strategy here is to show that once $\lambda_{3,1} < \lambda_{3,2} < \lambda_{2,2} < \lambda_{3,3}$ are fixed, we can find a $\lambda_{2,1} \in (\lambda_{3,1}, \lambda_{3,2})$ so that the matrix that realizes the leading principal eigenvalues $\{\lambda_{3,1}, \lambda_{3,2}\lambda_{3,3}, \lambda_{2,1}, \lambda_{2,2}\}$ also realizes $\lambda_{1,1} \in (\lambda_{3,1}, \lambda_{3,2})$ as the eigenvalue of A_1. The characteristic polynomial of A_1 is a linear polynomial. So we find that

$$\begin{aligned} \lambda_{1,1} &= \frac{E_3(\{\lambda_{3,i}\}) + E_2(\{\lambda_{2,i}\})(E_1(\{\lambda_{2,i}\}) - E_1(\{\lambda_{3,i}\}))}{E_2(\{\lambda_{3,i}\}) - E_2(\{\lambda_{2,i}\}) + E_1(\{\lambda_{2,i}\})(E_1(\{\lambda_{2,i}\}) - E_1(\{\lambda_{3,i}\}))} \\ &= \frac{\lambda_{3,1}\lambda_{3,2}\lambda_{3,3} + (\lambda_{2,1}\lambda_{2,2})(\lambda_{2,1} + \lambda_{2,2} - \lambda_{3,1} - \lambda_{3,2} - \lambda_{3,3})}{\lambda_{3,1}\lambda_{3,2} + \lambda_{3,1}\lambda_{3,3} + \lambda_{3,2}\lambda_{3,3} - \lambda_{2,1}\lambda_{2,2} + (\lambda_{2,1} + \lambda_{2,2})(\lambda_{2,1} + \lambda_{2,2} - \lambda_{3,1} - \lambda_{3,2} - \lambda_{3,3})}. \end{aligned}$$

We can think of the above expression for $\lambda_{1,1}$ as a function of the variable $\lambda_{2,1}$ whose domain is $(\lambda_{3,1}, \lambda_{3,2})$. Note that both the numerator and denominator of the function are continuous functions of $\lambda_{2,1}$ and that Theorem 1 guarantees the existence of a matrix A_3 with spectrum $\{\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,3}\}$ and with $\{\lambda_{2,1}, \lambda_{2,2}\} = \sigma(A_2)$. Thus, $a_{23} \neq 0$ and the denominator of the expression is never 0.

If we take the limit of the function as $\lambda_{2,1}$ approaches $\lambda_{3,1}$ we find that

$$\begin{aligned}\lim_{\lambda_{2,1}\to\lambda_{3,1}} \lambda_{1,1} &= \lambda_{3,1}\frac{\lambda_{2,2}(\lambda_{2,2}-\lambda_{3,2}-\lambda_{3,3})+\lambda_{3,2}\lambda_{3,3}}{\lambda_{2,2}(\lambda_{2,2}-\lambda_{3,2}-\lambda_{3,3})+\lambda_{3,2}\lambda_{3,3}} \\ &= \lambda_{3,1}.\end{aligned}$$

If we take the limit of the function as $\lambda_{2,1}$ approaches $\lambda_{3,2}$ we find that

$$\begin{aligned}\lim_{\lambda_{2,1}\to\lambda_{3,2}} \lambda_{1,1} &= \lambda_{3,2}\frac{\lambda_{2,2}(\lambda_{2,2}-\lambda_{3,1}-\lambda_{3,3})+\lambda_{3,1}\lambda_{3,3}}{\lambda_{2,2}(\lambda_{2,2}-\lambda_{3,1}-\lambda_{3,3})+\lambda_{3,1}\lambda_{3,3}} \\ &= \lambda_{3,2}.\end{aligned}$$

We see that the function is onto the entire interval $(\lambda_{3,1}, \lambda_{3,2})$, by the Intermediate Value Theorem. This proves the lemma for the ordering $\lambda_{3,1} < \lambda_{1,1} < \lambda_{3,2} < \lambda_{2,2} < \lambda_{3,3}$. The proof of the lemma for the ordering $\lambda_{3,1} < \lambda_{2,1} < \lambda_{3,2} < \lambda_{1,1} < \lambda_{3,3}$ uses an analogous argument. □

4. New results

An inverse eigenvalue problem similar to the ones resolved by known results is the problem that asks that the $2n-1$ values from a list be assigned to the set composed of the n eigenvalues of A, any $n-2$ eigenvalues of A_{n-1}, and any 1 eigenvalue of A_{n-2}. This problem is more accessible when we choose the 1 eigenvalue of A_{n-2} to be either the largest or smallest eigenvalue of A_{n-2}. The problem is then resolved by the following result.

Lemma 2. *Let $\{\lambda_{n,i}\}_{i=1}^{n}$, $\{\lambda_{n-1,i}\}_{i=2}^{n-1}$, and $\lambda_{n-2,1}$ be $2n-1$ numbers such that*

$$\lambda_{n,1} < \lambda_{n-2,1} < \lambda_{n,2}$$

and

$$\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1}, \forall i = 2, \ldots, n-1.$$

Then there exists an n-by-n real symmetric tridiagonal matrix A such that $\sigma(A) = \{\lambda_{i,n}\}_{i=1,\ldots n}$, $\{\lambda_{n-1,i}\}_{i=2,\ldots,n-1} \subset \sigma(A_{n-1})$, and $\lambda_{n-2,1} \in \sigma(A_{n-2})$.

Proof. The classical result allows us to choose the sets $\{\lambda_{n,i}\}$ and $\{\lambda_{n-1,i}\}$ so that the members of the latter strictly interlace the members of the former and be certain that a matrix A exists with $\{\lambda_{n,i}\} = \sigma(A)$ and $\{\lambda_{n-1,i}\} = \sigma(A_{n-1})$. Given the numbers assigned to us in the statement of Theorem 1, all we must show is that there exists a $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$ so that the matrix A realizes $\lambda_{n-1,2} \in (\lambda_{n,1}, \lambda_{n,2})$ as an eigenvalue of A_{n-2}.

Without loss of generality, we may assume that $\lambda_{n,1} = 0$, for if we prove the existence of a matrix for this case, then we have proven it for all $\lambda_{n,1}$ by a shift of the matrix by a scalar multiple of the identity matrix. Lemma 1 gives:

$$\begin{aligned}&E_{k-2}(\{\lambda_{n-2,i}\})\\&= \frac{E_k(\{\lambda_{n,i}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}\end{aligned}$$

for $k = 2, \ldots, n$. When $k = n$, we have that: $E_n(\{\lambda_{n,i}\}) = 0$, since $\lambda_{n,1} = 0$; and $E_n(\{\lambda_{n-1,i}\}) = 0$, by convention. Thus,

$$\begin{aligned}&E_{n-2}(\{\lambda_{n-2,i}\})\\&= \frac{E_{n-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}\\&\prod_{i=1}^{n-2}(\lambda_{n-2,i}) = \frac{\left(\prod_{i=1}^{n-1}(\lambda_{n-1,i})\right)(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}.\end{aligned}$$

The denominator of the right hand side of the above equation never vanishes because it is equal to $-a_{n-1,n}^2$, the negative square of the bottom right super-diagonal entry of A. We note that $\{E_{k-2}(\{\lambda_{n-2,i}\})\}$ are the coefficients (plus or minus) of $P_{n-2}(\lambda)$, the characteristic polynomial of A_{n-2}. These vary continuously as a function of $\lambda_{n-1,1}$ when the rest of the spectra of A_n and A_{n-1} is fixed. Since the coefficients of the polynomial vary continuously, the roots of the polynomial vary continuously. In particular, $\lambda_{n-2,1}$ varies continuously as a function of $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$. We see that

$$\begin{aligned}&\lim_{\lambda_{n-1,1}\to\lambda_{n,1}}(\prod_{i=1}^{n-2}(\lambda_{n-2,i}))\\&= \lim_{\lambda_{n-1,1}\to 0}(\prod_{i=1}^{n-2}(\lambda_{n-2,i}))\\&= \frac{0(E_1(\{\lambda_{n-1,i}\}) - E_1(\lambda_{n,i}))}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}\\&= 0\end{aligned}$$

Since $\prod_{i=1}^{n-2}(\lambda_{n-2,i})$ approaches 0, at least one member of $\sigma(A_{n-2})$ approaches 0.

However, for $i = 2, \ldots, n-2$, the interlacing inequalities imply

$$0 < \lambda_{n,2} < \lambda_{n-2,i}.$$

Thus, $\lambda_{n-2,1}$ is the only eigenvalue of A_{n-2} which may approach 0 and

$$\lim_{\lambda_{n-1,1} \to \lambda_{n,1}} (\lambda_{n-2,1}) = 0.$$

Thus, 0 is the greatest lower bound of $\lambda_{n-2,1}$.

Now, the interlacing inequalities guarantee that $\lambda_{n-1,1} < \lambda_{n-2,1}$, so as $\lambda_{n-1,1}$ approaches $\lambda_{n,2}$, we still have that $\lambda_{n-1,1} < \lambda_{n-2,1}$. This gives us the result that the least upper bound for $\lambda_{n-2,1}$ is at least $\lambda_{n,2}$. So we have shown that $(\lambda_{n,1}, \lambda_{n,2})$ is contained within the interval between the greatest lower bound and least upper bound of $\lambda_{n-2,1}$. Thus, by the intermediate value theorem, for every $\lambda_{n-2,1} \in (\lambda_{n,1}, \lambda_{n,2})$, there exists some $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$ so that the matrix A realizes $\lambda_{n-2,1}$ as an eigenvalue of A_{n-2}. $\square$

Instead of targeting the smallest eigenvalue of A_{n-2} we could have targeted the largest by fixing $\sigma(A_n)$, $\{\lambda_{n-1,i}\}_{i=1,\ldots n-2} \subset \sigma(A_{n-1})$ and choosing $\lambda_{n-2,n-2} \in (\lambda_{n,n-1}, \lambda_{n,n})$. For that proof, we would assume $\lambda_{n,n} = 0$ and use steps analogous to those used in the above proof.

We can further generalize the result of Lemma 2 to show that we can use an eigenvalue of A_{n-1} to target an eigenvalue for A_k for any $k \leq n-2$.

Lemma 3. *For $k \leq n-2$, let $\{\lambda_{n,i}\}_{i=1}^{n}$, $\{\lambda_{n-1,i}\}_{i=2}^{n-1}$, and $\lambda_{k,1}$ be $2n-1$ numbers such that*

$$\lambda_{n,1} < \lambda_{k,1} < \lambda_{n,2}$$

and

$$\lambda_{n,i} < \lambda_{n-1,i} < \lambda_{n,i+1} \, \forall i = 2, \ldots, n-1.$$

Then there exists an nxn real symmetric tridiagonal matrix A such that $\sigma(A) = \{\lambda_{n,i}\}_{i=1,\ldots n}$, $\{\lambda_{n-1,i}\}_{i=2,\ldots,n-1} \subset \sigma(A_{n-1})$, and $\{\lambda_{k,1}\} \subset \sigma(A_k)$.

Proof. Without loss of generality, we will assume that $\lambda_{n,1} = 0$ and we will first show by induction that

$$\lim_{\lambda_{n-1,1} \to \lambda_{n,1}} (\lambda_{k,1}) = \lambda_{n,1} \, \forall k < n.$$

The case $k = n-1$ holds trivially and $k = n-2$ is the case covered by the previous lemma. We now assume that it holds for $k = m$ and show that this implies its validity for $k = m-1$. The following formula holds:

$$E_{m-1}(\{\lambda_{m-1,i}\})$$
$$= \frac{E_m(\{\lambda_{m,i}\})(E_1(\{\lambda_{m,i}\}) - E_1(\{\lambda_{m+1,i}\})}{E_2(\{\lambda_{m+1,i}\}) - E_2(\{\lambda_{m,i}\}) + E_1(\{\lambda_{m,i}\})(E_1(\{\lambda_{m,i}\}) - E_1(\{\lambda_{m+1,i}\}))}.$$

The inductive assumption gives

$$\lim_{\lambda_{n-1,1}\to\lambda_{n,1}} (\lambda_{m,1}) = 0.$$

Thus, $\lim_{\lambda_{n-1,1}\to\lambda_{n,1}}(E_m(\{\lambda_{m,i}\})) = 0$ and thus $\lim_{\lambda_{n-1,1}\to\lambda_{n,1}}(E_{m-1}(\{\lambda_{m-1,i}\})) = 0$. Thus we see that $\lim_{\lambda_{n-1,1}\to\lambda_{n,1}}(\lambda_{m-1,1}) = 0$.

Now each $\lambda_{k,1}$ is a continuous function of $\lambda_{n-1,1}$ when the rest of the spectra of A and A_{n-1} are fixed. We've shown that $\lambda_{n,1}$ is the greatest lower bound of $\lambda_{k,1}$. Also, it is obvious that the least upper bound for $\lambda_{k,1}$ is at least $\lambda_{2,n}$. Thus, by the intermediate value theorem, for each $\lambda_{k,1} \in (\lambda_{n,1}, \lambda_{n,2})$, there exists some $\lambda_{n-1,1} \in (\lambda_{n,1}, \lambda_{n,2})$ so that the matrix A guaranteed to exist by the classical result also realizes $\lambda_{k,1}$ as an eigenvalue of A_k. □

We would like to show that we can use the eigenvalues of A_{n-1} to simultaneously target multiple eigenvalues of A_{n-2}. In order to do this, we first have to analyze the behavior of the eigenvalues of A_{n-2} as a function of the eigenvalues of A_{n-1}. We would like to explore the partial derivatives of the expression for the coefficients of P_{n-2} as a function of $\lambda_{n-1,j}$, as $\lambda_{n-1,j}$ ranges from $\lambda_{n,j}$ to $\lambda_{n,j+1}$. The expression for the coefficients is:

$$E_{k-2}(\{\lambda_{n-2,i}\})$$
$$= \frac{E_k(\{\lambda_{ni}\}) - E_k(\{\lambda_{n-1,i}\}) + E_{k-1}(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\})}{E_2(\{\lambda_{n,i}\}) - E_2(\{\lambda_{n-1,i}\}) + E_1(\{\lambda_{n-1,i}\})(E_1(\{\lambda_{n-1,i}\}) - E_1(\{\lambda_{n,i}\}))}$$

We will use some notation to help save space and emphasize the role of $\lambda_{n-1,j}$. Let

$$\begin{aligned}
x &= \lambda_{n-1,j} \\
R &= \{\lambda_{n-2,i}\}_{i=1,..,n-2} \\
S &= \{\lambda_{n-1,i}\}_{i=1,...,n-1} \\
T &= \{\lambda_{n,i}\}_{i=1,...,n} \\
S - x &= \{\lambda_{n-1,i}\} \setminus \{x\} \\
N_k(x) &= E_k(T) - E_k(S) + E_{k-1}(S)(E_1(S) - E_1(T)) \\
D(x) &= E_2(T) - E_2(S) + E_1(S)(E_1(S) - E_1(T))
\end{aligned}$$

and note that $E_{k-2}(R)(x) = \dfrac{N_k(x)}{D(x)}$. The following facts may be quickly checked.

$E_k(S)(x) = xE_{k-1}(S - x) + E_k(S - x)$.

$\frac{d}{dx}E_k(S) = E_{k-1}(S - x)$.

$$\tfrac{d}{dx}N(x) = -E_{k-1}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T)) + E_{k-1}(S).$$
$$\tfrac{d}{dx}D(x) = -E_1(S-x) + 2E_1(S) - E_1(T) = 2x + E_1(S-x) - E_1(T).$$

Lemma 4. *For each $k = 3, \dots, n$, $E_{k-2}(R)(x)$ has a critical point if and only if*

$$\frac{1}{2}(-E_1(S-x) + E_1(T)) \in (\lambda_{n,j}, \lambda_{n,j+1}).$$

Proof.

$$\frac{d}{dx}E_{k-2}(R)(x) = \frac{D(x)N'(x) - N(x)D'(x)}{(D(x))^2}.$$

By using the above properties, we find that

$$\begin{aligned} N'(x) &= -E_{k-1}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T) + E_{k-1}(S) \\ &= xE_{k-2}(S-x) + E_{k-2}(S-x)(E_1(S) - E_1(T)) \\ &= E_{k-2}(S-x)(x + E_1(S) - E_1(T)) \\ &= E_{k-2}(S-x)(2x + E_1(S-x) - E_1(T)) \\ &= E_{k-2}(S-x)D'(x). \end{aligned}$$

Thus,

$$D(x)N'(x) - N(x)D'(x) = D'(x)(E_{k-2}(S-x)D(x) - N(x)).$$

$D'(x) = 0$ if and only if $x = \frac{1}{2}(-E_1(S-x) + E_1(T))$. So we have shown that $E_{k-2}(R)(x)$ has a critical point if $\frac{1}{2}(-E_1(S-x) + E_1(T)) \in (\lambda_{n,j}, \lambda_{n,j+1})$. To show that this is the only possible critical point, we will show that the expression $E_{k-2}(S-x)D(x) - N(x)$ is constant with respect to x.

We will first consider the expression $E_{k-2}(S-x)D(x)$ and collect from it only the terms non-constant with respect to x.

$$\begin{aligned} E_{k-2}(S-x)D(x) &= E_{k-2}(S-x)(E_2(T) - E_2(S) + E_1(S)(E_1(S) - E_1(T))) \\ &= E_{k-2}(S-x)(c_1 - E_2(S) + (E_1(S))^2 - E_1(S)E_1(T)) \\ &= E_{k-2}(S-x)(c_1 - (xE_1(S-x) + E_2(S-x)) \\ &\quad + (x + E_1(S-x))^2 - (x + E_1(S-x))E_1(T)) \end{aligned}$$

If we eliminate all constant terms from this last expression, we obtain:

$$E_{k-2}(S-x)(x^2 + x(E_1(S-x) - E_1(T))).$$

Next, consider the expression $N(x)$ and collect from it only the terms non-constant with respect to x.

$$\begin{aligned}
N(x) &= E_k(T) - E_k(S) + E_{k-1}(S)(E_1(S) - E_1(T)) \\
&= c_2 - E_k(S) + E_{k-1}(S)E_1(S) - E_{k-1}(S)E_1(T) \\
&= c_2 - (xE_{k-1}(S-x) + E_k(S-x)) \\
&\quad + (xE_{k-2}(S-x) + E_{k-1}(S-x))(x + E_1(S-x)) \\
&\quad - (xE_{k-2}(S-x) + E_{k-1}(S-x))E_1(T)
\end{aligned}$$

If we eliminate all constant terms from this last expression, we obtain:

$$\begin{aligned}
&E_{k-2}(S-x)x^2 + xE_{k-2}(S-x)E_1(S-x) - xE_1(T)E_{k-2}(S-x) \\
= \; &E_{k-2}(S-x)(x^2 + x(E_1(S-x) - E_1(T)))
\end{aligned}$$

Thus, $E_{k-2}(S-x)D(x) - N(x)$ is constant with respect to x. □

Lemma 5.

$$\frac{d}{dx}E_{k-2}(S)(x) = \sum_{i=1}^{n-2} \frac{d}{dx}\lambda_{n-2,i}E_{k-3}(R - \lambda_{n-2,i}).$$

Proof. This formula follows from the product rule. □

Lemma 6. *For any $k \in \{3, 4, \dots, n\}$, if x_0 is a critical point of $E_{k-2}(R)(x)$, it is also a critical point of $\lambda_{n-2,i}(x)$ for each $i = 1, .., n-2$.*

Proof. Let x_0 be a critical point of $E_{k-2}(R)(x)$ for some $k \in \{3, \dots, n\}$. Then by Lemma 5, x_0 is a critical point of each E_{k-2}. We obtain the following system of equations:

$$\begin{aligned}
\lambda'_{n-2,1} + \lambda'_{n-2,2} + \ldots + \lambda'_{n-2,n-2} &= 0 \\
\lambda'_{n-2,1}(\lambda_{n-2,2} + \lambda_{n-2,3} + \ldots + \lambda_{n-2,n-2}) + \ldots + \lambda'_{n-2,n-2}(\lambda_{n-2,1} + \lambda_{n-2,2} + \ldots + \lambda_{n-2,n-1}) &= 0 \\
\vdots \quad &\vdots \quad \vdots \\
\lambda'_{n-2,1}(\lambda_{n-2,2}\lambda_{n-2,3} \cdots \lambda_{n-2,n-2}) + \cdots + \lambda'_{n-2,n-2}(\lambda_{n-2,1}\lambda_{n-2,2} \cdots \lambda_{n-2,n-1}) &= 0
\end{aligned}$$

We may regard this as a system of linear equations in the variables $\{\lambda'_{n-2,i}\}$. The matrix of coefficients for this system is:

$$\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
E_1(R \setminus \lambda_{n-2,1}) & E_1(R \setminus \lambda_{n-2,2}) & E_1(R \setminus \lambda_{n-2,3}) & \cdots & E_1(R \setminus \lambda_{n-2,n-2}) \\
E_2(R \setminus \lambda_{n-2,1}) & E_2(R \setminus \lambda_{n-2,2}) & E_2(R \setminus \lambda_{n-2,3}) & \cdots & E_2(R \setminus \lambda_{n-2,n-2}) \\
\vdots & \vdots & \vdots & & \vdots \\
E_{n-3}(R \setminus \lambda_{n-2,1}) & E_{n-3}(R \setminus \lambda_{n-2,2}) & E_{n-3}(R \setminus \lambda_{n-2,3}) & \cdots & E_{n-3}(R \setminus \lambda_{n-2,n-2})
\end{bmatrix}$$

The determinant of this matrix has the same magnitude as the Vandermonde determinant:

$$\prod_{1\le s<t\le n-2}(\lambda_{n-2,s}-\lambda_{n-2,t}).$$

This determinant is nonzero by virtue of the distinct $\{\lambda_{n-2,i}\}$. Thus, our system of equations has only the trivial solution. We have shown that when x_0 is a critical point of some $E_{k-2}(R)$, it is a critical point of each member of R:

$$\lambda'_{1,n-2}(x_0)=\lambda'_{2,n-2}(x_0)=\cdots=\lambda'_{n-2,n-2}(x_0)=0. \qquad \square$$

Lemma 7. *If x_0 is a critical point of $\lambda_{n-2,i}(x)$, x_0 is also a critical point of $E_{k-2}(R)(x)$ for some $k=3,\dots,n$.*

Proof. Suppose x_0 is a critical point of $\lambda_{n-2,i}(x)$ but that x_0 is not a critical point of some $E_{k-2}(R)$. Then by Lemma 4, x_0 is not a critical point of any of the $\{E_{k-2}(R)\}$.

As in the proof of Lemma 6, we obtain a set of linear equations that may be represented by the following equation:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ E_1(R\setminus\lambda_{n-2,1}) & E_1(R\setminus\lambda_{n-2,2}) & \cdots & E_1(R\setminus\lambda_{n-2,n-2}) \\ E_2(R\setminus\lambda_{n-2,1}) & E_2(R\setminus\lambda_{n-2,2}) & \cdots & E_2(R\setminus\lambda_{n-2,n-2}) \\ \vdots & \vdots & & \vdots \\ E_{n-3}(R\setminus\lambda_{n-2,1}) & E_{n-3}(R\setminus\lambda_{n-2,2}) & \cdots & E_{n-3}(R\setminus\lambda_{n-2,n-2}) \end{bmatrix} \begin{bmatrix} \lambda'_{n-2,1} \\ \lambda'_{n-2,2} \\ \lambda'_{n-2,3} \\ \vdots \\ \lambda'_{n-2,n-2} \end{bmatrix}$$
$$= \begin{bmatrix} E_1(R)' \\ E_2(R)' \\ E_3(R)' \\ \vdots \\ E_{n-2}(R)' \end{bmatrix}$$

We can find a formula for the magnitude of the $\lambda'_{n-2,i}$ by applying Cramer's Rule:

$$|\lambda'_{n-2,i}| = \left|\frac{E_1(R)'\lambda^{n-3}_{n-2,i}-E_2(R)'\lambda^{n-4}_{n-2,i}+E_3(R)'\lambda^{n-5}_{n-2,i}-\cdots+(-1)^{n-3}E_{n-2}(R)'}{\prod\limits_{1\le s<t\le n-2}(\lambda_{n-2,s}-\lambda_{n-2,t})}\right|$$

If we evaluate this relationship at a point x_0 for which $\lambda'_{n-2,i}(x_0)=0$ but $E_k(R)'(x_0)\ne 0$, $\forall k=1,\dots,n-2$, we find that

$$E_1(R)'\lambda^{n-3}_{n-2,i}-E_2(R)'\lambda^{n-4}_{n-2,i}+E_3(R)'\lambda^{n-5}_{n-2,i}-\cdots+(-1)^{n-3}E_{n-2}(R)'=0,$$

and

$$\lambda^{n-3}_{n-2,i}-\frac{E_2(R)'}{E_1(R)'}\lambda^{n-4}_{n-2,i}+\frac{E_3(R)'}{E_1(R)'}\lambda^{n-5}_{n-2,i}-\cdots+(-1)^{n-3}\frac{E_{n-2}(R)'}{E_1(R)'}=0.$$

We wish to show that the roots of this polynomial are values that lie outside the range of $\lambda_{n-2,i}$. We will begin by examining the coefficients of the polynomial. Recall that we showed in the proof of Lemma 4 that

$$E_{k-2}(R)'(x) = \frac{D'(x)(E_{k-2}(S-x)D(x) - N_k(x))}{(D(x))^2}$$

Since x_0 is not a critical point of $E_{k-2}(R)(x)$, $D'(x_0) \neq 0$ and we find that

$$\frac{E_{k-2}(R)'}{E_1(R)'}(x_0) = \frac{E_{k-2}(S-x)D(x_0) - N_k(x_0)}{E_1(S-x)D(x_0) - N_3(x_0)}$$

Since D is a non-vanishing function, we may divide both the numerator and the denominator of the right hand side of the above equation by D to find that

$$\begin{aligned}\frac{E_{k-2}(R)'}{E_1(R)'}(x_0) &= \frac{E_{k-2}(S-x) - \frac{N_k}{D}(x_0)}{E_1(S-x) - \frac{N_3}{D}(x_0)} \\ &= \frac{E_{k-2}(S-x) - E_{k-2}(R)}{E_1(S-x) - E_1(R)}\end{aligned}$$

After multiplying by a common denominator of these coefficients, our polynomial takes the following form:

$$\begin{aligned}&(E_1(S-x) - E_1(R))\lambda_{i,n-2}^{n-3} - (E_2(S-x) - E_2(R))\lambda_{n-2,i}^{n-4} + \dots \\ &+(-1)^{n-3}(E_{n-2}(S-x) - E_{n-2}(R)) = 0\end{aligned}$$

We make the substitution

$$E_k(R) = \lambda_{n-2,i}E_{k-1}(R \setminus \lambda_{n-2,i}) + E_k(R \setminus \lambda_{n-2,i})$$

and note that

$$\begin{aligned}&(E_k(S-x) - \lambda_{n-2,i}E_{k-1}(R \setminus \lambda_{n-2,i}) - E_k(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m} \\ &\quad - (E_{k+1}(S-x) - \lambda_{n-2,i}E_k(R \setminus \lambda_{n-2,i}) - E_{k+1}(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m-1} \\ &= -E_{k-1}(R \setminus \lambda_{n-2,i})\lambda_{n-2,i}^{m+1} \\ &\quad + (E_k(S-x) + E_k(R \setminus \lambda_{n-2,i}) - E_k(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m} \\ &\quad - (E_{k+1}(S-x) + E_{k+1}(R \setminus \lambda_{n-2,i}) - E_{k+1}(R \setminus \lambda_{n-2,i}))\lambda_{n-2,i}^{m-1} \\ &= -E_{k-1}(R \setminus \lambda_{n-2,i})\lambda_{n-2,i}^{m+1} + E_k(S-x)\lambda_{n-2,i}^{m} - E_{k+1}(S-x)\lambda_{n-2,i}^{m-1}.\end{aligned}$$

This telescoping relationship allows us to write our polynomial as

$$\sum_{k=0}^{n-2}((-1)^k E_k(S-x)\lambda_{n-2,i}^{n-2-k}) = 0.$$

It is now clear that the set of roots of this polynomial is the set $(S-x) = \{\lambda_{n-1,i}\}\backslash\lambda_{n-1,j}$. However, this contradicts the bounds for $\lambda_{n-2,i}$ set by the strict interlacing inequalities. Therefore, there exists no point which is a critical point of some $\lambda_{n-2,i}$ but not a critical point of an $E_{k-2}(R)$. □

We would now like to consider the behavior of the function

$$\begin{aligned} F_k : (\lambda_{n,1}, \lambda_{n,2}) \times (\lambda_{n,2}, \lambda_{n,3}) \times \ldots \times (\lambda_{n,k}, \lambda_{n,k+1}) &\to (\lambda_{n,1}, \lambda_{n,3}) \\ &\quad \times (\lambda_{n,2}, \lambda_{n,4}) \times \ldots \times (\lambda_{n,k}, \lambda_{n,k+2}) \\ (\lambda_{n-1,1}, \lambda_{n-1,2}, \ldots, \lambda_{n-1,k}) &\mapsto (\lambda_{n-2,1}, \lambda_{n-2,2}, \ldots, \lambda_{n-2,k}) \end{aligned}$$

which, for $k \in \{1, \ldots, n-2\}$, maps the k smallest eigenvalues of A_{n-1} to the k smallest eigenvalues of A_{n-2} when the rest of the spectra of A and A_{n-1} are assumed to be fixed. Our goal is to show that the image of this function includes its domain. Our next task is to examine the Jacobian of the function F_k. In the proof of Lemma 7, we used Cramer's rule to find a formula for the jth partial derivative of $\lambda_{n-2,i}$. We discovered that

$$\begin{aligned} &\frac{\partial}{\partial \lambda_{n-1,j}} \lambda_{n-2,i} \\ &= (-1)^{i+1} \frac{E_1(R)_{\lambda_{n-1,j}} \lambda_{n-2,i}^{n-3} - E_2(R)_{\lambda_{n-1,j}} \lambda_{n-2,i}^{n-4} + \cdots + (-1)^{n-3} E_{n-2}(R)_{\lambda_{n-1,j}}}{\prod\limits_{1 \le s < t \le n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})} \\ &= (-1)^{i+1} \frac{E_1(R)_{\lambda_{n-1,j}} \sum\limits_{k=0}^{n-2} ((-1)^k E_k(\{\lambda_{n-1,i}\}_{i \ne j}) \lambda_{n-2,i}^{n-2-k}}{\prod\limits_{1 \le s < t \le n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})} \\ &= E_1(R)_{\lambda_{n-1,j}} (-1)^{i+1} \frac{\prod\limits_{r \ne j} (\lambda_{n-2,i} - \lambda_{n-1,r})}{\prod\limits_{1 \le s < t \le n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})}, \end{aligned}$$

assuming $E_1(R)_{\lambda_{n-1,j}} \ne 0$. In the above expressions, $E_1(R)_{\lambda_{n-1,j}}$ denotes $\frac{\partial}{\partial \lambda_{n-1,j}} E_1(R) \times (\lambda_{n-1,j})$. We now consider the magnitude of the determinant of the Jacobian, denoted $|J_k|$. After factoring common terms from rows and columns, we find that

$$|J_k| = \frac{\prod\limits_{j \le k} (E_1(R)_{\lambda_{n-1,j}}) \prod\limits_{1 \le i \le k} (\prod\limits_{r > k} (\lambda_{n-2,i - \lambda_{n-1,r}}))}{\prod\limits_{1 \le s < t \le n-2} (\lambda_{n-2,s} - \lambda_{n-2,t})} \left| \left\{ \prod\limits_{j \ne r \le k} (\lambda_{n-2,i} - \lambda_{n-1,r}) \right\}_{ij} \right|$$

$$= \frac{\prod_{j\le k}(E_1(R)_{\lambda_{n-1,j}}) \prod_{1\le i\le k}(\prod_{r>k}(\lambda_{n-2,i}-\lambda_{n-1,r}))}{\prod_{1\le s<t\le n-2}(\lambda_{n-2,s}-\lambda_{n-2,t})}$$
$$\times \prod_{1\le u<v\le k}(\lambda_{n-2,u}-\lambda_{n-2,v}) \prod_{1\le w<x\le k}(\lambda_{n-1,w}-\lambda_{n-1,x}).$$

Thus, the Jacobian is singular only when a partial derivative is zero for the set $\{\lambda_{n-2,i}\}$ or when interlacing bounds are violated. Since the partial derivatives are continuous, by the inverse function theorem we have that the function F_k is locally invertible at all points at which J_k is nonsingular. We have now accumulated enough information about the function to prove the following.

Theorem 3. *Let $\lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,n}$, $\lambda_{n-2,1} < \lambda_{n-2,2} < \cdots < \lambda_{n-2,k}$, and $\lambda_{n-1,k+1} < \lambda_{n-1,k+2} < \cdots < \lambda_{n-1,n-1}$ be $2n-1$ real numbers such that*

$$\lambda_{n,i} < \lambda_{n-2,i} < \lambda_{n,i+1},$$

for all $i = 1, \ldots, k$ and

$$\lambda_{n,j} < \lambda_{n-1,j} < \lambda_{n,j+1}$$

for all $j = k+1, \ldots, n-1$.. Then for $k = 1, \ldots, n-2$, there exists an n-by-n real symmetric tridiagonal matrix A such that the eigenvalues of A are $\lambda_{n,1}, \ldots, \lambda_{n,n}$, the k smallest eigenvalues of A_{n-2} are $\lambda_{n-2,1}, \ldots, \lambda_{n-2,k}$, and the $n-1-k$ largest eigenvalues of A_{n-1} are $\lambda_{n-1,k+1}, \ldots, \lambda_{n-1,n-1}$.

Proof. By the result of Theorem 1 we are able to ensure A realizes all the desired eigenvalues of A and A_{n-1} as long as those of A_{n-1} interlace those of A_n. We fix $\lambda_{n-1,k+1}, \ldots, \lambda_{n-1,n-1}$ as eigenvalues of A_{n-1}. We now need to show that there exist real numbers $x_1 < x_2 < \cdots < x_k$ interlacing the smallest $k+1$ eigenvalues of A such that

$$F_k(x_1, \ldots, x_k) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,k}).$$

We will show this by induction on k. The inductive assumption is that there exist real numbers $x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}$ such that $F_{m-1}(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1})$ and that, furthermore, $x_i^{(m-1)}$ lies in the interval $(\lambda_{n,i}, c_i)$, where c_i is the unique zero (if it exists) of $\frac{\partial}{\partial \lambda_{n-1,i}}\lambda_{n-2,i}$. c_i is a continuous function of the spectra of A and A_{n-1}, excluding the eigenvalue $\lambda_{n-1,i}$. Thus, c_i is a constant whenever the rest of the spectra is held fixed. We use the convention that $c_i = \lambda_{n,i+1}$ in case the zero is not contained within the interlacing boundaries. We know that, apart from the interlacing bounds, the only points where the Jacobian of F_k is singular are the zeroes of the partial derivatives of the function. The mention of c_i in our inductive assumption is motivated by the suspicion that we can restrict our function F_k to a subrectangle of its domain such that the function is invertible on this subrectangle.

For the $k = 1$ case, note that $F_1(\lambda_{n-1,1})$ has at most one critical point, which we call c_1, using the convention that $c_1 = \lambda_{n,2}$ if there is no critical point. Since the function approaches $\lambda_{n,1}$ and $\lambda_{n,2}$ at its boundaries, the function is monotone increasing on $(\lambda_{n,1}, c_1)$ and monotone decreasing on $(c_1, \lambda_{n,2})$. Thus, for any $\lambda_{n-2,1} \in (\lambda_{n,1}, \lambda_{n,2})$ in the image of F_1, there exists a unique preimage $x_1^{(1)} \in (\lambda_{n,1}, c_1)$. Thus, our claim holds for the case $k = 1$.

Now we assume that our proposition holds for $k = m-1$. We seek values $x_1^{(m)}, \ldots, x_m^{(m)}$ such that

$$F_m(x_1^{(m)}, \ldots, x_m^{(m)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m}).$$

By our inductive assumption, when we have fixed $x_m^{(m-1)} \in (\lambda_{n,m}, \lambda_{n,m+1})$, we are able to find values $x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}$ such that

$$F_{m-1}(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1}).$$

Thus,

$$F_m(x_1^{(m-1)}, \ldots, x_{m-1}^{(m-1)}, x_m^{(m-1)}) = (\lambda_{n-2,1}, \ldots, \lambda_{n-2,m-1}, y),$$

for some $y \in (\lambda_{n,m}, \lambda_{n,m+1})$.

We have assumed that the first $m-1$ components of this preimage lie on a subrectangle on which the function is devoid of critical points. Thus, the function is locally invertible in the first $m - 1$ components. This allows us to move $\lambda_{n-1,m}$ continuously along the interval of $(\lambda_{n,m}, \lambda_{n,m+1})$ while the smallest $m - 1$ eigenvalues of A_{n-2} are kept fixed as $\lambda_{n-1,1}, \ldots, \lambda_{n-1,m-1}$. We know from the argument used in the proof of Lemma 3 that

$$\lim_{\lambda_{n-1,m} \to \lambda_{n,m}} (\lambda_{n-2,m}) = \lambda_{n,m}$$

and that

$$\lim_{\lambda_{n-1,m} \to \lambda_{n,m+1}} (\lambda_{n-2,m}) = \lambda_{n,m+1}.$$

Thus, by the intermediate value theorem, we may conclude that we can realize any desired $\lambda_{n-2,m} \in (\lambda_{n,m}, \lambda_{n,m+1})$.

We still must show that the mth component of the preimage lies in the interval $(\lambda_{n,m}, c_m)$. We can show this by contradiction. Assume that the mth component of the preimage lies in the interval $[c_m, \lambda_{n,m+1})$. Then we may fix the other components of the preimage and think of $\lambda_{n-2,m}$ as a function of $\lambda_{n-1,m}$. On this interval, the function is monotone decreasing, but has a lower bound of $\lambda_{n,m+1}$. Thus, it takes on only values higher than $\lambda_{n,m+1}$ on this interval, which excludes the value $\lambda_{n-2,m}$. This is a contradiction. Hence, the mth component of the preimage lies in the interval $(\lambda_{n,m}, c_m)$. $\square$

We hope that further analysis of the behavior of eigenvalues of leading principal submatrices as a function of the eigenvalues of A and A_{n-1} will provide us with tools for proving more classes of cases of the general conjecture.

5. 4-by-4 cases

In Section 3 we showed that the conjecture is true for the 2-by-2 and 3-by-3 cases. Each case of the main conjecture can be identified with a sequence $z_1, \ldots, z_n$ majorized by $2k - 1$. There are 14 such sequences in the case $n = 4$. We are able to show that the conjecture holds for 8 of these cases, and we list them here.

$(0, 0, 3, 4)$: This is covered by Theorem 1.

$(0, 1, 2, 4)$ and $(0, 2, 1, 4)$: These are covered by Theorem 3.

$(1, 0, 2, 4)$: This is covered by Lemma 3.

$(1, 1, 1, 4)$: Given $L = l_1 < \ldots < l_7$, we choose l_1, l_3, l_5, l_7 to be the eigenvalues of A and l_6 to be the largest eigenvalue of A_3. By Theorem 3, we choose the middle eigenvalue of A_3 between l_3 and l_5 so that it fixes l_4 as the largest eigenvalue of A_2 while we vary the smallest eigenvalue of A_3 between l_1 and l_3. By the argument used in Lemma 3, we see that as $\lambda_{3,1}$ approaches l_1, $\lambda_{1,1}$ approaches l_1 and that as $\lambda_{3,1}$ approaches l_3, $\lambda_{1,1}$ approaches l_3. So by the Intermediate Value Theorem, there is a value for $\lambda_{3,1}$ between l_1 and l_3 so that $\lambda_{1,1} = l_2$. Thus, there exists a matrix for which $l_1, l_3, l_5, l_7 \in \sigma(A)$, $l_6 \in \sigma(A_3)$, $l_4 \in \sigma(A_2)$, and $l_2 \in \sigma(A_1)$.

$(1, 2, 2, 2)$, $(0, 2, 3, 2)$, and $(1, 1, 3, 2)$: Note that the sequences $(1, 2, 2)$, $(0, 2, 3)$, and $(1, 1, 3)$ are each resolved 3-by-3 cases of the conjecture. Thus, if L consists of the numbers $l_1 < l_2 < \cdots < l_7$, in each case we may choose as A_3 the matrix realizing $l_2, \ldots, l_6$ as the desired eigenvalues of upper left submatrices of A_3 and then use the method given in [7] for completing the matrix so that it realizes l_1 and l_7 as the largest and smallest eigenvalues of A_4.

6. Other problems

Instead of focusing on $2n-1$ eigenvalues of leading principal submatrices only, we may widen our scope to include inverse eigenvalue problems dealing with $2n-1$ eigenvalues of any of the principal submatrices. We do not currently have a conjecture about which of the many possible problems will always have a solution since, while leading principal matrices are nested, matrices in the more general problem may be nested, have overlapping entries, or be disjoint. We present two quick examples of these types of problems.

Problem 1. Let A_i^{i+1} denote the 2-by-2 submatrix of A obtained by crossing out all rows and columns except the ith and $(i+1)$st. Submatrices of consecutive indices, A_i^{i+1} and A_{i+1}^{i+2} overlap in one entry. Given a list of $2n-1$ distinct real numbers, there exists an n-by-n real symmetric tridiagonal matrix A which realizes the $2n-1$ numbers as: the single eigenvalue of A_1 and the 2 eigenvalues of each of the A_i^{i+1}, $i = 1, \dots, n-1$.

Proof. Suppose a_{11} has been chosen. Then we may choose the eigenvalues, $\lambda_1^{1,2}$ and $\lambda_2^{1,2}$ of A_1^2 to be any values such that

$$\lambda_1^{1,2} < a_{11} < \lambda_2^{1,2}.$$

This choice determines the matrix A_1^2, and we thus obtain the entry a_{22}, which is also the upper left entry of the matrix A_2^3. In this recursive manner, we may construct the matrix as long as we have chosen the eigenvalues so that the inequality

$$\lambda_1^{i,i+1} < a_{i,i} < \lambda_2^{i,i+1}$$

is always satisfied. Depending on the given list, there may be many ways to make this choice, but we note that the choice satisfying

$$\lambda_1^{i+1,i+2} < \lambda_1^{i,i+1} < \lambda_2^{i,i+1} < \lambda_2^{i+1,i+2}$$

for all $i = 1, \dots, n-2$ will always satisfy the required inequalities. $\square$

Problem 2. Let B^{n-k} denote the bottom right $(n-k)$-by-$(n-k)$ submatrix of A. Given a list $l_1 < \cdots < l_{2n-1}$ of $2n-1$ distinct real numbers, there exists an n-by-n real symmetric tridiagonal matrix A which realizes the $2n-1$ numbers as: All k eigenvalues of A_k, all $(k-1)$ eigenvalues of A_{k-1}, all $(n-k)$ eigenvalues of B^{n-k}, all $(n-k-1)$ eigenvalues of B^{n-k-1}, and 1 eigenvalue of the entire matrix A.

Proof. We note that A_k and B^{n-k} are disjoint. By the classical result, we will be able to choose the eigenvalues of A_k and A_{k-1} independently of those of B_{n-k} and B_{n-k-1}. There is a single independent entry of the matrix, $a_{k,k+1}$ not contained in either A_k or B^{n-k}. All we must determine is when we may choose this entry so that the matrix realizes a desired eigenvalue λ_0. If we let P_k and Q_{n-k} denote the characteristic polynomials of A_k and B^{n-k}, respectively, we may compute the characteristic polynomial of A by expanding along the kth row:

$$P_n(\lambda) = P_k(\lambda)Q_{n-k}(\lambda) - a_{k,k+1}^2 P_{k-1}(\lambda)Q_{n-k-1}(\lambda)$$
$$a_{k,k+1}^2 = -\frac{P_k Q_{n-k}}{P_{k-1}Q_{n-k-1}}(\lambda_0).$$

Thus, $a_{k,k+1}$ is real if and only if λ_0 is less than an odd number of eigenvalues in the union of the spectra of A_k, A_{k-1}, B^{n-k}, and B^{n-k-1}. Thus, there exists a matrix realizing the desired eigenvalues as long as we choose the single desired eigenvalue of A to be some even-indexed l_{2j} from the list. All other eigenvalues may be chosen freely, as long as they are chosen so that the spectra of A_{k-1} and B^{n-k-1} interlace the spectra of A_k and B^{n-k}. □

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