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Multiplicity Lists for Classes of Hermitian Matrices whose Graph is a Certain Tree

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1 Introduction

A *tree* is an undirected, connected, acyclic graph on n vertices, and a *forest* is a graph in which every component is a tree. We denote the set of all n-by-n Hermitian matrices whose graph is the tree T as $\mathcal{H}(T)$. Every $A \in \mathcal{H}(T)$ has a list of eigenvalue multiplicities which sum to n; we define a *multiplicity list* of T as a list that occurs for some $A \in \mathcal{H}(T)$, and the set of all such lists as $L(T)$ (and for a general graph, $L(G)$).

The first aspect we consider is the *minimum number of 1's* among the multiplicity lists in $L(T)$, which we denote $U(T)$. In Sections 2 and 3, we will obtain explicit formulas for $U(T)$ for two classes of trees by constructing multiplicity lists and using certain bounds that we give based on the structure of the tree.

In fact, the structure of a tree T can give a great deal of information about $L(T)$. In Section 5, we focus on the *path cover number* (denoted $p(T)$), which is the minimum number of non-intersecting paths that cover all vertices on the tree and also the maximum multiplicity of any eigenvalue among the matrices whose graph is T [**JL1**]. We explore how subdividing an edge between two vertices can affect $p(T)$.

In Section 6, we continue our exploration of the structure of trees, but this time we look at the specific vertices. Note that removal of a vertex of T is analogous to taking a principal submatrix of a matrix H whose graph is T, so we are interested in what we can determine about eigenvalue multiplicities of principal submatrices of H from subgraphs of T. We classify each vertex in T as parter, downer, or neutral with regard to an eigenvalue λ, depending on whether removal of that vertex causes the multiplicity of λ to increase, decrease, or remain the same, respectively, in the principal submatrix. Given v_i and v_j as vertices of T, we completely evaluate how the classification of v_i can change when we remove v_j.

Finally, in Section 7, we briefly explore what we can determine about $L(G)$ for certain non-trees given what we already know about $L(T)$. This analysis is somewhat fragmented, but, given how little is known about $L(G)$ in general, it seems like a worthwhile endeavor.

The point of this paper is to explore some different ways we can learn more about $L(T)$ and $L(G)$, and, other than that, the topics are not always directly related. But, together, the sections make steps to further our knowledge about this topic in general.

We begin with some background on matrices and graphs.

2 Background

Consider an n-by-n matrix A. Let $\alpha \subseteq \{1,\ldots,n\}$. We denote the principal submatrix resulting from deleting the rows and columns of A indexed by α as $A(\alpha)$. Since we will often instead consider G, the graph representation of A, we adopt the same notation: we denote the graph resulting from the deletion of the vertices of G indexed by α as $G(\alpha)$. Similarly, we denote the principle submatrix that lies in the rows and columns of A indexed by α as $A[\alpha]$; the analogous subgraph of G is denoted $G[\alpha]$.

Recall that the eigenvalues of a Hermitian matrix are always real. Assume that an n-by-n Hermitian matrix A has eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

and $A(i)$ has eigenvalues

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}.$$

Then, by the classical interlacing inequalities,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \ldots \leq \mu_{n-1} \leq \lambda_n.$$

We denote the multiplicity of λ_j as an eigenvalue of A as $m_A(\lambda_j)$. We see that, by the interlacing inequalities,

$$m_A(\lambda_j) - 1 \leq m_{A(i)}(\lambda_j) \leq m_A(\lambda_j) + 1$$

for $0 \leq i \leq n$. Thus we see that the multiplicity of any given eigenvalue can at most increase or decrease by 1 when we delete a single vertex from a graph representing a Hermitian matrix. This is an important result; we will often show that an eigenvalue has multiplicity $\geq k$ in A by forcing $m_{A(i)} = k+1$.

A quick aside: the *degree* of a vertex is the number of edges incident to that vertex. We denote the degree of a vertex v_i in a tree T as $\deg_T(i)$.

For trees, we can infer even more about how the deletion of certain vertices can affect the multiplicities of an eigenvalue from this next result ([**Pa**], [**Wi**], [**JLS1**]):

Theorem 2.1 (Parter's Theorem). *Let T be a tree and $A \in \mathcal{H}(T)$. Suppose that there exists an index i and a real number λ such that $\lambda \in \sigma(A)$ and $\lambda \in \sigma(A(i))$. Then,*

(a) there is an index j such that $m_{A(j)}(\lambda) = m_A(\lambda) + 1$;

(b) if $m_A(\lambda) \geq 2$, then j may be chosen so that $\deg_T(j) \geq 3$ and so that there are at least three components T_1, T_2, and T_3 of $T(j)$ such that $m_{A[T_k]}(\lambda) \geq 1$, $1 \leq k \leq 3$; and

(c) if $m_A(\lambda) = 1$, then j may be chosen so that there are two components T_1 and T_2 of $T(j)$ such that $m_{A[T_k]}(\lambda) \geq 1$, $1 \leq k \leq 2$.

Let H be a Hermitian matrix whose graph is a tree T, and let $\lambda \in \sigma(H)$ and $m_{H(j)} = m_H(\lambda) + 1$. Then we call $v_j \in T$ a *Parter vertex*. The above result is critical for us when we are creating matrices with certain multiplicity lists because it ensures that the graph of that matrix will not only have a Parter vertex, but we can also sometimes assume certain vertices must be Parter for a given eigenvalue depending on its multiplicity.

We also classify vertices that are not Parter vertices. If $\lambda \in \sigma(H)$ and $m_{H(j)} = m_H(\lambda)$, we call $v_j \in T$ a *neutral vertex*. If $\lambda \in \sigma(H)$ and $m_{H(i)} = m_H(\lambda) - 1$, we call $v_i \in T$ a *downer vertex*. Sometimes the classification of one vertex depends on the classification of another. For instance, if v_1 is a Parter vertex in T, then a vertex adjacent to v_1 is downer in $T(i)$; this relation will prove very useful in Section 6.

We will refer to a vertex v in T as *high-degree* if $\deg_T(v) \geq 3$ and as *low-degree* if $\deg_T(v) \leq 2$. If a vertex has degree 1 we will sometimes call it a *pendant vertex*.

For any tree T, we denote the *diameter of* T as $d(T)$ and define it as such: $d(T) =$ the maximum number of vertices in any path that is a subtree of T. We can use $d(T)$ as an important tool because of the following theorem:

Theorem 2.2 (JL2). *Let T be a tree. The minimum number of distinct eigenvalues for any $A \in \mathcal{H}(T)$ is at least $d(T)$.*

The previous result immediately gives us this corollary about $L(T)$ for a path:

Corollary 2.3. *Let T be a path on n vertices. Then $L(T)$ consists only of the list containing all 1's; that is, if the graph of a Hermitian matrix A is T, the eigenvalues of A are distinct.*

In the classes of trees for which we determine $U(T)$, removing one vertex will often create many paths, and it is critical to know that the eigenvalues in each path are distinct.

We should also discuss the *Inverse Eigenvalue Problem* for trees: given a tree T, what are all the possible spectra that occur among matrices in $\mathcal{H}(T)$? This is often a difficult question, but, luckily for us, the inverse eigenvalue problem is equivalent to the multiplicity lists for many trees, including paths, generalized stars, and double generalized stars [**JLSS**]. In other words, if a multiplicity list is in $L(T)$ when T is in any of these classes of trees, there exists a matrix H whose graph is T and has the specified eigenvalue multiplicities for any values we choose for those eigenvalues. We will put this fact to great use in the following sections.

3 $U(T)$ for a Generalized Star

The problem of finding an explicit formula for $U(T)$ given a tree T is not an easy one, and little was proven about it. Here we give and prove a formula for finding $U(T)$ for a fairly simple tree, the generalized star. We will do this by constructing a matrix whose graph is a generalized star S with a certain number of eigenvalues of multiplicity $= 1$ (*simple* eigenvalues). We then show that this number is a lower bound for $U(S)$

Definition 3.1. *A tree T is a generalized star if T has at most one vertex with degree greater than* 2*.*

For ease of calculation, we will assume that any generalized star S has exactly one vertex with degree greater than 2 (recall that we already know $U(T) = n$ for T a path on n vertices). We call that vertex with high degree the *center vertex*, denoted v_c. An *arm* of S is any path created by the deletion of v_c.

Lemma 3.2. *Let S be a generalized star on n vertices with arm lengths $l_1 \geq l_2 \geq \ldots \geq l_p$. Then $U(S) \geq l_1 + 1$.*

Proof. Consider S as above and H, a Hermitian matrix whose graph is S. Let the eigenvalues of multiplicity at least 2 in H be $\mu_1, \ldots, \mu_k$. We know that the number of simple eigenvalues in H equals $n - \sum_{i=1}^{k} m_H(\mu_i)$.

We remove the center vertex of S, leaving a forest on $n-1$ vertices consisting of paths. We refer to this forest as S' and the analogous principal submatrix of H as H'. Let the distinct eigenvalues of H' be

$$\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_b, \lambda_{b+1}, \ldots, \lambda_c$$

where $m_{H'}(\lambda_i) \geq 3$ if $1 \leq i \leq a$, $m_{H'}(\lambda_i) = 2$ if $a < i \leq b$ and $m_{H'}(\lambda_i) = 1$ if $b < i \leq c$. Since S' contains a path on l_1 vertices, and all the eigenvalues of any matrix whose graph is a path are distinct, we know that $c \geq l_1$. Also, since S' has $n-1$ vertices, $\sum_{j=1}^{c} m_{H'}(\lambda_j) = n-1$.

By Parter's Theorem, the center vertex must be parter for any eigenvalue in H of multiplicity at least 2. Thus, τ is a an eigenvalue in H of multiplicity at least 2 if and only if τ has multiplicity at least 3 in H'. By interlacing, $m_H(\tau) = m_{H'}(\tau) - 1$. Therefore,

$$\sum_{i=1}^{k} m_H(\mu_i) = \sum_{j=1}^{a} [m_{H'}(\lambda_j) - 1] = \sum_{j=1}^{a} m_{H'}(\lambda_j) - a.$$

As we stated before, the number of simple eigenvalues in H equals $n - \sum_{i=1}^{k} m_H(\mu_i)$. We substitute and subtract zero twice, obtaining

$$\begin{aligned} n - \sum_{i=1}^{k} m_H(\mu_i) &= n - [\sum_{j=1}^{a} m_{H'}(\lambda_j) - a] - [\sum_{j=a+1}^{b} 1 - (b-a)] \\ &\quad - [\sum_{j=b+1}^{c} m_{H'}(\lambda_j) - (c-b)] \\ &= n - [\sum_{j=1}^{a} m_{H'}(\lambda_j) + \sum_{j=a+1}^{b} 1 + \sum_{j=b+1}^{c} m_{H'}(\lambda_j)] + c \end{aligned}$$

We know that $c \geq l_1$ and that

$$\begin{aligned} &\sum_{j=1}^{a} m_{H'}(\lambda_j) + \sum_{j=a+1}^{b} 1 + \sum_{j=b+1}^{c} m_{H'}(\lambda_j) \\ &\leq \sum_{j=1}^{a} m_{H'}(\lambda_j) + \sum_{j=a+1}^{b} m_{H'}(\lambda_j) + \sum_{j=b+1}^{c} m_{H'}(\lambda_j) = n-1 \end{aligned}$$

Thus, if we denote the number of simple eigenvalues in H as X, $X = n - k_1 + k_2$, where $k_1 \leq n-1$ and $k_2 \geq l_1$. Thus,

$$X \geq n - (n-1) + l_1 = l_1 + 1$$

Since this is true for a general matrix H with graph S, $U(S) \geq l_1 + 1$. □

Lemma 3.3 (DS). *Let T be a tree on n vertices with diameter $d(T)$. Then $U(T) \geq 2d(T) - n$.*

Theorem 3.4. *Let S be a generalized star on n vertices with arm lengths $l_1 \geq l_2 \geq \ldots \geq l_p$ and diameter $d(S) = l_1 + l_2 + 1$. Then*

$$U(S) = \begin{cases} 2d(S) - n & \text{if } l_2 > \sum_{i=3}^{p} l_i \\ l_1 + 1 & \text{if } l_2 \leq \sum_{i=3}^{p} l_i \end{cases}$$

Proof. We consider S as above, with center vertex v_c. Let H be some Hermitian matrix whose graph is S. We refer to the forest $S(c)$ as S' and the analogous principal submatrix as H'. Again, we know that v_c is parter for any eigenvalue $\in \sigma(S)$ that has multiplicity at least 2. Let H_1 be the largest direct summand in H', H_2 be the second largest, and so on; since S' is a forest of paths, H_1 will be l_1 x l_1 with distinct eigenvalues, H_2 will be l_2 x l_2 with distinct eigenvalues, and so on.

Case 1: Let $l_2 > \sum_{i=3}^{p} l_i$.

Let $w = \sum_{i=3}^{p} l_i$. Since $l_1 \geq l_2$, we can create w real eigenvalues of multiplicity $= 3$ in H' by assigning each to H_1, H_2, and exactly one of $H_3, \ldots, H_p$. By Parter's Theorem and interlacing, each of these eigenvalues will have multiplicity $= 2$ in H. Thus, the number of simple eigenvalues in H is n - 2w. We know that

$$\begin{aligned} n - 2w = & \sum_{i=1}^{p} l_i + 1 - 2\sum_{i=3}^{p} l_i \\ = & \; l_1 + l_2 + 1 - \sum_{i=3}^{p} l_i \\ = & \; 2l_1 - l_1 + 2l_2 - l_2 + 2 - 1 - \sum_{i=3}^{p} l_i \\ = & \; 2(l_1 + l_2 + 1) - (l_1 + l_2 + \sum_{i=3}^{p} + 1) l_i \\ = & \; 2d(S) - n \end{aligned}$$

By our lemma, this is a lower bound, so $U(S) = 2d(S) - n$.

Case 2: Let $l_2 \leq \sum_{i=3}^{p} l_i$.

Let $\lambda_1,\ldots,\lambda_{l_2} \in \mathbb{R}$. We create a sequence s_{l_2} such that

$$s_{l_2} = \lambda_1,\lambda_2,\ldots,\lambda_{l_2},\lambda_1,\lambda_2,\ldots,\lambda_{l_2},\lambda_1,\ldots,\lambda_{l_2},\ldots$$

We assign the first l_2 terms of s_{l_2} to $\sigma(H_2)$, the next l_3 terms to $\sigma(H_3)$, the next l_4 terms to $\sigma(H_4)$, and so forth. Since $l_2 \geq l_3 \geq \ldots \geq l_p$, we know that, for $i \geq 2$, no H_i will contain the same eigenvalue twice (which is necessary, as the graph of each H_i is a path). Also, since $l_2 \leq \sum_{i=3}^{p} l_i$, we know that, for $1 \leq j \leq l_2$, λ_j is an eigenvalue in at least two direct summands of H'. Since $l_1 \geq l_2$, we can also place every λ_j in $\sigma(H_1)$. This gives us l_2 eigenvalues of multiplicity at least 3 in H'.

By our construction, the sum of the multiplicities of these eigenvalues in H' is $l_2 + l_2 + l_3 + \ldots + l_p$. By Parter's Theorem and interlacing, each eigenvalue will decrease in multiplicity by exactly 1 when we replace the center vertex, and will therefore have multiplicity at least 2 in H. Since there are l_2 eigenvalues, the sum of these multiplicities in H will be $l_2 + l_3 + \ldots + l_p$. H has $l_1 + l_2 + \ldots + l_p + 1$ total eigenvalues, and thus at most

$$l_1 + l_2 + \ldots + l_p + 1 - (l_2 + \ldots + l_p) = l_1 + 1$$

eigenvalues of multiplicity $= 1$. So we know $U(S) \leq l_1 + 1$. But, by our lemma, $U(S) \geq l_1 + 1$, so $U(S) = l_1 + 1$.

□

We note that, if S has no high-degree vertices, it is a path, and $U(S) = n$, since the eigenvalues of any matrix whose graph is a path are all distinct. By our theorem, since there are only two arms on a path and thus $l_2 > \sum_{i=3}^{p} l_i$, $U(S)$ should equal $2d(S) - n$. Since the diameter of a path on n vertices is n, $2d(S) - n = n$. Thus our theorem holds for any generalized star, regardless of whether it has a high-degree vertex.

4 $U(T)$ for a Double Generalized Star

The double generalized star is as it sounds: two single generalized stars connected at their centers to form a single tree. Thus, it is natural to progress to the question of $U(T)$ for this class of trees, which is of a much higher order of complexity than our previous question. Again, we will give and prove an explicit formula for $U(T)$ below.

Definition 4.1. *A tree T is a double generalized star if and only if it has two vertices, v_l and v_m, such that v_l and v_m are adjacent and the only vertices in T of degree ≥ 2.*

Let T be a double generalized star. We consider T as two separate generalized stars, S_l and S_m, centered and attached at v_l and v_m, respectively. We define the arm lengths of S_l as $l_1, \ldots, l_p$ such that $l_1 \geq l_2 \geq \ldots \geq l_p$, and we define the arm lengths of S_m as $m_1, \ldots, m_q$ such that $m_1 \geq m_2 \geq \ldots \geq m_q$. We denote the arms of S_l, from longest to shortest, as $L_1, L_2, \ldots, L_p$. Thus, $length(L_1) = l_1$, $length(L_2) = l_2$, and so forth. Similarly, we denote the arms of S_m as $M_1, M_2, \ldots, M_q$. We also define $\Gamma_l = \sum_{i=1}^{p} l_i$ and $\Gamma_m = \sum_{i=1}^{q} m_i$.

We classify S_l as Type 1 if $l_1 > l_2 + l_3 + ... + l_p$. We classify S_l as Type 2 if $l_1 \leq l_2 + l_3 + ... + l_p$. Similarly, we classify S_m as Type 1 or Type 2 based on $m_1, \ldots, m_q$.

For the rest of this section, we will blur the distinction between a matrix and its graph. For example, if we say that $\lambda \in \sigma(S_l)$, we mean that λ is an eigenvalue for M, a certain matrix whose graph is S. Also, if we say that we assign an eigenvalue to an arm L_i of S_l, we are constructing a matrix H whose graph is S_l such that λ is an eigenvalue of the direct sum in $H(l)$ analogous to L_1. We recall from Section 2 that, the inverse eigenvalue problem for paths and single and double generalized stars is equivalent to the multiplicity lists, so we can assign values for eigenvalues however we wish given the multiplicity list we create can occur.

Let $\lambda \in \sigma(S)$, where S is a generalized star. We say that λ is a Parter eigenvalue for S if and only if λ is an eigenvalue for at least two arms of S; that is, when we remove the center vertex of S, λ is a Parter eigenvalue if it is an eigenvalue of at least two of the paths remaining. Otherwise, we call λ a non-Parter eigenvalue. For a star S_k, we denote the number of Parter eigenvalues as r_k and the number of non-Parter eigenvalues as c_k.

Lemma 4.2. *Let S_k be a generalized star on n vertices. We denote the center vertex of S_k as v_k and $S_k(k)$, the star S_k without the vertex v_k, as S'_k. Let P be the set of eigenvalues with multiplicity at least 2 in S'_k (Parter eigenvalues) and N be the set of simple eigenvalues in S'_k (non-Parter eigenvalues). Then $c_k = |P|+|N|+1$, or the number of distinct eigenvalues in S'_k plus* 1.

Proof. Let $\mu \in P$; then $m_{S'_k}(\mu) \geq 2$. If $m_{S'_k}(\mu) \geq 3$, v_k must be parter for μ because v_k is the only high-degree vertex in S_k (Parter's Theorem). If $m_{S'_k}(\mu) = 2$, we know that v_k cannot be neutral or downer for μ, because then $m_{S_k}(\mu) \geq 2$, which implies v_k is Parter for μ. Thus v_k is Parter for every $\mu \in P$.

Therefore, since S_k has n total eigenvalues and every eigenvalue is either Parter or non-Parter,

$$c_k = n - \sum_{\lambda_i \in P} [m_{S'_k}(\lambda_i) - 1] = n + |P| - \sum_{\lambda_i \in P} m_{S'_k}(\lambda_i)$$

Since S'_k has $n-1$ eigenvalues and P and N contain all the eigenvalues in S'_k, we know that

$$\begin{aligned}
n-1 &= \sum_{\lambda_i \in |P|} m_{S'_k}(\lambda_i) + \sum_{\lambda_j \in |N|} m_{S'_k}(\lambda_j) \\
n-1 &= \sum_{\lambda_i \in |P|} m_{S'_k}(\lambda_i) + |N| \\
n - \sum_{\lambda_i \in |P|} m_{S'_k}(\lambda_i) &= |N| + 1 \\
c_k - |P| &= |N| + 1 \\
c_k &= |P| + |N| + 1
\end{aligned}$$

Thus c_k is the number of distinct eigenvalues in S'_k plus 1. □

Corollary 4.3. *For any generalized star S_k with longest arm length k_1, $c_k \geq k_1 + 1$.*

Proof. When we remove the center vertex v_k, the longest arm becomes a path with distinct eigenvalues. Each eigenvalue either has multiplicity ≥ 2 in $\sigma(S'_k)$ or multiplicity $= 1$ in

$\sigma(S'_k)$; that is, every eigenvalue belongs to either P or N, where P and N are as they were defined earlier. Since the eigenvalues are distinct, we know that $k_1 \leq |P|+|N| = c_k - 1$, and thus $c_k \geq k_1 + 1$. □

Our method for creating multiplicity lists will consist of creating assignments such that we can match Parter eigenvalues in S_l with non-Parter eigenvalues from S_m, and vice-versa. We see that, if λ is a Parter eigenvalue in S_l, by definition, it is present in at least two arms in S_l. Thus, if λ is also a non-Parter eigenvalue in S_m, when we remove v_l, λ will be present in at least three of the trees in the resulting forest. Therefore, by interlacing, $m_T(\lambda) \geq 2$.

For any matrix H whose graph is generalized star S_k, we see that there is an upper bound on the number of Parter eigenvalues in $\sigma(H)$ since we can only assign so many eigenvalues to two arms of S_k. We refer to this upper bound as R_k.

Lemma 4.4. *Consider a generalized star T comprised of two generalized stars, S_l and S_m, where l_1 and m_1 are the longest arm lengths of S_l and S_m, respectively. Then $U(T) \geq l_1 + 1 - R_m$ and $U(T) \geq m_1 + 1 - R_l$.*

Proof. We denote the number of vertices in S_l and S_m as n_l and n_m, respectively. We call the forest $S_l(l)$ F_l and the forest $S_m(m)$ F_m. Let $\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_x$ be the eigenvalues of F_l, where $m_{F_l}(\lambda_i) \geq 2$ if $1 \leq i \leq a$ and $m_{F_l}(\lambda_i) = 1$ if $a < i \leq x$. Similarly, let $\mu_1, \ldots, \mu_b, \mu_{b+1}, \ldots, \mu_y$ be the eigenvalues of F_m where $m_{F_m}(\mu_j) \geq 2$ if $1 \leq j \leq b$ and $m_{F_m}(\mu_j) = 1$ if $b < i \leq y$.

By Parter's Theorem, if $\tau \in \sigma(T)$ such that $m_T(\tau) \geq 2$, then $m_{T(l)}(\tau) \geq 3$ or $m_{T(m)}(\tau) \geq 3$. Without loss of generality, let $m_{T(l)}(\tau) \geq 3$. Then $m_{F_l)}(\tau) \geq 2$ or $m_{S_m}(\tau) \geq 2$. In the first case, τ is a Parter eigenvalue for S_l, and, in the second case, τ is a Parter eigenvalue for S_m; thus, if an eigenvalue has multiplicity at least 2 in T, that eigenvalue is a Parter eigenvalue for at least one of S_l and S_m.

Let $m_T(\tau) \geq 2$; without loss of generality, let τ be a Parter eigenvalue for S_l. If τ is a non-Parter eigenvalue on S_m, then

$$m_T(\tau) = m_{T(l)}(\tau) - 1 = m_{F_l}(\tau) + m_{S_m} - 1 = m_{F_l}(\tau) \tag{1}$$

If τ is a Parter eigenvalue on S_m, then

$$m_T(\tau) = m_{T(l)}(\tau) - 1 = m_{F_l}(\tau) + m_{S_m} - 1 = m_{F_l}(\tau) + m_{F_m}(\tau) - 2 \tag{2}$$

If τ is not an eigenvalue on S_m, then

$$m_T(\tau) = m_{T(l)}(\tau) - 1 = m_{F_l}(\tau) + m_{S_m} - 1 = m_{F_l}(\tau) - 1 \tag{3}$$

We consider the case when $c_m \geq a$, and thus we can match every Parter eigenvalue in S_l with a non-Parter eigenvalue from S_m. We see that using equation (1), the number of simple eigenvalues in T is at least:

$$\begin{aligned} n_l + n_m - [\sum_{i=1}^{a} m_{F_l}(\lambda_i) + \sum_{i=1}^{b} m_{F_m}(\mu_i)] \\ &\leq n_l + n_m - [\sum_{i=1}^{x} m_{F_l}(\lambda_i) - (x-a) + \sum_{i=1}^{y} m_{F_m}(\mu_i) - (y-b)] \\ &\leq n_l + n_m - [(n_l - 1) + (n_m - 1) - (x-b) - (y-a)] \\ &\leq (x+1-b) + (y+1-a) \end{aligned}$$

Since F_l contains a path on l_1 vertices, $x \geq l_1$. Also, by definition, $b \leq R_m$. So there are at least $(l_1 + 1 - R_m) + (y + 1 - a)$ simple eigenvalues in T. By Lemma 4.2, $c_k = y + 1$, and, since $c_k \geq a$, we know there are at least $l_1 + 1 - R_m$ simple eigenvalues in T.

We now consider the case when $c_m = a + z$, where $z \in N$. Therefore, there are z Parter eigenvalues from S_l that cannot be matched with non-Parter eigenvalues from S_m. So, by equations (1), (2), and (3), the number of simple eigenvalues in T is at least:

$$\begin{aligned} n_l + n_m - [\sum_{i=1}^{a} m_{F_l}(\lambda_i) - z + \sum_{i=1}^{b} m_{F_m}(\mu_i)] \\ &\leq n_l + n_m - [\sum_{i=1}^{x} m_{F_l}(\lambda_i) - z - (x-a) + \sum_{i=1}^{y} m_{F_m}(\mu_i) - (y-b)] \\ &\leq n_l + n_m - [(n_l - 1) + (n_m - 1) - (x-b) - (y-a-z)] \end{aligned}$$

$$\leq (x+1-b)+(y+1-a+z)$$

$$\leq (l_1+1-R_m)+(c_m-a+z)$$

Since $c_m = a+z$, l_1+1-R_m is again a lower bound for the number of simple eigenvalues in T. Therefore, $U(T) \geq l_1+1-R_m$, and, by the same argument, $U(T) \geq l_1+1-R_m$. $\square$

We note that, since each Parter eigenvalue must be present in at least two branches, $R_l = \sum_{i=2}^{p} l_i$ if S_l is Type 1 and $R_l = \lfloor \frac{\Gamma_l}{2} \rfloor$ if S_l is Type 2. Without loss of generality, the analogous results are true for S_m.

Lemma 4.5 (DS)**.** *Let T be a tree on n vertices with diameter $d(T)$. Then $U(T) \geq 2d(T)-n$.*

Theorem 4.6. *Let T be a double generalized star consisting of two generalized stars S_l and S_m, as described earlier. Then*

1.) If both S_l and S_m are Type 1, and, without loss of generality, $l_1 \leq m_1$,

$$U(T) = \begin{cases} m_1+1-\sum_{i=2}^{p} l_i & \text{if } m_2 \leq l_1+1 \leq \sum_{j=2}^{q} m_j \\ 2d(T)-n & \text{otherwise} \end{cases}$$

2.) If both S_l and S_m are Type 2, and, without loss of generality, $l_l \leq m_1$,

$$U(T) = \begin{cases} m_1+1-\sum_{i=2}^{p} l_i & \text{if } m_2 \leq l_1+1 \leq \sum_{j=2}^{q} m_j \\ 2d(T)-n & \text{otherwise} \end{cases}$$

3.) If S_l is Type 1, S_m is Type 2, and $l_1 < m_1$,

$$U(T) = \begin{cases} m_1+1-\sum_{i=2}^{p} l_i & \text{if } m_2 - \sum_{j=3}^{q} m_j \leq l_1+1 \\ 2d(T)-n & \text{otherwise} \end{cases}$$

4.) If S_l is Type 1, S_m is Type 2, and $l_1 \geq m_1$,

$$U(T)=\begin{cases} l_1+2-\sum_{i=2}^{p} l_i & if\ \lfloor\frac{\Gamma_m}{2}\rfloor+1>\sum_{j=2}^{p} l_j \\ l_1+1-\lfloor\frac{\Gamma_m}{2}\rfloor & if\ \sum_{i=2}^{p} l_i\geq\lfloor\frac{\Gamma_m}{2}\rfloor+1\geq l_2-\sum_{j=3}^{p} l_j \\ 2d(T)-n & otherwise \end{cases}$$

Proof. We consider three main cases.

Case 1: Let S_l and S_m both be Type 1.

By definition, $l_1 > l_2+l_3+...+l_p$ and $m_1 > m_2+m_3+...+m_q$; without loss of generality, let $l_1 \leq m_1$.

Consider $k \in \mathbb{N}$ such that $l_2 \leq k \leq \sum_{i=2}^{p} l_i$. Let $\lambda_1,\ldots,\lambda_k \in \mathbb{R}$. We create a sequence s_k such that

$$s_k = \lambda_1,\lambda_2,\ldots,\lambda_k,\lambda_1,\lambda_2,\ldots,\lambda_k,\lambda_1,\ldots,\lambda_k,\ldots$$

We assign the first l_2 terms of s_k to $\sigma(L_2)$, the next l_3 terms to $\sigma(L_3)$, the next l_4 terms to $\sigma(L_4)$, and so forth. Since $k \geq l_2 \geq l_3 \geq \ldots \geq l_p$, we know that no arm will contain the same eigenvalue twice (which is necessary, as each arm is a path). Also, since $k \leq \sum_{i=2}^{p} l_i$, we know that, for $1 \leq j \leq k$, λ_j is an eigenvalue in at least one arm of S_l. Since $l_1 > \sum_{i=2}^{p} l_i$, we can also place every λ_j in $\sigma(L_1)$. Thus we have obtained exactly k eigenvalues in at least two arms of S_l, and therefore we can obtain any r_l such that $l_2 \leq r_l \leq \sum_{i=2}^{p} l_i$. By our assignment, we also see that every Parter eigenvalue and every non-Parter eigenvalue in $\sigma(S_l)$ is in $\sigma(L_1)$. Thus, by Lemma 4.2, $c_l = l_1+1$ in all such cases.

By the same reasoning, we can obtain any r_m such that $m_2 \leq r_m \leq \sum_{i=2}^{q} m_i$, and, in such cases, $c_m = m_1+1$.

Let $x_m = m_2-(m_3+\ldots+m_q)$. We see that, if $x_m \geq 0$, the maximum number of eigenvalues of multiplicity ≥ 2 is $m_3+\ldots+m_q$. We can make x_m additional Parter eigenvalues, but they must all be simple.

(a) Consider $l_1+1 < x_m$. Note that, in this case, $d(T) = m_1+m_2+1$. We create an assignment as described earlier such that

$$r_l = \sum_{i=2}^{p} l_i \qquad r_m = m_2$$
$$c_l = l_1 + 1 \quad c_m = m_1 + 1$$

We see that, since $m_2 \geq x_m > l_1 + 1$ and $m_1 + 1 > \sum_{i=2}^{p} l_i$, we can assign numerical values such that none of the eigenvalues on S_l are simple. But we are left with $x_m - (l_1 + 1)$ unmatched Parter eigenvalues of multiplicity $= 1$ on S_m. Thus, our total number of simple eigenvalues is

$$\begin{aligned}
&(c_m - r_l) + (x_m - c_l) \\
&\qquad = m_1 + 1 - (l_2 + \ldots + l_p) + m_2 - (m_3 + \ldots + m_q) - (l_1 + 1) \\
&\qquad = 2m_1 + 2m_2 + 2 - (m_1 + m_2 + 1) - (m_3 + \ldots + m_q) - (l_1 + l_2 + \ldots + l_p + 1) \\
&\qquad = 2d(T) - n
\end{aligned}$$

We have achieved a lower bound from Lemma 4.5; therefore, $U(T) = 2d(T) - n$.

(b) Consider $l_1 + 1 > \sum_{i=2}^{q} m_i$. Note that, in this case, $d(T) = l_1 + m_1 + 2$.

We create an assignment as described earlier such that

$$r_l = \sum_{i=2}^{p} l_i \qquad r_m = \sum_{j=2}^{q} m_j$$
$$c_l = l_1 + 1 \quad c_m = m_1 + 1$$

Since $l_1 + 1 > \sum_{j=2}^{q} m_j$ and $m_1 + 1 > l_1 > \sum_{i=2}^{q} l_i$, we can assign numerical values such that neither S_l nor S_m will have unmatched Parter eigenvalues. Thus, our total number of simple eigenvalues is

$$\begin{aligned}
&(c_m - r_l) + (c_l - r_m) \\
&\qquad = [m_1 + 1 - (l_2 + \ldots + l_p)] + [l_1 + 1 - (m_2 + \ldots + m_q)] \\
&\qquad = 2l_1 + 2m_1 + 4 - (l_1 + m_1 + 2) + (l_2 + \ldots + l_p) - (m_2 + \ldots + m_q) \\
&\qquad = 2d(T) - n
\end{aligned}$$

We have thus achieved a lower bound; therefore, $U(T) = 2d(T) - n$.

(c) Consider $x_m \leq l_1 + 1 \leq m_2 + m_3 + \ldots m_q$.

If $l_1+1 \geq m_2$, we can create an assignment as described earlier such that

$$r_l = \sum_{i=2}^{p} l_i \quad r_m = l_1+1$$
$$c_l = l_1+1 \quad c_m = m_1+1$$

In this case, the total number of simple eigenvalues will be

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) \\ &= [m_1+1-\sum_{i=2}^{p} l_i] + [(l_1+1)-(l_1+1)] \\ &= m_1+1-\sum_{i=2}^{p} l_i\end{aligned}$$

If $l_1+1 < m_2$, we can create an assignment as described earlier such that

$$r_l = \sum_{i=2}^{p} l_i \quad r_m = m_2$$
$$c_l = l_1+1 \quad c_m = m_1+1$$

Since $l_1+1 < m_2$, we will not have any unmatched non-Parter eigenvalues from S_l. Also, since $x_m < l_1+1$, we can match the non-Parter eigenvalues from S_l with the Parter eigenvalues of S_m such that there are no unmatched Parter eigenvalues of multiplicity $= 1$ on S_m. Therefore, the total number of simple eigenvalues is

$$c_m - r_l = m_1+1-(l_2+\ldots+l_p),$$

just as when $l_1 \geq m_2$.

This is a lower bound, so $U(T) = m_1+1-\sum_{i=2}^{p} l_i$.

Therefore, in sum,

$$U(T) = \begin{cases} m_1+1-\sum_{i=2}^{p} l_i & \text{if } x_m \leq l_1+1 \leq \sum_{i=2}^{q} m_i \\ 2d(T)-n & \text{otherwise} \end{cases}$$

Case 2: Let S_l and S_m both be Type 2.

By definition, $l_1 \leq l_2+l_3+\ldots+l_p$ and $m_1 \leq m_2+m_3+\ldots+m_q$; without loss of generality, let $l_1 \leq m_1$.

Recall that $\Gamma_l = l_1 + l_2 + \ldots + l_p$ and $\Gamma_m = m_1 + m_2 + \ldots + m_q$. Consider $k \in \mathbb{N}$ such that $l_1 \leq k \leq \lfloor \frac{\Gamma_l}{2} \rfloor$. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. We create a sequence s_k such that

$$s_k = \lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_1, \ldots, \lambda_k, \ldots$$

We assign the first l_1 terms of s_k to $\sigma(L_1)$, the next l_2 terms to $\sigma(L_2)$, the next l_3 terms to $\sigma(L_3)$, and so forth. Since $k \geq l_1 \geq l_2 \geq \ldots \geq l_p$, we know that no arm will contain the same eigenvalue twice (which is necessary, as each arm is a path). Also, since $k \leq \lfloor \frac{\Gamma_l}{2} \rfloor$, we know that, for $1 \leq i \leq k$, λ_i is an eigenvalue in at least two arms of S_l. Thus we see that we can obtain any r_l such that $l_1 \leq r_l \leq \lfloor \frac{\Gamma_l}{2} \rfloor$. By our assignment, we know that $S_l(v_l)$ has no simple eigenvalues, and therefore, in such cases, $c_l = r_l + 1$. Similarly, the same is true for S_m, so we can choose any r_l and r_m such that

$$l_1 \leq r_l \leq \lfloor \frac{\Gamma_l}{2} \rfloor, \;\; m_1 \leq r_m \leq \lfloor \frac{\Gamma_m}{2} \rfloor$$

and, in such cases, $c_l = r_l + 1$ and $c_m = r_m + 1$.

Consider $j \in \mathbb{N}$ such that $l_2 \leq j \leq l_1$. Let $\lambda_1, \ldots, \lambda_j \in \mathbb{R}$. We create a sequence s_j such that

$$s_j = \lambda_1, \lambda_2, \ldots, \lambda_j, \lambda_1, \lambda_2, \ldots, \lambda_j, \lambda_1, \ldots, \lambda_j, \ldots$$

We assign the first l_2 terms of s_j to $\sigma(L_2)$, the next l_3 terms to $\sigma(L_3)$, the next l_4 terms to $\sigma(L_4)$, and so on. Since $j \geq l_2 \geq l_3 \geq \ldots \geq l_p$, we know that no arm will contain the same eigenvalue twice. Also, since $j \leq l_1 \leq l_2 + l_3 + \ldots + l_p$, we know that, for $1 \leq i \leq j$, λ_i is an eigenvalue in at least one of $L_2, \ldots, L_p$. We then assign $\lambda_1, \ldots, \lambda_j$ each exactly once to $\sigma(L_1)$, which we can do because $j \leq l_1$. The rest of the eigenvalues in $\sigma(L_1)$ must be only in that arm, since no space remains in any other arm after our assignment. Thus we have j eigenvalues present in two or more arms and $l_1 - j$ eigenvalues present in only one arm. Therefore, by our lemma, $c_l = j + (l_1 - j) + 1 = l_1 + 1$. Without loss of generality, the same reasoning holds for S_m, so we can choose any r_l and r_m such that

$$l_2 \le r_l \le l_1, \;\; m_2 \le r_m \le m_1$$

and, in such cases, $c_l = l_1 + 1$ and $c_m = m_1 + 1$.

(a) Consider $m_1 \le \lfloor \frac{\Gamma_l}{2} \rfloor + 1$. Thus we can choose r_l such that $r_l = m_1$ or $r_l = m_1 - 1$. We choose $r_m = m_1$, and thus $c_m = m_1 + 1$.

If $r_l = m_1$, then $c_l = m_1 + 1$, so

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) = \;& [(m_1+1) - m_1] + [(m_1+1) - m_1] \\ = \;& 2\end{aligned}$$

If $r_l = m_1 - 1$, then $c_l = m_1$, so

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) = \;& [(m_1+1) - (m_1 - 1)] + [m_1 - m_1] \\ = \;& 2\end{aligned}$$

Since $U(T)$ is always ≥ 2, we know that $U(T) = 2$.

(b) Consider $m_1 > \lfloor \frac{\Gamma_l}{2} \rfloor + 1$ and $x_m \le \lfloor \frac{\Gamma_l}{2} \rfloor + 1$.

If $\lfloor \frac{\Gamma_l}{2} \rfloor + 1 \ge m_2$, we can create an assignment as described earlier such that

$$\begin{aligned}r_l &= \lfloor \frac{\Gamma_l}{2} \rfloor & r_m &= \lfloor \frac{\Gamma_l}{2} \rfloor + 1 \\ c_l &= \lfloor \frac{\Gamma_l}{2} \rfloor + 1 & c_m &= m_1 + 1\end{aligned}$$

In this case, the total number of simple eigenvalues will be

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) \;\; &= [(m_1+1) - \lfloor \frac{\Gamma_l}{2} \rfloor] + [(\lfloor \frac{\Gamma_l}{2} \rfloor + 1) - (\lfloor \frac{\Gamma_l}{2} \rfloor + 1)] \\ &= m_1 + 1 - \lfloor \frac{\Gamma_l}{2} \rfloor\end{aligned}$$

This is the lower bound in Lemma 4.4, so $U(T) = m_1 + 1 - \lfloor \frac{\Gamma_l}{2} \rfloor$.

If $\lfloor \frac{\Gamma_l}{2} \rfloor < m_2$, we can create an assignment such that

$$\begin{aligned}r_l &= \lfloor \frac{\Gamma_l}{2} \rfloor & r_m &= m_2 \\ c_l &= \lfloor \frac{\Gamma_l}{2} \rfloor + 1 & c_m &= m_1 + 1\end{aligned}$$

Since $x_m \leq \lfloor \frac{\Gamma_l}{2} \rfloor + 1$, we can match the non-Parter eigenvalues from S_l with the Parter eigenvalues of S_m such that no Parter eigenvalues of multiplicity $= 1$ from S_m are left unmatched. But, since $c_l < r_m$, we will not have any unmatched non-Parter eigenvalues from S_l, either. Thus, our total number of simple eigenvalues will be

$$(c_m - r_l) \quad = (m_1 + 1) - \lfloor \frac{\Gamma_l}{2} \rfloor$$

So, again, by Lemma 4.4, $U(T) = m_1 + 1 - \lfloor \frac{\Gamma_l}{2} \rfloor$.

(c) Consider $m_1 > \lfloor \frac{\Gamma_l}{2} \rfloor$ and $x_m > \lfloor \frac{\Gamma_l}{2} \rfloor$. Thus we note that $m_2 \geq x_m > \lfloor \frac{\Gamma_l}{2} \rfloor \geq l_1$, and so $d(T) = m_1 + m_2 + 1$.

We choose an assignment such that

$$\begin{array}{ll} r_l = \lfloor \frac{\Gamma_l}{2} \rfloor & r_m = m_2 \\ c_l = \lfloor \frac{\Gamma_l}{2} \rfloor + 1 & c_m = m_1 + 1 \end{array}$$

We see that, even after we have matched all non-Parter eigenvalues from S_l with Parter eigenvalues of multiplicity $= 1$ from S_m, we will still have Parter eigenvalues of multiplicity $= 1$ remaining. Thus the total number of simple eigenvalues will be

$$\begin{aligned} (c_m - r_l) + (x_m - c_l) &= [(m_1 + 1) - \lfloor \frac{\Gamma_l}{2} \rfloor] + [(m_2 - \sum_{i=3}^{q} m_i) - (\lfloor \frac{\Gamma_l}{2} \rfloor + 1)] \\ &= (m_1 + m_2 + 1) - [(2\lfloor \frac{\Gamma_l}{2} \rfloor + 1) + (\sum_{i=3}^{q} m_i)] \\ &= 2(m_1 + m_2 + 1) - [(2\lfloor \frac{\Gamma_l}{2} \rfloor + 1) + (\Gamma_m + 1)] \end{aligned}$$

If Γ_l is even, $2\lfloor \frac{\Gamma_l}{2} \rfloor = \Gamma_l$, so we have $2d(T) - n$ eigenvalues of multiplicity $= 1$. If Γ_l is odd, we return to our assignment. To make $r_l = \lfloor \frac{\Gamma_l}{2} \rfloor$ and $c_l = \lfloor \frac{\Gamma_l}{2} \rfloor + 1$, we created exactly one eigenvalue of multiplicity $= 3$ in $S_l(l)$ so as not to have any eigenvalues of multiplicity $= 1$. If we instead make all Parter eigenvalues appear in only two arms, we will still have $\lfloor \frac{\Gamma_l}{2} \rfloor$ Parter eigenvalues, but we will now have an additional simple eigenvalue in $S_l(l)$. So

$$\begin{aligned} c_l &= |P| + |N| + 1 \\ &= \lfloor \frac{\Gamma_l}{2} \rfloor + 1 + 1 \\ &= \lfloor \frac{\Gamma_l}{2} \rfloor + 2 \end{aligned}$$

From our previous calculation where $c_l = \lfloor \frac{\Gamma_l}{2} \rfloor + 1$ and Γ_l is odd, $(c_m - r_l) + (x_m - c_l) = 2d(T) - n + 1$. But, with our new assignment, r_l, x_m, and c_m have remained constant while c_l increased by one. Thus we now have $2d(T) - n$ simple eigenvalues, just as when Γ_l is even.

This is a lower bound by Lemma 4.5, so $U(T) = 2d(T) - n$.

So, in sum,

$$U(T) = \begin{cases} 2 & \text{if } m_1 \leq \lfloor \frac{\Gamma_l}{2} \rfloor + 1 \\ m_1 + 1 - \lfloor \frac{\Gamma_l}{2} \rfloor & \text{if } m_1 > \lfloor \frac{\Gamma_l}{2} \rfloor \text{ and } m_2 - \sum_{i=3}^{q} m_i > \lfloor \frac{\Gamma_l}{2} \rfloor + 1 \\ 2d(T) - n & \text{otherwise} \end{cases}$$

Case 3: Let one of S_l and S_m be Type 1, and let the other be Type 2. Without loss of generality, let S_l be Type 1.

(a) Consider $m_1 > l_1$. Since S_l is Type 1, $m_1 > l_1 > l_2 + l_3 + \ldots + l_p$.

1.) Let $x_m \leq l_1 + 1$ and $m_2 \leq l_1 + 1$. Then we can choose an assignment such that

$$r_l = \sum_{i=2}^{p} l_i \qquad r_m = l_1 + 1$$

$$c_l = l_1 + 1 \quad c_m = m_1 + 1$$

In this case, the total number of simple eigenvalues can be

$$\begin{aligned} (c_m - r_l) + (c_l - r_m) &= [(m_1 + 1) - \sum_{i=2}^{p} l_i] + [(l_1 + 1) - (l_1 + 1)] \\ &= m_1 + 1 - \sum_{i=2}^{p} l_i \end{aligned}$$

This is a lower bound by Lemma 4.4, so $U(T) = m_1 + 1 - \sum_{i=2}^{p} l_i$.

2.) Let $x_m \leq l_1 + 1$ and $m_2 > l_1 + 1$. Then we will not have any unmatched non-Parter eigenvalues from S_l nor any unmatched simple eigenvalues from S_m. So, by creating an assignment where $r_l = \sum_{i=2}^{p} l_i$ and $c_m = l_1 + 1$, we can obtain a total number of simple eigenvalues of

$$c_m - r_l \quad = m_1 + 1 - \sum_{i=2}^{p} l_i$$

This is a lower bound by Lemma 4.4, so $U(T) = m_1 + 1 - \sum_{i=2}^{p} l_i$.

3.) Let $x_m > l_1 + 1$. We note that, in this case, $d(T) = m_1 + m_2 + 1$. Then we can create an assignment such that

$$r_l = \sum_{i=2}^{p} l_i \qquad r_m = m_2$$

$$c_l = l_1 + 1 \quad c_m = m_1 + 1$$

We see that we will have exactly $x_m - c_l$ unmatched Parter eigenvalues of multiplicity $= 1$ from S_m, so the total number of simple eigenvalues is

$$\begin{aligned}(c_m - r_l) + (x_m - c_l) \quad &= [(m_1+1) - (l_2 + l_3 + \ldots + l_p)] + [(m_2 - (m_3 + m_4 + \ldots + m_q) - (l_1 + 1)] \\ &= (m_1 + m_2 + 1) - (\sum_{i=1}^{p} l_i + 1) - (\sum_{j=3}^{q} m_j) \\ &= 2(m_1 + m_2 + 1) - [(\sum_{i=1}^{p} l_i + 1) + (\sum_{j=1}^{q} m_j + 1)] \\ &= 2d(T) - n\end{aligned}$$

This is a lower bound by Lemma 4.5, so $U(T) = 2d(T) - n(T)$.

(b) Consider $m_1 \leq l_1$.

1.) Let $\lfloor \frac{\Gamma_m}{2} \rfloor + 1 > \sum_{i=2}^{p} l_i$. If $\lfloor \frac{\Gamma_m}{2} \rfloor > l_1 (\geq m_1)$, then we can make an assignment such that

$$r_l = \sum_{i=2}^{p} l_i \qquad r_m = l_1 + 1$$

$$c_l = l_1 + 1 \quad c_m = l_1 + 2$$

Thus we can make the total number of simple eigenvalues be

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) \quad &= [(l_1 + 2) - \sum_{i=2}^{p} l_i] + [(l_1 + 1) - (l_1 + 1)] \\ &= l_1 + 2 - \sum_{i=2}^{p} l_i\end{aligned}$$

If $\lfloor\frac{\Gamma_m}{2}\rfloor + 1 > \sum_{i=2}^{p} l_i$. If $\lfloor\frac{\Gamma_m}{2}\rfloor \leq l_1$, then we can make an assignment such that

$$r_l = \sum_{i=2}^{p} l_i \quad r_m = \lfloor\frac{\Gamma_m}{2}\rfloor$$
$$c_l = l_1 + 1 \quad c_m = \lfloor\frac{\Gamma_m}{2}\rfloor + 1$$

Thus we can make the number of eigenvalues of multiplicity $= 1$ be

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) &= [(\lfloor\frac{\Gamma_m}{2}\rfloor + 1) - \sum_{i=2}^{p} l_i] + [(l_1 + 1) - \lfloor\frac{\Gamma_m}{2}\rfloor] \\ &= l_1 + 2 - \sum_{i=2}^{p} l_i\end{aligned}$$

So it is left to show that $l_1 + 2 - \sum_{i=2}^{p} l_i$ is a lower bound.

From Corollary 4.3, $c_l \geq l_1 + 1$. As we have established, if λ is one of the non-Parter eigenvalues, we will have a simple eigenvalue in the double generalized star unless λ is a parter eigenvalue for S_m. So we will have at least $l_1 + 1 - r_m$ eigenvalues of multiplicity $= 1$. Since $c_m = |P| + |N| + 1$, there are at least $|P| + 1 = r_m + 1$ non-Parter eigenvalues on S_m. Just as with the non-Parter eigenvalues on S_l, these eigenvalues will be simple unless they are matched with Parter eigenvalues from S_l, so we have $r_m + 1 - r_l$ simple eigenvalues. So we find that

$$U(T) \geq (l_1 + 1 - r_m) + (r_m + 1 - r_l) = l_1 + 2 - r_l$$

Since $r_l \leq \sum_{i=2}^{p} l_i$, we can say that $U(T) \geq l_1 + 2 - \sum_{i=2}^{p} l_i$.

2.) Let $\sum_{i=2}^{p} l_i \geq \lfloor\frac{\Gamma_m}{2}\rfloor + 1 \geq l_2 - \sum_{j=3}^{p} l_j (= x_l)$. If $\lfloor\frac{\Gamma_m}{2}\rfloor + 1 \geq l_2$, then we can create an assignment such that

$$r_l = \lfloor\frac{\Gamma_m}{2}\rfloor + 1 \quad r_m = \lfloor\frac{\Gamma_m}{2}\rfloor$$
$$c_l = l_1 + 1 \qquad c_m = \lfloor\frac{\Gamma_m}{2}\rfloor + 1$$

Thus we can make the number of simple eigenvalues be

$$\begin{aligned}(c_m - r_l) + (c_l - r_m) &= [(\lfloor\frac{\Gamma_m}{2}\rfloor + 1) - (\lfloor\frac{\Gamma_m}{2}\rfloor + 1)] + [(l_1 + 1) - \lfloor\frac{\Gamma_m}{2}\rfloor] \\ &= l_1 + 1 - \lfloor\frac{\Gamma_m}{2}\rfloor\end{aligned}$$

This is a lower bound, so $U(T) = l_1 + 1 - \lfloor \frac{\Gamma_m}{2} \rfloor$.

If $\lfloor \frac{\Gamma_m}{2} \rfloor + 1 < m_2$, we create an assignment such that

$$\begin{array}{ll} r_l = l_2 & r_m = \lfloor \frac{\Gamma_m}{2} \rfloor \\ c_l = l_1 + 1 & c_m = \lfloor \frac{\Gamma_m}{2} \rfloor + 1 \end{array}$$

Since $x_l \leq \lfloor \frac{\Gamma_m}{2} \rfloor + 1$ we can match the Parter eigenvalues from S_l and the non-Parter eigenvalues from S_m such that no Parter eigenvalues of multiplicity $= 1$ are left unmatched. Thus we can make the number of simple eigenvalues be

$$c_l - r_m = l_1 + 1 - \lfloor \frac{\Gamma_m}{2} \rfloor$$

This is a lower bound, so $U(T) = l_1 + 1 - \lfloor \frac{\Gamma_m}{2} \rfloor$.

(c) Let $\lfloor \frac{\Gamma_m}{2} \rfloor + 1 < x_l$. We create an assignment such that

$$\begin{array}{ll} r_l = l_2 & r_m = \lfloor \frac{\Gamma_m}{2} \rfloor \\ c_l = l_1 + 1 & c_m = \lfloor \frac{\Gamma_m}{2} \rfloor + 1 \end{array}$$

We know that we will have $x_l - c_m$ unmatched simple eigenvalues from S_l, so we can make the number of simple eigenvalues be

$$\begin{aligned} (c_l - r_m) + (x_l - c_m) &= [(l_1 + 1) - (\lfloor \frac{\Gamma_m}{2} \rfloor)] + [(l_2 - \Sigma_{i=3}^{p} l_i) - (\lfloor \frac{\Gamma_m}{2} \rfloor + 1)] \\ &= (l_1 + l_2 + 1) - (2\lfloor \frac{\Gamma_m}{2} \rfloor + 1) - (\Sigma_{i=3}^{p} l_i) \\ &= 2(l_1 + 1_2 + 1) - [(2\lfloor \frac{\Gamma_m}{2} \rfloor + 1) - (\Sigma_{i=i}^{p} l_i + 1)] \end{aligned}$$

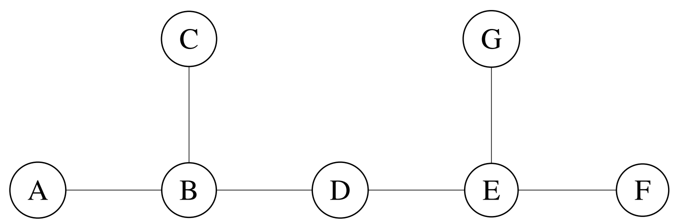
If Γ_m is even, this expression equals $2d(T) - n$, which is a lower bound. If Γ_m is odd, we see that this expression equals $2d(T) - n + 1$. But if we return to our assignment, we see that, using the same strategy as in Case 2(c), we can increase c_m to $\lfloor \frac{\Gamma_m}{2} \rfloor + 2$ without changing r_m, x_l, or c_l. This results in a loss of exactly one eigenvalue of multiplicity

$= 1$, so we can obtain $2d(T)-n$ simple eigenvalues. This is a lower bound, so $U(T) = 2d(T)-n$. □

Thus we have obtained an explicit formula for $U(T)$ when T is a double generalized star. This case is much more complicated than when T is a single generalized star, and we came to see that the complexity lies mainly in the number of high-degree vertices. Therefore, we do not continue to find $U(T)$ for the triple generalized star, but rather turn our attention to a different area.

5 Edge Subdivision

We recall that the *path cover number*, denoted $p(T)$, is the minimum number of non-intersecting paths that cover all vertices on the tree. For example, if

$T =$ 

,

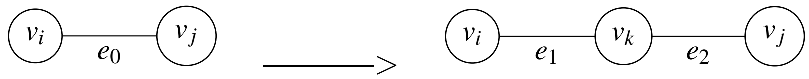
then $p(T) = 3$. A tree can have multiple *minimum path covers*, or sets of cardinality $p(T)$ containing paths that cover all vertices of T; for example, $\{ABC, DEG, F\}$ and $\{ABDEF, C, G\}$ are both minimum path covers of T.

We use $p(T)$ in this powerful result:

Theorem 5.1 (JL1)**.** *Let H be a Hermitian matrix whose graph is a tree T. Then, for any $\lambda \in \sigma(H)$, $m_H(\lambda) \leq p(T)$. Moreover, there exists a matrix $\hat{H}$ whose graph is T such that, for some $\mu \in \sigma(\hat{H})$, $m_{\hat{H}}(\mu) = p(T)$.*

Because of its significance, we want to increase our understanding of the path cover number. In this section, we explore *edge subdivision*, which we define as such:
Let T be a tree with adjacent vertices v_i and v_j connected by an edge e_0. We remove e_0

from T and insert a new vertex v_k, which is connected to v_i by a new edge e_1 and connected to v_j by a new edge e_2.



Thus we have subdivided e_0 with v_k.

We will explore how certain edge subdivisions affect the path cover number. To do this, we will use an alternate definition of $p(T)$ and the lemmas listed below.

Definition 5.2. *Let T be a tree on n vertices. We remove q vertices from T until only paths remain; we denote the number of paths remaining as b. Then $p(T) = max[b-q]$.*

Lemma 5.3. *Let T be a tree and e be an edge in T. Let T' be the tree formed when we subdivide e. Then $p(T') \geq p(T)$.*

Proof. Let $p(T) = b - q$, where q is the number of vertices removed from T to obtain b paths. We denote the vertices removed to obtain this maximum as $v_1, \ldots, v_q$ and the paths created as $p_1, \ldots, p_b$. We subdivide any edge e in T with v_k, then remove $v_1, \ldots, v_q$.

If v_q is a disconnected, single vertex in $T(v_1, \ldots, v_q)$, then there are now $b+1$ paths remaining, so $p(T') \geq b+1-q > p(T)$, and we are done. We thus assume that v_k is connected to some path p_k such that $p_k \in \{p_1, \ldots, p_b\}$; we denote the graph $p_k + v_k$ as p'_k.

If p'_k is a path, we are done, as $p(T') \geq b - q = p(T)$. If p'_k is not a path, $p(p'_k) > 1$, so we can remove q_k additional vertices from it to create b_k additional components (all of which are paths), where $b_k - q_k \geq 1$. So $p(T') \geq (b-q) + (b_k - q_k) \geq p(T) + 1$. □

Lemma 5.4. *Let T be a tree and e be an edge in T. Then subdividing e will not increase the path cover number if and only if e lies in a path that is part of some minimum path cover of T.*

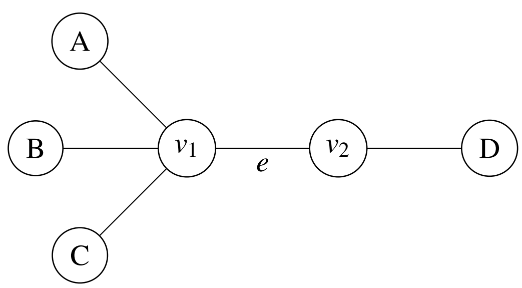
Proof. ($\Rightarrow$): We use a proof by contrapositive. Let e in T be such that e does not lie in a path that is part of any minimum path cover of T, and let $p(T) = l$. When we subdivide e

with v_k to create T', we essentially place v_k directly in the middle of e. So, if there is no set of l non-intersecting paths such that those paths cover all the vertices in T and e, there is no set of l non-intersecting paths that will cover all the vertices in T and v_k (which is, in short, T'). Thus $p(T') > l = p(T)$.

($\Leftarrow$): Let e lie in some path in C, where $C = \{p_1, p_2, \ldots, p_l\}$ is a minimum path cover of T. Without loss of generality, let e lie in p_1. We subdivide e with v_k to create T'. We denote the path that includes only p_1 and v_k as q_1. We see that $C' = \{q_1, p_2, \ldots, p_l\}$ is a path cover of T'; since $|C'| = |C|$, we are done. □

Theorem 5.5. *Let T be a tree with an edge e such that e is incident to a vertex of degree 2. Let T' be the tree created by subdividing e. Then $p(T') = p(T)$.*

Proof. Let e be an edge in T such that e is incident to v_1 and v_2 and $\deg(v_2) = 2$. Then, if we focus on v_1 and v_2, the graph looks something like this:



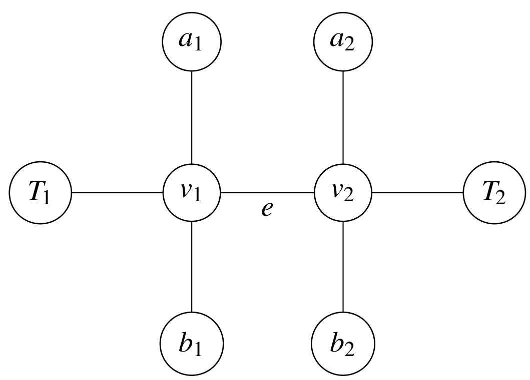
Let $C = \{p_1, p_2, \ldots, p_l\}$ be a minimum path cover of T; thus $p(T) = l$. By definition, v_2 must be present in a path in C; without loss of generality, let v_2 be present in p_1. If v_1 is also in p_1, then e is in p_1, and we are done.

Therefore we assume that v_1 is not in p_1. Since $\deg(v_2) = 2$, p_1 must terminate in v_2. We subdivide e with v_k, creating a new tree T'. Since p_1 terminates in v_2, we can append v_k to p_1 and still have a path; we denote this path as q_1. So $C' = \{q_1, p_2, \ldots, p_l\}$ is a path cover of T' and $|C'| = |C|$. Since edge subdivision can never decrease the path cover number (Lemma 5.3), C' is a minimum path cover of T', and $p(T') = p(T)$. □

The above theorem applies to all trees; next, we prove a result that applies specifically to *diametric* trees. A tree T is diametric if and only if there exists a diameter of T such that all high-degree vertices lie on that diameter (note that this does not preclude the existence of other diameters that do not meet this condition).

Theorem 5.6. *Let T be a diametric tree with an edge e such that e is incident to two vertices whose degree is at least* 4*. Then subdividing e will always increase the path cover number.*

Proof. Let T be such that there are adjacent vertices v_1 and v_2 in T such that $\deg(v_1) \geq 4$ and $\deg(v_2) \geq 4$. We denote the edge that connects v_1 and v_2 as e. The graph of T looks something like this:



The branches T_1 and T_2 represent the rest of the tree and contain all high-degree vertices in T other than v_1 and v_2. The branches a_1, a_2, b_1, and b_2 are paths attached to v_1 and v_2; there may be many more of these branches, but, for our purposes, we only need at least 2 for each. We now explore the possible path covers for C.

We calculate $p(T_1)$ and $p(T_2)$ as if they were independent trees. We also note that there are exactly $\deg(v_i) - 2$ pendant paths connected to v_i for $i \in \{1, 2\}$. Let C_1 be a path cover such that e lies in some path $p_k \in C_1$. We see that p_k will contain vertices from neither T_1 nor T_2, one of T_1 and T_2, or both of T_1 and T_2; we address these as separate cases.

Case 1: Let p_k contain no vertices from T_1 or T_2. Since both v_1 and v_2 lie in p_k, at most one pendant path for each vertex lies in p_k. We will thus need at least [deg$(v_1)-3$] + [deg$(v_2)-3$] additional paths to cover these pendant paths. We also need at least $p(T_1)+p(T_2)$ paths to cover both T_1 and T_2, so

$$\begin{aligned}|C_1| &\geq [deg(v_1)-3]+[deg(v_2)-3]+p(T_1)+p(T_2)+1\\ &\geq deg(v_1)+deg(v_2)+p(T_1)+p(T_2)-5\end{aligned}$$

Case 2: Let p_k contain vertices from both T_1 and T_2. Since both v_1 and v_2 lie in p_k, no pendant path for either vertex lies in p_k. We will thus need at least [deg$(v_1)-2$] + [deg$(v_2)-2$] additional paths to cover those pendant paths. We will also need at least $p(T_1)-1$ paths that only contain vertices from T_1 and $p(T_2)-1$ paths that only contain vertices from T_2 to cover the vertices in those branches not covered by p_k. Therefore,

$$\begin{aligned}|C_1| &\geq [deg(v_1)-2]+[deg(v_2)-2]+[p(T_1)-1]+[p(T_2)-1]+1\\ &\geq deg(v_1)+deg(v_2)+p(T_1)+p(T_2)-5\end{aligned}$$

Case 3: Let p_k contain vertices from exactly one of T_1 and T_2. Since both v_1 and v_2 lie in p_k, at most one pendant path for at most one of v_1 and v_2 lies in p_k. We will thus need at least [deg$(v_1)-2$] + [deg$(v_2)-2$] -1 additional paths to cover those pendant paths. We will also need $p(T_1)+p(T_2)-1$ paths that only contain vertices from T_1 or T_2 to cover the vertices in those branches not covered by p_k. Therefore,

$$\begin{aligned}|C_1| &\geq [deg(v_1)-2]+[deg(v_2)-2]-1+[p(T_1)+p(T_2)-1]+1\\ &\geq deg(v_1)+deg(v_2)+p(T_1)+p(T_2)-5\end{aligned}$$

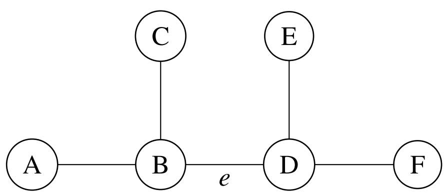
So, if e lies in a path of a path cover C_1, then we know that $|C_1| \geq deg(v_1)+deg(v_2)+p(T_1)+p(T_2)-5$.

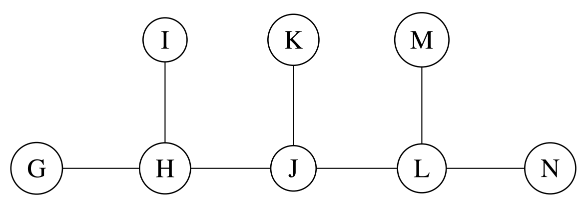
We consider a specific path cover C_2 of T that does not include a path that contains e. First, we let the $p(T_1)$ paths that independently cover T_1 and the $p(T_2)$ paths that independently cover T_2 be in C_2. Now we need only assign paths to cover v_1, v_2, and the pendant paths of each. Let p_1 cover a_1, v_1, and b_1, and let p_2 cover a_2, v_2, and b_2. We can cover the remaining pendant paths with [deg$(v_1) - 4$] + [deg$(v_2) - 4$] additional paths. Therefore,

$$\begin{aligned}|C_2| &= [deg(v_1) - 4] + [deg(v_2) - 4] - 1 + p(T_1) + p(T_2) + 2 \\ &= deg(v_1) + deg(v_2) + p(T_1) + p(T_2) - 6\end{aligned}$$

Thus $|C_2| < |C_1|$, where C_1 is any path cover of T that includes e. So e cannot lie in a path that is part of any minimum path cover of T, and therefore, by Lemma 5.4, we cannot subdivide e without increasing the path cover number. □

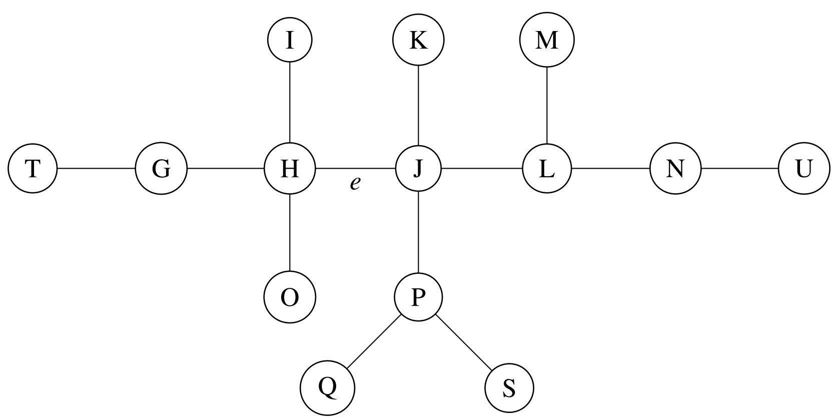
Unfortunately, other than these two results, we have been unable to prove anything about edge subdivision and $p(T)$ using only local characteristics of T. For example, vertices of degree 3, even in the diametric case, pose a major problem:



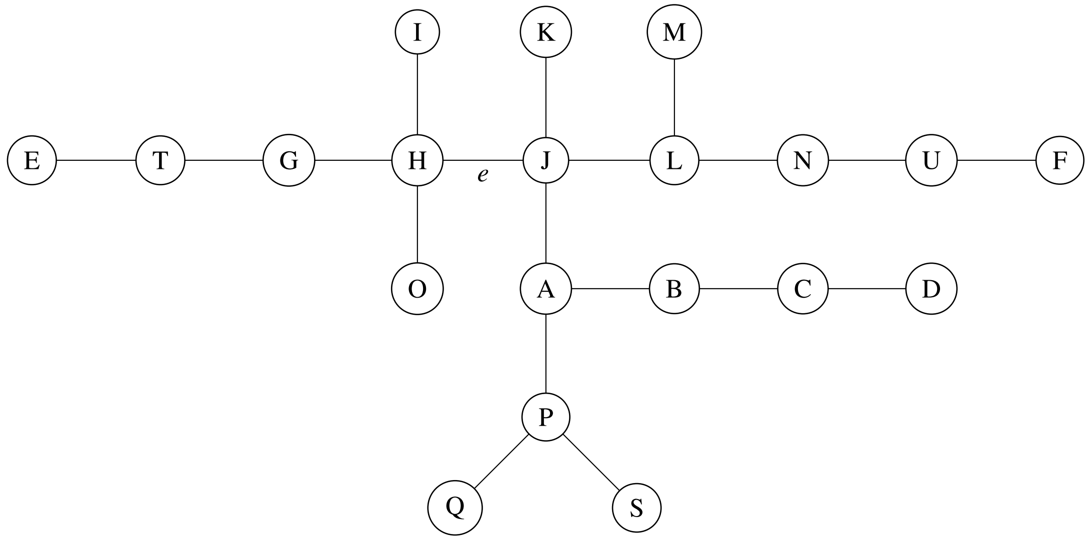


For the graph on the left, we cannot subdivide e without increasing the path cover number. But we can subdivide any edge, including those between vertices of degree 3, in the graph on the right without increasing its path cover number.

We also run into problems in non-diametric cases. For example, consider

$T =$ 

Here, $p(T) = 5$, so $C = \{TG, O, IHJK, MLNU, QPS\}$ is a minimum path cover. We see that e lies in $IHJK$, and thus can be subdivided without increasing the path cover number, despite lying between two vertices of degree 4. We can still further muddle the issue by examining this non-diametric tree, denoted, T':



Here, $p(T') = 5$; $C = \{FUNLM, KJABCD, QPS, IHO, ETG\}$ is a minimum path cover. But, try as we may, we cannot find a minimum path cover that covers e, so we cannot subdivide e without increasing the path cover number.

Because of these findings, it appears that the effects of edge subdivision on $p(T)$ in general require global information about T.

6 Parter, Downer, and Neutral Vertices

Consider a Hermitian matrix H whose graph is a tree T. Again, we blur the distinction between the graph and the matrix and consider $\sigma(T) = \sigma(H)$. Let v_i be a vertex in T and $\lambda \in \mathbb{R}$. Recall that v_i is a *Parter vertex* if $m_{T(i)}(\lambda) = m_T(\lambda) + 1$, a *neutral vertex* if $m_{T(i)}(\lambda) = m_T(\lambda)$, or a *downer vertex* if $m_{T(i)}(\lambda) = m_T(\lambda) - 1$. By interlacing, the multiplicity of λ can increase or decrease by at most 1, so all vertices can be classified as parter, neutral, or downer.

As we've seen in our discussions about $U(T)$, recognizing which vertices are Parter is very important. One useful way of doing this involves downer vertices. We remove a vertex v_j from T, creating a forest. We look at the vertices in this forest that were adjacent to v_j; if a neighbor v_i of v_j is a downer vertex in $T(j)$, we call the tree in $T(j)$ that contains v_i a *downer branch* of T at v_j. Identifying downer branches helps us recognize Parter vertices through the following lemma:

Lemma 6.1 (JLS1). *For a tree T, a vertex v_j in T is a Parter vertex for λ if and only if there is a downer branch at v_j for λ*

If there is exactly one downer branch at v_j, we call v_j *singly Parter*. If v_j has more than one downer branch, we call it *multiply Parter*.

Consider a tree T with vertices v_1 and v_2. We classify each of v_1 and v_2 as Parter, downer, or neutral. The table below details what the classification of v_1 can become when we remove v_2, given the original classifications of each. We consider the case when v_1 and v_2 are adjacent, and also the case when they are non-adjacent.

	v_1	v_2	$\Delta(v_1)$ when v_2 is removed	Possible? (Adjacent)	Possible? (Non-Adjacent)
1.	P	P	P	Yes	Yes
2.	P	P	N	No	No
3.	P	P	D	Yes*	Yes*
4.	P	N	P	Yes	Yes
5.	P	N	N	Yes*	Yes*
6.	P	N	D	No	No
7.	P	D	P	Yes**	Yes
8.	P	D	N	No	No
9.	P	D	D	No	No
10.	N	P	P	No	No
11.	N	P	N	Yes	Yes
12.	N	P	D	Yes*	Yes*
13.	N	N	P	No	Yes*
14.	N	N	N	Yes	Yes
15.	N	N	D	No	No
16.	N	D	P	No	No
17.	N	D	N	No	Yes
18.	N	D	D	No	No
19.	D	P	P	No	No
20.	D	P	N	No	No
21.	D	P	D	Yes**	Yes
22.	D	N	P	No	No
23.	D	N	N	No	No
24.	D	N	D	No	Yes
25.	D	D	P	No	Yes*
26.	D	D	N	Yes	Yes
27.	D	D	D	No	Yes

*Only occurs if all Parter vertices are singly Parter.
**Only occurs if the Parter vertex is multiply Parter in T.

We use a variety of arguments to reach these results, some of which apply to multiple cases. We detail the arguments used more frequently below:

Argument 1: The multiplicity of λ as an eigenvalue in $T(1,2)$ is constant; thus, if we remove v_1 and then v_2, the multiplicity of λ in the resulting subgraph should be the same as if we removed v_2 and then v_1. This rules out quite a few of the combinations in the the table. For example, consider line 6, where v_1 is Parter, v_2 is neutral, and $\Delta(v_1)$ is downer. Then, if $m_T(\lambda) = k$, the multiplicity of λ in $T(1,2)$ is

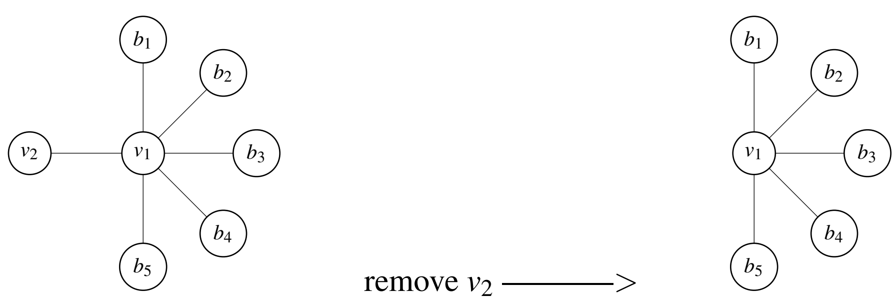
$$k \xrightarrow[\text{remove } v_2]{} k \xrightarrow[\text{remove } v_1]{} k-1$$

$$k \xrightarrow[\text{remove } v_1]{} k+1 \xrightarrow[\text{remove } v_2]{} k+2,\ k+1, \text{ or } k$$

We thus obtain different values for $m_{T(1,2)}(\lambda)$ depending on the order in which we remove v_1 and v_2, which is a contradiction.

Argument 2: We remember from Lemma 6.1 that a vertex v_i is Parter if and only if there is a downer branch at v_i. Consider line 15 in the adjacent case, where v_1 and v_2 are both neutral, and $\Delta(v_1)$ is downer. So when we remove v_2 from T, an adjacent vertex becomes downer, meaning that there is a downer branch at v_2. Thus v_2 must have been originally Parter, which is a contradiction.

Argument 3: This is similar to Argument 2, and again uses Lemma 6.1. Consider line 13 in the adjacent case, where v_1 and v_2 are neutral, and $\Delta(v_1)$ is Parter. Thus, since neither v_1 nor v_2 is originally parter, there cannot be a downer branch at either vertex. If we focus on v_1, the graph looks something like this:



There may be more or fewer branches connected to v_1, but, for convenience's sake, we consider five branches other than the one containing v_2, denoted $b_1,\ldots,b_5$. For v_1 to become Parter with the removal of v_2, one of $b_1,\ldots,b_5$ must become a downer branch. But, since v_1 is originally neutral, if we remove it from the graph on the left, none of the disconnected branches $b_1,\ldots,b_5$ is a downer branch. When we remove v_1 from the graph on the right, the same disconnected branches $b_1,\ldots,b_5$ remain, none of which can be downer. Thus we have a contradiction.

Argument 4: Here we show that some cases are identical to each other. Consider line 17, where v_1 is neutral, v_2 is downer, and $\Delta(v_1)$ is neutral. So, if $m_T(\lambda) = k$, the multiplicity of λ in $T(1,2)$ is

$$k \xrightarrow[\text{remove } v_2]{} k-1 \xrightarrow[\text{remove } v_1]{} k-1$$

Since v_1 is originally neutral, v_2 must stay downer with the removal of v_1 for $m_{T(1,2)} = k-1$. Thus, this case is equivalent to the case in line 24, where v_1 is downer, v_2 is neutral, and $\Delta(v_1)$ is downer (simply switch the labels on v_1 and v_2).

Argument 5: Let v_i be a multiply Parter vertex for λ in T, and let v_j be any other vertex in T. By definition, there are multiple downer branches at v_i, so v_i will still have at least one downer branch in $T(j)$. Thus, by Lemma 6.1, v_i is still Parter for λ in $T(j)$. Similarly, if v_i is singly Parter but v_j is not in the downer branch at v_i, v_i will be Parter in $T(j)$.

Therefore, if $m_T(\lambda) = k$ and $m_{T(j)}(\lambda) = k + t_0$ where $t_0 \in \{-1, 0, 1\}$, the multiplicity of λ in $T(i, j)$ will be

$$k \xrightarrow[\text{remove } v_j]{} k+t_0 \xrightarrow[\text{remove } v_i]{} k+1+t_0$$

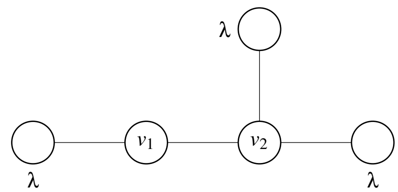
If we assume that the classification of v_j changes in $T(i)$, then the removal of v_j will change the multiplicity of λ by t_1, where $t_1 \in \{-1, 0, 1\}$ and $t_1 \neq t_0$. So $m_{T(i,j)}(\lambda)$ would equal

$$k \xrightarrow[\text{remove } v_i]{} k+1 \xrightarrow[\text{remove } v_j]{} k+1+t_1$$

This is a contradiction, so v_j must retain the same classification in $T(i)$ when v_i is a multiply Parter vertex in T.

Proofs of Adjacent Cases:

For each case, we give an example of a tree for which that case occurs, or we prove that such a tree does not exist. In the examples, if a vertex v_i is labeled with a value a, it means that $\sigma(T[i]) = \{a\}$; additionally, if v_i is not labeled with a, we assume $\sigma(T[i]) \neq \{a\}$. The classifications of each vertex are also with respect to the eigenvalue λ. We will refer often to the four arguments above.

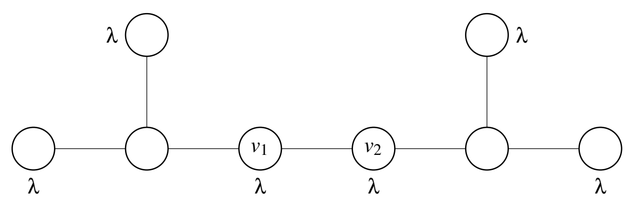
1. 

2. By Lemma 6.1, v_1 is Parter if and only if there is a downer branch at v_1. When we delete v_2, v_1 becomes neutral, so the downer branch at v_1 must have been the branch including v_2, and thus v_2 becomes a downer vertex when you delete v_1. But now we have a discrepancy in $m_{T(1,2)}(\lambda)$:

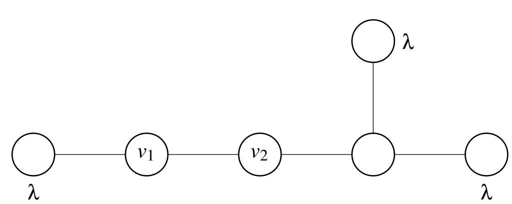
$$k \xrightarrow[\text{remove } v_2]{} k+1 \xrightarrow[\text{remove } v_1]{} k+1$$

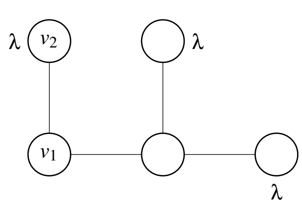
$$k \xrightarrow[\text{remove } v_1]{} k+1 \xrightarrow[\text{remove } v_2]{} k$$

This is impossible, so we have a contradiction.

3. 

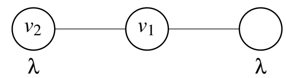
By Argument 5, both v_1 and v_2 must be singly Parter.

4. 

5. 

By Argument 5, v_1 must be singly Parter.

6. See Argument 1

7. 

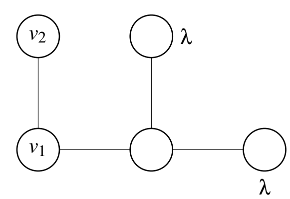
From entries 18, 19, and 20, we see that, v_2 is always downer in $T(1)$. If we assume that v_1 is singly Parter, then the branch at v_1 including v_2 must be the one downer branch at v_1. Therefore, v_1 has no downer branch in $T(2)$ and, by Lemma 6.1, cannot be Parter. So v_1 must always be multiply Parter in this case.

8., 9. and 10. See Argument 1

11. By Argument 4, this line is equivalent to line (4). Thus we can switch the labels for v_1 and v_2 on the example graph for line (4) and obtain an example here.

12. By Argument 4, this line is equivalent to line (5). Thus we can switch the labels for v_1 and v_2 on the example graph for line (5) and obtain an example here.

13. See Argument 3

14. 

15. See Argument 2

16. See Argument 3

17. By Argument 4, this line is equivalent to line (24).

18., 19. and 20. See Argument 1

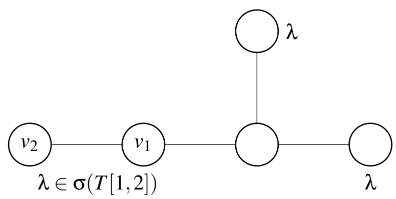
21. By Argument 4, this line is equivalent to line (7). Thus we can switch the labels for v_1 and v_2 on the example graph for line (7) and obtain an example here.

22. See Argument 1

23. By Argument 4, this line is equivalent to line (16)

24. See Argument 2

25. See Argument 3

26. 

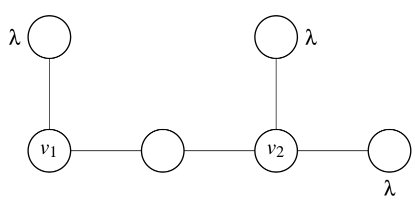
27. See Argument 2

From the table, we can infer an important lemma that will be useful in our proofs of the non-adjacent cases.

Lemma 6.2. *If v_i is a downer vertex in T and v_j is either singly parter or neutral in T, then v_i and v_j are not adjacent.*

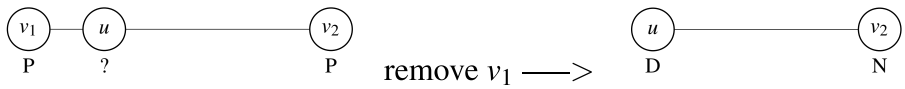
Proofs of Non-Adjacent Cases:

We use the same methods as in our proofs of the adjacent cases.

1. 

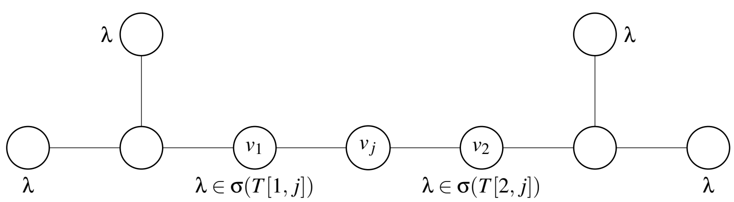
2. We assume that this case can occur: let there be a tree T with non-adjacent vertices v_1 and v_2 such that v_1 is Parter in T, v_2 is Parter in T, and v_1 is neutral in $T(2)$. Since T is a tree, it is connected and acyclic, so there is a unique path connecting v_1 and v_2. By

Argument 5, v_1 must be singly Parter, and the one downer branch at v_1 must be the branch including v_2. Also, by Argument 4, we know that v_2 is neutral in $T(1)$, or we will obtain different multiplicities of λ for $T(1,2)$ depending on the order we remove vertices. Thus, if we focus on the path connecting v_1 and v_2 and denote the vertex adjacent to v_1 as u, the graph looks something like

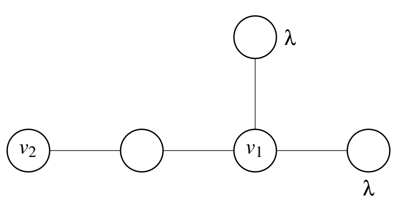


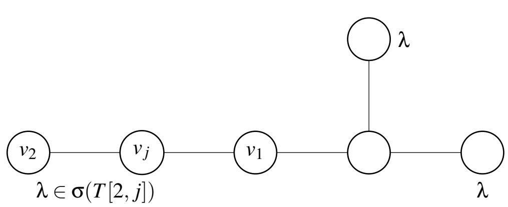
When we remove v_2 on the right, we know that u will remain downer by lines 22 - 24 in the table. Thus u is downer in $T(1,2)$.

If we remove v_2 first, however, v_1 becomes neutral by our hypothesis. So, when we remove v_1 from $T(2)$, the vertex u cannot become downer; if it did, v_1 would have a downer branch in $T(2)$ and thus be Parter instead of neutral (Lemma 6.1). Therefore, we have a contradiction.

3. 

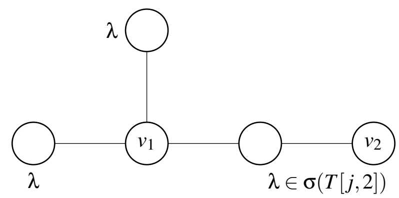
By Argument 5, both v_1 and v_2 must be singly Parter.

4. 

5. 

By Argument 5, v_1 must be singly Parter.

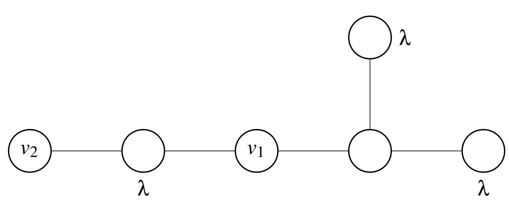
6. See Argument 1

7. 

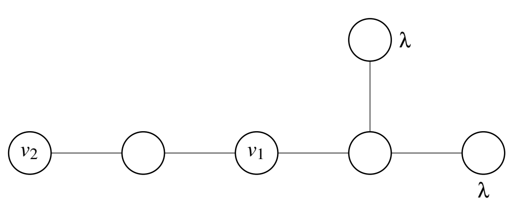
8., 9. and 10. See Argument 1

11. By Argument 4, this line is equivalent to line (4). Thus we can switch the labels for v_1 and v_2 on the example graph for line (4) and obtain an example here.

12. By Argument 4, this line is equivalent to line (5). Thus we can switch the labels for v_1 and v_2 on the example graph for line (5) and obtain an example here.

13. 

Since v_1 has no downer branch in T, it can have at most one downer branch in $T(2)$ (the branch containing v_2). So in this case, v_1 is always singly Parter in $T(2)$.

14. 

15. We assume that this case can occur: let there be a tree T with non-adjacent vertices v_1 and v_2 such that v_1 and v_2 are both neutral in T, and v_1 is downer in $T(2)$. By

Argument 4, we know that v_2 is downer in $T(1)$, or we will obtain different multiplicities of λ for $T(1,2)$ depending on the order we remove vertices. Since T is a tree, it is connected and acyclic, so there is a unique path connecting v_1 and v_2.

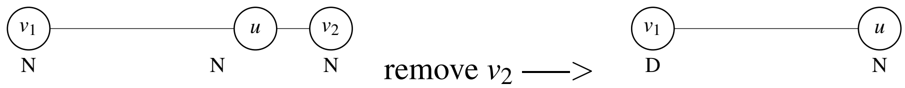
We assume that a multiply Parter vertex v_m lies on this unique path. By Argument 5, when we remove v_m, v_1 remains neutral. Since v_2 lies in a branch disconnected from v_1 in $T(m)$, we know that v_1 will stay neutral when we remove v_2 from $T(m)$; thus v_1 is neutral in $T(2,m)$. If we remove v_2 first, however, v_1 becomes downer in $T(2)$ by our hypothesis while v_m remains Parter by Argument 5. By lines 19 - 21 in the table, v_1 will remain downer when we remove any Parter vertex, so v_1 is downer in $T(2,m)$, which is a contradiction.

Thus there are no multiply Parter vertices on the unique path connecting v_1 and v_2; by Lemma 6.2, we also know there are no downer vertices, as downer vertices cannot be adjacent to neutral or singly Parter vertices.

We next assume that a singly Parter vertex v_s lies on this unique path. If the downer branch at v_s does not contain v_1 or v_2, we can treat v_s as if it were multiply Parter by Argument 5, and we are done. So, without loss of generality, we assume v_2 lies in the downer branch at v_s; we can assume this because v_1 is downer in $T(2)$ and v_2 is downer in $T(1)$. We remove v_s; by Argument 5, v_1 remains neutral. Since v_1 and v_2 are now in disconnected components in $T(s)$, we know that v_1 remains neutral when we remove v_2; thus v_1 is neutral in $T(2,s)$.

If we remove v_2 from T first, however, v_1 becomes downer by our hypothesis. By lines 4 - 6 in the table, v_s is either Parter or neutral in $T(2)$. By lines 19 - 24 in the table, v_1 will remain downer when we remove v_s from $T(2)$; thus v_1 is downer in $T(2,s)$, which is a contradiction.

Thus the path connecting v_1 and v_2 contains no Parter or downer vertices, and thus consists only of neutral vertices. Let u be the vertex on that path adjacent to v_2. By our hypothesis, v_1 is downer in $T(2)$, and, by lines 13 - 15 in the table, u is neutral in $T(2)$.



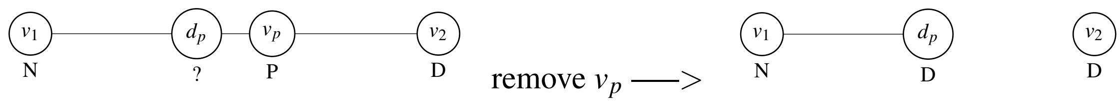
If v_1 and u are adjacent, we have a contradiction by Lemma 6.2. So we let $u_1, \ldots, u_k$ ($k \geq 1$) be the vertices between v_1 and u. But, for $1 \leq i \leq k$, u_i is neutral in T, and thus, by line 13 in the table, cannot be multiply Parter in $T(2)$. So, at some point on the path connecting v_1 and v_2, a downer vertex will be adjacent to a singly Parter or neutral vertex, which is a contradiction. Therefore, this case is impossible.

16. We assume that this case can occur: let there be a tree T with non-adjacent vertices v_1 and v_2 such that v_1 is neutral in T, v_2 is downer in T, and v_1 is downer in $T(2)$. Since T is a tree, it is connected and acyclic, so there is a unique path connecting v_1 and v_2. From Lemma 6.2, we know that a downer vertex cannot be adjacent to a neutral or singly Parter vertex, so there must be a multiply Parter vertex on this path. We consider v_p, the multiply Parter on this path closest to v_1.

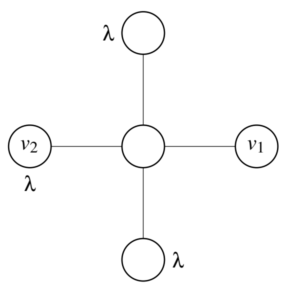
Since v_p is multiply Parter, it has a downer branch that does not include v_1. So, by Argument 5, v_1 remains neutral in $T(p)$. Since v_1 and v_2 are vertices in different, disconnected components in $T(p)$, v_1 is also neutral in $T(2,p)$.

We consider $T(2)$; by our hypothesis, v_1 is a Parter vertex in this graph. If v_p has a downer branch in $T(2)$ that does not include v_1, we know by Argument 5 that v_1 remains Parter in $T(2,p)$, which is a contradiction. We therefore only need to consider the case when v_p has exactly two downer branches in T: one including v_1, and another including v_2.

We remove v_p from T. We know that v_1 remains neutral in $T(p)$, and, since the branch including v_1 is a downer branch at v_p, the vertex adjacent to v_p and on the path connecting v_1 and v_p (denoted d_p) must be downer in $T(p)$.



Since v_p is multiply Parter in T, by Argument 5, the classification of d_p cannot change when we remove v_p from T. Thus, since d_p is downer in $T(p)$, it is also downer in T. If $d_p \neq v_1$, then some other multiply Parter vertex must separate d_p from v_1 in T, which is a contradiction, since v_p is the multiply Parter vertex closest to v_1. If $d_p = v_1$, we also have a contradiction, as v_1 is neutral in T. Therefore, this case is impossible.

17. 

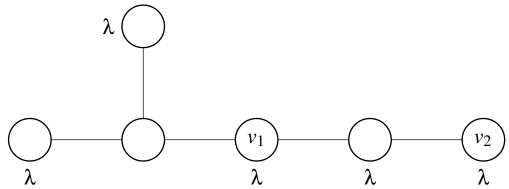
18., 19. and 20. See Argument 1

21. By Argument 4, this line is equivalent to line (7). Thus we can switch the labels for v_1 and v_2 on the example graph for line (7) and obtain an example here.

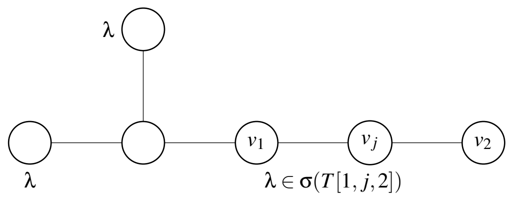
22. See Argument 1

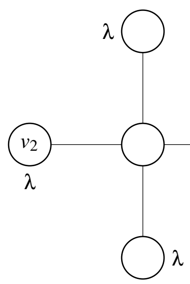
23. By Argument 4, this line is equivalent to line (16).

24. By Argument 4, this line is equivalent to line (17). Thus we can switch the labels for v_1 and v_2 on the example graph for line (17) and obtain an example here.

25. 

Since v_1 has no downer branch in T, it can have at most one downer branch in $T(2)$ (the branch containing v_2). So in this case, v_1 is always singly Parter in $T(2)$.

26. 

27. 

Thus we have proven every line in the table on page 32. From those results, we can infer a powerful theorem about the relationship between downer and neutral vertices that would be otherwise unclear.

Theorem 6.3. *Let H be a Hermitian matrix whose graph is a tree T, and let $m_H(\lambda) = k$ where $k \geq 1$. Let v_i and v_j be vertices in T such that v_i is neutral for λ and v_j is downer for λ. Then v_i is neutral for λ in $T(j)$, v_j is downer for λ in $T(i)$, and $m_{H(i,j)}(\lambda) = k - 1$.*

7 Creating Non-Trees from Trees

For a tree T, we have many tools to help us discover multiplicity lists in $L(T)$; unfortunately, for a general graph G, much less is known about the multiplicity lists in $L(G)$. In this section, we explore one fairly basic way to discover multiplicity lists for certain graphs that are ”nearly” trees (that is, connected graphs on n vertices with n or $n+1$

edges).

We recall that a p x p matrix U is *unitary* if and only if the columns of U form an orthonormal basis of $\mathbb{C}_p$. We also recall that the inverse of U is U^*, the Hermitian adjoint of U. For our purposes, we will use unitary matrices with real entries, so $U^{-1} = U^T$, and the columns of U will form an orthonormal basis of $\mathbb{R}_p$. We say that a matrix M' is *unitarily similar* to a matrix M if $M' = U^T MU$ for some U; in other words, we can obtain M' by performing a unitary similarity on M. If this is possible, $\sigma(M') = \sigma(M)$, as in all similarities.

We define 2 x 2 (a,b) unitary similarity on M as a unitary similarity where, for $|k| < 1$,

$$U[a,b] = \begin{pmatrix} k & -\sqrt{1-k^2} \\ \sqrt{1-k^2} & k \end{pmatrix}$$

and every other column c_i in U is e_i, the vector consisting of all zeros except for a 1 in the ith position.

Consider real symmetric matrices $A_1 \in M_m$ and $A_2 \in M_n$ with graphs T_1 and T_2, where both T_1 and T_2 are trees. Let A be the direct sum of A_1 and A_2. We label the vertices of T_1 and T_2 to match the indices in A; that is, we label the vertices in T_1 as $v_1, \ldots, v_m$ the vertices of T_2 as $v_{m+1}, \ldots, v_{m+n}$. Since T_1 and T_2 are both trees, each have at least two pendant vertices. We select one pendant vertex from each: let v_i and v_j be pendant, such that $1 \le i \le m$ and $m < j \le m+n$. We denote the vertex adjacent to v_i as v_x and the vertex adjacent to v_j as v_y.

We perform a 2 x 2 (i, j) similarity on A to obtain a new matrix, which we denote B. With some calculation, we find that all nonzero entries in A are still nonzero in B. Additionally, we will have at most six new nonzero entries in B that were zero in A, and they are:

$$b_{x,j} = b_{j,x} = [-\sqrt{1-k^2}]a_{i,x}$$

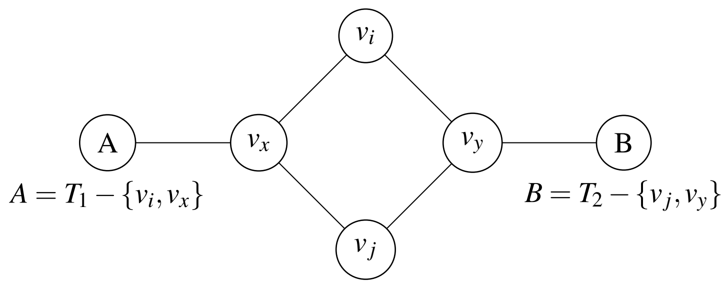
$$b_{i,y} = b_{y,i} = [\sqrt{1-k^2}]a_{j,y}$$

$$b_{i,j} = b_{j,i} = k[\sqrt{1-k^2}](a_{j,j} - a_{i,i})$$

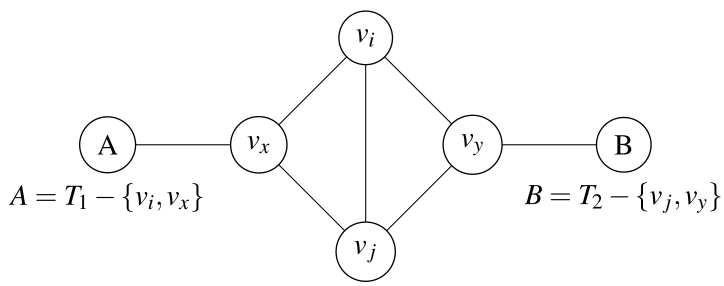
Since $a_{i,x}$ and $a_{j,y}$ are nonzero in A (by definition, v_x and v_i are adjacent and v_y and v_j are adjacent in T_1 and T_2, respectively), the first four entries will always be nonzero. The last two entries are zero if and only if $a_{j,j} = a_{i,i}$.

Thus we have two separate cases; we denote the graph of B when $a_{j,j} = a_{i,i}$ as G_1 and the graph of B when $a_{j,j} \neq a_{i,i}$ as G_2. Then

$G_1 =$



$G_2 =$



Now we can make a few observations:

1. Let T_1 and T_2 be the same tree on n vertices; thus $L(T_1) = L(T_2)$. We double the multiplicity of every eigenvalue in every multiplicity list in $L(T_1)$ and call the set of these new multiplicity lists $2L(T_1)$ (for example, if the list $\{3,2,1,1\} \in L(T_1)$, then $\{6,4,2,2\} \in 2L(T_1)$. We choose any matrix H such that T_1 and T_2 are the graph of H. We then create A as the direct sum of H with itself; we know that A has the same distinct eigenvalues as H, with each eigenvalue having twice the multiplicity in A as it did in H. Therefore, since there is a B similar to A by some 2 x 2 unitary similarity $(i, n+i)$ for $i \leq n$ and G_1 is the graph of B in this instance, $2L(T_1) \subseteq L(G_1)$.

2. Let T_1 be a tree on m vertices and T_2 be a tree on n vertices. We take all the multiplicity lists in $L(T_1)$ and append n 1's to each, and we denote this new set $L_n(T_1)$. Similarly, we denote the set consisting of all the multiplicity lists of $L(T_2)$ appended with m 1's as $L_m(T_2)$. Then $L_n(T_1) \subseteq L(G_2)$ and $L_m(T_1) \subseteq L(G_2)$.

Proof. We consider any multiplicity list m_i in $L(T_1)$; let H_1 be a matrix whose graph is T_1 and eigenvalue multiplicity list is m_i. Since the list of all 1's is in $L(T)$ for any tree T, we can choose a matrix H_2 with distinct eigenvalues whose graph is T_2. Let λ be the largest eigenvalue in magnitude in H_1 and μ be the smallest eigenvalue in magnitude in H_2. Thus, if we let $H_2' = \frac{2|\lambda|}{|\mu|} H_2$, H_2' will be a matrix whose graph is T_2 and has eigenvalues that are all larger in magnitude than any eigenvalue in H_1. We can similarly avoid equality with any diagonal entries in H_1 and H_2'. We let A be the direct sum of H_1 and H_2'; the eigenvalue multiplicity list for A is m_i appended with n 1's. We use a unitary similarity to obtain the matrix B, which has the same eigenvalues as A and whose graph is G_2. Thus m_i appended with n 1's is in $L(G_2)$, so $L_n(T_1) \subseteq L(G_2)$. By a similar argument, $L_m(T_2) \subseteq L(G_2)$. □

We could make many more observations if we define T_1 or T_2 more specifically, especially if we define them such that the inverse eigenvalue problem is equivalent to the multiplicity lists, but it does not seem worthwhile to list all such possibilities. Still, it seems that we have only just scratched the surface in this area, and it may be very useful to continue trying to determine $L(G)$ from trees.

8 References

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