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# Trees and the Implicit Construction of Eigenvalue Multiplicities

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**Trees and the Implicit Construction of Eigenvalue Multiplicities**

A thesis submitted in partial fulfillment of the requirement  
for the degree of Bachelors of Sciences in **Mathematics** from  
The College of William and Mary

by

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**April 25, 2008**

# 1 Introduction

Let  $G$  be an undirected graph on  $n$  vertices. A real symmetric matrix  $A = (a_{ij})$  is said to have a graph  $G$  provided  $a_{ij} \neq 0$  if and only if vertices  $i$  and  $j$  are adjacent in  $G$  (no restriction is placed on the diagonal entries of  $A$ — they may be zero or any nonzero real numbers). We denote by  $S(G)$  the set of all real symmetric matrices whose graph is  $G$ . For each  $A$  in  $S(G)$ , there is a list of multiplicities for the distinct eigenvalues of  $A$  (a partition of  $n$ ), and we denote by  $L(G)$  the set of all such partitions (by convention, place in non-increasing order). We are interested generally in the fundamental problem of understanding  $L(G)$  and in particular in the *minimum number* of 1's,  $U(G)$ , appearing among elements of  $L(G)$ . It is known that if  $G = T$  is a tree, then  $U(T) \geq 2$ . It can be much greater, and insight into  $U(T)$ , based upon the combinatorial structure of the tree, has proven elusive (some is known, based upon the diameter bound for the minimum number of distinct eigenvalues [JL2], but the diameter can be “short” while  $U(T)$  is still bigger than 2).

From the known multiplicity lists (all trees on fewer than 12 vertices, which we have compiled into an electronic database for ease of research, and the several known infinite families [JSW,JLS2]), it has been conjectured that

$$U(T) \leq 2 + D_2(T) \tag{1.1}$$

in which  $D_2(T)$  is the number of degree 2 vertices in  $T$ . In particular, if there are no degree 2 vertices in  $T$ , we would have  $U(T) = 2$ . It can happen that  $U(T) = 2$ , even when degree 2 vertices are present, but it has not been clear from prior work why trees with no degree 2 vertices should be cases of equality in the known lower bound  $U(T) \geq 2$ .

Call a vertex  $v$  of a tree  $T$  *high degree* if the degree of  $v$  in  $T$ ,  $deg_T(v)$ , is at least

3 (otherwise, *low degree*). It has also been conjectured (the “degree conjecture”) that for any tree  $T$ , there is a multiplicity list in  $L(T)$  containing the multiplicity  $\deg_T(v) - 1$ , counting multiplicities for degrees, for every high degree vertex  $v$  in  $T$ ; all remaining multiplicities are 1’s. This natural conjecture has been beyond the reach of simple construction techniques that have succeeded for many special kinds of trees. Here, we show that the degree conjecture implies (1.1) about  $U(T)$  and prove the conjecture for “diametric” trees using the implicit function theorem based technique, pioneered in [JSW]; this advance requires substantial technical extension of that technique. In the process, we also generalize the result of [JSW] (characterization of  $L(T)$  for “vines”) by characterizing  $L(T)$  for “binary, diametric, depth one” trees.

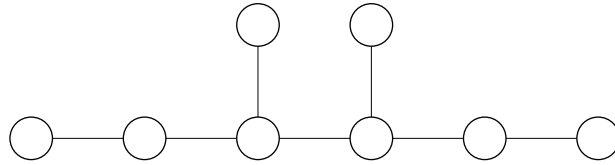
We describe the implicit function theorem based technique for designing matrices in  $S(T)$  with given multiplicities. Then we show how to use it to describe all possible multiplicity lists for binary, diametric, depth one trees. In this case, the structure of the collection of all lists is especially nice as it consists of all lists falling below a single majorization maximum; in general this is quite far from the case. Then, we use the implicit function theorem technique to prove the degree conjecture for diametric trees. We use the degree theorem to prove the conjectured bound for  $U(T)$ . Finally, we describe a way to prove the degree conjecture in general and discuss the barriers that prevent us from completely verifying it.

## 2 Background

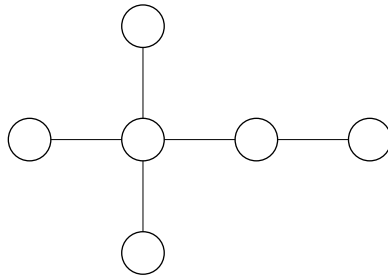
Previous work has shown that there are several limitations on the possible eigenvalue multiplicities which can occur for a given tree  $T$  based on the combinatorial structure of  $T$ . Let  $p(T)$  denote the *path cover number* of  $T$ , or the minimum cardinality of any set of vertex-disjoint paths that cover the vertices of  $T$ . It was shown in [JL1] that  $p(T)$  can also be characterized as  $\Delta(T) = \max [p - q]$  over all ways in

which  $q$  vertices can be deleted from  $T$  to form  $p$  paths. If  $M(T)$  is the maximum multiplicity of any eigenvalue among symmetric matrices whose graph is  $T$ , then [JL1] showed that, in fact,  $M(T) = p(T)$ .

Another important restrictive characteristic of  $T$  is the size of its longest path, which is called its *diameter*, denoted  $d(T)$ . It was shown in [JL2] that for a symmetric matrix  $A$  whose graph is  $T$ ,  $A$  has at least  $d(T)$  distinct eigenvalues. These results can be used to rule out possible multiplicity lists. For example, consider the following tree  $T$ :



We see that  $p(T) = 2$ , so the multiplicity list  $3, 1, 1, 1, 1, 1$  cannot occur. We also see that  $d(T) = 6$ , which means  $2, 2, 2, 1, 1$  cannot be a multiplicity list. In fact, for this tree, all multiplicity lists not eliminated by the above restrictions can occur, and can be constructed explicitly. It would be nice if that were the case for all trees, or even for some particular classes of trees. However, construction of matrices with certain eigenvalue multiplicities can be quite difficult for large trees, and it can also happen that a list passing the restrictions can still be impossible. For instance, consider the following tree  $T$ :



Note  $p(T) = 3$  and  $d(T) = 4$ . However, the multiplicity list  $2, 2, 1, 1$  cannot occur, so other techniques have been introduced to construct or eliminate multiplicity lists, such as the explicit manipulation of polynomials. Here we focus on expanding the

technique based on the implicit function theorem presented in [JSW]. To do so, we mention three facts that will be helpful.

Deleting the  $i$ th row and column of a matrix  $A$  naturally corresponds to deleting the  $i$ th vertex from the graph of  $A$ . We denote the removal of vertex  $i$  from tree  $T$  as  $T - i$ . Note that the removal of any vertex  $i$  from a tree  $T$  leaves a number of *branches at  $i$* , or connected components of the vertex-deleted tree, each of which is a tree. For convenience, we may say that we delete the  $i$ th “vertex” of  $A$  to form the principal submatrix of  $A$ , denoted  $A(i)$  (the branches at  $i$  correspond to direct summands of  $A(i)$ ). Similarly, we let  $A[S]$  denote the principal submatrix of  $A$  lying in rows and columns indexed by  $S \subseteq \{1, \dots, n\}$ . Sometimes, we may use a collection of vertices of the graph of  $A$  to describe  $S$ . If  $A$  is symmetric, the eigenvalues of  $A$  and the eigenvalues of  $A(i)$  are related by what is known as the *eigenvalue interlacing inequalities*. Specifically, if  $A$  has eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , and  $A(i)$  has eigenvalues  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-1}$ , then the  $\mu$ 's *interlace* the  $\lambda$ 's, i.e.  $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n$ .

The interlacing inequalities imply that the multiplicity of  $\lambda$  after deleting the same row and column can only increase or decrease by 1, or stay the same. Therefore, to show that  $m_A(\lambda) \geq k$ , it suffices to find an  $i$  for which  $m_{A(i)}(\lambda) \geq k + 1$ .

A key instrument for what we do here revolves around an index  $i$  for which the multiplicity of  $\lambda$  increases by 1 in  $A(i)$ . For historical purposes [Pa,Wi], we call the  $i$ th vertex in the graph of  $A$  a *Parter vertex* (for  $A$  and  $\lambda$ ). The following theorem has proven incredibly useful in this field, and will be used throughout this paper.

**Theorem 2.1** (JLS1). *Let  $T$  be a tree and  $A \in S(T)$ . Suppose that there exists an index  $i$  and a real number  $\lambda$  such that  $\lambda \in \sigma(A) \cap \sigma(A(i))$ . Then,*

(i) *there is an index  $j$  such that  $m_{A(j)}(\lambda) = m_A(\lambda) + 1$ ;*

(ii) *if  $m_A(\lambda) \geq 2$ , then  $j$  may be chosen so that  $\deg_T(j) \geq 3$  and so that there are at least three components  $T_1, T_2$ , and  $T_3$  of  $T - j$  such that  $m_{A[T_k]}(\lambda) \geq 1, k =$*

1, 2, 3; and

(iii) if  $m_A(\lambda) = 1$ , then  $j$  may be chosen so that there are two components  $T_1$  and  $T_2$  of  $T - j$  such that  $m_{A[T_k]}(\lambda) = 1, k = 1, 2$ .

This theorem is useful because it says that if  $\lambda$  is an eigenvalue of  $A$  and a principal submatrix of  $A$  (which must be true if  $m_A(\lambda) \geq 2$ ), then there exists a Parter vertex for  $\lambda$ . We call the Parter vertex in (ii) a *strong Parter vertex*. In our construction, we will force certain vertices to be Parter for  $\lambda$  by making  $\lambda$  an eigenvalue of the direct summands of a matrix that result from deleting the vertices. We will then use the implicit function theorem to perturb some entries of the matrix from zero to nonzero, which will modify the graph while preserving our eigenvalue multiplicities and Parter vertices. A common statement of the implicit function theorem is the following.

**Theorem 2.2** (Implicit Function Theorem). *Let  $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be a continuously differentiable function. Suppose that, for  $x_0 \in \mathbb{R}^n$  and  $y_0 \in \mathbb{R}^m$ ,  $f(x_0, y_0) = 0$  and the Jacobian  $\det(\partial f / \partial x)(x_0, y_0) \neq 0$ . Then there exists a neighborhood  $U \subset \mathbb{R}^m$  around  $y_0$  such that  $f(x, y) = 0$  has a solution  $x$  for any fixed  $y \in U$ . Furthermore, there is a solution  $x$  arbitrarily close to  $x_0$  associated with a  $y$  sufficiently close to  $y_0$ .*

### 3 The Implicit Function Theorem Technique

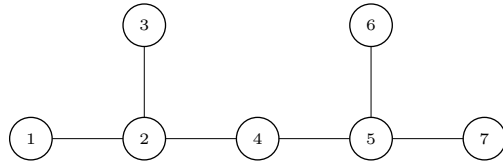
We follow the same general method as [JSW] in implementing the implicit function theorem to construct matrices with a given graph and eigenvalue constraints. The process includes two steps:

- (i) Construct an “initial point”—a matrix that satisfies the eigenvalue constraints and whose graph is a subgraph of the desired graph (in terms of edge containment).

- (ii) Fix the graph using the implicit function theorem, perturbing the necessary entries from zero to nonzero.

We have found that the technique is more easily understood by example, so we offer the following to illustrate the idea.

Consider the following tree  $T$ :



We want to find all possible eigenvalue multiplicities for a symmetric matrix whose graph is  $T$ . Note that  $p(T) = 3$  and  $d(T) = 5$ . Thus, the multiplicity of any eigenvalue can be at most 3, and there must be at least 5 distinct eigenvalues, which means that if there is an eigenvalue of multiplicity 3, then all other eigenvalues must have multiplicity 1. In fact, the adjacency matrix of  $T$  ( $A = (a_{ij})$ , where  $a_{ij} = 1$  if and only if there is an edge between vertex  $i$  and vertex  $j$ ) has these eigenvalue multiplicities. The adjacency matrix for  $T$  is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

If we delete rows and columns 2 and 5, we are left with the 5-by-5 zero matrix, and the interlacing inequalities force  $m(0) \geq 5 - 2 = 3$ . It turns out that  $\sigma(A) = \{-2, -\sqrt{2}, 0, 0, 0, \sqrt{2}, 2\}$ .

Now, does there exist a symmetric matrix  $B = (b_{ij}) \in S(T)$  having two eigenvalues,  $\lambda \neq \mu$ , such that  $m(\lambda) = m(\mu) = 2$ ? There are no known conditions precluding



this, but it appears to be more difficult than the previous case to construct explicitly. We can, however, construct this matrix implicitly using determinant conditions which are sufficient for having  $m_B(\lambda) = m_B(\mu) = 2$ . These determinant conditions are

$$b_{11} - \lambda = 0, \tag{3.1}$$

$$b_{33} - \lambda = 0, \tag{3.2}$$

$$\det(B[4, 5, 6, 7] - \lambda I_4) = 0, \tag{3.3}$$

$$\det(B[1, 2, 3, 4] - \mu I_4) = 0, \tag{3.4}$$

$$b_{66} - \mu = 0, \tag{3.5}$$

$$b_{77} - \mu = 0. \tag{3.6}$$

These conditions are sufficient because they would imply that  $m_{B(2)}(\lambda) \geq 3$  and  $m_{B(5)}(\mu) \geq 3$ , and the interlacing inequalities would give  $m_B(\lambda) \geq 3 - 1 = 2$  and  $m_B(\mu) \geq 3 - 1 = 2$ . In the previous case, we saw that if an eigenvalue has multiplicity 3, then it is the only multiple eigenvalue. Thus, we would have  $m_B(\lambda) = m_B\mu = 2$ .

Now, from the above determinant conditions, we see that certain entries in  $B$  must be specified. For instance,  $b_{11}$  must equal  $\lambda$ . We can then think of  $B$  as a matrix-valued function of variables  $x_1, x_2, b_{12}, b_{23}, b_{24}, b_{45}, b_{56}, b_{57}$ , letting  $a \neq \lambda, \mu$ :

$$B = \begin{pmatrix} \lambda & b_{12} & 0 & 0 & 0 & 0 & 0 \\ b_{12} & x_1 & b_{23} & b_{24} & 0 & 0 & 0 \\ 0 & b_{23} & \lambda & 0 & 0 & 0 & 0 \\ 0 & b_{24} & 0 & x_2 & b_{45} & 0 & 0 \\ 0 & 0 & 0 & b_{45} & a & b_{56} & b_{57} \\ 0 & 0 & 0 & 0 & b_{56} & \mu & 0 \\ 0 & 0 & 0 & 0 & b_{57} & 0 & \mu \end{pmatrix}$$

Note that if all  $b_{ij} \neq 0$ , then  $B \in S(T)$ . Since conditions (3.1, 3.2, 3.5, 3.6) hold for all choices of  $b_{ij}$ , we let

$$F = (\det(B[4, 5, 6, 7] - \lambda I_4), \det(B[1, 2, 3, 4] - \mu I_4)).$$

The Jacobian matrix of  $F$  is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \det(B[5, 6, 7] - \lambda I_3) \\ \det(B[1, 3, 4] - \mu I_3) & 0 \end{pmatrix} \end{aligned}$$

We now construct our initial matrix  $B^{(0)}$  so that  $F(B^{(0)}) = (0, 0)$  and  $\det J(B^{(0)}) \neq 0$ . By trial and error, we find that the diagonal matrix  $B^{(0)} = \text{diag}(\lambda, \mu, \lambda, \lambda, a, \mu, \mu)$  works, since

$$\det J(B^{(0)}) = \begin{vmatrix} 0 & (a - \lambda)(\mu - \lambda)^2 \\ (\lambda - \mu)^3 & 0 \end{vmatrix} \neq 0$$

because  $a$ ,  $\lambda$ , and  $\mu$  are all distinct.

Now, since the determinant is a polynomial, and thus continuously differentiable, we can use the implicit function theorem and choose  $y = (b_{12}, b_{23}, b_{24}, b_{45}, b_{56}, b_{57})$  with sufficiently small nonzero entries so that  $F$  is satisfied, and hence equations (3.1 - 3.6), for some pair  $(x_1, x_2)$ .

Thus, the matrix  $B((x_1, x_2), y) \in S(T)$  has eigenvalues  $\lambda$  and  $\mu$ , both of which have multiplicity 2.

It was proven in [JSW] that the multiplicity lists for  $T$  are precisely the sequences *majorized* by  $p(T), 1, \dots, 1 = 3, 1, 1, 1, 1$ . A non-increasing sequence  $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k$  is majorized by  $\beta = \beta_1, \beta_2, \dots, \beta_k$  if  $\sum_{i=1}^l \alpha_i \leq \sum_{i=1}^l \beta_i$  for  $l = 1, \dots, k-1$ , and  $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i$  (append  $\alpha$  or  $\beta$  with zeros to make both the same length, if necessary).

So, the general technique is to enforce some eigenvalue constraints on an  $n$ -by- $n$  symmetric matrix  $A = (a_{ij})$  by requiring that  $\det(A[S] - \lambda I) = 0$  for various choices of  $S \in \{1, \dots, n\}$  and  $\lambda \in \mathbb{R}$ . For convenience, if  $f(A) = \det(A[S] - \lambda I)$ , in which  $f$  is viewed as a function of “variables” in  $A$ , we will abuse notation and write  $f(A[R]) = \det(A[S \cap R] - \lambda I)$ . We follow the convention that the determinant of an empty matrix is 1, so that, in particular,  $f(A[R]) = 1$  if  $S \cap R = \emptyset$ .

Given a tree  $T$  on  $n$  vertices and a *vector of determinant conditions*  $F = (f_k)$ ,  $f_k(A) = \det(A[S_k] - \lambda_k I)$ , we wish to show the existence of a symmetric matrix  $A$  with graph  $T$  that satisfies  $F(A) = 0$ .

To do this, we will construct an initial  $n$ -by- $n$  matrix  $A^{(0)} = (a_{ij}^{(0)})$  for which  $F(A^{(0)}) = 0$  and the graph of  $A^{(0)}$  is a subgraph of  $T$  (in terms of edge containment). Then we will perturb some entries of  $A^{(0)}$  as we see fit, and the implicit function theorem will perturb the remaining entries in order to maintain the eigenvalue constraints specified by  $F$ . We will designate the entries that we perturb as *manual entries*, and the entries perturbed by the implicit function theorem as *implicit entries*. In the above example, the manual entries were  $b_{12}, b_{23}, b_{24}, b_{45}, b_{56}$ , and  $b_{57}$ , and the implicit entries were  $x_1$  and  $x_2$ . Because of the Jacobian requirement in the implicit function theorem, if  $F$  is a vector of length  $r$ , then precisely  $r$  of the entries in  $A^{(0)}$  must be designated as implicit. Note that because of the prevailing symmetry requirement, a symmetrically placed pair of off-diagonal entries is not independent.

The most difficult aspect of implementing the implicit function theorem is, of course, determining whether the Jacobian is nonsingular. The following two lemmas are helpful.

**Lemma 3.1** (JSW). *Let  $T$  be a tree and  $F = (f_k)_{k=1, \dots, r}$  be a vector of determinant conditions with  $r$  implicit entries identified. Suppose that a symmetric matrix  $A^{(0)}$ , whose graph is a subgraph of  $T$ , is the direct sum of irreducible matrices*

$A_1^{(0)}, A_2^{(0)}, \dots, A_p^{(0)}$ . Let  $J(A^{(0)})$  be the Jacobian matrix of  $F$  with respect to the implicit entries evaluated at  $A^{(0)}$ , and suppose

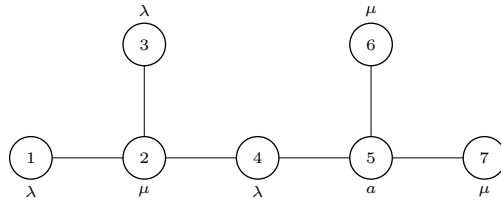
- (i) Every off-diagonal implicit entry in  $A^{(0)}$  has a nonzero value.
- (ii) For every  $k = 1, \dots, r$ ,  $f_k(A_l^{(0)}) = 0$  for precisely one  $l \in \{1, \dots, p\}$ .
- (iii) For every  $l = 1, \dots, p$ , the columns of  $J(A^{(0)})$  associated with the implicit entries of  $A_l^{(0)}$  are linearly independent.

Then  $J(A^{(0)})$  is nonsingular.

If our initial matrix is a diagonal matrix, then the determining the non-singularity of the Jacobian of determinant conditions at the initial matrix is straightforward.

**Lemma 3.2** (JSW). *Let  $F = (f_k)$  be a vector of  $r$  determinant conditions, and let  $A^{(0)}$  be a diagonal matrix. Suppose that for every  $k = 1, \dots, r$ ,  $f_k(A^{(0)}[l]) = 0$  for precisely one  $l \in \{1, \dots, n\}$ . Take  $a_{ii}$  to be an implicit entry if and only if  $f_k(A^{(0)}[l]) = 0$  for some  $k$ . If there are then  $r$  implicit entries, the Jacobian of  $F$  with respect to the implicit entries evaluated at  $A^{(0)}$  is nonzero.*

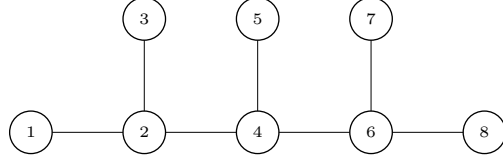
It would be nice if we could use diagonal initial matrices to implicitly construct all multiplicity lists. Unfortunately, that is not the case. To demonstrate this, we adopt a visual technique of labeling vertices to help us construct the initial matrix. In the previous example, our labeling would be the following:



We then use this labeling to construct our initial matrix by letting  $a_{ii}^{(0)} = k$  if and only if vertex  $i$  is labeled with the value  $k$ . Every other entry will be zero. The graph of  $A^{(0)}$ , then, is the subgraph of  $T$  with no edges. Note that it is easy to tell

if the initial matrix satisfies the determinant conditions based on our labeling. For instance,  $\det(A^{(0)}[1, 2, 3, 4] - \mu) = 0$  is satisfied because vertex 2 is labeled with  $\mu$ . In other words,  $\mu$  is an eigenvalue of one of the direct summands of  $A^{(0)}[1, 2, 3, 4]$ .

Now, consider the following tree  $T$ :



Note that this tree is very similar to the tree we used in the previous example, with only one additional vertex. Let us attempt to implicitly construct a symmetric matrix,  $A \in S(T)$ , with three eigenvalues,  $\lambda$ ,  $\mu$ , and  $\nu$ , such that  $m(\lambda) = m(\mu) = m(\nu) = 2$ . To do so, we must satisfy the following determinant conditions:

$$a_{11} - \lambda = 0, \tag{3.7}$$

$$a_{33} - \lambda = 0, \tag{3.8}$$

$$\det(A[4, 5, 6, 7, 8] - \lambda I_5) = 0, \tag{3.9}$$

$$\det(A[1, 2, 3] - \mu I_3) = 0, \tag{3.10}$$

$$a_{55} - \mu = 0, \tag{3.11}$$

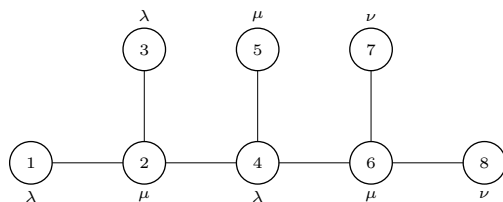
$$\det(A[6, 7, 8] - \mu I_3) = 0, \tag{3.12}$$

$$\det(A[1, 2, 3, 4, 5] - \nu I_5) = 0, \tag{3.13}$$

$$a_{66} - \nu = 0, \tag{3.14}$$

$$a_{88} - \nu = 0. \tag{3.15}$$

Let us now attempt our labeling technique to construct an initial matrix. Note that we must label vertex 1 with  $\lambda$  because of constraint (3.7), etc. We can satisfy constraints (3.10) and (3.12) by labeling vertices 2 and 6 with  $\mu$ . We can then label vertex 5 with  $\lambda$ , satisfying constraint (3.9). Our labeling gives the following picture:

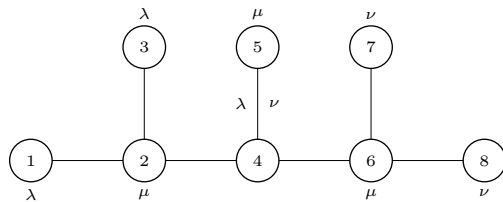


We see that all of our vertices have been labeled, but when we construct our initial matrix, constraint (3.13) will not be satisfied. It would be convenient if this meant that our desired multiplicity list were not possible, but in fact it is. We offer a resolution to this problem.

### 3.1 Second Order Initial Matrix

If a diagonal initial matrix can be constructed, we call it a *first order* initial matrix because its largest direct summand has size 1-by-1. As we have seen, a first order initial matrix makes the application of the implicit function theorem not too difficult, but it is not always possible to construct one satisfying all desired determinant conditions.

However, in the previous example, there was only one determinant condition unsatisfied. Instead of labeling vertex 4 with  $\lambda$ , let us label the edge connecting vertices 4 and 5 with  $\lambda$  and  $\nu$ . We have the following picture:



Now, we construct our initial matrix  $A^{(0)}$  such that  $a_{ii}^{(0)} = k$  if and only if vertex  $i$  is labeled with  $k$ ,  $A^{(0)}[i, j]$  has eigenvalues  $l$  and  $m$  if and only if the edge connecting vertices  $i$  and  $j$  is labeled with  $l$  and  $m$ , and all other entries are zero. Note that the entries of  $A^{(0)}[i, j]$  are easy to find using the fact that the sum of the eigenvalues is equal to the trace, and the product of the eigenvalues is equal to the

determinant. If neither diagonal entry is restricted, there are an infinite number of choices for the entries of  $A^{(0)}[i, j]$ . If both diagonal entries are specified, then the eigenvalues of  $A^{(0)}[i, j]$  are restricted based on the trace and determinant. For our constructions, we will specify only one of the diagonal entries, as in the above example, which makes calculating the entries of  $A^{(0)}[i, j]$  straightforward. If  $a_{ii}^{(0)} = k$ , then  $a_{jj}^{(0)} = l + m - k$ , and  $a_{ij}^{(0)} = \sqrt{ka_{jj}^{(0)} - lm} = \sqrt{kl + km - k^2 - lm}$ . Here the ordering of the eigenvalues becomes important, since, because of the interlacing inequalities, we must have  $l < k < m$  assuming, without loss of generality, that  $l < m$ .

To construct the *second order* initial matrix from the above example, let us choose numerical values for our desired eigenvalues. Let  $\lambda = -2$ ,  $\mu = 1$ , and  $\nu = 5$ . Then we have:

$$A^{(0)} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \sqrt{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

The difficulty, again, in implementing the implicit function theorem is determining whether the Jacobian is nonsingular. We prove the following about the Jacobian of a function of determinant conditions evaluated at a second order initial matrix.

**Lemma 3.3.** *Let  $F = (f_k)$  be a vector of  $r$  determinant conditions, and let  $A^{(0)}$  be the direct sum of 1-by-1 and 2-by-2 symmetric irreducible matrices  $A_1^{(0)}, A_2^{(0)}, \dots, A_p^{(0)}$ . Suppose that for every  $k = 1, \dots, r$ ,  $f_k(A_l^{(0)}) = 0$  for precisely one  $l \in \{1, \dots, p\}$ . If*

(i) Both diagonal entries,  $a_{m_1}$  and  $a_{m_2}$ , and the off-diagonal entry,  $b_m$ , of each 2-by-2 direct summand of  $A^{(0)}$  is implicit.

(ii) For any 2-by-2 direct summand  $A_m^{(0)} = A^{(0)}[m_1, m_2]$ , there is at least one determinant condition  $f_i = \det(A[S_i] - \lambda_i I)$  such that  $\{m_1\} \subseteq S_i$  and  $\{m_2\} \not\subseteq S_i$ , and at least two determinant conditions  $f_j = \det(A[S_j] - \lambda_j I)$  and  $f_k = \det(A[S_k] - \lambda_k I)$ , such that  $\lambda_j \neq \lambda_k$ ,  $\{m_1, m_2\} \subseteq S_j$ , and  $\{m_1, m_2\} \subseteq S_k$ .

(iii)  $\lambda_i$ ,  $\lambda_j$ , and  $\lambda_k$  are not eigenvalues of  $A^{(0)}[S_i \setminus \{m_1\}]$ ,  $A^{(0)}[S_j \setminus \{m_1, m_2\}]$ , or  $A^{(0)}[S_k \setminus \{m_1, m_2\}]$ , respectively.

(iv) There are  $r$  implicit entries total.

Then the Jacobian of  $F$  with respect to the implicit entries evaluated at  $A^{(0)}$  is nonsingular.

*Proof.* We apply Lemma 3.1 and Lemma 3.2. Lemma 3.2 tells us that if  $A_l^{(0)}$  is a 1-by-1 direct summand, then the columns of  $J(A^{(0)})$  associated with the implicit entries in  $A_l^{(0)}$  (if any) are linearly independent. So, we need only show that the columns of  $J(A^{(0)})$  associated with the implicit entries in any 2-by-2 direct summand  $A_m^{(0)}$  are linearly independent. To do so, we consider  $f_i$ ,  $f_j$ , and  $f_k$  satisfying condition (ii). We then consider the following submatrix of the Jacobian of  $F$  with respect to  $a_{m_1}$ ,  $a_{m_2}$ , and  $b_m$ :

$$\begin{pmatrix} \frac{\partial f_i}{\partial a_{m_1}} & \frac{\partial f_i}{\partial a_{m_2}} & \frac{\partial f_i}{\partial b_m} \\ \frac{\partial f_j}{\partial a_{m_1}} & \frac{\partial f_j}{\partial a_{m_2}} & \frac{\partial f_j}{\partial b_m} \\ \frac{\partial f_k}{\partial a_{m_1}} & \frac{\partial f_k}{\partial a_{m_2}} & \frac{\partial f_k}{\partial b_m} \end{pmatrix}$$

We then evaluate it at  $A^{(0)}$ :

$$\begin{pmatrix} \det(A^{(0)}[S_i \setminus \{m_1\}] - \lambda_i I) & 0 & 0 \\ (a_{m_2} - \lambda_j) \det(A^{(0)}[S_j^*] - \lambda_j I) & (a_{m_1} - \lambda_j) \det(A^{(0)}[S_j^*] - \lambda_j I) & -2b_m \det(A^{(0)}[S_j^*] - \lambda_j I) \\ (a_{m_2} - \lambda_k) \det(A^{(0)}[S_k^*] - \lambda_k I) & (a_{m_1} - \lambda_k) \det(A^{(0)}[S_k^*] - \lambda_k I) & -2b_m \det(A^{(0)}[S_k^*] - \lambda_k I) \end{pmatrix}$$



Where  $S_p^* = S_p \setminus \{m_1, m_2\}$ . Because of condition (iii), we can reduce this to:

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{m_2} - \lambda_i & a_{m_1} - \lambda_i & -2b_m \\ a_{m_2} - \lambda_j & a_{m_1} - \lambda_j & -2b_m \end{pmatrix}$$

This, then, can be reduced to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{m_1} - \lambda_i & -2b_m \\ 0 & a_{m_1} - \lambda_j & -2b_m \end{pmatrix}$$

To show that these columns are linearly independent, we assume the opposite and set the determinant equal to zero:

$$(1)[-2b_m(a_{m_1} - \lambda_i) - (-2b_m)(a_{m_1} - \lambda_j)] = 0$$

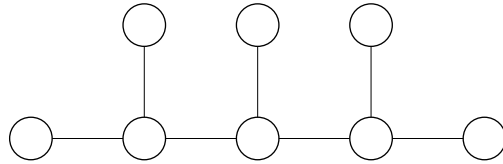
Since  $A_m^{(0)}$  is not a diagonal matrix,  $b_m \neq 0$ , which implies:

$$(a_{m_1} - \lambda_i) - (a_{m_1} - \lambda_j) = 0 \quad \Rightarrow \quad \lambda_i = \lambda_j$$

This contradicts condition (ii). Thus Lemma 3.1 applies. □

## 4 Multiplicity Lists for a Class of Trees

A *binary tree* is a tree in which no vertex has degree greater than 3. A tree is called *diametric* if there is a longest path along which all vertices of degree  $\geq 3$  lie. If every vertex is at most one edge from this path, the tree is called *depth one*. The tree from the previous example is binary, diametric, and depth one:



Recall that, if  $A$  is a symmetric matrix whose graph is a tree  $T$ , a Parter vertex for  $\lambda \in \sigma(A)$  is a vertex  $i$  in  $T$  such that  $m_{A(i)}(\lambda) - m_A(\lambda) = 1$ . As mentioned before, if  $\lambda$  is an eigenvalue of both  $A$  and  $A(j)$  for some  $k$ , then  $\lambda$  must have a Parter vertex in  $T$ . Furthermore, if  $m_A(\lambda) \geq 2$ , then  $\lambda$  must have a strong Parter vertex in  $T$ , or a Parter vertex  $i$  of degree  $\geq 3$  such that  $\lambda$  is an eigenvalue of at least 3 direct summands of  $A(i)$ . We use strong Parter vertices to prove the next lemma.

**Lemma 4.1.** *If the graph of a symmetric matrix  $A = (a_{ij})$  is a binary, diametric, depth one tree  $T$  and  $\lambda_1, \dots, \lambda_l$  are the distinct eigenvalues of  $A$ , then  $T$  has at least  $\sum_{i=1}^l (m_A(\lambda_i) - 1)$  degree 3 vertices.*

A proof of this for a similar class of trees is given in [JSW], but we offer a new proof.

*Proof.* We induct on the multiplicity of  $\lambda$  in  $A$ ,  $m_A(\lambda)$ . If  $m_A(\lambda) = 2$ , then there exists a strong Parter vertex  $i$  for  $\lambda$ , with  $\lambda$  being an eigenvalue of the three branches of  $A(i)$ . Note that one of the branches corresponds to a single vertex, where  $\lambda$  must have multiplicity 1. Therefore, the multiplicity of  $\lambda$  in each of the other two branches is less than  $m_A(\lambda)$ , but the sum of the multiplicities in both branches is  $m_A(\lambda)$ . Now, we assume the result to be true whenever  $2 \leq m(\lambda) < n$ , and let  $m_A(\lambda) = n$ . We know that  $\lambda$  has a strong Parter vertex. Then  $\lambda$  is an eigenvalue in all three branches, and the sum of the multiplicities in two of them is  $m_A(\lambda)$ . If the multiplicity of  $\lambda$  in one the remaining branches is 1, then it has multiplicity  $m_A(\lambda) - 1$  in the other. By the induction hypothesis, there are  $m_A(\lambda) - 2$  strong Parter vertices in that branch, making  $m_A(\lambda) - 1$  total. If, on the other hand, the multiplicity of  $\lambda$  in both remaining branches is greater than 1, they have a total of  $m_A(\lambda) - 2$  strong Parter vertices by the induction hypothesis, since the sum of the multiplicities in these branches is  $m_A(\lambda)$ . Then, including the original strong Parter vertex, there are a total of  $m_A(\lambda) - 1$ .  $\square$

**Lemma 4.2.** *If the graph of a symmetric matrix  $A = (a_{ij})$  is a binary, diametric, depth one tree  $T$  and  $\lambda$  is an eigenvalue of  $A$  such that  $m_A(\lambda) \geq 2$ , then no two strong Parter vertices for  $\lambda$  can be adjacent.*

*Proof.* Since  $m_A(\lambda) \geq 2$ , there exists a strong Parter vertex  $i$  for  $\lambda$ . If the multiplicity of  $\lambda$  in any branch at  $i$  is at least 2, then within that branch there is a strong Parter vertex  $j$  for  $\lambda$ , which means there must be 3 branches at  $j$ , which cannot be true if vertex  $j$  is adjacent to vertex  $i$ .  $\square$

For the next result, we describe  $S_k(T)$  as the maximum cardinality of all sets of non-adjacent degree  $k$  vertices, which, here, is very important in determining whether a multiplicity list can occur.

**Lemma 4.3.** *Let  $T$  be a binary, diametric, depth one tree on  $n$  vertices and suppose that*

$$m_1, \dots, m_l, 1, \dots, 1$$

*is an unordered list that partitions  $n$ , with  $m_1 \geq m_2 \geq \dots \geq m_l \geq 2$ . If*

$$(i) \sum_{i=1}^l (m_i - 1) \leq D_3(T).$$

$$(ii) \text{ For } i = 1, \dots, l, m_i - 1 \leq S_3(T).$$

*Then there exists a symmetric matrix  $A \in S(T)$  with the given multiplicities.*

*Proof.* Choose any distinct numerical values  $\lambda_1, \dots, \lambda_l$ .

Identify a diameter of  $T$ , placing one end on the “left” and the other on the “right.” We will identify  $m_k - 1$  separated degree 3 vertices which will be Parter for each  $\lambda_k$  in the above list. For convenience, we will immediately refer to these as Parter vertices, even though we have not yet constructed a matrix. The set of Parter vertices for  $\lambda_i$  will be denoted  $V_i$ . For each  $\lambda_i$ , we partition the degree 3 vertices by distributing  $V_i$  as equally as possible. We begin by finding  $V_1$ . We let the left-most

degree 3 vertex be Parter for  $\lambda_1$ , unless it is not adjacent to a degree 3 vertex and the right-most degree 3 vertex is, in which case we let the right-most degree 3 vertex be Parter for  $\lambda_1$ . Then, we skip  $\lceil \frac{D_3^*(T) - (m_1 - 1)}{m_1 - 1} \rceil$  unassigned degree 3 vertices, where  $D_3^*(T)$  denotes one more than the number of unassigned degree 3 vertices to the right (or left, if we began with the right-most degree 3 vertex), and let the next degree 3 vertex be Parter for  $\lambda_1$ . We then skip  $\lceil \frac{D_3^*(T) - (m_1 - 1) - 1}{(m_1 - 1) - 1} \rceil$  unassigned degree 3 vertices and let the next degree 3 vertex be Parter for  $\lambda_1$ . We continue skipping  $\lceil \frac{D_3^*(T) - (m_1 - 1) - k}{(m_1 - 1) - k} \rceil$  unassigned degree 3 vertices, where  $k$  is the number of degree 3 vertices we have already assigned, until we have assigned  $m_1 - 1$  degree 3 vertices as Parter for  $\lambda_1$ . We then repeat this process for  $\lambda_2$ , and so on. Note that there are several different ways of assigning Parter vertices, all of which will work for our construction.

The vector of determinant conditions  $F(A)$  has  $\sum_{i=1}^l (2m_i - 1)$  entries, since deleting  $m_i - 1$  Parter vertices for  $\lambda_i$  from  $T$  will increase the multiplicity of  $\lambda_i$  by  $m_i - 1$ . Each entry of  $F(A)$  is of the form  $\det(A[S] - \lambda I)$ , in which  $\lambda$  is a desired eigenvalue of  $A$  and  $S$  identifies one of the branches obtained from the deletion of the Parter vertices for  $\lambda$ .

The initial matrix  $A^{(0)} = (a_{ij}^{(0)})$  is the direct sum of 1-by-1 and 2-by-2 irreducible matrices. In constructing this matrix, it is helpful to label certain vertices of  $T$ . For  $i, \dots, l$ , find the  $m_i - 1$  Parter vertices for  $\lambda_i$ . For each, label the neighbor on the diameter immediately to the right and also the adjacent pendant vertex with  $\lambda_i$ . Next label the left-most vertex on the diameter with  $\lambda_1$ . Finally, for  $i = 2, \dots, l$ , label the next Parter vertex to the left with  $\lambda_i$ , unless it is also Parter for  $\lambda_i$ . In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with two distinct eigenvalues,  $\lambda_i$  and  $\lambda_j$ , and is Parter for some other eigenvalue  $\lambda_k$ . Then, instead of labeling the vertex twice, we can label the edge connected the Parter vertex to its adjacent pendant vertex with  $\lambda_i$  and  $\lambda_j$ . Now, construct the

initial matrix  $A^{(0)}$  by setting  $a_{kk} = \lambda_i$  if vertex  $k$  is labeled with  $\lambda_i$ , and ensuring  $A^{(0)}[u, v]$  has eigenvalues  $\lambda_w$  and  $\lambda_x$  if the edge connecting vertices  $u$  and  $v$  is labeled with  $\lambda_w$  and  $\lambda_x$ . Note that this construction requires a particular ordering of some of the eigenvalues. Since one of the diagonal entries,  $a_{uu}$ , of  $A^{(0)}[u, v]$  is equal to some eigenvalue,  $\lambda_y$ , that is not equal to  $\lambda_w$  or  $\lambda_x$ , we know that  $\lambda_w < \lambda_y < \lambda_x$  by interlacing. It is easy to find the entries of  $A^{(0)}[u, v]$ . The other diagonal entry,  $a_{vv}$ , can be calculated using the trace condition, i.e.  $a_{vv} = \lambda_w + \lambda_x - a_{uu} = \lambda_w + \lambda_x - \lambda_y$ . The off-diagonal entry,  $a_{uv}$ , can then be calculated using the determinant condition, i.e.  $a_{uv} = \sqrt{a_{uu}a_{vv} - \lambda_w\lambda_x} = \sqrt{\lambda_w\lambda_y + \lambda_x\lambda_y - \lambda_y^2 - \lambda_w\lambda_x}$ .

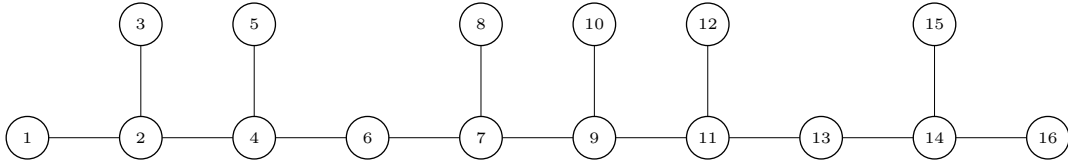
The implicit entries are precisely those corresponding to labeled vertices, the diagonal entries of the 2-by-2 matrices corresponding to vertices on the diameter, and the off-diagonal entries of the 2-by-2 matrices. Thus there are a total of  $\sum_{i=1}^l (2m_i - 1)$  implicit entries.

Because  $F(A^{(0)}) = 0$ , there are as many implicit entries in  $A^{(0)}$  as determinant conditions in  $F$ , and the Jacobian is nonsingular at  $A^{(0)}$  (by Lemma 3.3), we know that there exists a matrix  $A = (a_{ij})$  with graph  $T$  such that  $F(A) = 0$ . Thus,  $\lambda_i$  is an eigenvalue of each of the  $2m_i - 1$  direct summands of  $A(V_i)$ . By the interlacing inequalities,  $m_A(\lambda_i) \geq (2m_i - 1) - \#V_i = m_i$ .

The proof will be complete after placing upper bounds on the multiplicities. First consider  $\lambda_i$ . If  $m_A(\lambda_i)$  were greater than  $m_i$ , then  $\lambda_i$  would be a multiple eigenvalue of one of the direct summands of  $A(V_i)$ . However, the multiplicity of  $\lambda_i$  in each direct summand of  $A^{(0)}(V_i)$  is at most 1, so by choosing a small enough perturbation,  $\lambda_i$  can be guaranteed not to be a multiple eigenvalue of any direct summand of  $A(V_i)$ . Next, consider the remaining eigenvalues, that are intended to have multiplicity 1. To see that they must, in fact, be singletons, it suffices to show that no eigenvalue other than  $\lambda_1, \dots, \lambda_l$  has a strong Parter vertex. For a binary tree, no two eigenvalues may share a Parter vertex, so consider a degree 3 vertex

$v$  that is not Parter for any  $\lambda_i$ .  $v$  is adjacent to a hanging pendant  $u$ , whose corresponding entry is neither implicit nor manual, i.e., it remains equal to  $a_{uu}^{(0)}$  even after applying the implicit function theorem. By choosing the perturbation to be sufficiently small,  $A$  can be guaranteed not to have  $a_{uu}$  as an eigenvalue of any other direct summand of  $A(v)$ . This guarantees that  $v$  is not a Parter vertex for any eigenvalue.  $\square$

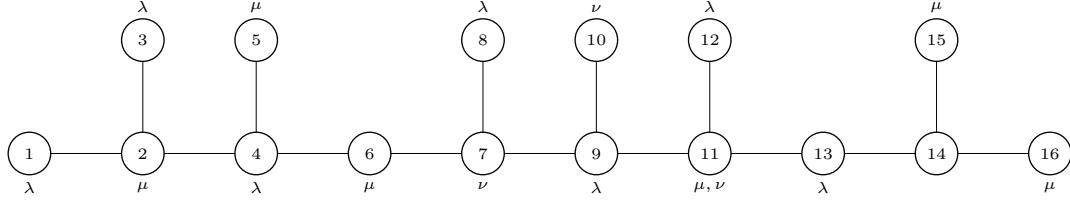
An example of our construction method will help. Consider the following tree  $T$ :



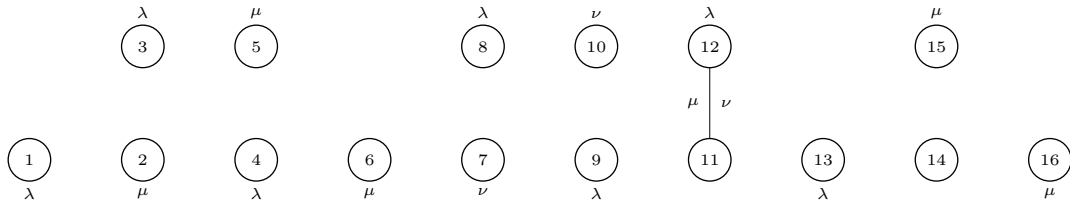
According to the previous lemma, there is a symmetric matrix  $A \in S(T)$  with eigenvalue multiplicities 4, 3, 2, 1, 1, 1. Let our three multiple eigenvalues be denoted  $\lambda$ ,  $\mu$ , and  $\nu$ , where  $m(\lambda) = 4$ ,  $m(\mu) = 3$ , and  $m(\nu) = 2$ . We use our labeling technique to construct an initial matrix satisfying the necessary determinant conditions. To do so, we first find the Parter vertices for  $\lambda$ , since it has the highest multiplicity. Vertex 2 is the left-most degree 3 vertex, so it is Parter for  $\lambda$ . We then skip  $\lceil \frac{6-(4-1)}{4-1} \rceil = 1$  degree 3 vertices, and make the next, vertex 7, Parter for  $\lambda$ . We skip  $\lceil \frac{4-((4-1)-1)}{(4-1)-1} \rceil = 1$  degree 3 vertices, and make vertex 11 Parter for  $\lambda$ . This gives us  $3 = m(\lambda) - 1$  Parter vertices for  $\lambda$ . We then find the Parter vertices for  $\mu$ . Vertex 4 is the left-most unassigned degree 3 vertex, so we make it Parter for  $\mu$ . We then skip  $\lceil \frac{3-(3-1)}{3-1} \rceil = 1$  unassigned degree 3 vertices, and make vertex 14 Parter for  $\mu$ . This gives us  $2 = m(\mu) - 1$  Parter vertices for  $\mu$ . Finally, we find the  $m(\nu) - 1 = 1$  Parter vertex for  $\nu$ , vertex 9.

Then, for each vertex that is Parter for  $\lambda$ , we label the vertex directly above, directly to the right, and the next Parter vertex to the left with  $\lambda$ . We do the same for  $\mu$  and  $\nu$ . Finally, we label the left-most vertex with  $\lambda$ . This gives us the following

picture:



We see that vertex 11 is labeled twice, so we remove  $\mu$  and  $\nu$  from the vertex and instead label the edge connecting vertices 11 and 12 with  $\mu$  and  $\nu$ , and remove the unlabeled edges:



We then use this labeling to construct a second order initial matrix  $A^{(0)}$  whose graph is the subgraph of  $T$ . Note that  $a_{12,12}^{(0)} = \lambda$  and  $A^{(0)}[11, 12]$  has eigenvalues  $\mu$  and  $\nu$ . We then can use this construction and the implicit function theorem to show that there does, in fact, exist a matrix  $A \in S(T)$  with the given multiplicities.

Now, we will show that the multiplicity lists that can occur among symmetric matrices whose graph is a binary, diametric, depth one tree  $T$  are nicely described by characteristics of  $T$ . First, we find a formula for  $p(T)$ .

**Lemma 4.4.** *Let  $T$  be a binary, diametric, depth one tree. Then  $p(T) = S_3(T) + 1$ .*

*Proof.* We use the fact that  $p(T) = \max[p - q]$  over all ways in which  $q$  vertices can be deleted from  $T$  to form  $p$  paths. We locate a maximal set of non-adjacent degree 3 vertices in  $T$ , which has  $S_3(T)$  vertices. Note that the deletion of the set leaves only paths, since any degree 3 vertex not in the set must be adjacent to at least one vertex in the set, or the set would not be maximal. Also note that not deleting any of these vertices would leave branches that are not paths. The number of these paths is  $2S_3(T) + 1$ , since there is a path to the left and above each deleted

vertex, and one path to the right of the right-most deleted vertex. Thus, for this set of vertices,  $p - q = S_3(T) + 1$ .

It only remains to show that deleting any other vertices will not increase this number. Deleting any degree 1 vertex will not increase this number, since it can only make an existing path shorter. Deleting any degree 2 vertex will also not increase this number, since it can only either make an existing path shorter or divide an existing path into two paths. Now, since the deletion of our maximal set leaves only paths, deleting any other vertex will not increase  $p - q$ . Therefore,  $p(T) = S_3(T) + 1$ .  $\square$

**Theorem 4.5.** *The possible unordered multiplicities for a binary, diametric, depth one tree  $T$  on  $n$  vertices are the sequences of positive integers that are majorized by  $p(T), d(T) - p(T) - D_2(T), 1, \dots, 1$ , a partition of  $n$ .*

*Proof.* First, we show that this list satisfies the conditions of Lemma 4.3. For condition (i), we have

$$\begin{aligned} \sum_{i=1}^l (m_i - 1) &= (p(T) - 1) + (d(T) - p(T) - D_2(T) - 1) \\ &= d(T) - D_2(T) - 2. \end{aligned}$$

Since  $d(T) = D_3(T) + D_2(T) + 2$ , we have

$$(D_3(T) + D_2(T) + 2) - D_2(T) - 2 = D_3(T).$$

Thus, condition (i) is satisfied. For condition (ii), we use Lemma 4.4. Since  $p(T) = S_3(T) + 1$ , for  $m_1$  we have

$$\begin{aligned} m_1 - 1 &= p(T) - 1 \\ &= S_3(T). \end{aligned}$$



Next, for  $m_2$  we have

$$\begin{aligned}
m_2 - 1 &= d(T) - p(T) - D_2(T) - 1 \\
&= (D_3(T) + D_2(T) + 2) - (S_3(T) + 1) - D_2(T) - 1 \\
&= D_3(T) - S_3(T).
\end{aligned}$$

To show that this satisfies condition (ii), we consider a tree that has  $2S_3(T) + 1$  degree 3 vertices, where  $S_3(T)$  is the the maximum cardinality of all sets of non-adjacent degree 3 vertices. Then we can choose every other degree 3 vertex, making a set of  $S_3(T) + 1$  non-adjacent degree 3 vertices, which is a contradiction. Thus,  $D_3(T) \leq 2S_3(T)$ , which implies that  $D_3(T) - S_3(T) \leq S_3(T)$ . Therefore, condition (ii) is satisfied.

Any other list,  $b$ , majorized by  $a = p(T), d(T) - p(T) - D_2(T), 1, \dots, 1$  will also satisfy the conditions of Lemma 4.3, since, by the definition of majorization,  $\sum_{i=1}^j b_i \leq \sum_{i=1}^j a_i$  for all  $j$ . □

**Corollary 4.6.** *Let  $T$  be a binary, diametric, depth one tree on  $n$  vertices. Then*

$$U(T) = n - 2D_3(T).$$

*Proof.* Since there is a single majorizing maximum list,  $U(T)$  will be attained with the maximally majorized list in which all multiple eigenvalues have multiplicity 2. In that list, there are  $k = \frac{n-U(T)}{2}$  2's. Since the list is maximally majorized, the sum of the first  $k$  elements in the majorizing list is  $k - 2 + p(T) + (d(T) - p(T) - D_2(T)) = \frac{n-U(T)}{2} - 2 + p(T) + (d(T) - p(T) - D_2(T))$ . This, of course, equals the sum of all

the 2's:

$$\begin{aligned}
& \frac{n - U(T)}{2} - 2 + p(T) + (d(T) - p(T) - D_2(T)) = n - U(T) \\
\Rightarrow & \frac{U(T) - n}{2} = 2 + D_2(T) - d(T) \\
\Rightarrow & U(T) = n - 2(d(T) - D_2(T) - 2)
\end{aligned}$$

Since the diameter of  $T$  includes only degree 2 vertices, degree 3 vertices, and two pendant vertices, we have

$$U(T) = n - 2D_3(T).$$

□

## 5 Verification of the Degree Conjecture

**Theorem 5.1.** *Let  $T$  be a diametric tree with high degree sequence  $d_1 \geq d_2 \geq \dots \geq d_k > 2$ . Then there exists a symmetric matrix  $A \in S(T)$  with the unordered multiplicity list  $d_1 - 1, d_2 - 1, \dots, d_k - 1, 1, \dots, 1$ .*

*Proof.* Here we construct an initial matrix and use the implicit function theorem, but we also account for all of the single eigenvalues to show that we can always get exactly this multiplicity list. To do so, we specify all but two eigenvalues, the largest and smallest, which must have multiplicity 1.

Choose any distinct numerical values  $\lambda_1, \dots, \lambda_k$  to be the multiple eigenvalues. Identify a diameter of  $T$ , placing one end on the “left” and the other on the “right.” Each  $\lambda_i$  will have exactly one Parter vertex, which can be easily identified. If  $\lambda_i$  has multiplicity  $m_i$ , then its Parter vertex will be the left-most vertex with degree  $d_i = m_i + 1$ , which we denote  $v_i$ .

The vector of determinant conditions has  $\sum_{i=1}^k (d_i)$  entries corresponding to the multiple eigenvalues. These entries will be of the form  $\det(A[S] - \lambda I)$ , in which

$\lambda$  is a desired eigenvalue of  $A$  and  $S$  identifies one of the branches obtained from the deletion of the Parter vertex for  $\lambda$ . We will also have  $n - \sum_{i=1}^k (d_i - 1) - 2 = n - \sum_{i=1}^k (d_i) + k - 2$  determinant conditions corresponding to all but the largest and smallest single eigenvalues. These entries will be of the form  $\det(A - \lambda I)$ , in which  $\lambda$  is a desired single eigenvalue of  $A$ . Thus, there are a total of  $\sum_{i=1}^k (d_i) + [n - \sum_{i=1}^k (d_i) + k - 2] = n + k - 2$  determinant conditions.

To construct the initial matrix  $A^{(0)} = (a_{ij}^{(0)})$ , which is a direct sum of 1-by-1 and 2-by-2 matrices, for  $i = 1, \dots, k$ , identify the Parter vertex for  $\lambda_i$ . Label every adjacent vertex off the diameter with  $\lambda_i$ . Then label the next Parter vertex to the left and the next Parter vertex to the right each with  $\lambda_i$ . In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with two distinct eigenvalues,  $\lambda_i$  and  $\lambda_j$ , and is Parter for some other eigenvalue  $\lambda_k$ . Then, instead of labeling the vertex twice, we can label the edge connecting the Parter vertex to any of its adjacent off-diameter vertices with  $\lambda_i$  and  $\lambda_j$ . We then use the remaining vertices to specify our single eigenvalues. Note that all Parter vertices except the left-most and the right-most were labeled twice. Thus there are  $n - [\sum_{i=1}^k (d_i) - (k - 2)] = n - \sum_{i=1}^k (d_i) + k - 2$  vertices that have not been labeled, which is equal to the number of single eigenvalues we need to specify. We then choose  $m = n - \sum_{i=1}^k (d_i) + k - 2$  distinct numerical values  $\mu_1, \dots, \mu_m$  for the single eigenvalues such that  $\min_{1 \leq i \leq l} \lambda_i < \mu_j < \max_{1 \leq i \leq l} \lambda_i$  for any  $j$ , and  $\mu_i \neq \lambda_j$  for any  $i$  and  $j$  and label the remaining vertices with them. Now, construct the initial matrix  $A^{(0)}$  by setting  $a_{kk} = \lambda_i$  if vertex  $k$  is labeled with  $\lambda_i$ ,  $a_{kk} = \mu_i$  if vertex  $k$  is labeled with  $\mu_i$ , and ensuring  $A^{(0)}[u, v]$  has eigenvalues  $\lambda_w$  and  $\lambda_x$  if the edge connecting vertices  $u$  and  $v$  is labeled with  $\lambda_w$  and  $\lambda_x$ . Note that this construction requires a particular ordering of some of the eigenvalues. Since one of the diagonal entries,  $a_{uu}$ , of  $A^{(0)}[u, v]$  is equal to some eigenvalue,  $\lambda_y$ , that is not equal to  $\lambda_w$  or  $\lambda_x$ , we know that  $\lambda_w < \lambda_y < \lambda_x$  by

interlacing. It is easy to find the entries of  $A^{(0)}[u, v]$ . The other diagonal entry,  $a_{vv}$ , can be calculated using the trace condition, i.e.  $a_{vv} = \lambda_w + \lambda_x - a_{uu} = \lambda_w + \lambda_x - \lambda_y$ . The off-diagonal entry,  $a_{uv}$ , can then be calculated using the determinant condition, i.e.  $a_{uv} = \sqrt{a_{uu}a_{vv} - \lambda_w\lambda_x} = \sqrt{\lambda_w\lambda_y + \lambda_x\lambda_y - \lambda_y^2 - \lambda_w\lambda_x}$ .

The implicit entries are those corresponding to labeled vertices, both diagonal entries of the 2-by-2 matrices, and the off-diagonal entries of the 2-by-2 matrices. There are a total of  $n + k - 2$  implicit entries.

Because  $F(A^{(0)}) = 0$ , there are as many implicit entries in  $A^{(0)}$  as determinant conditions in  $F$ , and the Jacobian is nonsingular at  $A^{(0)}$  (by Lemma 3.3), we know that there exists a matrix  $A = (a_{ij})$  with graph  $T$  such that  $F(A) = 0$ . Thus, for each  $i$ ,  $\lambda_i$  is an eigenvalue of each of the  $d_i$  direct summands of  $A(v_i)$ . By the interlacing inequalities,  $m_A(\lambda_i) \geq (d_i) - 1$ . However, for each  $j$ ,  $\mu_j$  is a single eigenvalue of  $A$ . This gives us at least  $\sum_{i=1}^k (d_i - 1) + n - \sum_{i=1}^k (d_i) + k - 2 = n - 2$  eigenvalues. Since we have not specified the largest and smallest eigenvalues, which must both be single eigenvalues, each  $\lambda_i$  must have multiplicity  $d_i - 1$  and each  $\mu_i$  must have multiplicity 1.  $\square$

## 6 U(T) Bound

**Theorem 6.1.** *Let  $T$  be a diametric tree. Then*

$$U(T) \leq 2 + D_2(T).$$

*Proof.* We use the degree list from Theorem 5.1 and count the number of 1's in the list. To do so, we label vertices in  $T$  to correspond to multiple eigenvalues, and then count the remaining unlabeled vertices.

Arrange  $T$  as a rooted tree, with some pendant vertex as the root. Place the root at the “top”, so that all other vertices fall “below” it. Find the first high

degree vertex below the root. If its degree is  $d_i$ , in each of the  $d_i - 1$  branches below it, label the next high degree vertex, or if there is none, the next pendant vertex, with  $\lambda_j$ , where  $m(\lambda_j) = d_i - 1$ . Do the same for each high degree vertex. This will label all pendant vertices except the root, and all high degree vertices except the first. Thus there are  $2 + D_2(T)$  unlabeled vertices. Since we have labeled a vertex for each multiple eigenvalue, the number of remaining unlabeled vertices is equal to the number of single eigenvalues, or 1's in the multiplicity list. Therefore,  $U(T) \leq 2 + D_2(T)$ .  $\square$

## 7 Generalization of Degree Conjecture

Of the 435 trees on fewer than 12 vertices, only 17 of them are not diametric. Thus, Theorem (5.1) applies for a vast number of trees. Furthermore, the proof we offer in this section for all trees is already true for any tree that requires only a second degree initial matrix. If a tree has a small number of high degree vertices off some diameter, or if it has enough degree 2 vertices, it may require only a second degree initial matrix even if it is not diametric. In fact, all trees on fewer than 12 vertices require only a second degree initial matrix.

However, as in the case of diagonal initial matrices, there are trees in which a second degree initial matrix cannot satisfy all necessary determinant conditions. One might think, in general, that needing the direct summands of an initial matrix to be larger than 2-by-2 would make the implementation of the implicit function theorem extremely difficult. As it turns out, for the purposes of proving the degree conjecture, we only need an initial matrix with tridiagonal direct summands. Our technique has led us to conjecture the following necessary statement, an inverse eigenvalue problem.

**Conjecture 7.1.** *Let  $S$  be a set of  $2n - 1$  distinct real numbers. Then there exists a*

symmetric tridiagonal matrix  $A$  such that  $\sigma(A) \subseteq S$ , and for each  $k, k = 1, \dots, n-1$ ,  $A[1, \dots, k]$  has an eigenvalue  $\lambda_k \in S \setminus \sigma(A)$ ,  $\lambda_i \neq \lambda_j$  if  $i \neq j$ .

The question of whether there exists such a matrix is an interesting one, and our research seems to indicate that it has yet to be answered. It also leads to the more general question of whether a symmetric tridiagonal matrix,  $A$ , exists given any  $2n - 1$  elements of  $A$ , either entries or eigenvalues of leading principal submatrices. A classical result states that given  $n$  real numbers  $\lambda_1, \dots, \lambda_n$ , and  $n - 1$  real numbers  $\mu_1, \dots, \mu_{n-1}$ , such that  $\lambda_1 < \mu_1 < \lambda_2 < \dots < \lambda_{n-1} < \mu_{n-1} < \lambda_n$ , there exists a symmetric tridiagonal matrix whose eigenvalues are  $\lambda_1, \dots, \lambda_n$ , and whose  $n - 1$ -by- $n - 1$  leading principle submatrix has eigenvalues  $\mu_1, \dots, \mu_{n-1}$ . Note that in the case that  $n = 2$ , our conjecture and the classical result are equivalent. We point out that, like the classical result, there will be some ordering restriction on the eigenvalues. We believe that the  $n - 1$  eigenvalues of the leading principal submatrices will interlace the eigenvalues of the who matrix in some way, but it is still unclear exactly how.

Unfortunately, our attempts to extend the classical result to apply to our general conjecture have proven unsuccessful. However, the 3-by-3 case can be proven with an analytical argument based on the classical result.

**Lemma 7.2.** *Let  $S$  be a set of 5 distinct real numbers,  $\lambda_1, \lambda_2, \lambda_3, \mu, \nu$ , such that  $\lambda_1 < \mu < \lambda_2 < \nu < \lambda_3$ . Then there exists a 3-by-3 symmetric tridiagonal matrix  $A$  such that  $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$ , and if  $\mu$  is an eigenvalue of  $A[1, 2]$ , then  $a_{11} = \nu$ , and vice versa.*

*Proof.* We begin by finding an algebraic formula for  $a_{11}$ . Without loss of generality, let  $\mu$  be an eigenvalue of  $A[1, 2]$ , whose other eigenvalue is  $k$ . We use the fact that the  $k$ th elementary symmetric function of the eigenvalues of  $A$  is the sum of the

$k$ -by- $k$  principal minors of  $A$ , which gives us the following system of equations:

$$\begin{aligned}
a_{11} + a_{22} &= \mu + k \\
a_{11}a_{22} - a_{12}^2 &= \mu k \\
a_{11} + a_{22} + a_{33} &= \lambda_1 + \lambda_2 + \lambda_3 \\
a_{11}a_{22} - a_{12}^2 + a_{11}a_{33} + a_{22}a_{33} - a_{23}^2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\
a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{33}a_{13}^2 &= \lambda_1\lambda_2\lambda_3
\end{aligned}$$

We can use these to solve for  $a_{11}$  in terms of  $k$ . After some algebraic calculations, we find that

$$a_{11}(k) = \frac{\lambda_1\lambda_2\lambda_3 + \mu k^2 + k\mu^2 - \lambda_1\mu k - \lambda_2\mu k - \lambda_3\mu k}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \mu^2 + k^2 + \mu k - \lambda_1\mu - \lambda_1k - \lambda_2\mu - \lambda_2k - \lambda_3\mu - \lambda_3k}.$$

Now, the classical result tells us that  $k$  can be any real number such that  $\lambda_2 < k < \lambda_3$ . Thus, we can let  $k$  range from  $\lambda_2$  to  $\lambda_3$ , and take the limit of  $a_{11}$  as  $k$  approaches each.

$$\begin{aligned}
\lim_{k \rightarrow \lambda_3} a_{11}(k) &= \frac{\lambda_1\lambda_2\lambda_3 + \mu\lambda_3^2 + \lambda_3\mu^2 - \lambda_1\mu\lambda_3 - \lambda_2\mu\lambda_3 - \lambda_3^2\mu}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \mu^2 + \lambda_3^2 + \mu\lambda_3 - \lambda_1\mu - \lambda_1\lambda_3 - \lambda_2\mu - \lambda_2\lambda_3 - \lambda_3\mu - \lambda_3^2} \\
&= \frac{\lambda_3(\lambda_1 - \mu)(\lambda_2 - \mu)}{(\lambda_1 - \mu)(\lambda_2 - \mu)} = \lambda_3,
\end{aligned}$$

since  $\mu$  must strictly interlace  $\lambda_1$  and  $\lambda_2$  because  $\mu$  is an eigenvalue of a leading principle submatrix of  $A$ . Similarly,

$$\lim_{k \rightarrow \lambda_2} a_{11}(k) = \frac{\lambda_2(\lambda_1 - \mu)(\lambda_3 - \mu)}{(\lambda_1 - \mu)(\lambda_3 - \mu)} = \lambda_2.$$

All that is left is to show that  $a_{11}(k)$  is continuous in the interval  $(\lambda_2, \lambda_3)$ . To do so, we will show that the denominator of  $a_{11}(k)$  equals zero only outside  $(\lambda_2, \lambda_3)$ . So

we set the denominator equal to zero:

$$k^2 + (\mu - \lambda_1 - \lambda_2 - \lambda_3)k + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \mu^2 - \lambda_1\mu - \lambda_2\mu - \lambda_3\mu) = 0.$$

We then use the quadratic formula to solve for  $k$ :

$$k = \frac{\lambda_1 + \lambda_2 + \lambda_3 - \mu \pm \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\mu\lambda_1 + 2\mu\lambda_2 + 2\lambda_3 - 3\mu^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3}}{2}.$$

Now we show that the larger root,  $k_2$ , is greater than  $\lambda_3$ . This implies:

$$\frac{2\lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3 - \mu)}{2} < \frac{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\mu\lambda_1 + 2\mu\lambda_2 + 2\lambda_3 - 3\mu^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3}}{2}$$

$$\Rightarrow \lambda_3 + \mu - \lambda_1 - \lambda_2 < \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\mu\lambda_1 + 2\mu\lambda_2 + 2\lambda_3 - 3\mu^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3}$$

$$\Rightarrow \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \mu^2 + 2\lambda_1\lambda_2 + 2\lambda_3\mu - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_1\mu - 2\lambda_2\mu$$

$$< \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\mu\lambda_1 + 2\mu\lambda_2 + 2\lambda_3 - 3\mu^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3$$

$$\Rightarrow 4\mu^2 - 4\lambda_1\mu - 4\lambda_2\mu + \lambda_1\lambda_2 < 0$$

$$\Rightarrow \mu^2 - (\lambda_1 + \lambda_2)\mu + \lambda_1\lambda_2 < 0.$$

Note that this is a quadratic equation in  $\mu$  with roots  $\lambda_1$  and  $\lambda_2$ . Since the leading coefficient is positive, the function is negative on the interval  $(\lambda_1, \lambda_2)$ . Thus  $k_2 > \lambda_3$ .

The proof that  $k_1$  is less than  $\lambda_2$  follows in the same way.

Therefore,  $a_{11}(k)$  is continuous on the interval  $(\lambda_2, \lambda_3)$ . So we can choose for  $a_{11}$  any value on the interval.  $\square$

As always, the most difficult aspect of using the implicit function theorem is making sure the Jacobian at the initial matrix is nonsingular. The necessity of tridiagonal direct summands in the initial matrix also carries a need to make all of the entries in each direct summand implicit. The following conjecture is also needed to fully prove the main result in this section.

**Conjecture 7.3.** *Let  $F = (f_k)$  be a vector of  $r$  determinant conditions, and let  $A^{(0)}$*



be the direct sum of symmetric, tridiagonal, irreducible matrices  $A_1^{(0)}, A_2^{(0)}, \dots, A_p^{(0)}$ . Suppose that for every  $k = 1, \dots, r$ ,  $f_k(A_l^{(0)}) = 0$  for precisely one  $l \in \{1, \dots, p\}$ . If

(i) If  $A_l^{(0)}$  is a direct summand of size larger than 1-by-1, then every entry is implicit.

(ii) For any  $k$ -by- $k$  direct summand  $A_m^{(0)} = A^{(0)}[m_1, \dots, m_k]$ , there is at least one determinant condition  $f_i = \det(A[S_i] - \lambda_i I)$  such that  $\{m_1, \dots, m_q\} \subseteq S_i$  and  $\{m_{q+1}, \dots, m_k\} \not\subseteq S_i$  and  $\lambda_i$  is not an eigenvalue of  $A^{(0)}[S_i \setminus \{m_1, \dots, m_k\}]$  for every  $q = 1, \dots, k - 1$ .

(iii) For any  $k$ -by- $k$  direct summand  $A_m^{(0)} = A^{(0)}[m_1, \dots, m_k]$ , there are at least  $k$  determinant conditions,  $f_{j_1}, \dots, f_{j_k}$ , each of the form  $f_{j_l} = \det(A[S_{j_l}] - \lambda_{j_l} I)$ , such that  $\lambda_{j_r} \neq \lambda_{j_s}$  if  $r \neq s$ ,  $\{m_1, \dots, m_k\} \subseteq S_{j_l}$ , and  $\lambda_{j_l}$  is not an eigenvalue of  $A^{(0)}[S_i \setminus \{m_1, \dots, m_k\}]$  for all  $l = 1, \dots, k$ .

(iv) There are  $r$  implicit entries total.

Then the Jacobian of  $F$  with respect to the implicit entries evaluated at  $A^{(0)}$  is nonsingular.

The fact that each direct summand is tridiagonal seems convenient, since the determinant of a tridiagonal matrix  $A$  follows a recursive pattern:

$$\det(A) = a_{ii} \det(A(i)) - a_{ij}^2 \det(A(i, j))$$

where  $j = i + 1$ . Therefore, if we consider a determinant condition  $f_k = \det(A[S_k] - \lambda I)$ , we can find formulas for entries of the Jacobian matrix:

$$\frac{\partial f_k}{\partial a_{ii}} = \begin{cases} \det(A[S_k \setminus i] - \lambda I) & \text{if } i \in S_k \\ 0 & \text{otherwise} \end{cases}$$

for diagonal entries. For off-diagonal entries, we have:

$$\frac{\partial f_k}{\partial a_{ij}} = \begin{cases} \det(A[S_k \setminus \{i, j\}] - \lambda I) & \text{if } i, j \in S_k \\ 0 & \text{otherwise} \end{cases}$$

We can use these formulas to prove the above conjecture in which the largest direct summand of  $A^{(0)}$  is 3-by-3.

**Lemma 7.4.** *Let  $F = (f_k)$  be a vector of  $r$  determinant conditions, and let  $A^{(0)}$  be the direct sum of 1-by-1, 2-by-2, and 3-by-3 symmetric, tridiagonal, irreducible matrices  $A_1^{(0)}, A_2^{(0)}, \dots, A_p^{(0)}$ . Suppose that for every  $k = 1, \dots, r$ ,  $f_k(A_l^{(0)}) = 0$  for precisely one  $l \in \{1, \dots, p\}$ . If*

- (i) *If  $A_l^{(0)}$  is a direct summand of size larger than 1-by-1, then every entry is implicit.*
- (ii) *For any  $k$ -by- $k$  direct summand  $A_m^{(0)} = A^{(0)}[m_1, \dots, m_k]$ , there is at least one determinant condition  $f_i = \det(A[S_i] - \lambda_i I)$  such that  $\{m_1, \dots, m_q\} \subseteq S_i$  and  $\{m_{q+1}, \dots, m_k\} \not\subseteq S_i$  and  $\lambda_i$  is not an eigenvalue of  $A^{(0)}[S_i \setminus \{m_1, \dots, m_k\}]$  for every  $q = 1, \dots, k - 1$ .*
- (iii) *For any  $k$ -by- $k$  direct summand  $A_m^{(0)} = A^{(0)}[m_1, \dots, m_k]$ , there are at least  $k$  determinant conditions,  $f_{j_1}, \dots, f_{j_k}$ , each of the form  $f_{j_i} = \det(A[S_{j_i}] - \lambda_{j_i} I)$ , such that  $\lambda_{j_r} \neq \lambda_{j_s}$  if  $r \neq s$ ,  $\{m_1, \dots, m_k\} \subseteq S_{j_i}$ , and  $\lambda_{j_i}$  is not an eigenvalue of  $A^{(0)}[S_i \setminus \{m_1, \dots, m_k\}]$  for all  $l = 1, \dots, k$ .*
- (iv) *There are  $r$  implicit entries total.*

*Then the Jacobian of  $F$  with respect to the implicit entries evaluated at  $A^{(0)}$  is nonsingular.*

*Proof.* We apply Lemma 3.1, Lemma 3.2, and Lemma 3.3. Lemmas 3.2 and 3.3 tell us that if  $A_l^{(0)}$  is a 1-by-1 or 2-by-2 direct summand, then the columns of  $J(A^{(0)})$  associated with the implicit entries in  $A_l^{(0)}$  (if any) are linearly independent. So, we need only show that the columns of  $J(A^{(0)})$  associated with the implicit

entries in any 3-by-3 direct summand  $A_m^{(0)}$  are linearly independent. To do so, we consider  $f_{i_1}, \dots, f_{i_5}$ , two of which satisfy condition (ii), and three of which satisfy condition (iii). We then consider the submatrix of the Jacobian of  $F$  with respect to  $a_{m_{11}}, a_{m_{12}}, a_{m_{22}}, a_{m_{23}}$ , and  $a_{m_{33}}$ , evaluate it at  $A_l^{(0)}$ , and row reduce to get the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2a_{m_{12}} & a_{m_{11}} - \lambda_{i_2} & 0 & 0 \\ 0 & -2a_{m_{12}}(a_{m_{33}} - \lambda_{i_3}) & a_{m_{11}}a_{m_{33}} - a_{m_{11}}\lambda_{i_3} - a_{m_{33}}\lambda_{i_3} + \lambda_{i_3}^2 & -2a_{m_{23}}(a_{m_{11}} - \lambda_{i_3}) & a_{m_{11}}a_{m_{22}} - a_{m_{11}}\lambda_{i_3} - a_{m_{22}}\lambda_{i_3} + \lambda_{i_3}^2 - a_{m_{12}}^2 \\ 0 & -2a_{m_{12}}(a_{m_{33}} - \lambda_{i_4}) & a_{m_{11}}a_{m_{33}} - a_{m_{11}}\lambda_{i_4} - a_{m_{33}}\lambda_{i_4} + \lambda_{i_4}^2 & -2a_{m_{23}}(a_{m_{11}} - \lambda_{i_4}) & a_{m_{11}}a_{m_{22}} - a_{m_{11}}\lambda_{i_4} - a_{m_{22}}\lambda_{i_4} + \lambda_{i_4}^2 - a_{m_{12}}^2 \\ 0 & -2a_{m_{12}}(a_{m_{33}} - \lambda_{i_5}) & a_{m_{11}}a_{m_{33}} - a_{m_{11}}\lambda_{i_5} - a_{m_{33}}\lambda_{i_5} + \lambda_{i_5}^2 & -2a_{m_{23}}(a_{m_{11}} - \lambda_{i_5}) & a_{m_{11}}a_{m_{22}} - a_{m_{11}}\lambda_{i_5} - a_{m_{22}}\lambda_{i_5} + \lambda_{i_5}^2 - a_{m_{12}}^2 \end{pmatrix}$$

We can further row reduce this to:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_{m_{11}} - \lambda_{i_2} & 0 & 0 \\ 0 & a_{m_{33}} - \lambda_{i_3} & a_{m_{11}}a_{m_{33}} - a_{m_{11}}\lambda_{i_3} - a_{m_{33}}\lambda_{i_3} + \lambda_{i_3}^2 & a_{m_{11}} - \lambda_{i_3} & a_{m_{11}}a_{m_{22}} - a_{m_{11}}\lambda_{i_3} - a_{m_{22}}\lambda_{i_3} + \lambda_{i_3}^2 - a_{m_{12}}^2 \\ 0 & \lambda_{i_4} - \lambda_{i_3} & (\lambda_{i_4} - \lambda_{i_3})(\lambda_{i_4} + \lambda_{i_3} - a_{m_{11}} - a_{m_{33}}) & a_{m_{11}} - \lambda_{i_4} & (\lambda_{i_4} - \lambda_{i_3})(\lambda_{i_4} + \lambda_{i_3} - a_{m_{11}} - a_{m_{22}}) \\ 0 & \lambda_{i_5} - \lambda_{i_3} & (\lambda_{i_5} - \lambda_{i_3})(\lambda_{i_5} + \lambda_{i_3} - a_{m_{11}} - a_{m_{33}}) & a_{m_{11}} - \lambda_{i_5} & (\lambda_{i_5} - \lambda_{i_3})(\lambda_{i_5} + \lambda_{i_3} - a_{m_{11}} - a_{m_{22}}) \end{pmatrix}$$

Continuing to row reduce, we finally get the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{m_{11}} - \lambda_{i_2} & a_{m_{12}} & 0 & 0 \\ 0 & a_{m_{12}} & a_{m_{33}} - a_{m_{11}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{i_4} - \lambda_{i_5} \end{pmatrix}$$

To show these columns are linearly independent, we assume, on the contrary, that the determinant is equal to zero:

$$[(a_{m_{11}} - \lambda_{i_2})(a_{m_{33}} - a_{m_{11}}) - a_{m_{12}}^2](\lambda_{i_4} - \lambda_{i_5}) = 0$$

Since each eigenvalue is distinct, we reduce this to:

$$\begin{aligned} & (a_{m_{11}} - \lambda_{i_2})(a_{m_{33}} - \lambda_{i_3}) - a_{m_{12}}^2 = 0 \\ \Rightarrow & a_{m_{11}}a_{m_{33}} + \lambda_{i_2}a_{m_{33}} - \lambda_{i_2}a_{m_{11}} - a_{m_{11}}^2 - a_{m_{12}}^2 = 0 \end{aligned}$$

Using the elementary symmetric function characterizations of the eigenvalues of  $A_l^{(0)}$ , we find that this implies:

$$(a_{m_{11}} - \lambda_{i_2})(a_{m_{33}} + \lambda_{i_2} - \lambda_{i_3} - a_{m_{22}}) = 0$$

Since  $a_{m_{11}} = \lambda_{i_2}$  we can reduce to:

$$a_{m_{33}} + \lambda_{i_2} - \lambda_{i_3} - a_{m_{22}} = 0$$

Again, using elementary symmetric functions, we find that this implies:

$$a_{m_{33}} - \lambda_{i_3} = \mu - a_{m_{11}}$$

where  $\mu$  is the other eigenvalue of  $A_l^{(0)}$  [1, 2]. This, though, is a contradiction due to ordering.  $\square$

Now, for the construction technique we use in our main proof, we define a characteristic of high degree vertices in a tree  $T$ . For a high degree vertex  $v$ , let the  $i$ th branch degree of periphery of  $v$  in the branch  $T_i$  at  $v$ , denoted  $r_{i_i}(v)$ , be the maximum number of high degree vertices in any path in  $T_i$ , including  $v$ . Let  $r(v)$ , the degree of periphery of  $v$ , mean the second largest branch degree of periphery of  $v$  over all  $T_i$  at  $v$ . If  $r(v) = k$ , then we say  $v_i$  is on the  $k$ th level of periphery. Denote the set of vertices on the  $k$ th level of periphery by  $R_k(T)$ .

**Lemma 7.5.** *Let  $T$  be a tree. Then there is at most one high degree vertex,  $v_k$ , in*

$T$  such that  $r(v_k) = \max_i r_{T_i}(v_k)$ .

*Proof.* Assume that there are two vertices,  $v_i$  and  $v_j$ , such that  $r(v_i) = \max_k r_{T_k}(v_i)$  and  $r(v_j) = \max_k r_{T_k}(v_j)$ . Identify two paths,  $P_{i_1}$  and  $P_{i_2}$ , in different branches at  $v_i$ , each having  $r(v_i)$  high degree vertices. If  $v_j$  is on one of those paths, say  $P_{i_1}$ , then  $r(v_j) > r(v_i)$ , since there is a path starting at  $v_j$  that includes  $v_i$  and  $P_{i_2}$ . But that means that  $P_{i_1}$  has more than  $r(v_i)$  high degree vertices, since it includes one of the paths of  $v_j$  with  $r(v_j)$  high degree vertices. Thus,  $v_j$  cannot be on  $P_{i_1}$  or  $P_{i_2}$ . However, if  $v_j$  is in some other path, then, again,  $r(v_j) > r(v_i)$ , since some path starting at  $v_j$  contains  $v_i$  and both  $P_{i_1}$  and  $P_{i_2}$ . But then there is a path starting at  $v_i$  containing  $v_j$  and one of its paths containing  $r(v_j)$  high degree vertices. Thus  $r(v_i)$  is not maximal.  $\square$

If there exists a vertex  $v$  such that  $r(v_i) = \max_i r_{T_i}(v)$  equals the largest number of high degree vertices along any path starting at  $v$ , let us call it the *center* vertex, denoted  $v_c$ .

The following we prove as a conjecture, since we assume that Conjectures 7.1 and 7.3 are true.

**Conjecture 7.6.** *Let  $T$  be a tree with high degree sequence  $d_1 \geq d_2 \geq \dots \geq d_k > 2$ . Then there exists a symmetric matrix  $A \in S(T)$  with the unordered multiplicity list  $d_1 - 1, d_2 - 1, \dots, d_k - 1, 1, \dots, 1$ .*

*Proof.* Choose any distinct numerical values  $\lambda_1, \dots, \lambda_k$  to be the multiple eigenvalues. Each  $\lambda_i$  will have exactly one Parter vertex, which can be easily identified. If  $\lambda_i$  has multiplicity  $m_i$ , then its Parter vertex will be the any vertex with degree  $d_i = m_i + 1$ , which we denote  $v_i$ .

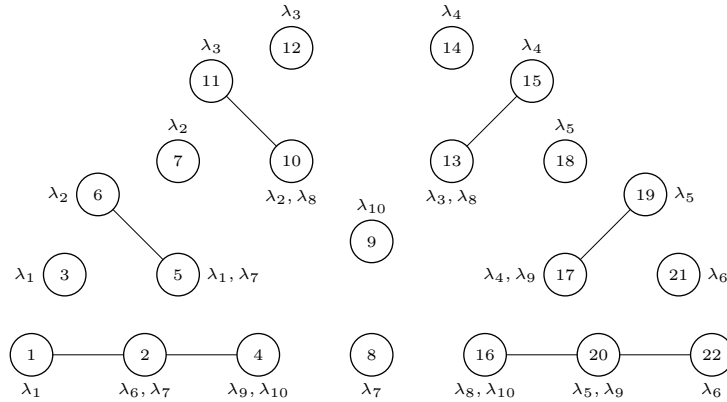
The vector of determinant conditions has  $\sum_{i=1}^k (d_i)$  entries corresponding to the multiple eigenvalues. These entries will be of the form  $\det(A[S] - \lambda I)$ , in which  $\lambda$  is a

desired eigenvalue of  $A$  and  $S$  identifies one of the branches obtained from the deletion of the Parter vertex for  $\lambda$ . We will also have  $n - \sum_{i=1}^k (d_i - 1) - 2 = n - \sum_{i=1}^k (d_i) + k - 2$  determinant conditions corresponding to all but the largest and smallest single eigenvalues. These entries will be of the form  $\det(A - \lambda I)$ , in which  $\lambda$  is a desired single eigenvalue of  $A$ . Thus, there are a total of  $\sum_{i=1}^k (d_i) + [n - \sum_{i=1}^k (d_i) + k - 2] = n + k - 2$  determinant conditions.

To construct the initial matrix  $A^{(0)} = (a_{ij}^{(0)})$ , which is a direct sum of 1-by-1 and 2-by-2 matrices, for  $i = 1, \dots, k$ , identify the Parter vertex for  $\lambda_i$ . If  $v_i \neq v_c$ , then in every branch that does not contain the path with more than  $r(v_i)$  high degree vertices, label the closest high degree vertex, or the vertex adjacent to  $v_i$  if there is no high degree vertex, with  $\lambda_i$ . Then, moving clockwise, label the next high degree vertex on the same level of periphery as  $v_i$  with  $\lambda_i$ . If  $v_i = v_c$ , then in every branch, label the closest high degree vertex with  $\lambda_i$ , or, if  $r(v_i) = 1$ , label the vertex adjacent to  $v_i$  with  $\lambda_i$ . Finally, in any of the  $v_c$ 's branches, remove the labeled eigenvalue on the high degree vertex closest to  $v_c$  whose Parter vertex is not  $v_c$ , and label  $v_c$  with it. This is to prevent a contradiction in the numerical ordering of the eigenvalues. In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with two distinct eigenvalues,  $\lambda_i$  and  $\lambda_j$ , and is Parter for some other eigenvalue  $\lambda_k$ . We then use the remaining vertices to specify our single eigenvalues. All but two Parter vertices are labeled twice. Thus there are  $n - [\sum_{i=1}^k (d_i) - (k - 2)] = n - \sum_{i=1}^k (d_i) + k - 2$  vertices that have not been labeled, which is equal to the number of single eigenvalues we need to specify. We then choose  $m = n - \sum_{i=1}^k (d_i) + k - 2$  distinct numerical values  $\mu_1, \dots, \mu_m$  for the single eigenvalues such that  $\min_{1 \leq i \leq l} \lambda_i < \mu_j < \max_{1 \leq i \leq l} \lambda_i$  for any  $j$ , and  $\mu_i \neq \lambda_j$  for any  $i$  and  $j$ , and label the remaining vertices with them. Now, construct the initial matrix  $A^{(0)}$  by setting  $a_{kk} = \lambda_i$  if vertex  $k$  is labeled with  $\lambda_i$ ,  $a_{kk} = \mu_i$  if vertex  $k$  is labeled with  $\mu_i$ , and ensuring that the tridiagonal matrix  $A^{(0)}[u, v, \dots, w]$  has the following



Here,  $n = 22$ , and there are 10 high degree vertices, all of degree 3. Thus, we would like to know if there exists a symmetric matrix  $A \in S(T)$  with eigenvalue multiplicities  $2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1$ . First, note that  $T$  does have a center, vertex 8, since every path starting at vertex 8 has 2 high degree vertices. Now, we see that  $R_1(T) = \{2, 5, 10, 13, 17, 20\}$ . Thus, making vertex 2 Parter for  $\lambda_1$ , we label vertices 1, 3, and 5 with  $\lambda_1$ . Similarly, if we say vertex 5 is Parter for  $\lambda_2$ , then we label vertices 6, 7, and 10 with  $\lambda_2$ . Next,  $R_2(T) = \{4, 9, 16\}$ , so, for example, if vertex 4 is Parter for  $\lambda_7$ , we label vertices 2, 5, and 9 with  $\lambda_7$ . Vertex 8 is the center, so if it is Parter for  $\lambda_{10}$ , we label vertices 4, 9, and 16 with  $\lambda_{10}$ . Now, since there is a center in  $T$ , we must remove a labeled eigenvalue from some adjacent vertex, say vertex 9, and label the center with it. Thus, we remove  $\lambda_7$  from vertex 9 and instead label vertex 8 with  $\lambda_7$ . This construction yields the following picture, in which we keep the edges that correspond to tridiagonal direct summands in the initial matrix:



Using the picture, we see that our third order initial matrix  $A^{(0)}$  includes, for example, a 3-by-3 direct summand  $A^{(0)}[1, 2, 4]$  such that  $a_1^{(0)}1 = \lambda_1$ , the principal submatrix  $A^{(0)}[1, 2]$  has  $\lambda_7$  as an eigenvalue, and the eigenvalues of  $A^{(0)}[1, 2, 4]$  are  $\lambda_6$ ,  $\lambda_9$ , and  $\lambda_{10}$ .

Note that, though we have not proven the degree conjecture in general, given Lemmas 7.2 and 7.4, we can say that this multiplicity list does occur for some  $A \in S(T)$ .



## References

- [JL1] C. R. Johnson and A. Leal-Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra* 46:139-144 (1999).
- [JL2] C. R. Johnson and A. Leal-Duarte. On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree. *Mathematical Inequalities and Applications* 5(2):175-180 (2002).
- [JL3] C. R. Johnson and A. Leal-Duarte. On the possible multiplicities of the eigenvalues of an Hermitian matrix whose graph is a given tree. *Linear Algebra and its Applications* 348:7-21 (2002).
- [JLS1] C. R. Johnson, A Leal-Duarte and C. M. Saiago. The Parter-Wiener theorem: refinement and generalization. *SIAM Journal on Matrix Analysis and Applications* 25(2):352-361 (2003).
- [JLS2] C. R. Johnson, A Leal-Duarte and C. M. Saiago. Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices who graph is a tree: the case of generalized stars and double generalized stars. *Linear Algebra and its Applications* 373:311-330 (2003).
- [JSW] C. R. Johnson, B.D. Sutton and A. Witt. Implicit construction of multiple eigenvalues for trees. Preprint.
- [Pa] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *Journal of the Society for Industrial and Applied Mathematics* 8:376-388 (1960).
- [Wi] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra and Applications* 61:15-29 (1984).