

5-2016

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# Normal Matrices Subordinate to a Graph

A thesis submitted in partial fulfillment of the requirement  
for the degree of Bachelor of Science in Mathematics from  
The College of William and Mary

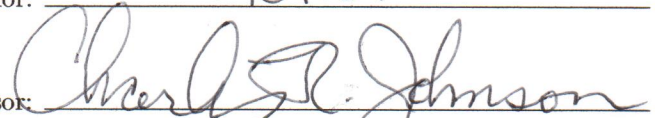
By

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Accepted for: \_\_\_\_\_

Honors

Faculty Advisor: \_\_\_\_\_



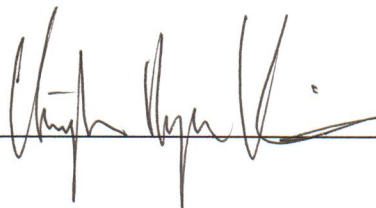
Charles Johnson

Panel Member 1: \_\_\_\_\_



Marc Sher

Panel Member 2: \_\_\_\_\_



Ryan Vinroot

Williamsburg, VA

May 2, 2016

## Abstract

Hence, we explore how the normality of a complex matrix relates to its graph. In particular, our focus is upon how the graph structure can simplify the relations among entries that characterize normality. We begin by observing what it means to be normal for a matrix subordinate to a tree, and then generalize the results to 34 - graphs. A key issue is the relation between absolute symmetry and normality. Real normal matrices are also considered.

# 1 Introduction

We begin with some useful views of what it means for a matrix to be normal, and some properties of trees and other special graphs. These will provide a foundation and be referenced later in the text. The topic of when a normal matrix is subordinate to a path is already well studied in [1]. Some of the results of this paper can be seen in [5].

**Definition 1.1.** A matrix,  $A \in M_n(\mathbb{C})$ , is **normal** if and only if  $AA^* = A^*A$ . Let  $A^*A$  be denoted by  $N = [n_{ij}]$ . Throughout this paper, when we say  $A^*$ , we mean the conjugate transpose of  $A$ . Another well known notation is  $A^\dagger$ .

**Remark 1.2.** Although this definition of normal is relatively simple to state, checking the conditions results in large amounts of calculations. Thus, one of our major objectives will be to create a more easily calculable notion of normality. As we will show, when a matrix is subordinate to certain graphs, one arrives at computationally simpler definitions of normality.

For purposes of reference and future discussion, we have included several equivalent conditions for a matrix,  $A \in M_n$ , to be normal.

**Theorem 1.3.** For  $A \in M_n(\mathbb{C})$ , the following are equivalent.

- 1)  $A$  is normal;
- 2)  $A$  is diagonalizable by a unitary similarity;
- 3) The space  $\mathbb{C}^n$  is spanned by an orthonormal set of eigenvectors of  $A$ .

Additional equivalent conditions for normality can be seen in [3].

We will now introduce some basic terms concerning graphs. Additional terminology can be seen in [8].

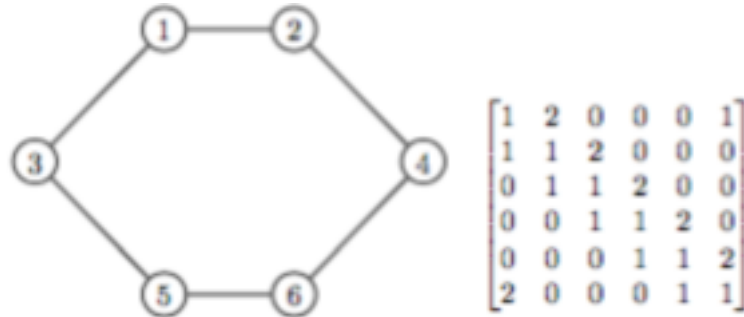
**Definition 1.4.** A **graph** is an ordered pair  $G = (V, E)$ , in which  $V$  is the set of **vertices** and  $E$  is the set of **edges**, which are ordered pairs of elements in  $V$ . An edge connecting vertices  $i, j \in V$  is denoted  $(i, j)$ .

**Definition 1.5.** Let  $G = (V, E)$ . Vertices  $i, j \in V$  are **adjacent** if there exists an edge  $(i, j) \in E$ .

**Definition 1.6.** For some vertex  $i \in V$ , the **neighborhood** of  $i$  is the set of all vertices adjacent to  $i$ . These adjacent vertices are called **neighbors**. The number of neighbors of vertex  $i$  is known as the **degree** of  $i$ .

**Definition 1.7.** We say that a matrix  $A \in M_n$  is **subordinate** to a graph,  $G$ , if and only if for every edge  $(i, j) \in E$ ,  $a_{ij} \neq 0$ , and the cardinality of  $V$  is equal to  $n$ . An edge is **undirected** if  $(i, j), (j, i) \in E$  denoted by  $\{i, j\}$ .

**Example 1.8.** Below is an example of a graph,  $G$ , and a subordinate matrix,  $A \in M_n(\mathbb{C})$ .

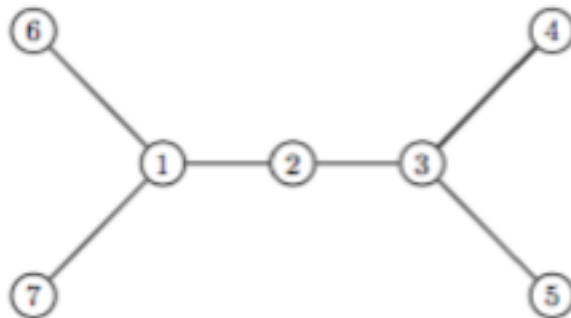


**Definition 1.9.** If a matrix  $A$  is subordinate to a graph  $G$ , and its entries are in some field  $\mathbb{F}$ , we will write  $A \in \mathbb{F}(G)$ .

**Definition 1.10.** A Vertex  $i$  in a graph  $G$  is **pendant** if vertex  $i$  has exactly one neighbor.

**Definition 1.11.** A graph  $G$  is a **tree** if  $G$  contains no cycles and is connected.

**Example 1.12.** The following is an example of a tree.



**Definition 1.13.** If a graph  $G$  contains no cycles, then  $G$  is a **forest**.

A forest can also be viewed as a direct sum of trees.

**Definition 1.14.** A **path** is a sequence of adjacent vertices in  $G$  from one vertex in  $G$  to another vertex in  $G$ . We will denote a path from vertex  $i$  to vertex  $j$  as  $P_{ij}$ .

**Definition 1.15.** A graph  $G$  is **connected** if for any two vertices in  $G$ , there exists at least one path between them.

**Definition 1.16.** A graph,  $G$ , has **girth  $N$** , if it contains at least one cycle, and its minimum cycle length is  $N$ .

Although we can describe graphs in terms of their girth, we have created a term to more simply describe this relationship.

**Definition 1.17.** A graph,  $G$ , is called a **34 - graph** if  $G$  has girth of at least 5.

We will ultimately be concerned with 34-graphs; the weakest condition on graphs for which our characterization of normality applies. In addition, we introduce the following due its ubiquity when referring to graph structures.

**Definition 1.18.** A graph  $G$  is called **triangle-free**, if  $G$  has girth 4.

**Definition 1.19.** Let  $A \in M_n$  and  $R_i(A)$  be the  $i$ th row of  $A$ . Define  $\hat{R}_i(A)$  to be  $R_i(A)$  with the diagonal entry deleted and  $\overline{R_i(A)}$  to be the complex conjugate of  $R_i(A)$ . Let  $C_j(A)$  be the  $j$ th column of  $A$ . Define  $\hat{C}_j(A)$  and  $\overline{C_j(A)}$  similarly.

**Lemma 1.20 (3).** If  $A \in M_n$  and  $A$  is a direct sum of matrices  $A_1, A_2, \dots, A_k$ , then  $A$  is normal if and if each summand is normal.

We have included a proof for demonstrative purposes of an already established result. This will be referenced to prove new results detailed later.

**Theorem 1.21.** If  $T$  is a tree on vertices with degrees  $d_1, d_2, \dots, d_n$ , then  $T$  has  $2 + \sum_{d_i \geq 2} (d_i - 2)$  pendant vertices.

*Proof.* First we will show that  $T$  has at least two pendant vertices. We will assume that  $G$  has  $m$  vertices. We know that the sum of the degree of each vertex in  $T$  is  $2(m - 1)$ . In addition, since  $T$  is a tree,  $T$  is connected, and every vertex has at least one neighbor. If there are less than 2 pendant vertices, then at least  $m - 1$  vertices have degree of at least 2 and the remaining vertices have a degree equal to one. Then the sum of the degree of each vertex is at least  $2(m - 1) + 1$ , so  $T$  is not a tree, which is a contradiction.

We are assuming that  $T$  is a tree, so once again the sum of the degree of each vertex in  $T$  is  $2(m-1)$ . If vertex  $i \in T$  has degree of  $d_i > 2$ , then we claim that there must exist an additional  $d_i - 2$  pendant vertices. Since  $T$  has 2 pendant vertices of degree one, one vertex of degree  $d_i > 2$ , and all other  $m - 3$  vertices of degree 2, the sum of the degree of each vertex is  $1 + 1 + d_i + 2(m - 3) = 2(m - 2) + d_i > 2(m - 2) + 2 = 2(m - 1)$ . Each vertex in  $T$  is connected, so each vertex must have degree of at least one. Thus, the additional  $d_i - 2$  must be subtracted from  $m - 2$  unique vertices. Otherwise, at least one vertex would have degree equal to 0. Vertex  $i$  was arbitrary, thus the argument holds for any vertex of degree greater than 2 as required.  $\square$

## 2 Basic Lemmas

In order to begin our discussion, we must first introduce a number of lemmas. In addition, new notation is introduced. We also explore a characterization of normal matrices when they are subordinate to a tree. Although the characterization was created to describe trees, it can be applied to a weaker extent to triangle-free graphs and to its full extent for 34 - graphs.

**Definition 2.1.** Let  $A \in M_n(\mathbb{C})$ .  $A$  is **absolutely symmetric** if for all  $1 \leq i, j \leq n$ ,  $|a_{ij}| = |a_{ji}|$ .

**Lemma 2.2.** A matrix  $A \in M_n$  is normal if and only if  $R_i(A) \cdot \overline{R_j(A)} = \overline{C_i(A)} \cdot C_j(A)$  for all  $1 \leq i, j \leq n$ .

*Proof.* We have  $AA^* = A^*A = N = [n_{ij}]$ . Then for each  $n_{ij}$ , we have  $R_i(A) \cdot C_j(A^*) = R_i(A^*) \cdot C_j(A)$ . Since  $C_j(A^*) = \overline{R_j(A)}$  and  $R_i(A^*) = \overline{C_i(A)}$ , the statement above is equivalent to the following: for each  $n_{ij}$ , we have  $R_j(A) \cdot \overline{R_j(A)} = \overline{C_i(A)} \cdot C_j(A)$ . Every statement for the forward direction of the proof was biconditional, therefore, the converse is the reversal of the previous argument.  $\square$

**Lemma 2.3.** If a given matrix  $A \in \mathbb{C}(G)$  is normal, then  $\|\hat{R}_i(A)\|^2 = \|\hat{C}_i(A)\|^2$  for  $i \in G$ .

*Proof.* If  $A$  is normal, then for each  $n_{ii}$  such that  $i \in G$ ,  $(A^*A)_{ii} = (AA^*)_{ii}$ . Therefore,  $R_i(A) \cdot \overline{R_i(A)} = \overline{C_i(A)} \cdot C_i(A)$ . In addition,  $R_i(A) \cdot \overline{R_i(A)} = \|R_i(A)\|^2$  and  $\overline{C_i(A)} \cdot C_i(A) = \|C_i(A)\|^2$ . Hence, we can conclude that  $\|\hat{R}_i(A)\|^2 = \|\hat{C}_i(A)\|^2$ .  $\square$

**Lemma 2.4.** Let  $A \in \mathbb{C}(G)$ . If  $A$  is absolutely symmetric, then  $(A^*A)_{ii} = (AA^*)_{ii}$ .

*Proof.* If  $A \in \mathbb{C}(G)$  is absolutely symmetric then the equalities for  $n_{ii}$ ,  $i = 1, \dots, n$ , are satisfied because  $R_i(A) \cdot \overline{R_i(A)} = \overline{C_i(A)} \cdot C_i(A)$  is trivially true.  $\square$

Lemmas 2.3 and 2.4 demonstrate that absolute symmetry is a sufficient but not necessary condition for  $(A^*A)_{ii} = (AA^*)_{ii}$ . In Section 4, we will show when absolute symmetry is a necessary condition for normality.

As an alternative description of 34 - graphs, we present the following characterization to enumerate the 3 possible relations between vertices.

**Lemma 2.5.** *A graph  $G$  is a 34-graph if and only if for any  $i, j \in G$ , one of following is true.*

- 1) *Vertex  $i$  and vertex  $j$  are adjacent, and they have no common neighbors.*
- 2) *Vertex  $i$  and vertex  $j$  have exactly one common neighbor and are not adjacent.*
- 3) *Vertex  $i$  and vertex  $j$  are not adjacent and their neighborhoods are disjoint.*

*Proof.* We will begin by showing that if  $G$  is a 34-graph, then for any vertices  $i, j \in G$  one of the conditions must be true. If  $G$  is a 34-graph, then  $G$  contains no cycles of length less than 5. Since  $G$  is triangle-free, if vertices  $i, j$  are adjacent, then they can not have a common neighbor. Thus, condition 1 is met. If vertices  $i$  and  $j$  are not adjacent, then they have at most one common neighbor. Otherwise,  $G$  contains a cycle of length 4. If vertex  $i$  and vertex  $j$  have no common neighbors, then condition 3 is true. If vertex  $i$  and vertex  $j$  have one common neighbor, then condition 2 is true. Thus, for any  $i, j \in G$ , one of the conditions is true.

In order to show the converse, we will show that if  $G$  is not 34-graph, then all 3 conditions are false. If  $G$  is not a 34-graph, then  $G$  contains a triangle or a cycle of length 4. If  $G$  contains a triangle, then there exists adjacent vertices  $i$  and  $j$  that have a one common neighbor. Thus, all 3 conditions are not met. If  $G$  contains a cycle of length 4, then there exists vertices  $i$  and  $j$ , which have 2 common neighbors. Hence, all 3 conditions are false.  $\square$

**Definition 2.6.** *Let  $A \in \mathbb{C}(G)$ . If  $j, k \in G$ , let  $\frac{\overline{a_{jj}} - \overline{a_{kk}}}{a_{jj} - a_{kk}}$  be denoted as  $e^{i\sigma_{jk}}$ . If  $a_{jj} = a_{kk}$  we will say that  $\sigma_{jk} = \infty$ .*

**Definition 2.7.** *Let  $A \in \mathbb{C}(G)$ . If  $j, k \in G$ , let  $\frac{\overline{a_{kj}}}{a_{jk}}$  be denoted as  $e^{i\rho_{jk}}$ .*

**Remark 2.8.** *We will define all  $\sigma_{jk}$  so that  $\sigma_{jk} \in [0, 2\pi]$ . Then, there exists a one to one mapping from  $e^{i\sigma_{jk}}$  to  $\sigma_{jk}$ . For simplicity, we will write  $s_{jk}$  instead of  $e^{i\sigma_{jk}}$ . Similarly, we will write  $r_{jk}$*



instead of  $e^{i\rho_{jk}}$ .

**Lemma 2.9.** *Let  $\{i, j\}$  be an undirected edge in  $G$  where  $a_{jj} \neq a_{kk}$  and vertices  $i$  and  $j$  are adjacent and have no common neighbors. If  $A \in \mathbb{C}(G)$  is normal, then  $s_{jk} = r_{jk}$ .*

*Proof.* Without loss of generality, let  $j < k$  and  $|G| = n$ . If  $A$  is normal, then  $R(j) \cdot \overline{R(k)} = \overline{C(j)} \cdot C(k)$  by Lemma 2.2. Let  $R(j) = [a_{j1}, \dots, a_{jj}, \dots, a_{jk}, \dots, a_{jn}]^t$ ,  $\overline{R(k)} = [\overline{a_{k1}}, \dots, \overline{a_{kj}}, \dots, \overline{a_{kk}}, \dots, \overline{a_{kn}}]^t$ ,  $\overline{C(j)} = [\overline{a_{1j}}, \dots, \overline{a_{jj}}, \dots, \overline{a_{kj}}, \dots, \overline{a_{nj}}]^t$ , and  $C(k) = [a_{1k}, \dots, a_{jk}, \dots, a_{kk}, \dots, a_{nk}]^t$ . Since vertex  $j$  and  $k$  are adjacent with no common neighbors, the equalities reduce to  $a_{jj}\overline{a_{kj}} + a_{jk}\overline{a_{kk}} = a_{jk}\overline{a_{jj}} + a_{kk}\overline{a_{kj}}$ . This can be rearranged to  $\frac{a_{jj} - \overline{a_{kk}}}{a_{jj} - a_{kk}} = \frac{\overline{a_{kj}}}{a_{jk}}$ , which is equivalent to  $s_{ij} = r_{ij}$ .  $\square$

**Lemma 2.10.** *Suppose that  $\{i, j\}$  is an undirected edge in  $G$  where vertices  $i$  and  $j$  are adjacent and have no common neighbors. If  $r_{ij} = s_{ij}$  where  $s_{ij} \neq \infty$ , then  $(AA^*)_{ij} = (A^*A)_{ij}$ .*

*Proof.* If  $r_{ij} = s_{ij}$  where  $s_{ij} \neq \infty$ , then  $\frac{\overline{a_{ii}} - \overline{a_{jj}}}{\overline{a_{ii}} - a_{jj}} = \frac{\overline{a_{ji}}}{a_{ij}}$ , which can be restated as  $a_{ii}\overline{a_{ji}} + a_{ij}\overline{a_{jj}} = a_{ij}\overline{a_{ii}} + a_{jj}\overline{a_{ji}}$ . This is equivalent to  $R_i(A) \cdot \overline{R_j(A)} = \overline{C_i(A)} \cdot C_j(A)$ . Therefore,  $(AA^*)_{ij} = (A^*A)_{ij}$ .  $\square$

**Lemma 2.11.** *Suppose  $G$  is a graph for which vertex  $i$  and vertex  $j$  are not adjacent and have a common neighbor vertex  $k$ . If  $A \in \mathbb{C}(G)$  is normal, then  $r_{ik} = r_{kj}$ .*

*Proof.* Since  $A$  is normal, we know that  $R(i) \cdot \overline{R(j)} = \overline{C(i)} \cdot C(j)$  by Lemma 2.2. We will assume without loss of generality that  $i < j < k$  and  $|G| = n$ . Then let  $R(i) = [a_{i1}, \dots, a_{ii}, \dots, a_{ik}, \dots, a_{in}]^t$ ,  $\overline{R(j)} = [\overline{a_{j1}}, \dots, \overline{a_{jj}}, \dots, \overline{a_{jk}}, \dots, \overline{a_{jn}}]^t$ ,  $\overline{C(i)} = [\overline{a_{1i}}, \dots, \overline{a_{ii}}, \dots, \overline{a_{ki}}, \dots, \overline{a_{ni}}]^t$ , and  $C(j) = [a_{j1}, \dots, a_{jj}, \dots, a_{jk}, \dots, a_{jn}]^t$ . Both  $R(i)$  and  $R(j)$ , have a nonzero  $k$ th entry, and for any other  $i \in 1, \dots, n$ , either  $R(i)$  or  $R(j)$  has a 0 entry. Thus,  $R(i) \cdot \overline{R(j)}$  results in only one nonzero summand  $a_{ik}\overline{a_{jk}}$ . By symmetry,  $\overline{C(i)} \cdot C(j)$  has only one nonzero summand  $\overline{a_{ki}}a_{kj}$ . Therefore,  $a_{ik}\overline{a_{jk}} = \overline{a_{ki}}a_{kj}$ , which can be written as  $\frac{\overline{a_{ki}}}{a_{ik}} = \frac{\overline{a_{jk}}}{a_{kj}}$ . Hence,  $r_{ik} = r_{kj}$ .  $\square$

**Lemma 2.12.** *Suppose that  $A \in \mathbb{C}(G)$  for which vertex  $i$  and vertex  $j$  are not adjacent and have a common neighbor  $k$ . If  $r_{ik} = r_{kj}$ , then  $(AA^*)_{ij} = (A^*A)_{ij}$ .*

*Proof.* We will consider  $i, j \in G$  such that vertex  $i$  and vertex  $j$  have a common neighbor  $k$ . If  $r_{ik} = r_{kj}$ , then  $\frac{\overline{a_{ki}}}{a_{ik}} = \frac{\overline{a_{jk}}}{a_{kj}}$ , which is equivalent to  $a_{ik}\overline{a_{jk}} = \overline{a_{ki}}a_{kj}$ . Therefore,  $R(i) \cdot \overline{R(j)} = \overline{C(i)} \cdot C(j)$ .  $\square$

**Lemma 2.13.** *Let  $A \in \mathbb{C}(G)$ . If vertex  $i$  and vertex  $j$  are not adjacent and their neighborhoods are disjoint, then  $(AA^*)_{ij} = (A^*A)_{ij} = 0$ .*

*Proof.* The equality  $R(i) \cdot \overline{R(j)} = \overline{C(i)} \cdot C(j)$  is trivially satisfied because for any position either  $R(i)$  or  $\overline{R(j)}$  has 0 entry so  $R(i) \cdot \overline{R(j)} = 0$ , and similarly,  $\overline{C(i)} \cdot C(j) = 0$ .  $\square$

**Lemma 2.14.** *Suppose  $A \in \mathbb{C}(G)$ ,  $\{i, j\}, \{j, k\} \in G$ , and  $a_{ii} = a_{jj}$ . If  $A$  is normal, then  $s_{ik} = s_{jk}$ .*

*Proof.* We know that vertex  $j$  and vertex  $k$  are neighbors, so by Definition 2.6,  $s_{jk} = \frac{\overline{a_{jj}} - \overline{a_{kk}}}{a_{jj} - a_{kk}}$  and  $s_{ik} = \frac{\overline{a_{ii}} - \overline{a_{kk}}}{a_{ii} - a_{kk}}$ . Since  $a_{ii} = a_{jj}$ , we conclude that  $s_{jk} = \frac{\overline{a_{jj}} - \overline{a_{kk}}}{a_{jj} - a_{kk}} = \frac{\overline{a_{ii}} - \overline{a_{kk}}}{a_{ii} - a_{kk}} = s_{ik}$ .  $\square$

**Lemma 2.15.** *Let  $A \in \mathbb{C}(G)$ . If  $A$  is absolutely symmetric, then  $r_{ij} = r_{ji}$ .*

*Proof.* We know that  $r_{ij} = \frac{\overline{a_{ji}}}{a_{ij}}$  and  $r_{ji} = \frac{\overline{a_{ij}}}{a_{ji}}$ . In addition,  $A$  is absolutely symmetric. Thus,  $|a_{ij}|^2 = |a_{ji}|^2$ , which is true if and only if  $\overline{a_{ij}}a_{ij} = \overline{a_{ji}}a_{ji}$ . Therefore,  $\frac{\overline{a_{ij}}}{a_{ji}} = \frac{\overline{a_{ji}}}{a_{ij}}$ . Thus,  $r_{ij} = r_{ji}$ .  $\square$

**Lemma 2.16.** *Let  $A \in \mathbb{C}(G)$  where  $G$  is a 34-graph and  $A$  is normal. If vertex  $i \in G$  is pendant and adjacent to vertex  $j \in G$ , then  $|a_{ij}| = |a_{ji}|$ .*

*Proof.* Since  $A$  is normal, we know that  $(A^*A)_{ii} = (AA^*)_{ii}$ . Without loss of generality, we will assume that  $i < j$ . By assumption,  $i$  is pendant in  $G$ , so  $R_i = [0, \dots, 0, a_{ii}, \dots, a_{ij}, 0, \dots, 0]$ ,  $C_i = [0, \dots, 0, a_{ii}, \dots, a_{ji}, 0, \dots, 0]$ ,  $\overline{R}_i = [0, \dots, 0, \overline{a_{ii}}, \dots, \overline{a_{ij}}, 0, \dots, 0]$ , and  $\overline{C}_i = [0, \dots, 0, \overline{a_{ii}}, \dots, \overline{a_{ij}}, 0, \dots, 0]$ . Thus,  $R(i) \cdot \overline{R(i)} = C(i) \cdot \overline{C(i)}$  reduces to  $|a_{ii}|^2 + |a_{ij}|^2 = |a_{ii}|^2 + |a_{ji}|^2$ , which is true if and only if  $|a_{ij}|^2 = |a_{ji}|^2$ . Now we can conclude that  $|a_{ij}| = |a_{ji}|$ .  $\square$

**Lemma 2.17.** *Suppose that  $A \in \mathbb{C}(G)$  is normal and  $G$  is a 34-graph, then  $r_{ij} = r$ , a constant, for all vertices  $i, j$  along a path in  $G$ .*

*Proof.* We will show that for any path in  $T$ ,  $r_{ij}$  is a constant by induction. Begin with a path of length 3 and vertices  $a, b, c$  such that  $a$  is neighbors with  $b$  and  $b$  is neighbors with  $c$ , but  $a$  is not neighbors with  $c$ . From Lemma 2.12, we know that  $r_{ab} = r_{bc}$ . Now assume for a path,  $\mathcal{P}_n$  of length  $n$ , and neighbors  $p_i, p_{i+1} \in P$ , that  $r_{p(p+1)}$  is a constant, namely  $r$ . Consider a path,  $\mathcal{P}_{n+1}$ , of length  $n+1$ , then  $r_{n(n+1)}$  is equal to  $r_{(n-1)n}$  by the base case and  $r = r_{(n-1)n}$  by assumption, therefore,  $r = r_{n(n+1)}$ . Now we can conclude that  $r = r_{ij}$  for any vertices  $i$  and  $j$  in some path,  $P$ , where  $i$  and  $j$  share a common neighbor.  $\square$

**Lemma 2.18.** *Let  $A \in \mathbb{C}(C)$  where  $C$  is any cycle. If  $A$  is normal, then  $|a_{(i-1)i}|^2 - |a_{(i+1)i}|^2 = |a_{(j-1)j}|^2 - |a_{(j+1)j}|^2 = c$  for all  $i, j = 1, \dots, m$*

*Proof.* Suppose that  $C$  has  $m$  vertices. We know that  $AA^* = A^*A = N = [n]_{ij}$ . Therefore the equalities necessary for  $n_{ii} = AA_{ii}^* = A^*A_{ii}$  are as follows:  $|a_{i(i-1)}|^2 + |a_{ii}|^2 + |a_{i(i+1)}|^2 = |a_{(i-1)i}|^2 + |a_{ii}|^2 + |a_{(i+1)i}|^2$ , and therefore  $|a_{i(i-1)}|^2 + |a_{i(i+1)}|^2 = |a_{(i-1)i}|^2 + |a_{(i+1)i}|^2$ . Hence,  $|a_{i(i-1)}|^2 - |a_{(i-1)i}|^2 = |a_{(i+1)i}|^2 - |a_{i(i+1)}|^2$ . This holds for all  $i = 1, 2, \dots, m$  where sums and differences are taken to be mod  $m$ . Hence, we can conclude that  $|a_{i(i-1)}|^2 - |a_{(i-1)i}|^2 = |a_{(i+1)i}|^2 - |a_{i(i+1)}|^2 = c$  for all  $i, j = 1, \dots, m$  where  $c \in \mathbb{R}$ .  $\square$

### 3 Trees

We begin with a characterization for normal matrices when they are subordinate to a tree and by Lemma 1.20, forests. In fact, we will show that the absolute symmetry is a necessary condition for normality when a matrix is subordinate to a forest, and we will show demonstrate that this is the widest class of graphs where this is the case. In addition, several corollaries concerning real matrices will be discussed.

**Lemma 3.1.** *Let  $A \in \mathbb{C}(T)$  where  $T$  is a tree. If  $A$  is normal, then  $A$  is absolutely symmetric.*

*Proof.* We will show that the system of equations defined by  $\|\hat{R}_i(A)\|^2 = \|\hat{C}_i(A)\|^2$  for  $i \in T$  results in absolute symmetry. We know that each equality must be true by Lemma 2.2. Since  $A$  is a tree, from Theorem 1.21, we know that  $A$  has 2 pendant vertex and let one of these vertices be vertex  $i$  adjacent to vertex  $j$ . Then  $\|\hat{R}_i(A)\|^2 = \|\hat{C}_i(A)\|^2$  implies that  $|a_{ij}| = |a_{ji}|$  by Lemma 2.16. Now we will consider vertex  $k$  that is neighbors with vertex  $j$ . We can assume that vertex  $k$  has degree greater than or equal to 2; otherwise,  $T$  is a simple path of length 3 and the above argument would imply that  $A$  is absolutely symmetric. Now,  $\|\hat{R}_j(A)\|^2 = \|\hat{C}_j(A)\|^2$  is equivalent  $|a_{ji}|^2 + |a_{jk}|^2 = |a_{ij}|^2 + |a_{kj}|^2$ , and since  $|a_{ij}|^2 = |a_{ji}|^2$ , we know that  $|a_{jk}|^2 = |a_{kj}|^2$ . This process will continue along any path where each vertex has a degree less than or equal to 2. If there is a vertex  $l$  along the path of degree  $m > 2$ , then by Theorem 1.21, there are  $m - 2$  other pendant vertices. For each pendant vertex there exists a path to vertex  $l$ , so for each neighbor,  $p$ , we have  $|a_{pl}|^2 = |a_{lp}|^2$ . Let  $N(l)$  denote the neighborhood of vertex  $l$ . Thus, the equality  $\sum_{p \in N(l)} |a_{lp}|^2 = \sum_{p \in N(l)} |a_{pl}|^2$  is equivalent to  $\|\hat{R}_l(A)\|^2 = \|\hat{C}_l(A)\|^2$ . This equality holds because we have shown that each term is equal. This process can be repeated for any vertex of degree greater than 2. Thus,  $A$  absolutely

symmetric. □

**Lemma 3.2.** *Let  $A \in \mathbb{C}(T)$  where  $T$  is a tree. If  $A$  is normal then,*

- 1)  $A$  is absolutely symmetric;
- 2)  $r_{ij} = r$ , a constant for all edges;
- 3)  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$

*Proof.* We have already shown that if  $A$  is normal and subordinate to a tree, then  $A$  is absolutely symmetric in Lemma 3.1.

We now want to show that  $r_{ij} = r$  for all  $i, j \in T$ . We will begin by fixing some path  $P_{kl}$  where for every vertex  $p, q \in P_{kl}$ ,  $r_{pq} = r_{kl}$  a constant by Lemma 3.1. Since  $A$  is subordinate to a tree, there exists a path,  $P_{ij}$ , from vertex  $i$  to vertex  $j$  for any  $i, j \in T$ . Then consider the paths from vertex  $i$  to vertex  $k$ , from vertex  $k$  to vertex  $l$ , and from vertex  $l$  to vertex  $j$ . Since the union of these paths,  $P_{ij}^*$ , is a path we can assume for any  $p^*, q^* \in P_{ij}^*$ ,  $r_{p^*q^*}$  is equal to a constant  $r^*$ . Note that  $A$  is absolutely symmetric, so we know that  $r_{i^*j^*} = r_{j^*i^*}$  by Lemma 2.15. Thus, we can consider a path along undirected edges as well. Because  $P_{kl} \subseteq P_{ij}^*$ , we can conclude that  $r^* = r_{kl}$ . The vertices selected were arbitrary for any vertices in  $T$  with a common neighbor. Thus,  $r_{ij} = r$  for  $i, j \in T$ .

Now we will show that  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$ . If  $s_{ij}$  is defined then  $a_{ii} \neq a_{jj}$ . By Lemma 2.9,  $r_{ij} = s_{ij}$ . Then, for any  $a_{ij} \in AA^*$  and  $b_{ij} \in A^*A$ ,  $a_{ij} = b_{ij} = n_{ij}$ . Therefore, the matrix must be normal. □

**Lemma 3.3.** *Let  $A \in \mathbb{C}(G)$  where  $T$  is a tree. If the following conditions are true, then  $A$  is normal.*

- 1)  $A$  is absolutely symmetric;
- 2)  $r_{ij} = r$ , a constant for all edges;
- 3)  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$

*Proof.* We will show that the 3 conditions listed are sufficient conditions for  $A \in \mathbb{C}(T)$  to be normal. We will begin by considering the characterization for a given vertex in a 34 - graph given by Definition 2.5. Then we will show that in each case, the 3 conditions are sufficient for normality.

First we will consider any  $(A^*A)_{ij}$  and  $(AA^*)_{ij}$  where vertex  $i$  is neighbors with vertex  $j$  in the graph  $G$ . Since  $A$  is a tree,  $i$  and  $j$  have no common neighbors; otherwise, the graph  $T$  would contain a triangle. Then, by Lemma 2.10,  $(A^*A)_{ij} = (AA^*)_{ij}$ .

We have shown that  $(A^*A)_{ij} = (AA^*)_{ij}$  where vertex  $i$  and vertex  $j$  are not adjacent and have a common neighbor  $k$  and  $r_{ik} = r_{kj}$  in Lemma 2.12. When  $i, j \in T$  and the neighborhoods of  $i, j$  are disjoint then  $(AA^*)_{ij} = (A^*A)_{ij}$  is true by Lemma 2.13.

Now we have shown that if the 3 conditions are true, then for every  $i, j \in T$  where  $i \neq j$ ,  $(A^*A)_{ij} = (AA^*)_{ij}$ . We have also shown that if  $A$  is absolutely symmetric, then  $(AA^*)_{ii} = (A^*A)_{ii}$  for all  $i \in T$  in Lemma 2.4. Thus, for any  $i, j \in G$ ,  $(A^*A)_{ij} = (AA^*)_{ij}$ , so  $A$  is normal.  $\square$

**Theorem 3.4.** *Let  $A \in \mathbb{C}(G)$  where  $T$  is any tree. Then  $A$  is normal if and only if:*

- 1)  $A$  is absolutely symmetric;
- 2)  $r_{ij} = r$ , a constant for all edges;
- 3)  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$

*Proof.* We have shown the forward implication and its implication in Lemma 3.2 and Lemma 3.3 respectively.  $\square$

**Definition 3.5.** *A given matrix  $A \in \mathbb{C}(G)$  is **principally normal** if every principal submatrix of  $A$  is normal.*

**Example 3.6.** *Being principally normal is a much stronger condition than being normal. In fact, we will demonstrate that any matrix can be a principal submatrix of a normal matrix.*

*Consider some arbitrary matrix  $A \in M_n$ ,*

$$\text{Let } B = \begin{bmatrix} A & A^* \\ A^* & A \end{bmatrix} \text{ so that } B^* = \begin{bmatrix} A^* & A \\ A & A^* \end{bmatrix}.$$

$$\text{Then } BB^* = \begin{bmatrix} AA^* + A^*A & A^2 + (A^*)^2 \\ (A^*)^2 + A^2 & A^*A + AA^* \end{bmatrix} \text{ and } B^*B = \begin{bmatrix} A^*A + AA^* & (A^*)^2 + A^2 \\ A^2 + (A^*)^2 & AA^* + A^*A \end{bmatrix}.$$

*Since matrix addition is commutative, we can see that  $BB^* = B^*B$ . Therefore,  $B$  is normal and has an arbitrary principal submatrix,  $A$ .*

**Corollary 3.7.** *Let  $A \in \mathbb{C}(T)$  where  $T$  is a tree. If  $A$  is normal, then  $A$  is principally normal.*

*Proof.* We will show that in the 3 necessary and sufficient conditions for a matrix subordinate to a 34 - graph to be normal stated in Theorem 3.4 are inherited by every principal submatrix. Absolute

symmetry is necessarily a property of every principal submatrix. In addition, the remaining two properties stated are local conditions that will hold as long as the graph remains a forest. Since the subgraph of a tree is at worst a forest, both properties hold. Therefore all 3 properties are inherited by any principal submatrix of  $A$ . Hence,  $A$  is principally normal.  $\square$

In the real case, we can see that a normal matrix subordinate to a tree functions similarly to the already well studied  $M_2(\mathbb{R})$  case.

**Corollary 3.8.** *Let  $A \in \mathbb{R}(T)$  where  $T$  is a tree. Then,  $A$  is normal if and only if  $A$  is symmetric or  $A$  is skew symmetric added to a scalar multiple of the identity.*

*Proof.* If  $A$  is normal and subordinate to a tree, then by Lemma 3.2, for all  $\{i, j\} \in T$ ,  $r_{ij} = r$  where  $r$  is a constant. Since every entry in  $A$  is real,  $\frac{\overline{a_{ij}}}{a_{ji}} = \frac{a_{ij}}{a_{ji}}$ . Once again by Lemma 3.2, we know that  $A$  is absolutely symmetric, so  $|\frac{a_{ij}}{a_{ji}}| = 1$ . Since both entries are real,  $\frac{a_{ij}}{a_{ji}} = 1$  or  $\frac{a_{ij}}{a_{ji}} = -1$ . In addition, for every  $i, j \in G$ ,  $\frac{a_{ij}}{a_{ji}} = r$ , and therefore  $r = 1$  or  $r = -1$ . If  $r = 1$ , then  $A$  is symmetric because  $\frac{a_{ij}}{a_{ji}} = 1$ , so  $a_{ij} = a_{ji}$  for all  $\{i, j\} \in E$ . If  $r = -1$ ,  $A$  is skew symmetric added to a scalar multiple of the identity because  $\frac{a_{ij}}{a_{ji}} = -1$  and therefore,  $a_{ij} = -a_{ji}$  for all  $\{i, j\} \in T$ .  $\square$

## 4 Absolute Symmetry

We generalize the notion of absolute symmetry seen in the previous section. We describe the conditions for which being normal implies absolute symmetry when a matrix is subordinate to a triangle-free graph. We will begin with a strong condition on the diagonal entries of the matrix and proceed to introduce a weaker condition. As we will later show, absolute symmetry is integral to many of the major results.

**Example 4.1.** *In the case of trees, we showed that if a matrix was subordinate to a tree and normal, then the matrix was absolutely symmetric. This is not the case for more general graphs. The following will show that a normal matrix does not need to be absolutely symmetric.*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

One can confirm that  $A$  is normal and not absolutely symmetric. As we will later show, the uniqueness of the diagonal entries play an integral role. In order to demonstrate this, we will show that it is necessary that all of the diagonal entries of  $A$  are equal in order for  $A$  to be normal. We will begin by showing that even an arbitrarily small change in the value of  $a_{11}$  will cause  $A$  to no longer be normal. Let  $a_{11} = 1 + \delta x$ , and then observe the following calculations:

$$\delta x \neq 2\delta x$$

$$1 + \delta x + 2 \neq 2 + 2\delta x + 1$$

$$(1 + \delta x)1 + 2(1) \neq (1 + \delta x)2 + 1$$

$$(AA^*)_{12} = (1 + \delta x)1 + 2(1) + 0 + 0 + 0 \neq (1 + \delta x)2 + 1 + 0 + 0 + 0 = (A^*A)_{12}$$

Therefore, it is necessary that  $\delta x = 0$  in order for  $A$  to be normal. This is an example of a basic circulant: a matrix where each row is a permutation of the previous row where each entry is shifted by one entry to the right of the preceding row. In fact, any basic circulant, will have this property.

The case of how distinct the diagonal entries need to be in order for normality to guarantee absolute symmetry will be the primary focus of the remainder of the section. The following property will later allow for an algorithmic proof, in order to describe the distinctness of the diagonal entries of a normal matrix that is necessarily absolutely symmetric.

We will first begin with a strong condition on the distinctness of the diagonal entries. Later in the section, we will show that we can improve upon this result.

**Definition 4.2.** *If  $A \in \mathbb{C}(G)$ , and every edge vertex in  $G$  has a distinct diagonal entry with respect*

to every vertex in its neighborhood,  $A$  is said to be **adjacently diagonally distinct**.

This is a mildly weaker claim than every diagonal entry being distinct. For our purposes, this leads to equally strong theorems as we are only concerned with the distinctness of a diagonal entry associated with a given vertex and its neighbors, namely vertices  $i$  and  $j$ . In particular, this condition guarantees  $s_{ij}$  exists.

**Lemma 4.3.** *Let  $A \in \mathbb{C}(G)$  be normal and let  $G$  be triangle-free. If  $A$  is adjacently diagonally distinct, then  $A$  is absolutely symmetric.*

*Proof.* By assumption,  $A$  is adjacently diagonally distinct and normal, so we know that the equality presented in Lemma 2.9,  $s_{ij} = r_{ij}$ , exists for every  $i, j \in G$  where  $i$  and  $j$  are adjacent. Since  $|\frac{\overline{a_{ii} - a_{jj}}}{a_{ii} - a_{jj}}| = 1$ ; we can conclude that  $|\frac{\overline{a_{ji}}}{a_{ij}}| = |r_{ij}| = 1$ . Therefore,  $|\overline{a_{ji}}| = |a_{ij}|$ , and since this describes any adjacent edges in  $G$ , it holds that  $A$  is absolutely symmetric.  $\square$

Although the above theorem shows a sufficient condition for normality to imply absolute symmetry, we can improve upon this result.

**Definition 4.4.** *For some  $A \in \mathbb{C}(G)$ , we will say that a given edge  $\{i.j\} \in G$  is **cut** if  $a_{ii} \neq a_{jj}$ .*

Note that the notion of cut is a condition on the matrix not the graph. Thus, when we say that an undirected edge of a graph  $G$  has been cut, we mean that any matrix subordinate to  $G$  will have unique diagonal entries pertaining to the cut edge.

**Definition 4.5.** *For a given vertex  $i \in G$  with neighborhood  $N_i$ , we will call the set of vertices  $i \cup N_i$  the **city** of  $i$ , denoted  $C_i$ .*

**Definition 4.6.** *For a given edge  $\{i.j\} \in G$ , we will say that  $\{i.j\}$  has **cycle number  $n$**  if it is contained in  $n$  cycles.*

We will first begin with a statement that will be used for a proof involving triangle-free graphs.

**Lemma 4.7.** *Let  $A \in \mathbb{C}(G)$  and  $A' \in \mathbb{C}(G')$  where  $G = G' - \{i, j\}$ . The equalities associated with  $(AA^*)_{ii} = (A^*A)_{ii}$  where  $\{i.j\} \in G$  has been cut are equivalent to the equalities for  $(A'A'^*)_{ii} = (A'^*A')_{ii}$  when  $A' \in \mathbb{C}(G')$ .*



*Proof.* First we will describe the equality  $(A^*A)_{ii} = (AA^*)_{ii}$ , associated with  $A \in \mathbb{C}(G)$  where the edge  $\{i,j\}$  has been cut. We will assume that vertex  $i$  has city  $C_i$ . Then, the equality for  $(A^*A)_{ii} = (AA^*)_{ii}$  is as follows  $\sum_{v \in C_i} |a_{iv}|^2 = \sum_{v \in C_i} |a_{vi}|^2$  by Lemma 2.3. Since we have already show that if  $a_{ii} \neq a_{jj}$ , then  $|a_{ij}| = |a_{ji}|$ , and then  $|a_{ij}|^2 = |a_{ji}|^2$  by Lemma 4.3. The previous equality is reduced to  $\sum_{v \in (C_i - j)} |a_{iv}|^2 = \sum_{v \in (C_i - j)} |a_{vi}|^2$  where  $C_i - j$  is the city of vertex  $i$  not containing vertex  $j$ .

Now, we will consider the equality for  $A' \in \mathbb{C}(G')$ . The city of vertex  $i$ ,  $C'_i$ , in  $G'$  is precisely the city of vertex  $i$  in  $G$  minus vertex  $j$  by the definition of  $G'$ . Therefore  $(A'A'^*)_{ii} = \sum_{v \in C'_i} |a_{iv}|^2 = \sum_{v \in C'_i} |a_{vi}|^2 = (A'^*A')_{ii}$ , which is equivalent to  $\sum_{v \in C_i - j} |a_{iv}|^2 = \sum_{v \in C_i - j} |a_{vi}|^2$  as required.  $\square$

By symmetry this will be sufficient to demonstrate that equalities associated with  $AA^*_{jj} = A^*A_{jj}$  are also identical for both  $A' \in \mathbb{C}(G')$  and  $A \in \mathbb{C}(G)$  where  $\{i,j\} \in G$  has been cut.

We will only be considering the absolute symmetry of a matrix  $A \in \mathbb{C}(G)$ . Therefore, when we say that an edge  $\{i,j\}$  is cut, we can delete the edge  $\{i,j\}$  from  $G$  and consider the resulting subgraph  $G'$ . This will allow for an ease in notation as we continue the section.

**Lemma 4.8.** *Every edge in  $G$  has cycle number 0 if and only if  $G$  is a tree.*

*Proof.* We will begin by assuming the opposite, so that at least one edge,  $\{i,j\} \in G$  has cycle number  $k$ ,  $k > 0$ , and  $G$  is a tree. Since  $\{i,j\}$  has cycle number  $k$ , we know that there exists some minimal cycle in  $G$ . Therefore,  $G$  can not be a tree.

Now we will show the converse. If  $G$  is a tree, then no edge is contained in any cycle. Therefore, every edge in  $G$  has cut number 0.  $\square$

**Lemma 4.9.** *Let  $A \in \mathbb{C}(G)$  where  $A$  is normal. If every edge in  $G$  has cycle number 0, then  $A$  is absolutely symmetric.*

*Proof.* Since every edge in  $G$  has cycle number 0, we know that  $G$  is a tree. Then  $A$  is normal and subordinate to a tree. Therefore, by Lemma 3.1,  $A$  is absolutely symmetric.  $\square$

**Definition 4.10.** *We will say that  $A \in \mathbb{C}(G)$  is **trimmed** if we cut the edges of  $G$ , so that the resulting subgraph is a tree.*

As we have seen in Lemma 4.7, if we simply cut every edge in some graph  $G$ , then this is equivalent to considering the resulting matrix  $A^0 \in \mathbb{C}(G^0)$  where for every  $(i, j) \in G$ ,  $a_{ij} = 0$ . The graph  $G^0$  is simply the set of isolated vertices in  $G$ . Any matrix subordinate to such a graph is necessarily absolutely symmetric because it will be diagonal, and thus, symmetric, which is necessarily absolutely symmetric. We will now introduce an algorithmic proof to more efficiently trim a graph  $G$ .

**Lemma 4.11.** *Let  $A \in \mathbb{C}(G)$  where  $A$  is normal and  $G$  is triangle-free and connected. If  $G$  is trimmed, then  $A$  is absolutely symmetric.*

*Proof.* We will begin by creating a systematic method to cut the graph just enough, so that  $G$  will be trimmed. Consider every edge  $\{i.j\} \in G$  with its respective cycle number  $n$ . For notational purposes, we will refer to this as  $[\{i.j\}, n]$ . Now, we will remove the edge  $[(i_1.j_1), \max(n_1)]$  where  $\max(n)$  refers to the highest cycle number of any edge  $\{i.j\} \in G$ . Since cutting edge  $(i_1, j_1)$  is equivalent to deleting it from  $G$  as shown by Lemma 4.7, it now has a cycle number of 0, and we can consider the new graph  $G^1$ . We will continue this process for each resulting subgraph until we arrive at  $G^k$  where  $[(i_k, j_k), \max(n_k)]$  such that  $\max(n_k) = 0$ . Then every edge in  $G^k$  will have a cycle number of 0 because the maximum cycle number of any edge in  $G^k$  is equal to 0. Thus,  $G^k$  is a tree. Therefore, the matrix  $A_k \in \mathbb{C}(G^k)$  is absolutely symmetric and  $A_k$  is precisely  $A$  where the entries of  $A = [a_{ij}]$  for  $i_1, \dots, i_n$ , and their respective  $j_1, \dots, j_n$  are 0 instead of the original entries. Our definition of cut implies that for each edge that was cut namely,  $i_m, j_m$  for  $m = 1, \dots, n$ ,  $a_{ii} \neq a_{jj}$ , which forces  $|a_{ij}| = |a_{ji}|$ . We can conclude that original matrix  $A$  with the sufficiently trimmed entries must be absolutely symmetric.  $\square$

It is important to note that when cutting an edge  $[\{i.j\}, n] \in G$ , it not only reduces the cycle number of edge  $\{i.j\}$  to 0, but also reduces the cycle number of any edges contained in any of the cycles that originally contained  $[\{i.j\}, n]$ . Therefore, the number of necessary cuts to trim a tree is only  $k$ , which is less than the total number of edges of the vertex. In some instance, such as a simple cycle of length  $n$ , it requires only  $k = 1$  cuts instead of  $n$  cuts.

**Lemma 4.12.** *Suppose for every  $A \in \mathbb{C}(G)$  where  $G$  is connected, we have  $A$  is normal if and only if  $A$  is absolutely symmetric, then  $G$  is a tree.*

*Proof.* We will show the contrapositive: if  $G$  is not a tree, then there exists a matrix  $A \in \mathbb{C}(G)$  where  $A$  is normal and not absolutely symmetric. If  $G$  is not a tree, then it contains at least one

cycle,  $C$ , of length  $n$ . Since  $C$  is not an isolated component of  $G$ , we can assume that every other off diagonal entry is absolutely symmetric. Otherwise we know that  $A$  is normal and not absolutely symmetric and the proof is complete. As in Example 4.1, let  $C$  be an instance of the basic circulant, with some additional other off diagonal entries and respective counterparts that are equal in absolute value corresponding to the edges in  $G$ . Then, we can conclude that  $C$  is not absolutely symmetric, and therefore,  $A$  is not absolutely symmetric.  $\square$

We can equivalently extend the result to forests and disconnected graphs as it results from Lemma 1.20.

**Theorem 4.13.** *The graph  $G$  is a forest if and only if every normal matrix  $A \in \mathbb{C}(G)$  is absolutely symmetric.*

*Proof.* We have shown both directions of the proof in the preceding lemmas.  $\square$

**Theorem 4.14.** *Let  $A \in \mathbb{C}(G)$  where  $G$  is triangle-free and  $A$  is normal. Then  $A$  is absolutely symmetric if and only if  $A$  is trimmed.*

*Proof.* We have already shown that  $G$  being trimmed is sufficient to show that  $A$  is normal only if  $A$  is absolutely symmetric in Lemma 4.11. We will prove the contrapositive of the converse. We want to show that if  $G$  is not trimmed then there exists  $A \in \mathbb{C}(G)$  such that  $A$  is normal and not absolutely symmetric. If  $G$  is not trimmed, then there exists  $A^l \in \mathbb{C}(G^l)$  where there exists an edge  $\{i,j\} \in G^l$  and a cycle count greater than 0 after the algorithm described in Lemma 4.11 has terminated. Then, there exists a cycle count greater than 0 in  $A^l$ , so a cycle exists in  $G^l$ . If there exists a cycle in  $G^l$ , then  $G^l$  is not a tree implying that  $A^l \in \mathbb{C}(G^l)$  is not absolutely symmetric. Therefore,  $A$  is also not absolutely symmetric.  $\square$

## 5 34-Graphs

We now apply absolute symmetry from the previous section to 34 - graphs. As one can see, the theorems above are directly applicable to this case as being triangle-free is a stronger condition than being a 34 - graph. We will see that once absolute symmetry is necessitated by normality, a given matrix subordinate to a 34 - graph inherits many of the characteristics seen the in the tree case. These properties include principal normality and additional stronger properties in the real case.

**Lemma 5.1.** *Let  $A \in \mathbb{C}(G)$  where  $G$  is a connected graph and a 34 - graph. If  $A$  is normal, then  $r_{ij} = r$  where  $r$  is a constant.*

*Proof.* This follows from Lemma 3.2 as the proof is not reliant that  $G$  is a tree. □

**Lemma 5.2.** *Let  $A \in \mathbb{C}(G)$  where  $G$  is a connected graph a 34 - graph. If  $A$  is normal, then  $r_{ij} = s_{ij}$  where  $r$  is a constant if  $s_{ij}$  exists.*

*Proof.* Since  $A$  is a 34 - graph, any pair of adjacent vertices will have no common neighbors, thus the result follows from Lemma 2.4. □

**Theorem 5.3.** *Let  $A \in \mathbb{C}(G)$   $G$  is a 34- graph and connected and  $A$  is trimmed. Then,  $A$  is normal if and only if:*

- 1)  $A$  is absolutely symmetric;
- 2)  $r_{ij} = r$ , a constant for all edges;
- 3)  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$ .

*Proof.* We will first show that if  $A$  is normal, then the 3 conditions are true. We have shown that  $A$  is absolutely symmetric in Theorem 4.14. In addition, we have shown the second condition and third condition in Lemma 5.1 and 5.2 respectively.

The converse follows from Lemma 3.3 as the proof does not rely on the fact that  $A$  is subordinate to a tree. □

**Lemma 5.4.** *If  $G$  is a 34-graph, then any induced subgraph of  $G$  is a 34-graph.*

*Proof.* Suppose that  $G$  is 34-graph and there exists an induced subgraph,  $G'$ , that is not a 34-graph. Then  $G'$  contains a triangle or a square and  $G' \subseteq G$ . Therefore,  $G$  contains a square or triangle and is not 34-graph, which is a contradiction. □

**Corollary 5.5.** *Let  $A \in \mathbb{C}(G)$  where  $G$  is a 34- graph and  $A$  is trimmed. Consider the partition of  $G = G_1 \oplus \dots \oplus G_n$ . Then  $A$  is normal if and only if for each  $A_i \in \mathbb{C}(G_i)$ :*

- 1)  $A$  is absolutely symmetric;
- 2)  $r_{ij} = r$ , a constant for all edges;
- 3)  $r_{ij} = s_{ij}$  if  $s_{ij} \neq \infty$ .

*Proof.* We know from Lemma 1.20 that  $A$  is normal if and only if every direct summand of  $A$  is normal. If every  $A_i$  has the above properties, then by Theorem 5.3, every  $A_i$  is normal. Thus, since every direct summand of  $A$  is normal,  $A$  is normal. Now we will show the converse. If  $A$  normal then every direct summand must be normal by Lemma 1.20. Note that by Lemma 5.4, each component,  $G_i$ , is also a 34-graph. Therefore, by Theorem 5.3, each summand,  $A_i$ , has the above properties.  $\square$

**Corollary 5.6.** *Let  $A \in \mathbb{C}(G)$  where  $A$  is trimmed and  $G$  is a 34 - graph. Then  $A$  is normal if and only if  $A$  is principally normal.*

*Proof.* If  $A$  is principally normal, then  $A$  is normal by definition. Now we will show that if  $A$  is normal then  $A$  is principally normal. Consider any principal submatrix  $A' \in \mathbb{C}(G')$ . We know by Lemma 5.4 that  $G'$  is a 34-graph. Absolute symmetry is inherited by any principal submatrix, so  $A'$  must be absolutely symmetric. In addition, the condition that  $r_{ij} = s_{ij}$  for any  $i, j \in G$  is a local condition. Since the condition holds for  $A$ . it must hold for  $A'$ . Since  $r_{ij} = r$  for all  $i, j \in G$  or every connected component of  $G$ , it must be true that  $r_{i'j'} = r$  for any  $i'j' \in G'$  or every connected component of  $G'$  because  $G' \subseteq G$ .  $\square$

We will now consider real normal matrices subordinate to a 34 - graph.

**Corollary 5.7.** *Let  $A \in \mathbb{R}(G)$  where  $G$  is a 34-graph and  $A$  is trimmed. Then,  $A$  is normal if and only if  $A$  is symmetric or skew symmetric added to a scalar multiple of the identity.*

*Proof.* The argument follows from Corollary 3.8 as the conditions are not reliant on  $G$  being a tree.  $\square$

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