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Black Holes and Finite-Temperature Field Theory in AdS/CFT

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Physics from
The College of William and Mary

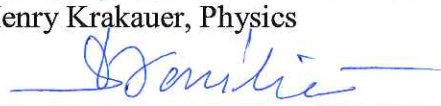
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Abstract

The Anti-de Sitter space/Conformal Field Theory (AdS/CFT) correspondence has motivated holographic models of physical phenomena at both zero and finite temperature. Zero-temperature results rely on one form of the correspondence, in which classical field theory calculations in a (usually modified) Anti-de Sitter space are said to be dual to calculations made for some quantum field theory at zero temperature. Finite-temperature calculations work similarly, except that the Anti-de Sitter space is taken to have a black hole horizon, and the correspondence is stated differently. In this work, we ask whether these two correspondences are in fact two distinct statements of some more general duality. We find that in certain limits calculations made at finite temperature using the zero-temperature statement of the correspondence match well-known finite-temperature results. Numerical results, while not conclusive, also do not rule out the possibility of a correspondence. This suggests that the two dualities may in fact be the same.

1 Introduction

For decades physicists have struggled to unify gravity and the other three known fundamental forces of nature. Out of this struggle emerged string theory, which offers the possibility of both a quantization scheme for gravity and the unification of all particles and forces. Attempts to produce a string theory consistent with the present quantum framework for particles and interactions, known as the Standard Model, have not been without obstacles, however. One such obstacle has been the fact that all known quantum string theories require more than four spacetime dimensions to be consistent. This development in recent years has led to a large interest in the study of extra dimensions.

Out of the study of gravity and extra dimensions emerged the notion of holography [1] [2]. Holography

suggests a duality between certain theories with different numbers of dimensions. In such a duality, a system that includes gravity is identical to a system in fewer dimensions without gravity. Using a “dictionary” between the two theories, it is possible to convert calculations made on one side of the duality into the results of calculations on the other side.

The development of holography culminated in 1997, when Juan Maldacena proposed a correspondence between Anti-de Sitter space (AdS) and a gauge theory called $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory, which is a conformal field theory (CFT) [3]. A few months later, the correspondence was made more formal and the beginnings of a dictionary between the two theories was created by Witten [4] [5] and by Gubser, Klebanov, and Polyakov [6].

Since then, the AdS/CFT correspondence has been used by physicists to make calculations with applications to many fields of physics. One advantage of the AdS/CFT correspondence is that it often manifests as a duality between a strongly-coupled system (usually the gauge theory) and a weakly-coupled system (usually the gravitational theory). Thus, strongly-correlated problems that usually lack attainable analytic solutions may be evaluated with AdS/CFT.

One application of AdS/CFT has been the construction of holographic models for hadrons, the bound states of strongly-interacting particles, which are normally described by Quantum Chromodynamics (QCD) [7]. Such models assume that particles exist far apart in a vacuum—in other words, the particles are at zero temperature. These calculations make use of the zero-temperature statement of the AdS/CFT correspondence. Much interest in recent years, however, has also been focused on the behavior of QCD and other quantum field theories at finite (nonzero) temperature, thereby motivating the development of a version of the AdS/CFT correspondence applicable at finite temperature.

Since the early days of the AdS/CFT correspondence, it has been believed that there is also a correspondence between an AdS spacetime with a black hole and a quantum field theory at finite (nonzero) temperature [3] [5]. Methods have been found for making calculations using such a duality [8] [9]; however, it has not been shown that the holographic QCD (AdS/QCD) models at zero temperature naturally extend to include the finite-temperature correspondence.

The goal of this project is to start with the statement of the AdS/CFT correspondence at zero temperature and attempt to extend this statement to also include phenomena at finite temperature. Our strategy is to use an AdS space with a black hole horizon (called AdS-Schwarzschild or AdS-Sch) and the zero-temperature form of the correspondence as used to construct AdS/QCD models to make a calculation for the current-current correlator of a QCD-like theory at finite temperature. Given the similarities between perturbative

QCD and Quantum Electrodynamics (QED), we shall tailor our calculations to reproduce the well-known results of finite-temperature QED.

By comparing this calculation to the result known from traditional finite-temperature quantum field theory, we will be able to determine whether or not our AdS/CFT-based calculation yields the correct result. In effect, we are starting with a zero-temperature field theory, slowly turning on a temperature, and exploring whether or not this leads to the appearance of a horizon on the AdS side of the correspondence. If successful, this would demonstrate that the correspondences already known at zero and finite temperature are in fact just different cases of a more general duality.

2 AdS/CFT at Zero Temperature

2.1 Anti-de Sitter space

Anti-de Sitter (AdS) space is a solution to the Einstein field equations for gravity in which the universe has a negative cosmological constant. AdS space in five spacetime dimensions is invariant under a particular isometry group; four-dimensional conformal field theories are invariant under this same group, partially motivating AdS/CFT [4] [10]. There are three normal spatial dimensions, one time dimension, and one extra spatial dimension, denoted by z . This space may be characterized by the metric [11]

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2). \quad (1)$$

In the above equation, $\eta_{\mu\nu}$ represents the spacetime metric for Minkowski (flat) space in 3+1 dimensions (with the mostly positive signature), and R is a constant that we set equal to 1. In these coordinates, AdS space exhibits a boundary at $z = 0$. The spacetime that we consider is described by $z > 0$.

2.2 The AdS/CFT Correspondence

The AdS/CFT correspondence connects two different theories, in two different numbers of dimensions, and motivates prescriptions that allow one to make calculations in one theory and then translate those calculations into results that are valid for the other theory. Therefore, the AdS/CFT correspondence states that two theories (in differing numbers of dimensions) are equivalent to one another, and then provides the mapping between those two theories. This allows solutions of normally difficult problems to be found easily on the other side of the duality and then transferred over.

A formal writing of the prototypical example of the AdS/CFT correspondence is as follows [4] [6] [10]:

$$Z_{AdS_5} = Z_{CFT}, \tag{2}$$

which is a relationship between the partition functions (or generating functionals) of two theories. From these partition functions, all of the information about the theories may be extracted.

Z_{CFT} is the generating functional for a conformal quantum field theory. A conformal field theory is a field theory with scale invariance (no such field theory has thus far been found to accurately describe the real world); in many cases of practical application the requirement of conformality is relaxed for this theory.

Z_{AdS_5} is the partition function for a string theory in $AdS_5 \times S^5$. This means that the spacetime under consideration looks like AdS space in 4+1 dimensions, where at each point in the spacetime there also lives a curled-up 5-dimensional sphere. When performing phenomenological model building, the S^5 aspect is often ignored. Furthermore, when the field theory is not conformal, the corresponding symmetry on the AdS side is broken by some alteration to the spacetime.

It is extremely difficult, however, to make calculations in string theory, and so one might be led to wonder why a correspondence with the partition function of a string theory is at all useful for making calculations easier. However, when certain limits are taken on the field theory side, Z_{AdS_5} may be taken to be the generating functional for the classical Euclidean action of a gravitational theory, with no appeal necessary to string theory.

A Euclidean action describes a theory in which space and time are weighted with the same sign in the spacetime metric. Recall the Minkowski metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. Notice that the time direction is weighted by -1 , while the spatial directions are weighted by $+1$. To produce a Euclidean metric, one takes $t \rightarrow i\tau$. It follows that the metric becomes $ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2$, where all terms now possess positive coefficients. Thus, if τ is taken to be the new time of the theory, this time is treated in exactly the same way as space. This new type of time is often referred to as Euclidean time or imaginary time.

The generating functional for a theory is given by $Z = e^{iS}$, where $S = \int dt d^3x \mathcal{L}$ is the action of the theory, and \mathcal{L} is the Lagrangian density. Making the substitution $t \rightarrow i\tau$ yields $S_E = i \int d\tau d^3x \mathcal{L}$, and $e^{iS} \rightarrow e^{-S_E}$, where S_E is the Euclidean action.

The correspondence we have been describing thus far is the prototypical example of the AdS/CFT correspondence, which relates Anti-de Sitter space to a conformal field theory. It is possible, however, to break the conformal symmetry on the AdS side, altering the correspondence so that it then matches to a

non-conformal field theory. All field theories presently known to describe nature lack conformal symmetry, making this a particularly useful extension of the correspondence.

Thus, the AdS/CFT correspondence is taken to be:

$$Z_{FT} = e^{-S_E}, \tag{3}$$

where S_E is the Euclidean action of a classical field theory in the AdS background, and Z_{FT} is the generating functional of some quantum field theory that is no longer necessarily conformal.

2.3 AdS/QCD

In this section, we review the techniques used to derive the hardwall AdS/QCD model of hadrons. These are the techniques that we want to extend to encompass finite-temperature effects. We follow the methods described in [11].

We use the metric for AdS space given by $ds^2 = \frac{1}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$. In order to reproduce results relevant for QCD, we include in this spacetime a classical vector field with the action

$$S = -\frac{1}{4g_5^2} \int d^4x dz \sqrt{|g|} F_{MN} F_{PQ} g^{MP} g^{NQ}. \tag{4}$$

Solving for the equations of motion for this theory in AdS space, we find

$$\frac{1}{z} \partial_\mu F^{\mu\nu} + \partial_z \left(\frac{1}{z} F^{z\nu} \right) = 0 \tag{5}$$

$$\frac{1}{z} \partial_\mu F^{\mu z} = 0. \tag{6}$$

Because QCD is not a conformal field theory, the extra-dimensional side of the correspondence should also lack a conformal invariance. This means that in AdS/QCD, the conformal invariance of the AdS space must be broken in some way. This is accomplished in the hardwall model of AdS/QCD by considering only a slice of the AdS space between the boundary of the AdS space (at $z = \epsilon$) and some other, infrared boundary that is imposed artificially at $z = z_m$. This infrared boundary breaks the conformal invariance. Our vector field needs a boundary condition at $z = z_m$, and we choose this to be $F_{\mu z}(x, z_m) = 0$.

The next step in the AdS/QCD procedure is to fix a gauge, $V_z = 0$. Given that the goal of the model is to derive results in 3+1 dimensions, this gauge choice imposes the condition that the 5-dimensional vector field

have only nonzero value in what will eventually be physical directions in the 4-dimensional theory. Thus, this gauge is chosen in part to gain physically comprehensible results. Making this gauge choice simplifies the equations of motion to

$$\frac{1}{z}\partial_\mu F^{\mu\nu} - \partial_z\left(\frac{1}{z}\partial_z V^\nu\right) = 0 \quad (7)$$

$$\partial_z\partial_\mu V^\mu = 0. \quad (8)$$

The second equation of motion now looks like a transverseness relation, indicating that $\partial_\mu V^\mu$ does not depend on z . Because when we set $V^z = 0$, we used a gauge transformation of the form $V'^z \rightarrow V^z = V'^z - \partial^z \epsilon(t, x, z) = 0$, this lack of z -dependence in $\partial_\mu V^\mu$ suggests that we still retain some gauge freedom, in the form of an additional gauge transformation $\tilde{\epsilon} = \tilde{\epsilon}(t, x)$ that does not depend on z . Because of this gauge freedom, we may make an additional gauge choice that sets $\partial_\mu V^\mu = 0$.

In order to make calculations using AdS/CFT, we need the vector field to be nonvanishing at the boundary $z = 0$. This may be realized by writing the expression for the vector field in terms of what is called the bulk-to-boundary propagator $V(q, z)$, where q is the Fourier conjugate variable to the 4-dimensional x directions. We write

$$V^\mu(x, z) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} V_0^\mu(q) V(q, z) \quad (9)$$

where $V_0^\mu(x)$ is the value of the field at $z = 0$. Therefore, we have $V(q, 0) = 1$ and $\partial_z V(q, z)|_{z=0} = 0$. From the expression for $V^\mu(x, z)$ and the form of the equations of motion, it is apparent that the equation of motion for $V(q, z)$ is

$$\frac{q^2}{z} V(q, z) + \partial_z\left(\frac{1}{z}\partial_z V(q, z)\right) = 0. \quad (10)$$

This equation of motion allows one to solve for the bulk-to-boundary propagator.

We now take advantage of the statement of the AdS/CFT correspondence, $Z_{AdS} = Z_{FT}$. We consider the action for the vector field in AdS space, given by $S = -\frac{1}{4g_5^2} \int d^4 x dz \sqrt{|g|} F_{MN} F_{PQ} g^{MP} g^{NQ}$, along with the transverseness condition $\partial_z \partial_\mu V^\mu = 0$. Integrating the action by parts, performing several Fourier transforms, and imposing this transverseness condition, we arrive at the expression [11].

$$S = -\frac{1}{2g_5^2} \int \frac{d^4 \bar{q}}{(2\pi)^4} \int d^4 \bar{x} d^4 \bar{x}' \left[e^{-i\bar{q} \cdot (\bar{x} - \bar{x}')} V_0^\mu(\bar{x}') (g_{\mu\nu} - \frac{\bar{q}_\mu \bar{q}_\nu}{\bar{q}^2}) V_0^\nu(\bar{x}) \frac{1}{z} \partial_z V(\bar{q}, z) \right] \Big|_{z=\epsilon} \quad (11)$$

The AdS/CFT correspondence states that this expression acts as the action on both sides of the corre-

spondence. This allows for the derivation of the current-current correlator on the field theory side of the correspondence.

The current-current correlator may be calculated by taking two functional derivatives of the action as follows [11]

$$\langle J^\mu(x)J^\nu(0) \rangle = \frac{\delta^2 S}{\delta V_\mu(x)\delta V_\nu(0)} \quad (12)$$

$$\frac{\delta^2 S}{\delta V_\mu(x)\delta V_\nu(0)} = -\frac{1}{2g_5^2} \int \frac{d^4 \bar{q}}{(2\pi)^4} \left[e^{i\bar{q}\cdot x} \left(g_{\mu\nu} - \frac{\bar{q}_\mu \bar{q}_\nu}{\bar{q}^2} \right) \frac{1}{z} \partial_z V(\bar{q}, z) \right]_{z=\epsilon}. \quad (13)$$

Fourier transforming the above expression, we are left with a current-current correlator of the form

$$\int d^4 x e^{-iq\cdot x} \langle J^\mu(x)J^\nu(0) \rangle = -\frac{1}{2g_5^2} \left(g_{\mu\nu} - \frac{\bar{q}_\mu \bar{q}_\nu}{\bar{q}^2} \right) \frac{1}{z} \partial_z V(\bar{q}, z)|_{z=\epsilon} \quad (14)$$

In the above expression, the bulk-to-boundary propagator can be calculated from the equations of motion, allowing one to map phenomenologically between the picture on the AdS side and field theory predictions. Matching to a perturbative QCD calculation fixes the value of g_5^2 . One finds a viable holographic model for hadrons with [7] [11]

$$g_5^2 = \frac{12\pi^2}{N_c}. \quad (15)$$

In the above expression, N_c is the number of colors of the theory. For simplicity, and because we will at times be looking at Quantum Electrodynamics (QED) rather than QCD, we take $N_c = 1$ in the remainder.

For a more complete picture of these calculations, including a description of the calculation of the bulk-to-boundary propagator and a treatment of the matching between the holographic model and perturbative QCD results, see Section 7.1 of [11].

The techniques we have reviewed here for zero-temperature AdS/QCD are well-understood in the literature. In this project, we wish to understand if it is possible to understand perturbatively the appearance of a black hole horizon in the Anti-de Sitter space by slowly turning on a temperature on the field theory side of the correspondence. In other words, we wish to determine if this statement of the correspondence also applies at finite-temperature. The approach we take is to apply the techniques described above to an AdS space with a black hole horizon (called AdS-Schwarzschild and abbreviated AdS-Sch). It has been long believed [3] [5] that such a spacetime corresponds to a finite-temperature field theory, and a dictionary of a different form from the zero-temperature version has been developed to handle such calculations [8] [9]. The difference in our approach is that we continue to use the zero-temperature dictionary at finite-temperature.

Should this evolution of the correspondence from zero to finite temperature prove successful, we would expect not only for our results to match calculations in finite-temperature field theory, but also for our expression for the coupling constant g_5^2 to be identical to the coupling constant at zero temperature.

3 Finite-Temperature Field Theory

Quantum field theory can be classified in two forms: zero temperature and finite temperature. Zero-temperature field theory is used for calculations of field theoretic observables expanded about the vacuum, as might apply to electron-electron collisions or the corrections to the electron $g - 2$. In contrast, finite-temperature field theory applies to a statistical ensemble of particles characterized by temperature, density, and chemical potential. Examples of systems where finite-temperature field theory applies include white dwarf stars, neutron stars, and the quark-gluon plasma produced by heavy-ion collisions [12]. All of these examples are characterized by densely-packed, interacting particles.

Note that in this section we switch our convention of the signature of the Minkowski metric $\eta^{\mu\nu}$ to mostly minus, in order to be consistent with the majority of the literature on finite-temperature field theory.

3.1 The Role of Time

We have discussed Euclidean time in terms of the AdS/CFT correspondence, but so far we have failed to provide a field-theoretic motivation for it.

There are two primary reasons for using Euclidean time. The first frequently arises in zero-temperature quantum field theory calculations, and has a simple motivation: to make calculations simpler [13] [14]. Often integrals in quantum field theory arise with the form $\int d^4p \frac{1}{(p^2+a)^n+i\epsilon}$. p^2 is some 4-vector, so $\int d^4p \frac{1}{(p^2+a)^n+i\epsilon} = \int d^4p \frac{1}{(p_0^2-\vec{p}^2+a)^n+i\epsilon}$, expanding the square with the Minkowski metric; the $i\epsilon$ factor carries the information about what contour to evaluate the integral along. To solve such an integral, it is helpful to cast it in a spherically symmetric form; in other words, it would be beneficial for the p_0^2 and \vec{p}^2 terms to have the same sign. Consider the p_0 integral:

$$\int dp_0 \frac{1}{(p_0^2 - \vec{p}^2 + a)^n + i\epsilon} \tag{16}$$

In the complex plane, this integral has poles slightly below $p_0 = \sqrt{\vec{p}^2 - a}$ and slightly above $p_0 = -\sqrt{\vec{p}^2 - a}$. A 90° counter-clockwise rotation of the integration contour from the real axis to the imaginary axis does not cross these poles; therefore, it yields the same solution to the integral as the original contour.

This is equivalent to taking $p_0 \rightarrow ip_4$, where p_4 is a new integration variable. Note that p^4 has the form of a Euclidean time. Now the integral is of the form [14]

$$\int d^4p \frac{1}{(p_0^2 - \vec{p}^2 + a)^n + i\epsilon} \rightarrow \int idp_4 d^3p \frac{1}{(-p_4^2 - \vec{p}^2 + a)^n + i\epsilon} \quad (17)$$

and we have achieved our goal of a spherically symmetric integrand.

We have seen that one use for Euclidean time is to simplify integrals that frequently arise in zero-temperature field theory. Such a rotation of the contour of integration in the complex plane is called a Wick Rotation. Effectively it is just a change of integration variables; any physical significance is hidden. The second frequent use for Euclidean time occurs in finite-temperature field theory, and has a much more apparent physical significance.

It is natural, in finite-temperature field theory, to imagine two different basic situations: one in which the system is in equilibrium, and one in which the system is not. Time plays a different role in each of these cases. In non-equilibrium calculations, time is used in the normal way (without the extra factor of i), leading to these calculations being referred to as “real-time” calculations. In calculations of systems in equilibrium, however, the usual notion of time plays no role, and it is useful to take $t \rightarrow i\tau$, and use Euclidean (imaginary) time. In this picture, τ becomes associated with the temperature of the system by the relation $\tau = \frac{1}{T}$, where T is a temperature. [12].

Thus, we often have expressions for the action of the form [12]

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L}. \quad (18)$$

In this case, $\beta = \frac{1}{T}$, where T is the temperature of the system in equilibrium. We also note that, for a scalar field $\phi(\vec{x}, \tau)$, the field possesses a periodicity with respect to the temperature: $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$. This requirement of periodicity in temperature can be derived by evaluation in the functional integral representation. Fermion fields have a similar periodicity, given by $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$ [12]. For a more complete discussion, see [12].

Oftentimes, field theory calculations are performed in momentum space rather than position space. In such cases, position integrals are replaced by integrals over momentum. In interacting theories, such integrals appear as part of series expansions around the coupling constant, generally expressed as Feynman diagrams. In zero-temperature field theory, these integrals would have bounds from $-\infty$ to ∞ over the 4-momentum without restriction. In finite-temperature field theory, however, this picture is complicated by Euclidean

time integral from 0 to β and by the periodicity of the fields. The p^0 integral at zero temperature is an integral over energies; thus, we would expect something similar in the finite-temperature case, altered by the Euclidean time association with temperature and the periodicity of the fields [12].

The periodicity of the fields turns the p^0 integral of zero-temperature field theory into a discrete summation at finite temperature. This sum adds integer multiples of the temperature $T = \frac{1}{\beta}$. The jump from zero to finite temperature can be summarized by the following [12]:

$$p^0 \rightarrow \omega_n \tag{19}$$

$$\int dp^0 \rightarrow T \sum_n \tag{20}$$

For bosons $\omega_n = 2\pi nT$, and for fermions $\omega_n = (2n + 1)\pi T$.

Most of the calculations of this project are performed in Euclidean time; therefore, they apply to finite-temperature calculations in equilibrium.

3.2 The Finite-Temperature Photon Self Energy

The quantity that we are going to match to at finite-temperature is called the current-current correlator. Given that we seek to extend AdS/QCD at zero temperature to include finite-temperature phenomena, it makes sense that we compare to the finite-temperature QCD current-current correlator. QCD, however, possesses several complicated features which we wish to ignore for the purposes of this project, such as the self-interaction of gluons. Therefore, we note here that, with $N_c = 3$, and without the contribution of gluon and ghost loops, QCD calculations of the current-current correlator possess the same structure and nearly the same values as the corresponding calculations for the current-current correlator in Quantum Electrodynamics (QED). In fact, the calculations only differ by a factor of 2. Therefore, in the following section, we shall provide a sketch of the calculation of the photon self energy at finite temperature and explain its relationship to the current-current correlator in QED. The current-current correlator in QCD differs by a factor of $\frac{1}{2}$ from the QED calculation, due to the normalization of the traces of group generators present in QCD given by $\text{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. Given that we are extending a zero-temperature model of QCD, we shall use the normalization for QCD.

The photon self energy arises as corrections to the photon propagator in quantum field theory. An informal, intuitive explanation for the self energy is to consider it as the correction to the electromagnetic field due to the existence of an interaction between electromagnetism and charged particles. It can also be

viewed as the physical consequence of photons possessing the ability to split into virtual electron-positron pairs which immediately recombine into a photon. The photon self energy is often denoted by $\Pi^{\mu\nu}(q)$ and arises from loops in the photon propagator terms in Feynman diagrams.

Turning our attention to finite-temperature field theory, we note that the self energy of the photon may be evaluated to one loop using the Feynman rules for finite-temperature field theory. It may then be split into a vacuum part (also found using zero-temperature field theory) and a matter part. The components of the matter part are as follows [12]:

$$\Pi_{mat}^{00} = -\frac{e^2}{\pi^2} Re \int_0^\infty \frac{dp p^2}{E_p} N_F(p) \left[1 + \frac{4E_p \omega - 4E_p^2 - k^2}{4p|\vec{k}|} \ln\left(\frac{R_+}{R_-}\right) \right] \quad (21)$$

$$\Pi_{mat\mu}^\mu = -2\frac{e^2}{\pi^2} Re \int_0^\infty \frac{dp p^2}{E_p} N_F(p) \left[1 - \frac{2m^2 + k^2}{4p|\vec{k}|} \ln\left(\frac{R_+}{R_-}\right) \right], \quad (22)$$

where $k^2 = \omega^2 - \vec{k}^2$, $E_p = (\mathbf{p}^2 + m^2)^{1/2}$, $N_F(p) = \frac{1}{e^{\beta(E_p - \mu)} + 1} + \frac{1}{e^{\beta(E_p + \mu)} + 1}$, and $R_\pm = k^2 - 2k_0 E_p \pm 2p|\vec{k}|$.

It is often beneficial to split the photon self energy into transverse and longitudinal parts, which can be considered separately. This will be useful in the limit $\vec{k} = 0$ which we shall consider later.

To split the photon self energy into transverse and longitudinal parts, we write it as [12]

$$\Pi^{\mu\nu} = GP_T^{\mu\nu} + FP_L^{\mu\nu}. \quad (23)$$

Here, F and G are momentum-dependent functions containing the functional form of the transverse and longitudinal parts, respectively. $P_T^{\mu\nu}$ and $P_L^{\mu\nu}$ carry the tensor structure of $\Pi^{\mu\nu}$ and are called the transverse and longitudinal projection operators. Thus, $GP_T^{\mu\nu}$ contains the information about transverse part of $\Pi^{\mu\nu}$, while $FP_L^{\mu\nu}$ carries the same for the longitudinal part.

The components of the projection operators are given in four dimensions by [12]

$$\begin{aligned} P_T^{0\mu} &= P_T^{\mu 0} = 0 \\ P_T^{ij} &= \delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \\ P_L^{\mu\nu} &= \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} - P_T^{\mu\nu}. \end{aligned}$$

A limit of the expression for $\Pi^{\mu\nu}$ that is of interest when considering plasmas is given when $|k^2| \ll T^2$. In this case, $G(\omega, \vec{k}) = m_P^2 - \frac{1}{2}F(\omega, \vec{k})$ and $F(\omega, \vec{k}) = -2m_P^2 \frac{k^2}{|\vec{k}|} \left[1 - \frac{\omega}{2|\vec{k}|} \ln\left(\frac{\omega + |\vec{k}|}{\omega - |\vec{k}|}\right) \right]$, where $m_P^2 = \frac{1}{6}e^2 T^2$ [12].

We now consider the limit of these expressions when $\vec{k} \rightarrow 0$. We will eventually also use this limit in the AdS/CFT calculation. Expanding around small \vec{k} and then taking the limit $\vec{k} \rightarrow 0$, we find that $F(\omega, \vec{k} \rightarrow 0) = \frac{1}{9}e^2T^2$. Using our expression for G , we find $G(\omega, \vec{k} \rightarrow 0) = \frac{1}{9}e^2T^2$ [12] [15]. We may use these expressions to calculate $\Pi^{\mu\nu}$ in the limit that $|k^2| \ll T^2$ and $\vec{k} \rightarrow 0$. When making this calculation, we do not immediately apply the $\vec{k} \rightarrow 0$ limit to the projection operators; this leads to nonsensical results. Rather, we evaluate $\Pi^{\mu\nu}$ using our expressions for F and G in this limit with the full expressions for the projection operators and apply the limit at the end of the evaluation.

The expression for Π^{00} is given by $GP_L^{00} + FP_T^{00} = GP_L^{00} = 0$, where the last equality follows after taking the limit $\vec{k} \rightarrow 0$. Likewise, $\Pi^{i0} = \Pi^{0i} = 0$. We find that the only nonzero expression is $\Pi^{ij} = GP_L^{ij} + FP_T^{ij} = G\delta^{ij}$. We now make the switch from QED to QCD, although we have suppressed the group generator indices and set $N_c = 1$. The self energy of QED is related to QCD by $\Pi_{QED}^{\mu\nu} = \frac{1}{2}\Pi_{QCD}^{\mu\nu}$. Thus, we have [16]

$$\Pi_{QCD}^{00} = \Pi_{QCD}^{i0} = \Pi_{QCD}^{0i} = 0 \quad (24)$$

$$\Pi_{QCD}^{ij} = \frac{1}{18}e^2T^2\delta^{ij}. \quad (25)$$

Similar expressions can be found in the $\vec{k} \rightarrow 0$ limit while also setting $\omega = 0$. These will be the easiest expressions to match to, and are as follows [16]

$$\Pi_{QCD}^{ij} = \Pi_{QCD}^{i0} = \Pi_{QCD}^{0i} = 0 \quad (26)$$

$$\Pi_{QCD}^{00} = \frac{1}{6}e^2T^2. \quad (27)$$

Note that in this case, Π^{ij} is now equal to 0, while Π^{00} no longer is. Thus, there appears to be a discontinuity between the cases when $\omega = 0$ and $\omega \neq 0$. In other words, there seems to be no smooth transition between the two cases, at least when $T \gg k^2$. This apparent discontinuity will also be shown to appear on the AdS side, through an additional gauge condition. Note that we will leave off the QCD subscript for results from now on; we will always mean QCD with $N_c = 1$, unless specifically stated otherwise. We also suppress the group theory labels.

So far, we have been discussing the calculations of the self energy. In AdS/CFT, however, we will be computing the current-current correlator. The current-current correlator looks like the one-loop self energy, except without the inclusion of vertices in the Feynman diagrams; these vertices contribute factors of the coupling e . Therefore, to convert our self-energy calculations to those of the current-current correlator, all

we need do is peel off the factors of e^2 .

4 AdS/CFT at Finite Temperature

4.1 AdS-Schwarzschild

We now seek to extend our AdS/CFT correspondence to include phenomena at finite temperature. In the 1970s, Hawking showed that black holes possess a temperature [17]. Due to this, in the past it has been conjectured that the gravitational dual to a conformal field theory at finite temperature is an asymptotically AdS spacetime that includes a black hole horizon [3] [5].

This motivates us to include a black hole horizon in our AdS spacetime. In order to do this, we must modify the AdS spacetime metric to include this extra feature [18]:

$$ds^2 = \frac{R^2}{z^2}[-f(z)dt^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j],$$

where i and j run over spatial components (not z), δ_{ij} is the Kronecker delta, z is the direction of the extra dimension, and $f(z)$ describes the black hole horizon. This spacetime is called AdS-Schwarzschild (AdS-Sch).

It is apparent that this metric has a similar form to the metric for regular AdS space, with the addition of the function $f(z)$. This function dictates the location of the horizon. We choose it such that $f(z_H) = 0$, so that the horizon occurs at $z = z_H$, and also so that it satisfies the Einstein field equations. One possible choice of this function is $f(z) = 1 - \frac{z^p}{z_H^p}$ in $p+1$ dimensions.

Notice that when $z = z_H$, the dt^2 term is multiplied by a factor of 0, and the dz^2 term is multiplied by a factor of infinity. What this means is that to an observer in these coordinates it takes an infinite amount of time to go an infinitesimal distance when they are very near to the horizon.

It is also worth noting that this function $f(z)$ breaks the Lorentz invariance of the metric, weighing differently the time component, the extra-dimensional component, and the normal spatial components. This will make calculating with this metric more difficult than when using the normal AdS metric.

Finally, consider the fact that the horizon occurs when $z = z_H$, but that there is no restriction on x^i , the normal spatial components. Thus, this black hole is not spherical. Rather, it is a planar black hole with a horizon at the plane of constant $z = z_H$ [10].

We now have a black hole in our AdS space, with a horizon at $z = z_H$. Black holes, however, have a temperature [17]. We shall now derive the Hawking temperature of the black hole in our AdS-Sch space,

following the method described in [18].

We begin with the metric $ds^2 = \frac{R^2}{z^2}[-f(z)dt^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j]$. We are interested in a system in equilibrium, so we switch to imaginary time $t \rightarrow i\tau$.

$$ds^2 = \frac{R^2}{z^2}[f(z)d\tau^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j] \quad (28)$$

Because we are now working in imaginary time, our time τ becomes periodic (recall the discussion of periodic time from finite-temperature field theory). We begin our calculation of the temperature with the assumption that the spacetime lacks a conical singularity at $z = z_H$. Very close to the horizon, then, our spacetime appears flat. Considering only the z and τ directions, we imagine making an analogy of polar coordinates around the horizon, with z as the radial direction and τ as the angular direction. This works nicely due to the periodicity of τ , and we imagine these polar coordinates being centered at the horizon, $z = z_H$.

We now “draw” a circle around the horizon. Given the nature of z and τ , we have two methods for calculating the circumference of this circle. Using both methods and then comparing them will provide us with our expression for the Hawking temperature of the black hole.

Our first method for calculating the circumference (which we shall call \mathcal{C}) focuses on the z direction, which we are thinking of as the radial direction. We say that $\mathcal{C} = 2\pi z^*$, where z^* is the distance from the circle to the horizon. Because our spacetime metric is nontrivial, we still need to take into account the geometry when evaluating \mathcal{C} in this way.

$$\mathcal{C} = 2\pi z^* = 2\pi \int_{z_H - \epsilon}^{z_H} \frac{Rdz}{z\sqrt{f(z)}} \quad (29)$$

The integral in this expression comes from the form of the AdS-Sch metric. We evaluate this integral assuming a small displacement from the horizon.

$$2\pi z^* = 2\pi \int_{z_H - \epsilon}^{z_H} \frac{Rdz}{z\sqrt{f(z)}} = 2\pi R \int_{z_H - \epsilon}^{z_H} \frac{dz}{z_H \sqrt{f'(z_H)} \sqrt{-(z_H - z)}} = 4\pi R \frac{\sqrt{\epsilon}}{z_H \sqrt{-f'(z_H)}} \quad (30)$$

Thus, we now have one expression for \mathcal{C} , which is $\mathcal{C} = 4\pi R \frac{\sqrt{\epsilon}}{z_H \sqrt{-f'(z_H)}}$.

We now turn to our second method for calculating \mathcal{C} . Recall that τ is periodic in the spacetime, and that $\tau = \frac{1}{T}$, where T is, in this case, the Hawking temperature of the spacetime. By calculating a complete cycle of τ , we will have our second expression for \mathcal{C} .

Once again, we must consider the fact that distances in the AdS-Sch spacetime are nontrivial. Drawing from the metric, we have $\beta^* = \sqrt{f(z)} \frac{R}{z} \tau$, where β^* is the distance along τ of one complete revolution of the circle. Considering a value for z of $z = z_H - \epsilon$, we find (once again assuming that our circle is close to the horizon)

$$\beta^* = \sqrt{f(z_H - \epsilon)} \frac{R}{T(z_H - \epsilon)} = \sqrt{f'(z_H)(-\epsilon)} \frac{R}{T(z_H - \epsilon)} = \sqrt{-f'(z_H)\epsilon} \frac{R}{Tz_H} \quad (31)$$

But $\beta^* = \mathcal{C}$, so we now have two expressions for \mathcal{C} . Setting them equal to one another, we find $\sqrt{-f'(z_H)\epsilon} \frac{R}{Tz_H} = 4\pi R \frac{\sqrt{\epsilon}}{z_H \sqrt{-f'(z_H)}}$. This yields the following expression for the Hawking temperature [18]:

$$T = \frac{-f'(z_H)}{4\pi} \quad (32)$$

Choosing the expression for $f(z)$ we specified earlier, we find that the Hawking temperature of a black hole horizon in a $p+1$ -dimensional AdS-Sch space is given by

$$T = \frac{p}{4\pi z_H}. \quad (33)$$

In our case, we are considering a 4+1-dimensional spacetime, so our Hawking temperature is

$$T = \frac{1}{\pi z_H}. \quad (34)$$

This expression gives us a relationship between the location of the horizon in the AdS-Sch space, z_H , and the temperature of the black hole, which will also be associated with the temperature on the field theory side of the correspondence.

4.2 Vector Fields in AdS-Sch

We are now in a position to begin our calculation of the current-current correlator at finite temperature using the AdS/CFT correspondence. We will apply the techniques described for a zero-temperature holographic model of hadrons to an AdS spacetime with a black hole horizon. This corresponds to a field theory at finite temperature. Via the calculation of the Hawking temperature, we have an expression for the temperature of the field theory in terms of the position of the horizon in the AdS space, allowing us to map temperature between the geometric and field theory sides of the correspondence. Given that classical vector fields on the AdS side of the correspondence are related to currents on the field theory side [11], we shall make a

calculation on the AdS side that corresponds to the current-current correlator on the field theory side. In this project we consider only systems in equilibrium, so our calculations are done in imaginary time. Finally, we have calculations on the field theory side in the limit where $\vec{k} \rightarrow 0$, and so we will eventually work in that limit on the AdS-Sch side, in order to make the matching easier.

We begin with the action for a vector field, given by

$$S = \frac{-1}{4g_5^2} \int d^4x dz \sqrt{|g|} F_{MN} F_{PQ} g^{MP} g^{NQ}, \quad (35)$$

where capitalized roman letters run over all spacetime components, $F_{MN} = \partial_M V_N - \partial_N V_M$, V_M is the vector field, and $\sqrt{|g|}$ is the square root of the absolute value of the determinant of the metric.

The metric for AdS-Sch is $ds^2 = \frac{1}{z^2} [-f(z)dt^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j]$, where δ_{ij} is the Kronecker delta, $i, j \in \{1, 2, 3\}$ for the three usual spatial dimensions, and z is the extra dimension. $f(z)$ specifies the location of the horizon, which we shall take to occur at $z = z_H$ but otherwise leave arbitrary for the moment. We shall now make the change to imaginary time $t \rightarrow i\tau$, to consider thermal effects at equilibrium. Making this change, the metric becomes

$$ds^2 = \frac{1}{z^2} [f(z)d\tau^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j]. \quad (36)$$

With this metric, it is apparent that

$$S = \frac{-1}{4g_5^2} \int d^4x dz \frac{1}{z^5} F_{MN} F_{PQ} g^{MP} g^{NQ} \quad (37)$$

It is desirable to find an expression for this action in terms of surface terms at the boundary of the AdS space (which we say occurs at $z = \epsilon$) [11]. In order to do this, we integrate by parts, noting that the fields drop off at infinity and go to zero at the horizon. This leaves us with an action of the form

$$S = \frac{-1}{4g_5^2} \left[\int d^4x \frac{2}{z} (V_0 \partial_z V_0 - V_0 \partial_0 V_z) + \int d^4x \frac{2}{z} f(z) (V_j \partial_z V_j - V_j \partial_j V_z) \right], \quad (38)$$

where this action is evaluated at $z = \epsilon$. This form of the action will eventually allow us to calculate the current-current correlator.

We now move on to calculating the classical equations of motion of the vector field in the AdS-Sch space with imaginary time. Beginning with the action for the theory in its original form, we have $S =$

$\frac{-1}{4g_5^2} \int d^4x dz \frac{1}{z^5} F_{MN} F_{PQ} g^{MP} g^{NQ}$. Using the Euler-Lagrange equations for a field theory, we find that the equations of motion for this theory are:

$$\frac{1}{z} f^{-1}(z) \partial_\ell F^{0\ell} + \partial_z \left(\frac{1}{z} F^{0z} \right) = 0 \quad (39)$$

$$\frac{1}{z} f^{-1}(z) \partial_0 F^{i0} + \frac{1}{z} \partial_\ell F^{i\ell} + \partial_z \left(\frac{1}{z} f(z) F^{iz} \right) = 0 \quad (40)$$

$$\partial_0 F^{0z} + f(z) \partial_\ell F^{z\ell} = 0. \quad (41)$$

The equations of motion split into three distinct equations due to the breaking of Lorentz symmetry by the function $f(z)$.

These equations admit the gauge choice $V^z = 0$. This is a logical gauge choice to make, as we would eventually like our solutions to relate to a four-dimensional theory; setting $V^z = 0$ naturally casts the vector field in a form that is simpler to understand from a four-dimensional perspective. This gauge choice leads to the simplified equations of motion:

$$\frac{1}{z} f^{-1}(z) \partial_\ell F^{0\ell} - \partial_z \left(\frac{1}{z} \partial^z V^0 \right) = 0 \quad (42)$$

$$\frac{1}{z} f^{-1}(z) \partial_0 F^{i0} + \frac{1}{z} \partial_\ell F^{i\ell} - \partial_z \left(\frac{1}{z} f(z) \partial^z V^i \right) = 0 \quad (43)$$

$$\partial_0 \partial^z V^0 + f(z) \partial_\ell \partial^z V^\ell = 0. \quad (44)$$

4.3 The Current-Current Correlator

In order to calculate the current-current correlator in imaginary time, we begin with the expression for the action that we derived previously:

$$S = \frac{-1}{4g_5^2} \left[\int d^4x \frac{2}{z} (V_0 \partial_z V_0 - V_0 \partial_0 V_z) + \int d^4x \frac{2}{z} f(z) (V_j \partial_z V_j - V_j \partial_j V_z) \right], \quad (45)$$

where this action is evaluated at $z = \epsilon$.

Next, we need to extract our bulk-to-boundary propagators. Because of the broken Lorentz symmetry of

the equations of motion for V^0 and V^i , we can no longer use a single bulk-to-boundary propagator; we now have four. These are given by $V^\mu(x, z) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} V_0^\mu(q) V_B^\mu(q, z)$, where there is no implied summation on the μ indices on the right hand side. Fourier transforming the action twice, we are left with an expanded form of the action

$$S = \frac{-1}{4g_5^2} \left[\int d^4 x \frac{2}{z} \frac{d^4 \bar{q}}{(2\pi)^4} e^{i\bar{q} \cdot x} V_{0t}(\bar{q}) V_{Bt}(\bar{q}, z) \left[\int \frac{d^4 \bar{q}'}{(2\pi)^4} e^{i\bar{q}' \cdot x} \partial_z (V_{0t}(\bar{q}') V_{Bt}(\bar{q}', z)) - \partial_t V_z \right] \right] \quad (46)$$

$$+ \int d^4 x \frac{2}{z} f(z) \int \frac{d^4 \bar{q}}{(2\pi)^4} e^{i\bar{q} \cdot x} V_{0i}(\bar{q}) V_{Bi}(\bar{q}, z) \left[\int \frac{d^4 \bar{q}'}{(2\pi)^4} e^{i\bar{q}' \cdot x} \partial_z (V_{0i}(\bar{q}') V_{Bi}(\bar{q}', z)) - \partial_i V_z \right] \Big]_{z=\epsilon} \quad (47)$$

In order to make this action more manageable, we impose the gauge choice $V^z = 0$ and split the action into two terms, S_1 and S_2 , with

$$S_1 = \frac{-1}{4g_5^2} \left[\int d^4 x \frac{2}{z} \int \frac{d^4 \bar{q}}{(2\pi)^4} e^{i\bar{q} \cdot x} V_{0t}(\bar{q}) V_{Bt}(\bar{q}, z) \int \frac{d^4 \bar{q}'}{(2\pi)^4} e^{i\bar{q}' \cdot x} \partial_z (V_{0t}(\bar{q}') V_{Bt}(\bar{q}', z)) \right]_{z=\epsilon} \quad (48)$$

and

$$S_2 = \frac{-1}{4g_5^2} \left[\int d^4 x \frac{2}{z} f(z) \int \frac{d^4 \bar{q}}{(2\pi)^4} e^{i\bar{q} \cdot x} V_{0i}(\bar{q}) V_{Bi}(\bar{q}, z) \int \frac{d^4 \bar{q}'}{(2\pi)^4} e^{i\bar{q}' \cdot x} \partial_z (V_{0i}(\bar{q}') V_{Bi}(\bar{q}', z)) \right]_{z=\epsilon} \quad (49)$$

Note that $S = S_1 + S_2$.

We shall now use similar techniques to rewrite the action as are used in Section 2. We will work with S_1 and S_2 separately.

$$S_1 = \frac{-1}{4g_5^2} \left[\int d^4 x \frac{2}{z} \frac{d^4 \bar{q}}{(2\pi)^4} \int \frac{d^4 \bar{q}'}{(2\pi)^4} e^{i(\bar{q} + \bar{q}') \cdot x} V_{0t}(\bar{q}) \partial_z V_{0t}(\bar{q}') V_{Bt}(\bar{q}', z) \right]_{z=\epsilon} \quad (50)$$

$$S_1 = \frac{-1}{4g_5^2} \left[\int d^4 \bar{q}' \frac{2}{z} \int \frac{d^4 \bar{q}}{(2\pi)^4} \delta(\bar{q}' + \bar{q}) V_{0t}(\bar{q}) V_{0t}(\bar{q}') \partial_z (V_{Bt}(\bar{q}', z)) \right]_{z=\epsilon} \quad (51)$$

$$S_1 = \frac{-1}{4g_5^2} \left[\int \frac{d^4 \bar{q}'}{(2\pi)^4} \frac{2}{z} V_{0t}(-\bar{q}') V_{0t}(\bar{q}') \partial_z (V_{Bt}(\bar{q}', z)) \right]_{z=\epsilon} \quad (52)$$

$$S_1 = \frac{-1}{4g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} \frac{2}{z} V_{0t}(-\bar{q}) V_{0t}(\bar{q}) \partial_z V_{Bt}(\bar{q}, z) \right]_{z=\epsilon} \quad (53)$$

$$S_1 = \frac{-1}{4g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} \int d^4 \bar{x} d^4 \bar{x}' e^{-i\bar{q} \cdot (\bar{x} - \bar{x}')} \frac{2}{z} V_{0t}(\bar{x}') V_{0t}(\bar{x}) \partial_z V_{Bt}(\bar{q}, z) \right]_{z=\epsilon} \quad (54)$$

We perform a similar calculation for S_2 , yielding

$$S_2 = \frac{-1}{4g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} \int d^4 \bar{x} d^4 \bar{x}' e^{-i\bar{q} \cdot (\bar{x} - \bar{x}')} \frac{2}{z} V_{0i}(\bar{x}') V_{0i}(\bar{x}) \partial_z V_{Bi}(\bar{q}, z) \right]_{z=\epsilon} \quad (55)$$

When we vary S , we can vary S_1 and S_2 separately, such that

$$\frac{\delta^2 S}{\delta V_\mu(x) \delta V_\nu(0)} = \frac{\delta^2 S_1}{\delta V_\mu(x) \delta V_\nu(0)} + \frac{\delta^2 S_2}{\delta V_\mu(x) \delta V_\nu(0)}. \quad (56)$$

Note that from the expressions for S_1 and S_2 , it is apparent that S_1 vanishes when varying with respect to spatial components, and S_2 vanishes when varying with respect to time components.

Performing these calculations separately yields

$$\frac{\delta^2 S_1}{\delta V_t(x) \delta V_t(0)} = \frac{-1}{2g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} (e^{i\bar{q} \cdot x} + e^{-i\bar{q} \cdot x}) \delta^{tt} \frac{1}{z} \partial_z V_{Bt}(\bar{q}, z) \right]_{z=\epsilon} \quad (57)$$

and

$$\frac{\delta^2 S_2}{\delta V_i(x) \delta V_j(0)} = \frac{-1}{2g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} (e^{i\bar{q} \cdot x} + e^{-i\bar{q} \cdot x}) \delta^{ij} \frac{1}{z} \partial_z V_{Bi}(\bar{q}, z) \right]_{z=\epsilon} \quad (58)$$

Thus, the expressions for the current-current correlators are:

$$\langle J^t(x) J^t(0) \rangle = \frac{-1}{2g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} (e^{i\bar{q} \cdot x} + e^{-i\bar{q} \cdot x}) \delta^{tt} \frac{1}{z} \partial_z V_{Bt}(\bar{q}, z) \right]_{z=\epsilon} \quad (59)$$

$$\langle J^i(x) J^j(0) \rangle = \frac{-1}{2g_5^2} \left[\int \frac{d^4 \bar{q}}{(2\pi)^4} (e^{i\bar{q} \cdot x} + e^{-i\bar{q} \cdot x}) \delta^{ij} \frac{1}{z} \partial_z V_{Bi}(\bar{q}, z) \right]_{z=\epsilon} \quad (60)$$

$$\langle J^i(x) J^t(0) \rangle = \langle J^t(x) J^j(0) \rangle = 0 \quad (61)$$

These expressions for the current-current correlators are written in terms of bulk-to-boundary propagators, $V_{Bt}(\bar{q}, z)$ and $V_{Bi}(\bar{q}, z)$. Note that we have written zero-component of the vector fields as $V_t(\bar{q}, z)$ rather than $V_0(\bar{q}, z)$, in order to avoid confusion. In order to solve for $V_\mu(\bar{q}, z)$ and obtain our results, we must now solve the equations of motion derived in Section 4.2.1.

4.4 The $\vec{k} \rightarrow 0$ Limit

We would like to match our expression for the current-current correlator derived via AdS/CFT at finite-temperature to the expression calculated using traditional finite-temperature quantum field theory. In order to do this, we need to solve our equations of motion for V^μ . The equations of motion we have derived, however, are difficult to solve, even with the $V^z = 0$ gauge choice. Recall, however, that we have expressions for the current-current correlator in the limit that $\vec{k} \rightarrow 0$ (in other words, with no spatial dependence). We make the same approximation in our equations on the AdS side, which corresponds to setting $\partial_\ell V^M = 0$ in the equations of motion, thus removing all (non- z) spatial dependence.

Our equations of motion become

$$\partial_z \left(\frac{1}{z} \partial^z V^0 \right) = 0 \tag{62}$$

$$\frac{1}{z} f^{-1}(z) \partial_0 \partial^0 V^i + \partial_z \left(\frac{1}{z} f(z) \partial^z V^i \right) = 0 \tag{63}$$

$$\partial_0 \partial^z V^0 = 0. \tag{64}$$

The equation of motion $\partial_0 \partial^z V^0 = 0$, in combination with the fact that $\vec{k} \rightarrow 0$, suggests that we have an additional gauge freedom, similar to that discussed in Section 2.3. Because $\partial_0 V^0$ is z -independent, we may set an additional gauge such that $\partial_0 V^0 = 0$. We are now left with two decoupled equations of motion, one for V^0 and one for V^i .

We pause to consider the implications of the gauge choice $\partial_0 V^0 = 0$ in momentum space. Fourier transforming, our gauge condition becomes $\omega V^0 = 0$. Such a condition leaves us with a choice. We may either choose $\omega = 0$ or $V^0 = 0$. It seems obvious that making one choice or the other will lead to very different forms for the current-current correlator. This corresponds to the discontinuity we noted when $\vec{k} \rightarrow 0$ on the finite-temperature field theory side.

4.4.1 The $\omega = 0$ Case

We consider first the case in which $\omega = 0$. In this scenario, we have a nonzero V^0 . Our equations of motion now have the form

$$\partial_z \left(\frac{1}{z} \partial^z V^0 \right) = 0 \quad (65)$$

$$\partial_z \left(\frac{1}{z} f(z) \partial^z V^i \right) = 0 \quad (66)$$

Fourier transforming V^0 and V^i and writing them in terms of their respective bulk-to-boundary propagators, we are left with

$$\partial_z \left(\frac{1}{z} \partial^z V_B^0 \right) = 0 \quad (67)$$

$$\partial_z \left(\frac{1}{z} f(z) \partial^z V_B^i \right) = 0 \quad (68)$$

We focus first on the equation of motion for V_B^0 . We can solve this equation analytically to yield the solution:

$$V_B^0(z) = Az^2 + B \quad (69)$$

We now have an expression for the current-current correlator in terms of two constants, A and B . In order to fully specify $V_B^0(z)$, we now consider boundary conditions.

We need two boundary conditions in order to fully determine $V_B^0(z)$. The first is given in the definition of the bulk-to-boundary propagator: $V_B^0(z)$ must go to 1 at the boundary of the AdS-Sch spacetime. Therefore, we have $V_B^0(0) = 1$, which sets $B = 1$. Next, we consider the behavior of the fields at the horizon of the spacetime. In real-time AdS/CFT, there are strange conditions at the horizon of the spacetime. We are, however, working in imaginary time, and we can make the straightforward assumption that the fields drop off at the horizon. Because $V_B^0(z)$ carries all of the z -dependence of the fields, we have a second condition, that $V_B^0(z_H) = 0$. This sets $A = -\frac{1}{z_H^2}$, and we arrive at our expression for the bulk-to-boundary propagator in imaginary time with $\vec{k} = 0$ and $\omega = 0$,

$$V_B^0(z) = -\frac{z^2}{z_H^2} + 1 \quad (70)$$

Having found an expression for $V_B^0(z)$, we now turn our attention to $V_B^i(z)$. Consider the equations of motion $\partial_z \left(\frac{1}{z} f(z) \partial^z V_B^i \right) = 0$. These may also be easily solved analytically in the following way:

$$\frac{1}{z}f(z)\frac{d}{dz}V_B^i = A \quad (71)$$

which leads to

$$\frac{d}{dz}V_B^i = \frac{Az}{f(z)} \quad (72)$$

It follows that the form of $V_B^i(z)$ may be found through integration. Recalling that $f(z) = 1 - \frac{z^4}{z_H^4}$, we find that $V_B^i = \frac{A}{2}z_H^2 \operatorname{arctanh}\left(\frac{z^2}{z_H^2}\right) + B$. Using the same boundary conditions as in the V_B^0 case yields

$$V_B^i = 1 \quad (73)$$

We may now insert our bulk-to-boundary propagators into our expressions for the current-current correlator:

$$\langle J^t(x)J^t(0) \rangle = \frac{-1}{2g_5^2} \left[\int \frac{d^4\bar{q}}{(2\pi)^4} (e^{i\bar{q}\cdot x} + e^{-i\bar{q}\cdot x}) \delta^{tt} \frac{1}{z} \partial_z V_B^0(\bar{q}, z) \right]_{z=\epsilon} \quad (74)$$

$$\langle J^i(x)J^j(0) \rangle = \frac{-1}{2g_5^2} \left[\int \frac{d^4\bar{q}}{(2\pi)^4} (e^{i\bar{q}\cdot x} + e^{-i\bar{q}\cdot x}) \delta^{ij} \frac{1}{z} \partial_z V_B^i(\bar{q}, z) \right]_{z=\epsilon}. \quad (75)$$

Noting that $\langle J^i(x)J^j(0) \rangle$ is proportional to the derivative of V_B^i , we see that it vanishes. Thus, we are left with only a nonzero $\langle J^t(x)J^t(0) \rangle$,

$$\int d^4x e^{-iq\cdot x} \langle J^t(x)J^t(0) \rangle = \frac{2\delta^{tt}}{g_5^2 z_H^2} \quad (76)$$

$$\int d^4x e^{-iq\cdot x} \langle J^i(x)J^j(0) \rangle = 0. \quad (77)$$

We almost have our current-current correlator in a form that we can match to calculations from traditional finite-temperature quantum field theory. All we need is an expression for z_H in terms of temperature. Recall from our calculation of the Hawking temperature of an AdS black hole that $T = \frac{1}{\pi z_H}$. Rearranging, we have $z_H = \frac{1}{\pi T}$, and we are now finally ready to write down a final form for our current-current correlator in imaginary time with $\vec{k} = 0$,

$$\int d^4x e^{-iq \cdot x} \langle J^t(x) J^t(0) \rangle = \frac{2T^2 \pi^2 \delta^{tt}}{g_5^2} \quad (78)$$

$$\int d^4x e^{-iq \cdot x} \langle J^i(x) J^j(0) \rangle = 0 \quad (79)$$

$$\langle J^i(x) J^t(0) \rangle = \langle J^t(x) J^j(0) \rangle = 0 \quad (80)$$

We may now compare these expressions to the results from finite-temperature field theory, which we reproduce here for convenience:

$$\Pi^{ij} = \Pi^{i0} = \Pi^{0i} = 0 \quad (81)$$

$$\Pi^{00} = \frac{1}{6} T^2 \delta^{ij}. \quad (82)$$

An initial inspection shows that the finite-temperature results have the same tensor structure as the AdS/CFT results. Next, we need to fix g_5 . Setting our two results for Π^{00} (one on the field theory side, one on the AdS side) equal to each other, we have $\frac{1}{6} T^2 \delta^{ij} = \frac{2T^2 \pi^2 \delta^{ij}}{g_5^2}$. Rearranging this inequality, we find that the two expressions match when

$$g_5 = \sqrt{12} \pi. \quad (83)$$

Thus, we now have a fixed value for the coupling on the AdS side of the correspondence. This coupling, furthermore, has precisely the value of the five-dimensional coupling that is found in the zero-temperature version of AdS/QCD (with $N_c = 1$). This indicates that the AdS/QCD picture at zero temperature may be extended to also include finite-temperature results.

4.4.2 The $\omega \neq 0$ Case

We have just considered the $\omega = 0$ case in the limit of $\vec{k} \rightarrow 0$. We now consider the alternative case where $\omega \neq 0$. We restate the expressions from finite-temperature field theory here for ease of comparison.

$$\Pi^{00} = \Pi^{i0} = \Pi^{0i} = 0 \quad (84)$$

$$\Pi^{ij} = \frac{1}{18} T^2 \delta^{ij}. \quad (85)$$

Recalling our gauge condition $\omega V^0 = 0$, notice that in order for ω to be nonzero, we must now have $V^0 = 0$. This causes the $\langle J^0(x)J^0(0) \rangle$ component of the current-current correlator to vanish, in agreement with the finite-temperature field theory calculations. We are now only interested in the $\langle J^i(x)J^j(0) \rangle$ part.

Setting $V^0 = 0$ leaves us with only one equation of motion, for V^i ,

$$\frac{1}{z}f^{-1}(z)\partial_0\partial^0V^i + \partial_z\left(\frac{1}{z}f(z)\partial^zV^i\right) = 0. \quad (86)$$

Fourier transforming and writing the fields in terms of the bulk-to-boundary propagator V_B^i , we arrive at the following equation for V_B^i :

$$-\frac{1}{z}f^{-1}(z)\omega^2V_B^i + \partial_z\left(\frac{1}{z}f(z)\partial^zV_B^i\right) = 0. \quad (87)$$

Unlike the equation of motion we considered for V^0 in the $\omega = 0$ case, this differential equation cannot be solved analytically. Instead, we are forced to use a numerical approach.

Using Mathematica's NDSolve function, we found solutions to the differential equation for given ranges of z and ω and for specific values of z_H . We took derivatives with these solutions, allowing us to plot the value of the current-current correlator versus ω . To agree with the field theory calculations, this plot should be constant, at least when $\omega^2 \ll T^2 = \frac{1}{\pi^2 z_H^2}$.

A plot of the AdS/CFT expression for the $\Pi^{11}(= \Pi^{22} = \Pi^{33})$ current-current correlator is shown in Figure 1 plotted with the calculation from finite-temperature field theory. The plot corresponds to the value $z_H = 1.2$ and $T = 0.265$, and the derivative of the vector field is evaluated at $z = 0.05$, because it diverges for exactly $z = 0$. Due to numerical difficulties when calculation at exactly $z = 0$ and $z = z_H$, we have replaced our $V(0, \omega) = 0$ boundary condition with $V(\epsilon, \omega) = 0$, where for the above plot $\epsilon = 0.01$. Rather than imposing $V(z_H, \omega) = 0$, we have set $V(z_H - 0.1, \omega) = 0$, and allowed our numerics to run from $z = \epsilon$ to $z = z_H - 0.1$. In this plot, we have taken $g_5^2 = 12\pi^2$, the value from both zero-temperature AdS/QCD and the $\omega = 0$ case of the finite-temperature AdS/CFT calculation. It is apparent that the AdS/CFT calculation of the current-current correlator at finite temperature does not agree with the value from finite-temperature field theory. Noteworthy, however, is the fact that the AdS/CFT calculation becomes flat for small ω , as would be expected. The two values also agree to about an order of magnitude.

We mentioned the numerical difficulties associated with this calculation. Fixing the boundary condition exactly at the horizon leads to numerical instability; moving the boundary condition too far from the horizon, however, jeopardizes the accuracy of the results. Using other values for the horizon and the location of the

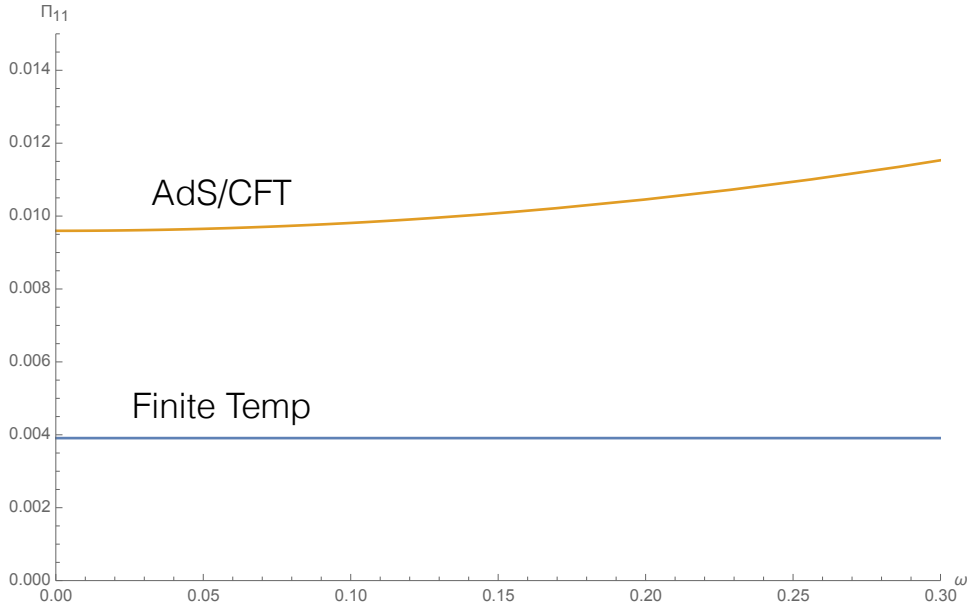


Figure 1: Comparison of Π_{11} calculations from AdS/CFT and finite-temperature field theory.

$V = 0$ boundary condition leads to results with smaller differences in the AdS/CFT prediction and the field theory calculation; such results, however, also still suffer from some numerical instability. Therefore, Figure 1 is not meant to be definitive proof that the AdS/CFT calculation does not work; on the contrary, the AdS/CFT approach may have a validity that is currently being clouded by numerical difficulties.

It is also noteworthy that while the numerics are inconclusive regarding the precise value of $\Pi^i j$ from AdS/CFT, we have accurately reproduced the tensor structure of $\Pi^{\mu\nu}$ when $\vec{k} \rightarrow 0$, $\omega \neq 0$.

5 Conclusion

In this project, we hoped to understand whether or not the AdS/CFT correspondence at zero-temperature could be extended to finite temperature by slowly turning on a temperature on the field theory side of the correspondence. Basing our approach on techniques used to construct holographic models of hadrons at zero-temperature, we applied the AdS/CFT correspondence to vector fields in an Anti-de Sitter space with a black hole horizon, producing expressions for the current-current correlator for a theory that could be hoped to be dual to Quantum Chromodynamics at finite temperature (with $N_c = 1$) that is very similar to

finite-temperature Quantum Electrodynamics. We performed our calculations for the equilibrium case, with imaginary time $t \rightarrow i\tau$. In order to evaluate our expressions for the current-current correlator, we took the limit on the AdS side that $\vec{k} \rightarrow 0$. There are two distinct cases for this on the field theory side, one when $\omega = 0$ and the other when $\omega \neq 0$. This distinction also appears on the AdS side; it became manifest in our calculations with the gauge choice $\omega V^0 = 0$.

The gauge condition ωV^0 gives us two choice: we may set $\omega = 0$ or we may set $V^0 = 0$. We examined both cases. In the $\omega = 0$ case, we solved the equations of motion analytically, allowing us to write down closed-form expressions for the current-current correlator. Fixing the five-dimensional coupling at $g_5^2 = 12\pi^2$, we exactly reproduced the finite-temperature field theory results, demonstrating an exact correspondence in the limit $\vec{k} \rightarrow 0$ when $\omega = 0$. Furthermore, this value of g_5 is precisely that found using zero-temperature AdS/QCD, suggesting that in these limits there is a smooth transition from zero- to finite-temperature AdS/QCD. The equations of motion when $V^0 = 0$ are more difficult, and may not be solved analytically. Using Mathematica, we solved these equations of motion numerically and subsequently plotted the numerical prediction of AdS/CFT for the nonzero components of the current-current correlator. Due to numerical difficulties, we are unable to make a precise statement about the agreement or disagreement of the values of the current-current correlator predicted by AdS/CFT and by finite-temperature field theory. We do, however, reproduce the tensor structure of finite-temperature expression for the current-current correlator, and our numerical results is within an order of magnitude of the field theory result.

We have shown that in certain limits the zero-temperature prescription of the AdS/CFT correspondence applies at finite temperature, suggesting that what seem to be two separate dualities—one for zero temperature and one for finite temperature—may in fact be particular cases of the same general correspondence. Future work might include generalizing the calculations performed here to results beyond the limit $\vec{k} \rightarrow 0$. There is also motivation to perform similar work in the nonequilibrium, real-time case. Past work [8] suggests that the simple boundary conditions for the fields at the horizon $z = z_H$ used in this work will no longer apply, and must be replaced by boundary conditions that are more difficult to use. This work, however, could provide further understanding of how the horizons of black holes emerge in Anti-de Sitter space.

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