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Row and Column Distributions of Letter Matrices

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Abstract

A *letter matrix* is an n-by-n matrix whose entries are n symbols, each appearing n times. The row (column) distribution of a letter matrix is an n-by-n nonnegative integer matrix that tells how many of each letter are in each row (column). A row distribution R and a column distribution C are compatible if there exits a letter matrix A whose row distribution is R and whose column distribution is C. We show that the matrix J of all ones is compatible with any C, and we also consider the the problem of when R and C pairs are compatible in terms of their values and patterns inside the distribution matrices.

1 Introduction

A *permutative matrix* is an n-by-n matrix whose entries are chosen from n distinct symbols in such a way that each row is a different permutation of the n symbols [HJDZ]. The topic was motivated by and a generalization of the latin-square, nonnegative interverse eigenvalue problems and other topics. By a *letter matrix* we mean a certain generalization of a permutative matrix in which n of each of the n symbols (called "letters") appear in the n-by-n matrix, with no restriction on their numbers in a row or column.

Example 1:

$$A = \begin{bmatrix} a & a & a & b \\ b & b & d & c \\ b & d & c & c \\ c & a & d & d \end{bmatrix}$$ is a 4-by-4 letter matrix, with 4 each of a, b, c and d.

For a given letter matrix, we may summarize the distribution of the letters by rows or by columns. The *row distribution* R is *rows-by-letters* and the *column distribution* C is *letters-by-columns*. Both of them are in *special doubly stochastic matrix* (SDSM) form, which is an integer matrix with row and column sums equal to a constant. In the case of distribution matrices, the constant is n—the dimension of the matrices.

Example 2: The row distribution of A above is

$$\text{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

and the column distribution is

$$C = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \end{array}$$

The 3 in [1,1] position of R means that A has 3 a's in its first row, and the 1 in [1,1] position of C means that there is 1 a in the first column of A.

Of course, the row and column distributions above are *compatible*, in that there is a (single) letter matrix A that explains both. As we shall see, not every pair of row and column distributions is compatible. A permutative matrix is a letter matrix whose row distribution is the n-by-n matrix J, each of whose entries is 1. Of course, other letter matrices may also have row distribution J. It turns out that this row distribution is compatible with any column distribution . (See Section 2.) The general problem of determining which pairs of row and column distributions are compatible seems very challenging, but we make some comments about it here, via the theory of 0,1 matrices with given row and column sums. (See Section 3.3)

Theorem 3: An n-by-n distribution matrix is a special doubly stochastic matrix with nonnegative integer entries and with row and column sums equal to n.

Proof: Each entry in a distribution matrix indicates the incidence of a letter in a certain row or column. Since the number of occurrences of each letter is n, the row sums of R and the column sums of C must each be n, and the entries are integers. Since each column and each row of a letter matrix has n letters appearing in it, the column suns of R (row sums of C) are also n. □

Remark: In a row distribution, each column indicates the distribution of a single letter over all the rows, and we may call it a *row-sum-vector*. In a column distribution, each row indicates the distribution of a certain letter over all the columns, and we call it a *column-sum-vector*. For letter a in A, its column-sum-vector is $\begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$ and row-sum-vector is $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

If there is a 0,1-matrix representing a placement of a that realizes a pair of row and column-sum-vectors, we call the pair *Ryser-compatible*. Here, we apply Ryser's theorem, which is presented in Section 1.1. Understanding those realizations is important for understanding the compatibility of a pair of distributions matrices with n pairs of row and column-sum-vectors, such as R and C above. (See Section 3.2.1)

1.1 Necessary Background

1.1.1 Hall's Matching Theorem:

Theorem: Suppose we have two sets of elements $K = [k_1, k_2, \dots k_n]$ and $L = [l_1, l_2 \dots l_m]$ $(n \leq m)$ and each element in K can be paired with each element in L. If, for any subset of K, say K_i, the size of the union of l's paired with K_i is at least $|K_i|$, then there is a matching from K to L. In this way, we can pair each element in K to a distinct element in L. [H]

In the first proof in Section 2, we are going to apply Hall's matching theorem to show that there is a letter matrix which is the realization of any distribution matrix with the J matrix. In the proof, the K represents the set of columns (or rows) and L stands for the set of distinct letters that can be placed in those columns. Also, $m = n$ in the context of distribution matrices.

1.1.2 Birkoff's Theorem:

Theorem: Every doubly stochastic matrix is a convex combination of permutation matrices. [B]

In the second proof in Section 2, we will show that each n-by-n distribution matrix can be written as the sum of n permutation matrices. The result is inspired by Birkoff's theorem and it directly demonstrates the relation between permutation matrices inside a distribution matrix and the permutations of each row in a letter matrix.

1.1.3 Gale-Ryser Theorem:

Theorem: Given two partitions (p and q) of an integer n, there is an n-by-n 0, 1-matrix A with row sums p and column sums q, if and only if the conjugate of p majorizes q.[R]

Here, the *conjugate* of a partition p is the set of counts of the number of elements in p which is at least 1, 2, $\ldots i$ (i is the largest element in p). For instance, if $p = [2, 4, 1]$, then the set of counts $\hat{p}$ will be $[3, 2, 1, 1]$. The *majorization* stands for that, ordering the partitions $\hat{p}$ and q descendingly, the sum of the first j elements in $\hat{p}$ is always greater or equal to the sum of the first j elements in q. Used in Section 3, the Gale-Ryser theorem is crucial in determining that, given the row and column-sum-vector of a letter, whether we can find a placement for the letter in a letter matrix.

2 Generalized Permutative Matrix

A generalized permutative matrix (gPM) is matrix whose rows are the permutations of the same set of symbols. A gPM is any letter matrix with row distribution matrix J. (As mentioned, a permutative matrix has row distribution J as well, but having row distribution J itself does not guarantee the distinctness of rows. We call any letter matrix with R being J a generalized permutative matrix)

Theorem 4: Any column distribution is gPM realizable (i.e. any C is compatible with the J matrix) and any row distribution is transpose gPM realizable (i.e. the J matrix is compatible with any R).

1. **Proof by Hall's Theorem:**
   Given C, we form a letter matrix A by placing each entry according to the column distribution without considering the row distribution. It is always possible, because the property of a special doubly stochastic matrix ensures that each column has enough slots for all its letters. (Notice that if we permute entries within the same column, the matrix will still be consistent with C.) Now we separate the A into two row parts, with the first p rows, A_1, being generalized permutative (i.e each letter appears once in every row) and the remaining $n - p$ rows, A_2, not. Then, the non-generalized permutative submatrix A_2 has a total of $(n - p)n$ entries (or "slots") and contains $(n - p)$ copies of each of the n letters. Since in A_2, the number of entries of each column is $n - p$, and we only have $n - p$ copies of each letter, then for any t columns in A_2, at least t distinct letters must appear. So, we have a matching between columns and letters and can form a new row with n different letters by only rearranging letters within each column. We increase the number of generalized permutative row to $p + 1$ in this way and can continue to apply this procedure. By reduction, we will eventually reach a matrix with rows being generalized permutative which is a realization of the original C with the J matrix, showing that they are compatible. □

2. **Proof Inspired by Birkoff's Theorem**
   From Birkoff's Theorem, we know that an n-by-n doubly stochastic matrix is a convex combination of permutation matrices. In our case, we want to show that if an n-by-n SDSM A has row and column sum n, then it is the sum of n permutation

matrices $P_1, \dots P_n$.

$$\sum_{i=1}^{n} P_i = A$$

First of all, we are able to find a permutation matrix within an n-by-n special doubly stochastic matrix, which is equivalent to finding a matching between rows and columns via nonzero entries in the matrix. In a submatrix formed by k rows $(0 \leq k \leq n)$ in SDSM, we know that each column sum is less than or equal to n and the total sum in the submatrix is kn because each row sum is n. Thus, at least k columns contains nonzero entries, and this holds true for any subset of rows in the SDSM. By Hall's theorem, we have a matching between rows and columns and can produce a permutation matrix. Subtract this permutation matrix from the SDSM, and get a new SDSM with row and column sum $n-1$. The same strategy applies and we can further delete a permutation matrix from the new SDSM. By reduction, after $n-1$ times, we will eventually reach an SDSM with row and column sum 1, which is exactly a permutation matrix. In this way, we have shown that an SDSM with row and column sum n can be written as the sum of n permutation matrices $\sum_{i=1}^{n} P_i$.

Note that some SDSMs can have more than 1 such set of n permutation matrices, since, at each step, there may be more than 1 matching. This phenomenon is strongly correlated with the fact that, given a C, there can be more than 1 realization by a gPM.

Now, we look at the correlation between the set of P's in C and the gPM A which realizes C. Each P_i represents a permutation of elements $\{a_1, a_2, \dots a_n\}$. Therefore, without loss of generality, we assume P_i represents the permutation of ith row, and have

$$[a_1 a_2 \dots a_n] P_i = e_i^T A$$

Since there are n permutation matrices in the representation of a distribution matrix, each representing a row's permutation, applying $P_1, P_2, \dots P_n$ on $\{a_1, a_2, \dots a_n\}$, we will arrive at a PM or gPM that realizes C. Since PM or gPM all have R being J, we have shown that any C is compatible with the J matrix. (Analogously, every R is compatible with the J matrix.) $\square$

Example 5:

$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which also corresponds to the transformation of $c \to b \to (c)$, $a \to c \to b \to (a)$ and $a \to d \to c \to b \to (a)$. In this way, we can assume that each row is transformed from the same set of letters (a, b, c and d), which guarantees that the R is J for the constructed letter matrix.

3. **Graph Representation of Column-Distributions' Realizability**
This method utilizes part of the previous proof and gives a graphical demonstration of distribution matrix's effects on each row of their gPM realizations. Suppose we have a gPM with each row being the same. Then, its column distribution is $C = n\mathbb{I}$. Each swap between two entries in a row will cause the $n\mathbb{I}$ to change the values in the two rows corresponding to the two entries.

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \; C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

change into

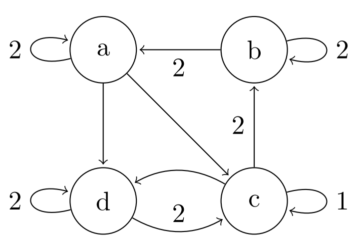
$$A' = \begin{bmatrix} b & a & c \\ a & b & c \\ a & b & c \end{bmatrix} \; C' = \begin{bmatrix} 2 & \underline{1} & 0 \\ \underline{1} & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In any column-distribution-matrix C, row 1 shows letter a_1's appearances at columns $1, 2, \ldots n$, and row 2 shows a_2's appearances at columns $1, 2, \ldots n$, etc. Assume originally, letter a_i only resides in ith column. Then, the off-diagonal entries in the column distribution show all the incidences of letters which move away from their original positions to the new positions. In a graph, we have n vertices corresponding to $a_1, \ldots a_n$ letters. We draw a directed edge from a_i to a_j if there is a nonzero entry at position $[i, j]$, denoting a_i moving from a_i's initial position to a_j's initial position via the C's transformation.

For example, the column-distribution-matrix C of a 4-by-4 permutative matrix with entries a, b, c, d is:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 2 & 0 & \underline{1} & \underline{1} \\ \underline{2} & 2 & 0 & 0 \\ 0 & \underline{2} & 1 & \underline{1} \\ 0 & 0 & \underline{2} & 2 \end{bmatrix} \end{array}$$

The corresponding directed graph is:



The edges between the nodes correspond to the off-diagonal entries and the self-loops correspond to the diagonal values. Here, we mainly care about the edges besides self-loops because they indicate a letter's change of positions. We can partition those edges into directed cycles. This is always achievable, because each distribution matrix, being an SDSM, can always be written as a sum of n permutation matrices. Permutation matrices' off-diagonal entries have graphical representations as cycles. Thus, we are always able to partition the subgraph into cycles which come from the set of permutation matrices whose sum is the distribution matrix. In this particular case, we have three cycles.

$$\begin{cases} c \to b \\ a \to c \to b \\ a \to d \to c \to b \end{cases}$$

Each line shows a permutation of entries from $[a\ b\ c\ d]$. Line $a \to c \to b$ describes a permutation such that a moves from its original position to column 3—c's original position and c moves from column 3 to column 2—b's original position, and b moves from column 2 to column 1—a's original position. These three cycles show that there are three distinct row permutations in the matrix. Therefore, in the 4-by-4 matrix, three shifts guarantee four distinct permutations of the entries a, b, c, d; thus, we have a permutative matrix. The corresponding permutative matrix is:

$$A = \begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ b & c & d & a \end{bmatrix}$$

Theorem 3: The matrix J is the only distribution matrix that enjoys universal compatibility.

Proof: We have proven that the J matrix has universal compatibility. Since column distribution $n\mathbb{I}$ has exactly one letter matrix realization, which describes a_1 appearing n times in column 1 and a_2 appearing n times in column 2, etc. This letter matrix has row distribution J, so we know that $n\mathbb{I}$ is compatible with the J matrix and nothing else. Therefore, we have shown that only the J matrix has universal compatibility. □

Example 4:

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ is only compatible with } J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and nothing else.}$$

3 Letter Matrices

A *letter matrice*, as defined in Section 1, is an n-by-n symbolic matrix in which each of n distinct letters appears n times. There is a total of $\frac{n^2!}{(n!)^n}$ distinct letter matrices for a given n. Since there are a total of n^2 slots, we first place each letter into the matrix counting the order. Then, we discount the order within each set of the same letters and obtain this count.

3.1 Letter matrices and distribution matrices

Given distribution matrices R and C, we can find each letter's placements in a matrix according to its row and column-sum-vector within R and C. However, we do not know immediately whether there is a letter matrix which realizes R and C for all the letters simultaneously. Some preliminary approaches were undertaken in an attempt to address this problem. For instance, given

$$R = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

we can form a matrix showing a, b, c, d's valid placements in the letter matrix by assigning each letter to a position if the row (column) index of the position is nonnegative in the letter's row (column)-sum-vector:

$$\begin{bmatrix} a & c, d & c, d & a, c, d \\ a, b & b, c & b, c & a, c \\ a, b & b, d & b, d & a, d \\ & c, d & c, d & c, d \end{bmatrix}.$$

From here, we are certain that there is no letter matrix consistent with R and C, because there are no letters which can be placed in position $[4, 1]$, since row 4 only contains c and d, and column 1 only contains a and b. Usually, we can get this information simply by multiplying R with C.

$$RC = \begin{bmatrix} 6 & 3 & 2 & 5 \\ 5 & 4 & 5 & 2 \\ 5 & 3 & 5 & 3 \\ \underline{0} & 6 & 4 & 6 \end{bmatrix}$$

Since the $[4, 1]$ entry is 0, there is no placement of letter in the $[4, 1]$ position.

However, RC may have no 0 entries and still not be compatible. In order to fully understand the compatibility of distribution matrices, we need more information by examining the placement of letter one-by-one, which will be discussed in Section 3.2 and Section 3.3.

3.1.1 Compatibility of Distribution Matrices

Here, we define $Z_n = \{B \in M_n(\{0, 1\}) : e^T Be = n\}$, the set of n-by-n 0,1-matrices with element sums equal to n. The placement of the letter l in the letter matrix A is simply the set of positions in which l resides in A. This may be viewed as a matrix $B_l \in Z_n$ in which the 1's in B_l occupy the positions of l in A. Thus, $A = \sum_l lB_l$. If R is the row distribution and C the column distribution of the letter matrix A, then $Re_l = B_l e$ and $e_l^T C = e^T B_l$. So, Re_l and $e_l^T C$ are Ryser-compatible. Given row-sum-vector r and column-sum-vector c, let $\mathcal{B}(r, c) = \{B \in Z_n : Be = r \text{ and } e^T B = c\}$. We say that row and column distribution R and C are compatible if there is at least one (there may be several) letter matrix A for which R is the row distribution and C is the column distribution.

Theorem 5: The n-by-n row and column distributions R and C are compatible if and only if for each $l = 1, \ldots, n$, there exist $B_e \in \mathcal{B}(Re_l, e_l^T C)$ such that $\sum_l B_l = J$ (or $B_l \circ B_k = 0$ for $k \neq l$).

Note, the compatibility of all row and column-sum-vector pairs in the two distributions does not guarantee the compatibility of two distribution matrices. Each pair's realization must be mutually orthogonal and we will discuss more in Section 3.3.

3.1.2 Latin-square Equivalence

Each letter matrix has a unique row distribution R and a unique column distribution C. But a pair of R and C can have zero, one or more letter matrix realizations. If the letter matrix contains a submatrix that is a latin square, then any permutations of rows or columns within the submatrix will generate a different (non-permutation equivalent) letter matrix with the same R and C pair. For this phenomenon, we call it *latin-square equivalence.*

The latin-square equivalence matrices all have the same pair of distribution matrices. A pair of distribution matrices generates all the latin-square equivalent letter matrices.

Example 6:

$$\begin{bmatrix} a & b & c & d \\ d & \underline{a} & b & \underline{c} \\ c & d & a & b \\ b & \underline{c} & d & \underline{a} \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c & d \\ d & \underline{c} & b & \underline{a} \\ c & d & a & b \\ b & \underline{a} & d & \underline{c} \end{bmatrix}$$

The underlined 2-by-2 submatrix in the second matrix is produced by permuting the rows (or columns) in the 2-by-2 submatrix in the first matrix. The first one is not permutation equivalent to the second one, which means, we cannot construct the second matrix by permuting any rows (or columns) or performing change of variables in the first letter matrix. However, both letter matrices share the same row and column distributions, in this case, the J matrix.

3.1.3 Permutation equivalence of compatibility

$C(\cdot)$ denotes the column distribution of a letter matrix. $R(\cdot)$ denotes the row distribution of a letter matrix. Then, given letter matrix A and permutation matrices P, Q, we have equivalent relations:

$$R(AP) = R(A)$$
$$C(QA) = C(A)$$
$$R(QA) = R(A)Q^T$$
$$C(AP) = C(A)P$$

3.2 The realization of a single letter from a row and column-sum-vector pair given by the distributions

Given a letter's row-sum-vector and column-sum-vector, if they are Ryser-compatible, we can find at least one 0,1-matrix B as it's distribution realization in the matrix. Then, we are able to generate all the other realizations via a sequence of *switches*, by performing latin-square transposition of 2-by-2 submatrices within B $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (if we can find such submatrices).

Not all entries in a B can be changed through such transpositions, and we can partition the positions in B into two parts: the fixed part and the movable part.

Definition 7: The *fixed* 1's (0's) of a letter's all realizations—$Z = \{B(r, c)\}$, are the positions that are always occupied by a 1 or a 0 in each $B \in Z$. Another way to say this is that, after taking the average over all Bs in Z, the values in these positions will be 1's (or 0's). The remaining positions may be called *movable* and are occupied by a 0 in some and a 1 in other placements, so that the averaged value is strictly between 0 and 1.

Example 8:

$$\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\text{'s positions all fixed, }\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\text{'s positions all movable.}$$

If we consider two or more row and column-sum-vector pairs, we may ask if there exists a representative realization of each, so that the sum of all realizations is still a 0,1-matrix (no two placements have overlapping positions). If this does not hold, we say the collection of row and column-sum-vector pairs has conflicts. Assuming that a R, C pair of distributions is nonrealizable, then some subsets of their pairs of row and column-sum-vectors have conflicts. We argue that the conflicts come from at least one of the following:

- **Conflict between letters' fixed parts**

  Example 9:

$$R=\begin{bmatrix}1&1&2&0\\2&0&1&1\\1&2&1&0\\0&1&0&3\end{bmatrix}\ C=\begin{bmatrix}2&0&1&1\\0&3&0&1\\1&0&1&2\\1&1&2&0\end{bmatrix}$$

  b's and d's placement are unique and are all fixed:

$$\begin{bmatrix}0&b&0&0\\0&0&0&0\\0&b&0&b\\0&\underline{b}&0&0\end{bmatrix}\text{ and }\begin{bmatrix}0&0&0&0\\0&0&d&0\\0&0&0&0\\d&\underline{d}&d&0\end{bmatrix}$$

  As we can see, the [4,2] position is occupied by both b and d, which causes incompatibility among R and C. (In this case, RC has no 0 entries, which is a good example showing that the condition is only necessary but not sufficient.)

- **Conflict between letters' movable parts**

  Example 10:

$$R=\begin{bmatrix}2&1&1&1&0\\2&1&1&1&0\\0&3&0&1&1\\0&0&3&1&1\\1&0&0&3&1\end{bmatrix}\ C=\begin{bmatrix}1&1&3&0&0\\2&0&0&2&1\\2&0&0&2&1\\0&2&2&0&1\\0&2&0&1&2\end{bmatrix}$$

  In this example, the movable parts of a, b, c conflict with each other at the upper left 2-by-2 submatrix. Using Theorem 18 explained in the latter section, we will know that the fixed positions of a, b and c are

$$\begin{bmatrix}0&0&a&0&0\\0&0&a&0&0\\b&0&0&b&b\\c&0&0&c&c\\0&0&a&0&0\end{bmatrix}.$$

  Now, we delete the fixed parts from a, b and c's sum-vectors and have new row-sum-vectors for a, b, c as

$$\begin{bmatrix}1&1&0&0&0\end{bmatrix}^T\begin{bmatrix}1&1&0&0&0\end{bmatrix}^T\begin{bmatrix}1&1&0&0&0\end{bmatrix}^T$$

and new column-sum-vectors for a, b, c as

$$\begin{bmatrix}1 & 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 1 & 0\end{bmatrix}.$$

Then we sum all the row-sum-vectors and column-sum-vectors of the 3 letters, and check their Ryser-compatibility and find $\begin{bmatrix}3 & 3 & 0 & 0 & 0\end{bmatrix}^T$'s conjugate—$\begin{bmatrix}2 & 2 & 2 & 0 & 0\end{bmatrix}$ does not majorize $\begin{bmatrix}3 & 2 & 1 & 0 & 0\end{bmatrix}$. So the two distribution matrices are not compatible, because there are not enough positions to place the movable parts of letter a, b and c at the same time.

- **Conflict between letters' fixed parts and other letters' movable parts**

  Example 11:

  Suppose letter a has row-sum-vector $\begin{bmatrix}2 & 2 & 0 & 0\end{bmatrix}^T$ and column-sum-vector $\begin{bmatrix}2 & 2 & 0 & 0\end{bmatrix}$ and letter b has row-sum-vector $\begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^T$ and column-sum-vector $\begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}$. Then,
  $\begin{bmatrix}a & a & 0 & 0\\ a & a & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ realizes a, which only contains fixed part, and $\begin{bmatrix}b & 0 & 0 & 0\\ 0 & 0 & b & 0\\ 0 & b & b & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ is one of the placements of b, which only contains movable part. One instance of letter b must be placed on the upper left 2-by-2 submatrix, which is in conflict with letter a because the fixed part of a occupies the whole submatrix. Therefore, a's and b's realizations have conflicts, and we cannot find a letter matrix that realizes a and b at the same time.

3.3 Fixed part of a letter

Now, we consider how to determine the fixed part in the set of realizations $Z = \{B(r, c)\}$ of a row and column-sum-vector pair (r, c). Since the fixed part is the same among all $B \in Z$ for a letter, if we can determine the fixed positions, it is easy to check the simultaneous realizability of multiple letters with respect to their fixed parts.

Definition 12: A block of 1's in a 0,1-matrix is maximal if and only if the adjacent block sharing the same rows has a 0 in every column and the adjacent block sharing the same columns has a 0 in every row.
A switchable submatrix is a 2-by-2 0,1 matrix ($\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$) by which we can perform latin-square transposition.

Lemma 13: If the complementary block of a block of 1's in a 0,1 matrix is all 0, then both the block of 1's and the block of 0's are fixed. Any maximal block of 1's containing such a block of 1's is all fixed.

Proof: Given a block of 1's with complementary block all 0, we cannot find a switchable submatrix which includes entries from both the blocks. We also cannot find a switchable submatrix using elements from the block of 1's and its adjacent blocks, since the two positions from the all 1's block are both ones which prevent the switching of the elements. We cannot find a switchable submatrix with the 0 block and its adjacent blocks either because the two positions from the 0 block are both 0, which do not qualify for the switchability. Thus, in such a matrix, both the block of 1's and its complementary 0 block are all fixed. If a maximal block of 1's containing such a block of 1's, then the maximal block has a complementary 0 block as well. Therefore, the maximal block is also fixed. □

Lemma 14: If every 1 in a maximal block of 1's of a 0,1-matrix is fixed, then the complementary block must be all 0.

Proof: Suppose there is a 1 in the complementary block of the fixed 1's. From Definition 12, we know that the adjacent block to the right of the maximal block of 1's must contain a 0 in every of its columns. The adjacent block to the south of the fixed block of 1' must contain a 0 in every of its rows. Therefore, if a 1's in $[p, q]$ position lies in the complementary block, then there exists some $[p, j]$ and $[i, q]$ being 0 with $[i, j]$ lies in the maximal fixed block which together form a switchable matrix. Thus, $[i, j]$ entry is movable which contradict with our assumption. So, If every 1 in a maximal block of 1's is fixed, its complementary block must be all 0. □

Theorem 15: A maximal block of 1's in a 0,1-matrix is fixed if and only if its complement is all 0. In this event, the complement 0 block will also be maximal and fixed.

Proof: If a block of 1's in a 0,1-matrix is maximal, with its complement being all 0, then we cannot find a switch position for all the 1's in the maximal block. Therefore, it is a fixed maximal block. If a maximal block of 1 is fixed, then by Lemma 14, its complementary block must be all 0. □

Definition 16: A $friend$ of a fixed 1 in a 0,1 matrix is another fixed 1 in the same row or column.

Lemma 17: In a 0,1 matrix with more than one 1, each fixed 1 has a friend.

Proof: Without loss of generality, we permute the fixed position to [1,1] position. If its complementary block is 0, then all the other 1's on the same row or column must be fixed, since all the nonzero elements have a complementary 0 block. (By Lemma 15.) Therefore, the 1's on the same row/column can form at least a fixed block and we are done.
If there is a movable 1 in the complementary block at position $[p, q]$, then there have to be a 1 at position $[1, q]$ or position $[p, 1]$ for the 1 at $[1, 1]$ position to be fixed. Since we assume this 1 at $[1, q]$ or $[p, 1]$ is movable, there must exist other 1's which together form a switchable submatrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, the matrix should be of the following form:

$$\begin{bmatrix} 1 & 0\dots0 & \underline{1} & 0\dots \\ 0 & 0\dots0 & \dots & \\ 0 & 0\dots0 & 1 & 0\dots \\ \dots & \dots & 0 & \underline{1}\dots \end{bmatrix},$$

and the 1 (at the lower-right corner) can make the 1 at position $[1, q]$ to 0. However, to achieve this, it also forms switchable submatrices with the other two entries and makes the 1 at $[1, 1]$ position movable. This contradicts to our assumption. Therefore, we have shown, the 1 at position $[1, q]$ or $[p, 1]$ must be fixed, so the 1 at $[1, 1]$ has a fixed friend lies in the same row or column.

If both $[1, q]$ and $[p, 1]$ positions are occupied by 1, then they are movable by assumption. Therefore, there have to be 1's at the complementary block except $[p, q]$ that can make them movable. ($[p, q]$ is in the same row or column with $[1, q], [p, 1]$, which prevents the formation of a switchable submatrix.) After making one of the $[1, q], [p, 1]$ movable, the problem reduces to the previous case with only one of $[p, 1]$ and $[1, q]$ being movable, which we have proven to be a contradiction to our assumption. Thus, one of them have to fixed and $[1, 1]$ must have a fixed friend lying on the same row or the same column.□

Note: from here, for clarity, we use R_i (instead of r) to denote the ith element in the row-sum-vector of a letter and C_j (instead of c) to denote the jth element in the column-sum-vector of a letter.

Theorem 18: If row and column-sum-vectors in a pair of distributions have the relation $R_1 + \ldots + R_k = \hat{C}_1 + \ldots \hat{C}_k$, the upper left k-by-$\hat{C}_k$ block of any realization B is all 1's and its complement is all 0's. Thus, both blocks are fixed.

Proof: Assume row and column-sum-vectors have non-increasing order. The number of 1's in the first k rows is $R_1 + \ldots + R_k$. The number of columns that have at least 1, 2, $\ldots k$ 1's are $\hat{C}_1, \hat{C}_2, \ldots \hat{C}_k$ in the submatrix formed by the first k rows by definition. Since the sum of the number of columns with at least 1, 2, $\ldots, k$ 1's is exactly the total number of 1's which appear in row 1,2,$\ldots k$ by assumption, the upper left k-by-$\hat{C}_k$ is a block with all 1's. Furthermore, there cannot be any 1's in its complementary block, because, otherwise, it will make some $\hat{C}_i$ larger, which violates our assumption of the equivalence relation. Therefore, the k-by-$\hat{C}_k$ block is a fixed block of all 1's with 0 complement. □

Lemma 19: If the upper left k-by-l block of a row and column-sum-vector realization is all 1's and has 0 complement, then $\hat{C}_k = l$ and $R_1 + \ldots + R_k = \hat{C}_1 + \ldots + \hat{C}_k$.

Proof: Since the upper left k-by-l block of a pair of row and column-sum-vector realization is all 1's with 0 complement, then there are at most k rows with at least l 1's, which means $\hat{C}_k = l$. Since $\hat{C}_1 + \ldots + \hat{C}_k$ counts all the columns with at least 1,2,$\ldots l$ 1's, then by our representation of the 0,1-matrix (the rows with non-increasing number of 1's), it is exactly $R_1 + R_2 + \ldots R_k$. Therefore, $R_1 + \ldots + R_k = \hat{C}_1 + \ldots + \hat{C}_k$. □

Theorem 20: The upper left k-by-l block of a pair of row and column-sum-vector realization B is maximal with all 1's fixed if and only if $l = \hat{C}_k$ and $R_1 + \ldots + R_k = \hat{C}_1 + \ldots + \hat{C}_k$ (similarly $k = \hat{R}_l$ and $\hat{R}_1 + \ldots + \hat{R}_l = C_1 + \ldots + C_l$).

Proof: If the upper left k-by-l submatrix is maximal with all 1's, then it must has a complementary block being 0. (By Lemma 14.) We know that $\sum_{i=1}^{k} R_i$ counts all the 1's in the first k rows above the 0 block. $\hat{C}_1, \ldots \hat{C}_k$ counts all the columns with the number of 1's less or equal to $1, 2, \ldots k$. If $\sum_{i=1}^{k} \hat{C}_i = \sum_{i=1}^{k} R_i$, then we know that all the lower right $(n-k)$-by-$(n-l)$ positions are all 0s, because otherwise, the equality wouldn't hold. Thus, we have a maximal block of 1's of size k-by-l. □

Algorithm 21: Applying Gale-Ryser's theorem to find the fixed part of a given letter k.

Firstly, we assume that the conjugate of the column-sum-vector C_k ($\hat{C}_k$) must majorize the row-sum-vector R_k (by Gale-Ryser's Theorem) in order to have a realization of a given letter.
Without loss of generality, we rearrange R_k and $\hat{C}_k$ in nonincreasing order and apply an algorithm to produce the fixed maximal blocks. For $R_k = \{r_1, r_2, \ldots, r_p\}$ and

$\hat{C}_k = \{\hat{c}_1, \hat{c}_2, \ldots \hat{c}_q\}$, we look at $r_j, \hat{c}_j$ pairs each time with $i = 1, \ldots, \min(p, q)$.

- If $\sum_{i=0}^{j} r_i = \sum_{i=0}^{j} \hat{c}_i$ $(\hat{c}_i \neq \hat{c}_{i-1})$
  or if $\sum_{i=0}^{j} r_i = \sum_{i=0}^{j} \hat{c}_i$ $(\hat{c}_i = \hat{c}_{i-1})$ and $\sum_{i=0}^{j-1} r_i \neq \sum_{i=0}^{j-1} \hat{c}_i$
  we record a fixed maximal block of the letter of size j-by-$\hat{c}_j$ on the upper left corner.
- If $\sum_{i=0}^{j} r_i = \sum_{i=0}^{j} \hat{c}_i$ $(\hat{c}_i = \hat{c}_{i-1})$ and $\sum_{i=0}^{j-1} r_i = \sum_{i=0}^{j-1} \hat{c}_i$
  we change the last fixed maximal block recorded to the current j-by-$\hat{c}_j$ block.
- If $\sum_{i=0}^{j} r_i \neq \sum_{i=0}^{j} \hat{c}_i$, we increment i by 1.

Knowing the fixed maximal blocks, we can mark those relative positions of the letter, map them back to the original matrix and compare them with other letters' placements.

It follows that, if a letter has $\sum_{i=1}^{k} \hat{C}_i = \sum_{i=1}^{k} R_i$ for $k = 1, 2 \ldots n$, then its the placement is all fixed and there is only one realization.

Remark 22: Compatibility of fixed positions of different letters
Given a pair of distribution matrices, if a subset of its letters have common fixed positions, they are not compatible and we cannot find a letter matrix which realizes those distribution matrices.
Like in Example 9, knowing the distribution of b and d, we cannot find a matrix which realizes both:

$$\begin{bmatrix} 0 & b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & b & 0 & b \\ d & \underline{bd} & d & 0 \end{bmatrix}.$$

For each letter, it is easy to locate all its fixed positions. So during the process of determining multiple letters' compatibility, it is recommended to start from checking their compatibility among the fixed positions.

3.4 Movable part of a letter

The movable part contains all the positions that are not fixed in the 0,1-matrix realizations of a pair of row and column-sum-vectors. Movable part of a given letter shows the positions of the letter, which, after performing switches, can be occupied by both 1's and 0's. If we have two different realizations of a movable part, then we can always perform a sequence of switches which transform the first one to the second.

Observation 23: Any letter's movable part is in the shape of a rectangle.

3.4.1 Compatibility between movable parts and fixed parts

Previously, we have discussed the conflicts among fixed positions of letters in a pair of distributions and how they prevent the compatibility of R and C. Now, looking at the movable part of a given letter, it can also have conflicts with the fixed parts of other letters. The ideas are illustrated as follows:

Conjecture 24: Given the movable part of letter a_1 and the fixed positions of letter a_2, we find their overlapping regions and increment the movable part's sum-vectors by the overlaps accordingly. If the updated sum-vectors are Ryser-compatible, then a_1's movable part and a_2's fixed part are compatible.

3.4.2 Compatibility between movable parts and movable parts

Conjecture 25: Suppose two letters from a pair of distributions only contain movable part. If the sum of two letters' row-sum-vectors and sum of their column-sum-vectors are Ryser-compatible, then those two letters are compatible.

Appendix: 4-by-4 Column Distributions and their Permutative Matrices Realizations

From Section 2, we know that, for any column distribution, we can find a gPM realization since the matrix J as the row distribution is compatible with any column distribution. However, it is not always the case of finding a permutative matrix realization, which requires each row being distinct. Here, we record the inventory of all column distributions (discounting permutation equivalence) and classify them into permutative-realizable and permutative-unrealizable. We list their permutative matrix realizations (discounting permutation equivalence) if they are permutative realizable.

1. **2 permutative-realizable column distributions with biggest value 4**

   (a) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ a & d & c & b \end{bmatrix}$

   (b) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ a & c & d & b \end{bmatrix}$

2. **7 permutative-unrealizable column distributions with biggest value 4**

(a) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

3. **8 permutative-realizable column distributions with biggest value 3**

(a) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & a & d & c \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & a & c & d \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ c & b & a & d \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ c & a & b & d \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ c & d & a & b \end{bmatrix}$

(3) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ a & d & b & c \\ b & a & d & c \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & d & b \\ b & c & a & d \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ c & a & b & d \end{bmatrix}$

(g) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & d & a & c \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ c & a & d & b \end{bmatrix}$

(i) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & c & d & a \end{bmatrix}$

(j) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & d & c & a \end{bmatrix}$

4. **7 permutative-unrealizable column distributions with biggest value 3**

(a) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ (g) $\begin{bmatrix} 0 & 3 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

5. **11 permutative-realizable column distributions with biggest value 2**

(a) $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & a & c & d \\ b & a & d & c \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & a & c & d \\ b & c & d & a \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ b & d & a & c \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ b & c & d & a \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrix: $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ b & d & c & a \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ d & c & a & b \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ b & c & a & d \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ c & d & a & b \\ c & d & b & a \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ b & d & c & a \end{bmatrix}$

(h) $\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ c & d & a & b \\ d & c & a & b \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ c & a & b & d \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ d & c & b & a \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ d & a & b & c \end{bmatrix}$

(j) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ c & a & d & b \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ c & d & a & b \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ d & a & c & b \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & a & c & d \\ c & d & b & a \end{bmatrix}$

(l) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ permutative matrices: (1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & a & c & d \\ c & d & a & b \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ d & b & a & c \end{bmatrix}$

(m) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ permutative matrices:

(1) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ c & d & b & a \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ c & d & a & b \end{bmatrix}$ (3) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & c & a & d \\ c & d & b & a \end{bmatrix}$

(n) $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ permutative matrices:

(1) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & d & a & c \\ c & d & b & a \end{bmatrix}$ (2) $\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$ (3) $\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ c & d & b & a \end{bmatrix}$

6. **1 permutative-unrealizable column distribution with biggest value 2**

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

7. **permutative-realizable J matrix** (Latin-squares)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ permutative matrix: (1) } \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \text{ (2) } \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & b & a \\ d & c & a & b \end{bmatrix}$$

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