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A DOUBLE SADDLE-NODE BIFURCATION THEOREM

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Abstract. In this paper, we consider an abstract equation $F(\lambda, u) = 0$ with one parameter λ, where $F \in C^p(\mathbb{R} \times X, Y)$, $p \geq 2$, is a nonlinear differentiable mapping, and X, Y are Banach spaces. We apply Lyapunov-Schmidt procedure and Morse Lemma to obtain a "double" saddle-node bifurcation theorem with a two-dimensional kernel. Applications include a perturbed problem and a semilinear elliptic equation.

1. Introduction.

Many examples of bifurcation can be found in the mathematical studies of models from physics, chemistry, biology and engineering. Analytical bifurcation theory in infinite dimensional spaces based upon the implicit function theorem, are most successful in problems with one dimensional kernels, typically leading to the existence of solution curves. Consider an abstract equation

$$F(\lambda, u) = 0, \tag{1}$$

where $F \in C^p(\mathbf{R} \times X, Y)$, $p \geq 1$, is a nonlinear differentiable mapping, and X, Y are Banach spaces. Let $F(\lambda_0, u_0) = 0$ so (λ_0, u_0) is a solution of (1). In [3, 4], Crandall and Rabinowitz proved two celebrated bifurcation theorems. In both theorems, it is assumed that 0 is a simple eigenvalue of the linearized operator, that is

(f1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$,

where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator F_u.

Theorem 1.1 (Saddle-node bifurcation theorem, [4] Theorem 3.2)**.** *Let U be a neighborhood of (λ_0, u_0) in $\mathbf{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, F satisfies $(f1)$ at (λ_0, u_0) and*

(f2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

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1. *If Z is a complement of span$\{w_0\}$ in X, then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $\{(\lambda_0 + \lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.*
2. *If F is $k-$times continuously differentiable, so are $\lambda(s)$ and $z(s)$.*
3. *If F is C^2 in u, then*
$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle}, \tag{2}$$
where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$.

Theorem 1.2 (Transcritical/Pitchfork bifurcation theorem, [3] Theorem 1.7)**.** *Let U be a neighborhood of (λ_0, u_0) in $\mathbf{R} \times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies $(f1)$ and*

(f3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of span$\{w_0\}$ in X. Then the solution set of (1) near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbf{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and
$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{3}$$
where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$.

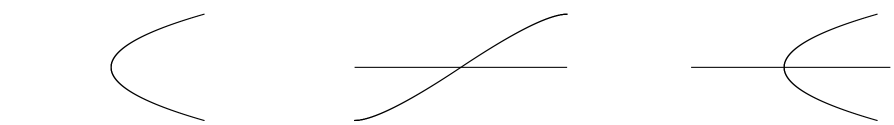

FIGURE 1. Illustration of bifurcations in Theorems 1.1 and 1.2.

When $\lambda'(0) \neq 0$ in Theorem 1.2, then a *transcritical bifurcation* occurs; and a *pitchfork bifurcation* occurs at (λ_0, u_0) if $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$. The saddle-node bifurcation (turning curve), and transcritical/pitchfork bifurcation (two crossing curves) illustrate the impact of different levels of degeneracy of the nonlinear mapping on the structure of local solution sets. In [10], the authors proved that Theorem 1.2 is a special case of a crossing-curve bifurcation theorem (see [10] Theorem 2.1).

While the bifurcations in Theorems 1.1 and 1.2 are the genetic ones occurring in numerous applications, bifurcations with higher degrees of degeneracy are also important in both theory and applications. It is the aim of this paper to discuss the bifurcation under the assumption of a two-dimensional kernel:

(F1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 2$,

and the same transversality condition as in saddle-node bifurcation theorem:

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Under an additional second order non-degeneracy condition, we prove that the solution set of (1) near the known solution (λ_0, u_0) is the union of two smooth parabola-like curves which both turn at the bifurcation point (see Section 2 for details). From the degeneracy of the curves at (λ_0, u_0), the two curves are tangent to each other. Moreover the turning directions of the two curves can be same or opposite (see Section 3 for examples). A finite-dimensional version of our main result was proved in Tiahrt and Poore [15], but our approach is more general and the proofs are different. An alternate approach is to use Lyapunov-Schmidt reduction which reduces the original problem to a finite dimensional one, then use the theory of singularities of differentiable maps and catastrophe theory (see Golubitsky and Schaeffer [6]). But our method is more direct and convenient for the infinite dimensional problem, as shown in [10].

Bifurcations with higher dimensional kernels have been considered in previous work. In [9], we obtained a generalized saddle node bifurcation theorem with finite-dimensional kernels by using the generalized inverse of the linearized operator. Krömer, Healey and Kielhöfer [8] proved a bifurcation result with two-dimensional kernel and a line of trivial solutions. Allgower et.al. [1], del Pino et.al. [5], Mei [11], Shi [13], Wang et.al. [16] all have considered the bifurcation of special semilinear elliptic equations on a square, and the kernel is a two-dimensional one. Other earlier work include Magnus [11], Rabier [12], Taliaferro [14], and more general approach to bifurcation theory can be found in Chow and Hale [2] and Kielhöfer [7].

In Section 2, we prove the main bifurcation result. In Section 3, we give two applications of our results. Throughout the paper, we use $||\cdot||$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range of L.

2. Double saddle-node bifurcation.

Suppose that F satisfies $(F1)$. Then we have the decompositions of X and Y: $X = N(F_u(\lambda_0, u_0)) \oplus Z$ and $Y = R(F_u(\lambda_0, u_0)) \oplus Y_1$, where Z is a complement of $N(F_u(\lambda_0, u_0))$ in X, and Y_1 is a complement of $R(F_u(\lambda_0, u_0))$ in Y. In particular, $F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism. Since $N(F_u(\lambda_0, u_0))$ is two-dimensional, we assume that $N(F_u(\lambda_0, u_0)) = span\{w_1, w_2\}$, for some $w_1, w_2 \in X$. Since $R(F_u(\lambda_0, u_0))$ is codimension two, there exists $v_1, v_2 \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{h \in Y : \langle v_1, h\rangle = 0 \text{ and } \langle v_2, h\rangle = 0\}$. If $(F2)$ is also satisfied, then $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$. Without loss of generality, we assume that $\langle v_1, F_\lambda(\lambda_0, u_0)\rangle \neq 0$ and $\langle v_2, F_\lambda(\lambda_0, u_0)\rangle = 0$. Indeed if the latter one is not satisfied for v_2, we can replace v_2 by $v_2 - \frac{\langle v_2, F_\lambda(\lambda_0, u_0)\rangle}{\langle v_1, F_\lambda(\lambda_0, u_0)\rangle} v_1$.

First we recall the well-known Lyapunov-Schmidt procedure under the condition $(F1)$. We sketch a proof for the completeness of presentation.

Lemma 2.1 (Lyapunov-Schmidt reduction)**.** *Suppose that $F : \mathbf{R} \times X \to Y$ is a C^p mapping ($p \geq 1$) such that $F(\lambda_0, u_0) = 0$, and F satisfies $(F1)$ at (λ_0, u_0). Then $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) can be reduced to*

$$\begin{aligned} &\langle v_1, F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2))\rangle = 0, \\ &\langle v_2, F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2))\rangle = 0, \end{aligned} \tag{4}$$

where $s_1, s_2 \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, δ is a small positive constant, $v_1, v_2 \in Y^*$ such that $\langle v_1, h\rangle = 0$ and $\langle v_2, h\rangle = 0$ if and only if $h \in R(F_u(\lambda_0, u_0))$, and g is a C^p function from a neighborhood of $(\lambda_0, 0, 0)$ into Z such that $g(\lambda_0, 0, 0) = 0$ and Z is a complement of $N(F_u(\lambda_0, u_0))$ in X.

Proof. We denote the projection from Y into $R(F_u(\lambda_0, u_0))$ by Q. Then $F(\lambda, u) = 0$ is equivalent to

$$Q \circ F(\lambda, u) = 0, \quad \text{and} \quad (I - Q) \circ F(\lambda, u) = 0. \tag{5}$$

We rewrite the first equation in the form

$$Q \circ F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g) = 0, \tag{6}$$

where $s_1, s_2 \in \mathbf{R}$ and $g \in Z$, from the fact that $N(F_u(\lambda_0, u_0)) = span\{w_1, w_2\}$ and $X = N(F_u(\lambda_0, u_0)) \oplus Z$. Since F satisfies $(F1)$ at (λ_0, u_0), we have $g = g(\lambda, s_1, s_2)$ is uniquely solvable from the implicit function theorem for (λ, s_1, s_2) near $(\lambda_0, 0, 0)$, and g is C^p. Hence $u = u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2)$ is a solution to $F(\lambda, u) = 0$ if and only if $(I - Q) \circ F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2)) = 0$. Since $R(F_u(\lambda_0, u_0))$ is codimension two, it becomes the two scalar equations in (4). □

Next we recall the following fundamental lemma about the zero level curves of a two-dimensional mapping.

Lemma 2.2 ([10], Lemma 2.5)**.** *Suppose that $(x_0, y_0) \in \mathbf{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f : U \to \mathbf{R}$ is a C^p function for $p \geq 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian $H = H(x_0, y_0)$ is non-degenerate. Then*

1. *If H is definite, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);*
2. *If H is indefinite, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of*

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0. \tag{7}$$

Our main result is the following theorem about the existence of two solution curves tangent to each other at the bifurcation point.

Theorem 2.3. *Let $F : \mathbf{R} \times X \to Y$ be a C^p mapping, where $p \geq 2$. Suppose that $F(\lambda_0, u_0) = 0$, and F satisfies $(F1)$ and $(F2)$. Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, $N(F_u(\lambda_0, u_0)) = span\{w_1, w_2\}$, and let $v_1, v_2 \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{h \in Y : \langle v_1, h\rangle = 0$ and $\langle v_2, h\rangle = 0\}$ so that $\langle v_1, F_\lambda\rangle \neq 0$ and $\langle v_2, F_\lambda\rangle = 0$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))*

$$H_2 = H_2(\lambda_0, u_0) \equiv \begin{pmatrix} \langle v_2, F_{uu}[w_1, w_1]\rangle & \langle v_2, F_{uu}[w_1, w_2]\rangle \\ \langle v_2, F_{uu}[w_1, w_2]\rangle & \langle v_2, F_{uu}[w_2, w_2]\rangle \end{pmatrix} \tag{8}$$

is non-degenerate, i.e., $\det(H) \neq 0$.

1. *If H_2 is definite, i.e.* $\det(H_2) > 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is $\{(\lambda_0, u_0)\}$.*
2. *If H_2 is indefinite, i.e.* $\det(H_2) < 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of two C^{p-1} curves, and the two curves are in form of $(\lambda_i(t), u_i(t)) = (\lambda_0 + t x_i(t), u_0 + \mu_i w_1 t + \eta_i w_2 t + t y_i(t))$, $i = 1, 2$, where $t \in (-\delta, \delta)$ for some $\delta > 0$, (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation*

$$\langle v_2, F_{uu}[w_1, w_1]\rangle\mu^2 + 2\langle v_2, F_{uu}[w_1, w_2]\rangle\eta\mu + \langle v_2, F_{uu}[w_2, w_2]\rangle\eta^2 = 0, \tag{9}$$

where $x_i(t), y_i(t)$ are some functions defined on $t \in (-\delta, \delta)$ which satisfy $x_i(0) = x_i'(0) = 0$, $y_i(0) = y_i'(0) = 0$, $y_i(t) \in Z$, and $i = 1, 2$.

Proof. From the proof of Lemma 2.1, we have

$$f_1(\lambda, s_1, s_2) \equiv Q \circ F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2)) = 0, \tag{10}$$

for (λ, s_1, s_2) near $(\lambda_0, 0, 0)$. Differentiating f_1 and evaluating at $(\lambda, s_1, s_2) = (\lambda_0, 0, 0)$, we obtain

$$0 = \nabla f_1 = (Q \circ (F_\lambda + F_u[g_\lambda]), Q \circ F_u[w_1 + g_{s_1}], Q \circ F_u[w_2 + g_{s_2}]). \tag{11}$$

Since $F_\lambda \notin R(F_u(\lambda_0, u_0))$ from $(F2)$ and $w_1, w_2 \in N(F_u(\lambda_0, u_0))$, we have $F_u(\lambda_0, u_0)[w_1] = 0$ and $F_u(\lambda_0, u_0)[w_2] = 0$. So we have

$$(Q \circ F_\lambda + Q \circ F_u[g_\lambda], Q \circ F_u[g_{s_1}], Q \circ F_u[g_{s_2}]) = 0. \tag{12}$$

Notice that $g_\lambda, g_{s_1}, g_{s_2} \in Z$ and $F_u(\lambda_0, u_0)|_Z$ is an isomorphism, thus $g_{s_1}(\lambda_0, 0, 0) = 0$, $g_{s_2}(\lambda_0, 0, 0) = 0$.

Next we define

$$G_i(\lambda, s_1, s_2) = \langle v_i, F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2))\rangle, \ i = 1, 2. \tag{13}$$

From Lemma 2.1, $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) is equivalent to $G_i(\lambda, s_1, s_2) = 0$, $i = 1, 2$ for (λ, s_1, s_2) near $(\lambda_0, 0, 0)$.

Since

$$\begin{aligned}\frac{\partial G_1}{\partial \lambda}(\lambda_0, 0, 0) &= \langle v_1, F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[g_\lambda(\lambda_0, 0, 0)]\rangle \\ &= \langle v_1, F_\lambda(\lambda_0, u_0)\rangle \neq 0,\end{aligned} \tag{14}$$

we have $G_1(\lambda, s_1, s_2) = 0$ is uniquely solvable for $\lambda = \lambda(s_1, s_2)$ near $(s_1, s_2) = (0, 0)$, λ is C^p, and $\lambda(0, 0) = \lambda_0$.

Now we define

$$\begin{aligned} f_2(s_1, s_2) &\equiv G_1(\lambda(s_1, s_2), s_1, s_2) \\ &= \langle v_1, F(\lambda(s_1, s_2), u_0 + s_1 w_1 + s_2 w_2 + g(\lambda(s_1, s_2), s_1, s_2))\rangle. \end{aligned} \tag{15}$$

Differentiating f_2 and evaluating at $(s_1, s_2) = (0, 0)$, we have

$$\begin{aligned} 0 =&\nabla f_2 \\ =&(\langle v_1, F_\lambda \lambda_{s_1} + F_u[w_1 + g_\lambda \lambda_{s_1} + g_{s_1}]\rangle, \langle v_1, F_\lambda \lambda_{s_2} + F_u[w_2 + g_\lambda \lambda_{s_2} + g_{s_2}]\rangle) \\ =&(\langle v_1, F_\lambda\rangle \lambda_{s_1}, \langle v_1, F_\lambda\rangle \lambda_{s_2}). \end{aligned}$$

Then $\lambda_{s_i}(0, 0) = 0$ holds for $i = 1, 2$ since $\langle v_1, F_\lambda(\lambda_0, u_0)\rangle \neq 0$.

Finally, to prove the statement in Theorem 2.3, we apply Lemma 2.2 to

$$G_2(s_1, s_2) = \langle v_2, F(\lambda(s_1, s_2), u_0 + s_1 w_1 + s_2 w_2 + g(\lambda(s_1, s_2), s_1, s_2))\rangle. \tag{16}$$

From Lemma 2.1 and above discussion of $G_1(\lambda, s_1, s_2)$, $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) is equivalent to $G_2(s_1, s_2) = 0$ for (s_1, s_2) near $(0, 0)$. To apply Lemma 2.2, we claim that

$$\nabla G_2(0, 0) = (\frac{\partial G_2}{\partial s_1}(0, 0), \frac{\partial G_2}{\partial s_2}(0, 0)) = 0, \tag{17}$$

and the Hessian matrix $Hess(G_2)$ at $(0, 0)$ is non-degenerate.

It is easy to see that

$$\nabla G_2(0, 0) = \begin{pmatrix} \langle v_2, F_\lambda(\lambda_0, u_0)\lambda_{s_1}(0, 0) + F_u(\lambda_0, u_0)[w_1 + g_\lambda \lambda_{s_1}(0, 0) + g_{s_1}(0, 0)]\rangle \\ \langle v_2, F_\lambda(\lambda_0, u_0)\lambda_{s_2}(0, 0) + F_u(\lambda_0, u_0)[w_2 + g_\lambda \lambda_{s_2}(0, 0) + g_{s_2}(0, 0)]\rangle \end{pmatrix}^T,$$

where P^T denotes the transpose of the matrix P. Thus $\nabla G_2(0,0) = 0$ from $\lambda_{s_i}(0,0) = 0$, for $i = 1, 2$. For the Hessian matrix, we have

$$Hess(G_2) = \begin{pmatrix} \dfrac{\partial^2 G_2}{\partial s_1^2} & \dfrac{\partial^2 G_2}{\partial s_1 \partial s_2} \\ \dfrac{\partial^2 G_2}{\partial s_1 \partial s_2} & \dfrac{\partial^2 G_2}{\partial s_2^2} \end{pmatrix}. \tag{18}$$

Here for $i = 1, 2$, we have

$$\begin{aligned} \frac{\partial^2 G_2}{\partial s_i^2}(0,0) =& \langle v_2, F_{\lambda\lambda}\lambda_{s_i}^2 + 2F_{\lambda u}[w_i + g_\lambda \lambda_{s_i} + g_{s_i}]\lambda_{s_i} + F_\lambda \lambda_{s_i s_i} \\ & + F_{uu}[w_i + g_\lambda \lambda_{s_i} + g_{s_i}, w_i + g_\lambda \lambda_{s_i} + g_{s_i}] \\ & + F_u[g_{\lambda\lambda}\lambda_{s_i s_i} + g_\lambda \lambda_{s_i s_i} + 2g_{s_i \lambda}\lambda_{s_i} + g_{s_i s_i}]\rangle \\ =& \langle v_2, F_{uu}[w_i, w_i]\rangle \end{aligned} \tag{19}$$

since $\lambda_{s_i}(0,0) = 0$, $g_{s_i}(\lambda_0, 0, 0) = 0$ and $\langle v_2, F_\lambda\rangle = 0$. And similarly we have

$$\begin{aligned} & \frac{\partial^2 G_2}{\partial s_1 \partial s_2}(0,0) = \frac{\partial^2 G_2}{\partial s_2 \partial s_1}(0,0) \\ =& \langle v_2, F_{\lambda\lambda}\lambda_{s_1}\lambda_{s_2} + F_{\lambda u}[w_2 + g_\lambda \lambda s_2 + g_{s_2}]\lambda_{s_1} \\ & + F_\lambda \lambda_{s_1 s_2} + F_{\lambda u}[w_1 + g_\lambda \lambda_{s_1} + g_{s_1}]\lambda_{s_2} \\ & + F_{uu}[w_1 + g_\lambda \lambda_{s_1} + g_{s_1}, w_2 + g_\lambda \lambda_{s_2} + g_{s_2}] \\ & + F_u[g_{\lambda\lambda}\lambda_{s_2}\lambda_{s_1} + g_{\lambda s_2}\lambda_{s_1} + g_\lambda \lambda_{s_1 s_2} + g_{\lambda s_1}\lambda_{s_2} + g_{s_1 s_2}]\rangle \\ =& \langle v_2, F_{uu}[w_1, w_2]\rangle. \end{aligned} \tag{20}$$

In summary, from our calculation,

$$Hess(G_2) = H_2(\lambda_0, u_0) \equiv \begin{pmatrix} \langle v_2, F_{uu}[w_1, w_1]\rangle & \langle v_2, F_{uu}[w_1, w_2]\rangle \\ \langle v_2, F_{uu}[w_1, w_2]\rangle & \langle v_2, F_{uu}[w_2, w_2]\rangle \end{pmatrix}. \tag{21}$$

Therefore from Lemma 2.2, we conclude that the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is a pair of intersecting curves if the matrix in (21) is indefinite, or is a single point if it is definite.

Now we consider the two curve case. We denote the two curves by $(\lambda_i(t), u_i(t)) = (\lambda_i(s_{1i}(t), s_{2i}(t)), u_0 + s_{1i}(t)w_1 + s_{2i}(t)w_2 + g(\lambda_i(s_{1i}(t), s_{2i}(t)), s_{1i}(t), s_{2i}(t)))$, with $i = 1, 2$. Then

$$\begin{aligned} & F(\lambda_i(s_{1i}(t), s_{2i}(t)), u_0 + s_{1i}(t)w_1 + s_{2i}(t)w_2 \\ & + g(\lambda_i(s_{1i}(t), s_{2i}(t)), s_{1i}(t), s_{2i}(t))) = 0. \end{aligned} \tag{22}$$

Differentiating (22), we obtain that from Lemma 2.2 the vectors $v_i = (s_{1i}'(0), s_{2i}'(0))$ are the solutions of $v^T H v = 0$, which are the solutions (μ, η) of (9). Also $\lambda_i'(0) = 0$ since $\lambda_i'(0) = \frac{\partial \lambda_i}{\partial s_1}(0)s_{1i}'(0) + \frac{\partial \lambda_i}{\partial s_2}(0)s_{2i}'(0) = 0$. $\square$

Note that the two curves in Theorem 2.3 are tangent to each other at (λ_0, u_0) since $\lambda_i'(0) = 0$. For each curve, $\lambda_i''(0)$ can also be calculated for C^p ($p \geq 2$) mapping F (see Proposition 2.4 next). In the case of a single curve $\lambda(s)$, if $\lambda''(0) > 0$ at the bifurcation point, then it is called supercritical; and if $\lambda''(0) < 0$, then it is called subcritical. Here we call the double saddle-node bifurcation at (λ_0, u_0) described in Theorem 2.3 to be supercritical if $\lambda_i''(0) > 0$ for $i = 1, 2$, and it is subcritical if $\lambda_i''(0) < 0$ for $i = 1, 2$. However it is also possible to have $\lambda_1''(0) \cdot \lambda_2''(0) < 0$, and we call it a transcritical double saddle-node bifurcation. The following proposition gives the calculation of $\lambda_i''(0)$ and determines the direction of the double saddle-node bifurcation:

Proposition 2.4. *Assume the conditions in Theorem 2.3 are satisfied, then the direction of the two solution curves are determined by*

$$\lambda_i''(0) = -\frac{\langle v_1, F_{uu}[\mu_i w_1 + \eta_i w_2, \mu_i w_1 + \eta_i w_2]\rangle}{\langle v_1, F_\lambda\rangle}, \tag{23}$$

for $i = 1, 2$. Moreover we consider the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_1 = H_1(\lambda_0, u_0) \equiv \begin{pmatrix} \langle v_1, F_{uu}[w_1, w_1]\rangle & \langle v_1, F_{uu}[w_1, w_2]\rangle \\ \langle v_1, F_{uu}[w_1, w_2]\rangle & \langle v_1, F_{uu}[w_2, w_2]\rangle \end{pmatrix},$$

then the double saddle-node bifurcation is supercritical or subcritical if H_1 is (positively or negatively) definite.

Proof. In Theorem 2.3, we have $\lambda_i(0) = \lambda_0$, $\lambda_i'(0) = 0$, $u_i(0) = u_0$, $u_i'(0) = \mu_i w_1 + \eta_i w_2$, for $i = 1, 2$. Differentiating $F(\lambda_i(t), u_i(t)) = 0$ twice with respect to t, we obtain

$$\begin{aligned} &F_{\lambda\lambda}(\lambda_i'(t))^2 + 2F_{\lambda u}[u_i'(t)]\lambda_i'(t) \\ &+F_\lambda \lambda_i''(t) + F_{uu}[u_i'(t), u_i'(t)] + F_u[u_i''(t)] = 0. \end{aligned} \tag{24}$$

Setting $t = 0$ in (24), we get

$$F_\lambda \lambda_i''(0) + F_{uu}[\mu_i w_1 + \eta_i w_2, \mu_i w_1 + \eta_i w_2] + F_u[u_i''(0)] = 0. \tag{25}$$

Applying v_1 to (25), we obtain (23). From (23), we have

$$\lambda_1''(0) \cdot \lambda_2''(0) = \frac{1}{(\langle v_1, F_\lambda\rangle)^2} \cdot [\mathbf{k}_1 H_1 \mathbf{k}_1^T] \cdot [\mathbf{k}_2 H_1 \mathbf{k}_2^T]$$

where $\mathbf{k}_i = (\mu_i, \eta_i)$, for $i = 1, 2$. If H_1 is positively or negatively definite, then $\mathbf{k}_1 H_1 \mathbf{k}_1^T$ and $\mathbf{k}_2 H_1 \mathbf{k}_2^T$ are both positive or negative, therefore $\lambda_1''(0) \cdot \lambda_2''(0) > 0$ and the direction of the two bifurcation curves are same. □

Remark 2.5.

1. A weaker version of Theorem 2.3 was proved in Tiahrt and Poore [15]. They prove similar crossing solution curves, but they only show that the curves are of class C^{p-2}, and their results are for finite dimensional spaces only. We prove that the curves are indeed of class C^{p-1}.
2. If the bifurcation is supercritical, then near the bifurcation point (λ_0, u_0), (1) has no solution when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, exactly one solution at $\lambda = \lambda_0$, and exactly four solutions when $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$; and it is similar for subcritical bifurcation. But if the bifurcation is transcritical, then near the bifurcation point (λ_0, u_0), (1) has exactly two solutions when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon)$, and exactly one solution at $\lambda = \lambda_0$ (see Figure 2 or Figure 3). Examples for each case will be shown in Section 3.
3. The reverse of the last statement in Proposition 2.4 is not true. If the double saddle-node bifurcation is sub(super)critical, then H_1 is not necessarily positively or negatively definite. An example that H_1 is indefinite but the double saddle-node bifurcation is subcritical is shown as part of Example 3.1.

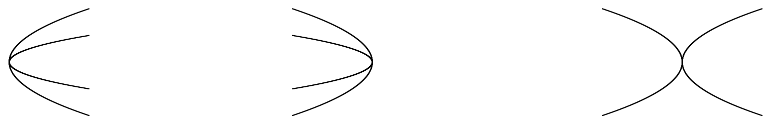

FIGURE 2. Three cases of double saddle-node bifurcations. This figure is for illustration only as the two curves cannot be in the same two-dimensional space, see Figure 3 for a real example.

3. **Examples.** We illustrate our result by several examples. The first one is a finite dimensional one which shows the canonical form of this double saddle-node bifurcation.

Example 3.1. *Define*

$$F\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \lambda - x^2 - 2axy - cy^2 \\ \lambda - y^2 - 2bxy - dx^2 \end{pmatrix}, \tag{26}$$

where $U = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$, $a, b, c, d \in \mathbf{R}$ *and* $\lambda \in \mathbf{R}$. *From simple calculations, we obtain*

$$F_U = \begin{pmatrix} -2x - 2ay & -2ax - 2cy \\ -2by - 2dx & -2y - 2bx \end{pmatrix}, \quad F_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$F_{UU} = \left(\begin{pmatrix} -2 & -2a \\ -2a & -2c \end{pmatrix}, \begin{pmatrix} -2d & -2b \\ -2b & -2 \end{pmatrix}\right). \tag{27}$$

We analyze the bifurcation at $\left(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. *It is easy to see that* $N(F_U) = \mathrm{span}\{w_1, w_2\}$, *where* $w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R(F_U) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$, *so obviously* $F_\lambda \notin R(F_U)$. *We can choose* $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *(here the functional is represented by elements in Hilbert space* $\mathbf{R}^2$*). Then* $\langle v_1, F_\lambda \rangle = 1$, $\langle v_2, F_\lambda \rangle = 0$. *Hence* $(F1)$, $(F2)$ *are satisfied. From the above calculation,*

$$F_{UU}[w_1, w_1] = \begin{pmatrix} -2 \\ -2d \end{pmatrix}, \quad F_{UU}[w_1, w_2] = \begin{pmatrix} -2a \\ -2b \end{pmatrix}, \quad F_{UU}[w_2, w_2] = \begin{pmatrix} -2c \\ -2 \end{pmatrix}.$$

We find the matrix H_2 *in* (8) *to be*

$$H_2 = \begin{pmatrix} -2 + 2d & -2a + 2b \\ -2a + 2b & -2c + 2 \end{pmatrix}$$

which is indefinite if

$$(b - a)^2 > (1 - c)(d - 1). \tag{28}$$

Thus we can apply Theorem 2.3 to this equation if (28) *hold, and near* $(\lambda, x, y) = \left(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$, *the solution set of* $F = (0, 0)^T$ *is the union of two touching curves. Moreover we can also apply the results in Proposition 2.4, and the associated matrix is*

$$H_1 = \begin{pmatrix} -2 & -2a \\ -2a & -2c \end{pmatrix}.$$

Hence if $det(H_1) = 4c - 4a^2 > 0$, *we have the double saddle-node bifurcation is subcritical or supercritical. We notice that if* $a = b = 0$, $c = d = -2$, *then* (28) *is satisfied,* H_1 *is indefinite, but the bifurcation is also subcritical. Thus the reverse of the last statement in Proposition 2.4 is not true (see Remark 2.5.3). On the other hand, if* $a = b = c = d = 0$, *then* (28) *holds and the bifurcation is supercritical (see Fig. 3.)*

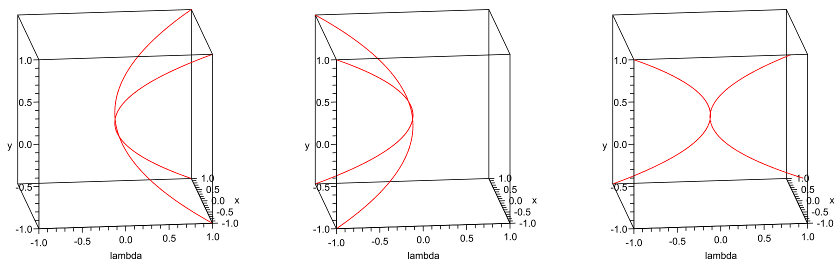

FIGURE 3. Three types of double saddle-node bifurcations in Example 4.1. (left) $\lambda - x^2 = 0$ and $\lambda - y^2 = 0$, supercritical; (middle) $\lambda - x^2 + 2y^2 = 0$ and $\lambda - y^2 + 2x^2 = 0$, subcritical; (right) $\lambda - x^2 - 2xy = 0$ and $\lambda - y^2 - 2xy = 0$, transcritical.

To obtain a transcritical bifurcation, we choose $(\mu_1, \eta_1) = (\mu_1, 1)$ *and* $(\mu_2, \eta_2) = (\mu_2, 1)$ *in Theorem 2.3 and Proposition 2.4, where* μ_1 *and* μ_2 *satisfy*

$$\mu_1\mu_2 = \frac{1-c}{d-1}, \quad \mu_1 + \mu_2 = 2 \cdot \frac{a-b}{d-1}.$$

From (23) *and* $\langle v_1, F_\lambda \rangle = 1$, *we can calculate that*

$$\lambda_1''(0) \cdot \lambda_2''(0) = \frac{4}{(d-1)^2}[c^2d^2 + 4b^2c + 4a^2d + 1 - 4abcd - 4ab - 2cd]. \tag{29}$$

From Remark 2.5, if a, b, c, d *satisfies* (28) *and* $\lambda_1''(0) \cdot \lambda_2''(0) > 0$, *then near the bifurcation point* $(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$, (26) *has no solution when* $\lambda \in (-\varepsilon, 0)$ *(or* $(0, \varepsilon)$*), exactly one solution at* $\lambda = 0$, *and exactly four solutions when* $\lambda \in (0, \varepsilon)$ *(or* $(-\varepsilon, 0)$*); if* a, b, c, d *satisfies* (28) *and* $\lambda_1''(0) \cdot \lambda_2''(0) < 0$, *then near the bifurcation point* $(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$, (26) *has exactly two solutions when* $\lambda \in (-\varepsilon, 0) \cup (0, \varepsilon)$, *and exactly one solution at* $\lambda = 0$.

It is easy to see that there are many values of (a, b, c, d) *so that* (28) *is satisfied and* $\lambda_1''(0) \cdot \lambda_2''(0) < 0$. *For example* $c = d = 0$ *and* $a = b = 1$ *(see Figure 3(right)).*

Next we consider an infinite dimensional example.

Example 3.2. *We consider a coupled logistic type semilinear elliptic equation:*

$$\begin{cases} \Delta u + \lambda_1 u + \lambda - v^2 = 0, & x \in \Omega, \\ \Delta v + \lambda_1 v + \lambda - u^2 = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{30}$$

where $\lambda \in \mathbf{R}$, Ω is a bounded domain in $\mathbf{R}^n$, and λ_1 is the principal eigenvalue of

$$\begin{cases} \Delta\phi + \lambda\phi = 0, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{31}$$

It is well-known that λ_1 is a simple eigenvalue, and its corresponding eigenfunction ϕ_1 does not change sign in Ω.

We define

$$F(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda_1 u + \lambda - v^2 \\ \Delta v + \lambda_1 v + \lambda - u^2 \end{pmatrix}, \tag{32}$$

where $\lambda \in \mathbf{R}$, and $U = (u, v) \in X \times X$, where $X = \{u \in C^{2,\alpha}(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}$.

From simple calculations, we obtain

$$F_U(\lambda, u, v)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda_1\phi - 2v\psi \\ \Delta\psi + \lambda_1\psi - 2u\phi \end{pmatrix}, \quad F_U(0,0,0)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda_1\phi \\ \Delta\psi + \lambda_1\psi \end{pmatrix},$$

$$F_\lambda(0,0,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F_{UU}(0,0,0)\left[\begin{pmatrix} \theta_1 \\ \psi_1 \end{pmatrix}\begin{pmatrix} \theta_2 \\ \psi_2 \end{pmatrix}\right] = \begin{pmatrix} -2\psi_1\psi_2 \\ -2\theta_1\theta_2 \end{pmatrix}.$$

Then

$$N(F_U(0,0,0)) = \operatorname{span}\left\{\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}\right\},$$

$$R(F_U(0,0,0)) = \left\{\begin{pmatrix} g \\ h \end{pmatrix} \in [C^\alpha(\overline{\Omega})]^2 : \int_\Omega g\phi_1 dx = 0 \text{ and } \int_\Omega h\phi_1 dx = 0\right\},$$

$\langle v_1, F_\lambda\rangle = \langle v_2, F_\lambda\rangle = \int_\Omega \phi_1 dx \neq 0$. *Thus $(F1)$ and $(F2)$ are satisfied. Moreover*

$$F_{UU}(0,0,0)\left[\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ -2\phi_1^2 \end{pmatrix}, \quad F_{UU}(0,0,0)\left[\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$F_{UU}(0,0,0)\left[\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}\right] = \begin{pmatrix} -2\phi_1^2 \\ 0 \end{pmatrix}.$$

Hence we find the matrix H_2 in (8) to be

$$H_2 = \begin{pmatrix} -2\int_\Omega \phi_1^3 dx & 0 \\ 0 & 2\int_\Omega \phi_1^3 dx \end{pmatrix} \tag{33}$$

which is indefinite.

Thus we can apply Theorem 2.3 to this equation, and near $(\lambda, x, y) = (0, 0, 0)$, the solution set is the union of two tangent curves in form of $\left(\lambda_i(t), \begin{pmatrix} u_i(t) \\ v_i(t) \end{pmatrix}\right) = \left(tx_i(t), \begin{pmatrix} \mu_i \\ \eta_i \end{pmatrix} t\phi_1 + ty_i(t)\right)$, where $(\mu_1, \eta_1) = (1, 1)$, $(\mu_2, \eta_2) = (1, -1)$, $x_i(0) = x_i'(0) = 0$, $y_i(0) = y_i'(0) = 0$, and $\lambda_i''(0) = 2\frac{\int_\Omega \phi_1^3 dx}{\int_\Omega \phi_1 dx} > 0$, $i = 1, 2$. Thus the double saddle-node bifurcation here is supercritical. It is easy to see that from the construction in Example 3.1, one can also modify (30) to include uv term to obtain a transcritical double saddle-node bifurcation.

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