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TP and TN Completability of Border Patterns

A thesis submitted in partial fulfillment of the requirement
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TP and TN Completability of Border Patterns

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Abstract

A matrix is called *totally nonnegative (TN)* if the determinant of **every** square submatrix is nonnegative and *totally positive (TP)* if the determinant of **every** square submatrix is positive. The *TP (TN) completion problem* asks which partial matrices have a TP (TN) completion. In this paper, a new class of TP- and TN- completable patterns, the border patterns, is identified. This answers an unpublished question about TP-completable patterns that has been outstanding for some time, and is the first case of completable patterns with all entries on the border specified. In the process, a new tool is developed: TP line insertion in the second or penultimate line when the first and last entries of the line are specified. Prior results about single unspecified entries are used and generalized.

Contents

Chapter 1

Introduction

A matrix is called *totally nonnegative (TN)* if the determinant of **every** square submatrix is nonnegative and *totally positive (TP)* if the determinant of **every** square submatrix is positive. Such matrices appear in many areas of pure and applied mathematics, including combinatorics, statistics, approximation theory, and other fields. Besides applications, they also display a wide array of very interesting mathematical properties. Many important properties and problems of TN and TP matrices are investigated and documented in [1], such as bidiagonal factorization, recognition, and the completion problem. Questions considered in this thesis fall into the category of completion problems and are extensions of previous results.

In this thesis we consider the following problems. Exact definitions of these problems will be given in later chapters.

- The TP- and TN- completability of the border patterns.

- TP- and TN- line insertions when the first and last entries of the line are specified.

- Possible TP- and TN- completable patterns that may occur inside a border.

The organization is as follows:

Chapter 1: Definitions and notations, examples of TN and TP matrices, recognition of TP matrices.

Chapter 2: We introduce TN and TP matrix completion problems and document previous results.

Chapter 3: We discuss two lemmas and a principle that are needed in later work.

Chapter 4: We prove the TN completability of the border pattern as well as the the possibility of TN line insertion with two extremal boundary conditions.

Chapter 5: We prove the TP completability of the border pattern as well as the possibility of TP line insertion with two extremal boundary conditions at the second row and the penultimate row.

1.1 Definitions and Notations

An m-by-n matrix is *totally positive* (TP) if **every** minor of it is positive and *totally negative* if **every** minor of it is negative. A **partial matrix** is a rectangular array in which some entries are specified, while the remaining entries are free to be chosen. A **completion** of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix.

In this section, a m-by-n matrix is denoted as $A = (a_{ij})$. Its submatrix lying in rows indexed by α and columns indexed by β is denoted as $A[\alpha, \beta]$, with $\alpha \subset \{1, 2, ..., m\}$ and $\beta \subset \{1, 2, ..., n\}$. A minor is the determinant of a square submatrix and the minor associated with the submatrix $A[\alpha, \beta]$ is denoted as $detA[\alpha, \beta]$, when $|\alpha| = |\beta|$. $|\alpha|$ is the cardinality of elements in the set α.

For $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\} \subset \{1, 2, ..., n\}$,$0 \leq \alpha_1 < ... < \alpha_k \leq n$, the *dispersion* of α is $d(\alpha) = \alpha_k - \alpha_1 - k + 1$. It measures how spread out the index set is relative to $\{1, 2, ..., n\}$. We call α a continuous index set if $d(\alpha) = 0$. If α and β are two continuous index sets and $|\alpha| = |\beta| = k$, then the submatrix $A[\alpha, \beta]$ is called a *contiguous submatrix* of A and its minor a *contiguous minor*. If either α or β is $\{1, 2, ...k\}$, we call the submatrix $A[\alpha, \beta]$ an *initial submatrix* and its minor *initial minor*.

An upper right *corner* minor of $A \in M_{m,n}$, $detA[\alpha, \beta]$ is defined such that α consists of the first k and β consists of the last k indices, $k = 1, 2, ..., min\{m, n\}$ and a lower left *cornor* minor is defined such that α consists of the last k and β consists of the first k. A *corner minor* is either a lower left or upper right minor.

1.2 Examples of TP and TN matrices

In this section we give one example of a TP matrix and one example of a TN matrix, both that arise in applications.

1.2.1 Vandermonde Matrices

Our example of TP matrices is the *Vandermonde matrices* that arise in the problem of interpolating n data points with a polynomial of degree $n - 1$ [1].

Given n that points $(x_i, y_i)_{i=1}^{n}$, we want to find the set of coefficients $\{a_0, a_1, ..., a_{n-1}\}$ such that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1}$ satisfies $p(x_i) = y_i, i = 1, 2, ..., n$. The coefficients are the solution to the following linear system:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ \vdots & & \vdots & ... & \vdots \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The determinant of the coefficient matrix is given by the formula $\prod_{i>j}(x_i - x_j)$ and the coefficient matrix is totally positive if $0 < x_1 < x_2 < ...x_n$ [3].

An example of a 3-by-3 Vandermonde matrix V with $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ is shown below:

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

It is a TP matrix by checking $detV = 2$, $detV[\{1,2\},\{1,2\}] = 1$, $detV[\{1,2\},\{2,3\}] = 2$, $detV[\{2,3\},\{2,3\}] = 1$ and all the entries (which are considered as a minor of a 1-by-1 submatrix). Notice how in here we check only some, instead of all, minors. It will be clear in later sections that in order to recognize a matrix as a TP matrix, we only need to check the initial minors of the matrix. However, this shortcut cannot be used to recognize TN matrices, due to the possible presence of zero minors.

1.2.2 Tridiagonal Matrices

We will use triadiagonal matrices as an example of TN matrices. Suppose the tridiagonal matrix A has the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & \dots & c_{n-1} & a_n \end{bmatrix}$$

A natural questions to ask is what constraints one has to put on non-zero entries to make the matrix TN. It is proven in [1] that if A is a nonnegative tridiagonal matrix that satisfies $a_i \geq b_i + c_{i-1}$ ($c_0 \equiv 0$ and $b_n \equiv 0$) then A is a TN matrix.

A sketch of the proof is as follows: first reduce the argument to an irreducible nonnegative tridiagonal matrix, and consider all possible patterns of submatrices that could occur and investigate the minors of those patterns. It turns out that there are only three patterns:

1. Principal minors, each of which can be written as a product of several principal minors with contiguous index sets, are nonnegative under the condition $a_i \geq b_i + c_{i-1}$.

2. Minors of the form $detA[\{i_1, ..., i_k\}, \{j_1, ..., j_k\}]$ are 0 whenever there exists an $s \in \{1, ..., k\}$ such that $|i_s - j_s| > 1$, a simple consequence of the property of tridiagonal matrices.

3. Minors of the form $detA[\{i_1, ..., i_k\}, \{j_1, ..., j_k\}]$ that satisfies $|i_s - j_s| \leq 1$, $s = 1, ..., k$ are nonnegative because it can be written as a product of principal minors with contiguous index set and nonprincipal minors from super- or subdiagonal.

The proof can be then completed by considering reducible matrices, which can be expressed as the direct sum of several irreducible matrices, and apply the same argument as above.

1.3 Recognition of TP matrices

We end this chapter by stating two important rules to recognize TP matrices. To determine whether a m-by-n matrix A is TP or TN is computationally complex and prohibitive, as it involves checking $\sum_{k=1}^{min\{m,n\}} \binom{m}{k}\binom{n}{k}$ minors. It is shown that one can check fewer than $\sum \binom{m}{k}\binom{n}{k}$ minors and, more importantly, the algorithm can check in polynomial time in n [1].

Theorem 1.1 (Thm 3.1.4 in [1]). *If all initial minors of $A \in M_{m,n}$ are positive then A is TP.*

This theorem states that in order to determine whether a matrix is TP, one only needs to check all the initial minors. We state one more theorem about the recognition of TP matrix if we know the original matrix is TN.

Theorem 1.2 (Thm 3.1.10 in [1])**.** *Suppose that* $A \in M_{m,n}$ *is TN. Then* A *is TP if and only if all corner minors of* A *are positive.*

Chapter 2

The Matrix Completion Problem

In this section, we introduce the TP and TN matrix completion problem. We will include a list of, by no means exhaustive, important TP- and TN- completable patterns that have been identified in previous literature.

2.1 Introduction

The *TP (TN) completion problem* asks which partial matrices have a TP (TN) completion. Obviously, a necessary condition is that in the partial matrix, all minors consisting entirely of specified entries are positive (nonnegative). We call such a partial matrix *partial TP (TN)*.

Not all partial TP (TN) matrices have a TP (TN) completion. For example, it can be shown that the following partial TP matrix from [1] is not TP completable:

$$\hat{A} = \begin{bmatrix} 1 & 1 & 0.4 & x \\ 0.4 & 1 & 1 & 0.4 \\ 0.2 & 0.8 & 1 & 1 \\ y & 0.2 & 0.4 & 1 \end{bmatrix}$$

. The determinant of the matrix can be expressed as

$$det(\hat{A}) = -0.0016 - 0.008x - 0.328y - 0.2xy$$

, which is always negative for positive x and y.

The example shows that sometimes additional conditions on the entries are necessary, and it is known that for each pattern of the specified entries there will be a finite list of such (polymonial) conditions [4]. What we are interested in this paper, however, is the case in which no additional conditions on the entries are needed. That is, which patterns of the specified entries of a partial TP (TN) matrix along imply that there is a TP (TN) completion. We call such patterns *TP (TN) completable patterns.*

Many such patterns have been identified, the monotonically labelled block clique patterns in the TN case [5] and in the TP case [6], and certain double echelon patterns in the TN case [7] (TP case open). In addition, such patterns with just one unspecified entry [8] and patterns with a full line of unspecified entries [9], the completable patterns have been identified. We will discuss some of them below.

2.2 Monotonically Labeled Block Clique

A clique is a subset of vertices of an graph such that its induced graph is complete. *A monotonically labeled block-clique graph* (an MLBC graph) is defined as a labeled, undirected graph, in which each maximal clique is consecutively labeled, and each two cliques that intersect intersect at the point that is the highest label for one clique and the lowest label for another. A example of the pattern with size 10-by-10 is shown below:

$$\begin{bmatrix} x & x & x & x & x & ? & ? & ? & ? & ? \\ x & x & x & x & x & ? & ? & ? & ? & ? \\ x & x & x & x & x & ? & ? & ? & ? & ? \\ x & x & x & x & x & ? & ? & ? & ? & ? \\ x & x & x & x & x & x & x & x & ? & ? \\ ? & ? & ? & ? & x & x & x & x & ? & ? \\ ? & ? & ? & ? & x & x & x & x & ? & ? \\ ? & ? & ? & ? & x & x & x & x & x & x \\ ? & ? & ? & ? & ? & ? & ? & x & x & x \\ ? & ? & ? & ? & ? & ? & ? & x & x & x \end{bmatrix}$$

Example of a partial matrix with a MLBC graph

It has been shown in [5] and [10] that such pattern is TN completable. We state this below as a theorem.

Theorem 2.1 (Thm 9.2.1 in [1])**.** *All square, combinatorially symmetric partial TN matrices with specified main diagonal and with G as the labeled undirected graph of its specified entries have a TN completion if and only if G is MLBC.*

A sketch of the proof will start with proving the TN completability of the partial TN matrix

$$\begin{bmatrix} A_{11} & a_{12} & ? \\ a_{21}^T & a_2 2 & a_{23}^T \\ ? & a_{32} & A_{33} \end{bmatrix}$$

,where a_{21} and a_{23} are vectors of size n_1-by-1 and n_3-by-1, and A_{11} and A_{33} are square matrices with size n_1-by-n_1 and n_3-by-n_3. Then one can identify the pattern repeatedly within the partial matrix with a MLBC graph and prove the TN competability of the partial matrix inductively.

It has also been shown in [6] that a partial matrix with a MLGB graph is TP completable. However, there is a difference between the result of TN completability and that of TP completability of the MLBC graph: one can shown that the MLGB graph is the only TN completable patterns among square, combinatorially symmetric patterns with a specified main diagonal, but an analogous result of the TP case is not known yet. We also state the theorem of the TP case below to compare with that of the TN case.

Theorem 2.2 (Thm 9.2.2 in [1])**.** *Let G be an MLBC graph on n vertices. Then every n-by-n partial matrix, with specified main diagonal, the graph of whose specified entries is G, has a TP completion.*

2.3 Echelon Patterns and Jagged Patterns

A pattern is called *echelon*, if, whenever a position is unspecified, either all positions north and east of it are unspecified or south and west of it are unspecified. An example of the echelon pattern is

$$\begin{bmatrix} * & * & * & * & ? & ? & ? & ? \\ * & * & * & * & ? & ? & ? & ? \\ ? & * & * & * & * & ? & ? & ? \\ ? & ? & ? & * & * & * & ? & ? \\ ? & ? & ? & * & * & * & * & ? \\ ? & ? & ? & ? & ? & * & * & * \end{bmatrix}$$

, in which $*$ denotes a specified entry and ? an unspecified entry. A pattern is called *jagged* if a position is unspecified, then either every position north and west of it, or every position south and east of it is specified. An example of a jagged patterns is a 90 rotation of an echelon pattern:

$$\begin{bmatrix} ? & ? & ? & ? & ? & * \\ ? & ? & ? & ? & * & * \\ ? & ? & ? & * & * & * \\ ? & ? & * & * & * & ? \\ * & * & * & * & * & ? \\ * & * & * & ? & ? & ? \\ * & * & * & ? & ? & ? \\ * & * & ? & ? & ? & ? \end{bmatrix}$$

[7] shows that a jagged pattern is always TP completable and an echelon pattern is TN completable if and only if the pattern contains no 4-by-4 P_1, P_1' or P_2 as a subpattern. The 4-by-4 P_1 and P_2 patterns are shown below. Again, $*$ denotes a specified entry and ? an unspecified entry.

$$P_1 = \begin{bmatrix} * & * & * & ? \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}, P_2 = \begin{bmatrix} * & * & * & ? \\ * & * & * & * \\ * & * & * & * \\ ? & * & * & * \end{bmatrix}$$

In addition, [7] also makes two conjectures: 1.an echelon pattern is TP competable if and only if it constains no P_1, P_1' and P_2 as a subpattern; 2. a jagged echelon pattern is TP completable if and only if it contains no P_1, P_1' and P_2 as a subpattern.

2.4 Single Entry Case

In this section, we consider a partial TP (TN) matrix with only one specified entry. In general, such pattern is not TP competable. Consider the case in [1]:

$$\begin{bmatrix} 100 & 100 & 40 & x \\ 40 & 100 & 100 & 40 \\ 20 & 80 & 100 & 100 \\ 3 & 20 & 40 & 100 \end{bmatrix}$$

.

x has to be smaller $\frac{-572}{7}$ to make the determinant positive. Thus no TP completion is possible for this partial matrix.

It turns out that among all the single unpspecified entry cases, the number of TP-completable patterns is rather limited. We now state two important theorems given in [8]. The first deals with a partial TP matrix that is 3-by-n and the second considers an m-by-n partial TP matrix, with $4 \leq m \leq n$, both of which have only one entry unspecified.

Theorem 2.3 (Thm 2.8 in [8])**.** *Let A be a 3-by-n, $n \geq 3$, partial TP matrix with exactly one unspecified entry. Then A is completable to a TP matrix.*

Theorem 2.4 (Thm 2.11 in [8])**.** *Let A be an m-by-n partial TP matrix in which $4 \leq m \leq n$ and in which the only unspecified entry lies in the (s,t) position. Any such A has a TP completion if and only if $s + t \leq 4$ or $s + t \geq m + n - 2$.*

Theorem 2.4 states that for an m-by-n partial TP matrix with only one unspecified entry, the positions of unspecified entries that always allow TP completability are those in the upper-left corner or lower-right corner. They are shown below as "x".

$$\begin{bmatrix} x & x & x & \dots & \dots & \dots & \dots \\ x & x & \dots & \dots & \dots & \dots & \dots \\ x & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & x \\ \dots & \dots & \dots & \dots & \dots & x & x \\ \dots & \dots & \dots & \dots & x & x & x \end{bmatrix}$$

The above two theorems play a fundamental role in our proof of the TP completability of the border pattern. An independent proof is given in the TN case. We will show how the theorems imply that a TP line insertion with two extremal boundary conditions at the second (or penultimate) row is always possible.

Chapter 3

Preliminaries

In this section, we describe some needed properties of TP/TN matrices and previous results.

3.1 (1,1) and (m,n) Entries

Lemma 3.1. ***(1,1) and (m,n) entries***

Given an m-by-n TP (TN) matrix:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & \dots & a_{m,n} \end{bmatrix},$$

then for any $\delta \geq 0$, both $A + \delta E_{11}$ and $A + \delta E_{mn}$ are TP (TN). E_{ij} is a matrix with the same dimensions as A. Its only nonzero entry is the entry (i,j), which is 1.

This lemma is a consequence of the fact that either of the entries, if included in a minor, will always contribute positively to the expansion of the determinant. This lemma implies that, whatever other entries are specified, if there is an $a^*_{1,1}$ (or $a^*_{m,n}$) for which there is a TP (TN) completion, then any value greater than $a^*_{1,1}$ ($a^*_{m,n}$) will also result in a TP (TN) completable partial matrix.

3.2 Normalizing First Row and First Column

Lemma 3.2. ***Normalizing a row and a column***
Given an m-by-n TP (TN) matrix:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & ... & ... & a_{1,n} \\ ... & ... & ... & ... & ... \\ a_{m,1} & a_{m,2} & ... & ... & a_{m,n} \end{bmatrix},$$

we can normalize any positive row or column by dividing row i by a_{i1} and column j by a_{1j} for all $i \in [1, m]$ and $j \in [1, n]$. The resulting matrix B is TP (TN) and has the following representation:

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & ... & ... & b_{1,n} \\ ... & ... & ... & ... & ... \\ b_{m,1} & b_{m,2} & ... & ... & b_{m,n} \end{bmatrix},$$

, where $b_{ij} = \frac{a_{ij}}{a_{i1}aj1}$. Then, there will be a row (column) of 1's in the resulting matrix.

This lemma is a consequence of the fact that multiplying a row or a column by a constant $c > 0$ does not change the sign of any minor. It is convenient to work with partial TN/TP matrices whose first column and row are normalized to 1, so we often assume this without loss of generality.

3.3 Northeast-Southwest Principle

In addition to theorems about partial TP matrices, we also need a theorem about partial TN matrices that is stated in [1] (Theorem 1.6.4). Instead of presenting it explicitly, we only introduce a consequence of this theorem. One of the consequences of this theorem states that if a 0 exists in a (partial) TN matrix that has no zero rows or columns, then either all entries to the northeast of the zero entry or all entries to the southwest of the zero entry must be zero. Consider the following 3-by-3 matrix as an example:

$$\begin{bmatrix} ? & ? & ? \\ ? & 0 & ? \\ ? & ? & ? \end{bmatrix}$$

If a zero occurs in the (2,2) position, then either positions (1,2), (1,3) and (2,3) have to be zero, or positions (2,1), (3,1) and (3,2) have to be zero, assuming there are no zero rows or columns in the partial TN matrix. To see why this is so, consider the northeast direction as an example. The matrix below has all entries except the (1,2), (1,3) and (2,3) entries specified. Among the specified entries, only the (2,2) entry is zero. All other specified entries are positive.

$$\begin{bmatrix} + & ? & ? \\ + & 0 & ? \\ + & + & + \end{bmatrix}$$

Consider submatrices $A[\{1,2\},\{1,2\}]$ and $A[\{2,3\},\{2,3\}]$. Because the main diagonal product in both submatrices is 0, both the (1,2) and (2,3) entries must be 0 to allow a TN completion.

$$\begin{bmatrix} + & 0 & ? \\ + & 0 & 0 \\ + & + & + \end{bmatrix}$$

Then consider the submatrix $A[\{1,2\},\{1,3\}]$. Again the product of main diagonal entries is 0, so the (1,3) entry must be 0 to allow a TN completion. This rule can generalized to any partial m-by-n TN matrix, which is part of the theorem 1.6.4 in [1]. We state this rule here as a principle. Later we note that for the border pattern to be TN completable, the partial TN matrix must obey this principle.

Definition 3.3. Northeast-Southwest (NS) principle states that if an entry, that does not lie on a zero line, in a pattern is zero, either all entries to the northeast or all entries to the southwest of the zero entry have to be zero. A partial TN pattern has to respect the Northeast-Southwest (NS) principle to be TN completable.

Chapter 4

TN Completability of the Border Pattern

In this section, we discuss the TN completability of the border pattern. Unfortunately, the border pattern is not TN completable if no additional constraint is imposed on the specified entries. However, the sufficient condition for a border pattern to be TN completable can be easily characterized. In short, the border pattern is TN completable only if it respects the NS principle. We will start the discussion by giving an example of a border pattern that is not TN completable. Then we will prove that given any TN matrix, it is possible to insert a line (between any two consecutive, existing lines) with its initial and final entries specified (two extremal boundary conditions), as long as they leave the resulting partial matrix partial TN and respect the NS principle. We then use this fact to prove that the border pattern is TN completable exactly when it respects the NS principle.

4.1 An Example of A Border Pattern That is not TN Completable

Example 4.1. An m-by-n border pattern need not be TN completable.

An example of a border pattern that is not TN completable is shown below. The (2,2) entry is the only unspecified entry. Although this matrix is partial TN, there does not exist a TN completion for it. Any value assigned to the (2,2) entry will cause $detA[\{2,3\},\{2,3\}]$ to be negative.

$$A_{3\times 3} = \begin{bmatrix} + & + & 0 \\ + & ? & + \\ 0 & + & 0 \end{bmatrix}$$

This 3-by-3 border pattern can be easily extended to a m-by-n border pattern:

$$A_{m\times n} = \begin{bmatrix} + & + & ... & + & 0 \\ + & ? & ... & ? & + \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ + & ? & ... & ? & + \\ 0 & + & ... & + & 0 \end{bmatrix}$$

Again, this pattern is partial TN but there does not exist a TN completion. In fact, as soon as one entry is specified, the pattern will cease to be partial TN.

The example shows that the border pattern is not TN completable if no additional constraint,besides being partial TN, is imposed on the specified entries. Our interest in the rest of this section is to find those constraints on the specified entries necessary to make a border pattern TN completable.

Observing the 3-by-3 partial matrix above, one will notice that the NS principle is violated. A zero appears in the (3,3) position but neither the (2,3) entry (northeast) nor the (3,2) entry (southwest) is zero. It turns out the NS principle is all that is necessary to ensure TN completability, beside the pattern being partial TN. We need a theorem of line insertions about two extremal boundary conditions. Then we will proceed to prove that a partial TN border pattern is TN completable as long as it respects the NS principle, which is defined on page 17.

A note before we start to avoid confusion: in the beginning of this paper, we defined a pattern TN being TN completable if the pattern along implies TN completability and no additional conditions on the specified entries are needed. In here, however, we will digress a little bit from our original definition: we still say that the border pattern is TN completable even thought it must obey the NS principle.

4.2 TN Line Insertion with Two Extremal Boundary Conditions

Theorem 4.2. *Given any m-by-n TN matrix, it is possible to insert a line between any two consecutive rows or columns with two extremal boundary conditions as long as the specified entries make the corresponding partial matrix partial TN and the extremal boundary conditions respect the Northeast-Southwest principle.*

Proof. Showing that a line insertion between any two consecutive rows (columns) is possible is equivalent to showing the pattern displayed below is TN completable. In the graph below, we try to insert a line with two boundary conditions between the $(i-1)$st row and the ith row. The entries to be chosen are between positions occupied by a_1^* and a_n^*. Any other entries in the partial TN matrix are specified. We call the row to be completed row* in the following proof.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & \dots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \dots & \dots & a_{3,n-1} & a_{3,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i-1,1} & a_{i-1,2} & \dots & \dots & \dots & a_{i-1,n} \\ a_1^* & ? & \dots & \dots & ? & a_n^* \\ a_{i,1} & a_{i,2} & \dots & \dots & \dots & a_{i,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m,1} & \dots & \dots & \dots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

To complete the row insertion, we view the initial (a_1^*) and final entries (a_n^*) of row* as a linear combination of the corresponding initial and final entries of the row above and the row below. Solving the system below will yield the coefficients of this linear combination. We denote c_1 as the coefficient for the row above and c_2 for the row below.

$$\underbrace{\begin{bmatrix} a_{i-1,1} & a_{i,1} \\ a_{i-1,n} & a_{i,n} \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1^* \\ a_n^* \end{bmatrix}$$

There are cases in which the determinant of A is zero, which we discuss at length later. For now we only consider cases with the determinant positive (it cannot be negative because the matrix is a partial TN matrix). Both c_1 and c_2 are guaranteed to be nonnegative by Cramer's Rule because numerators are nonnegative and denominators are positive:

$$c_1 = \frac{det(\begin{bmatrix} a_1^* & a_{i,1} \\ a_n^* & a_{i,n} \end{bmatrix})}{det(A)} \qquad \text{and} \qquad c_2 = \frac{det(\begin{bmatrix} a_{i-1,1} & a_1^* \\ a_{i-1,n} & a_n^* \end{bmatrix})}{det(A)}.$$

Having found the coefficients c_1 and c_2, we can complete row* such that each unspecified entry is expressed as:

$$a_j^* = c_1 a_{i-1,j} + c_2 a_{i,j}, j = 2, ..., n-1.$$

To prove that this yields a TN completion, we only need to check minors that contain row* and there are three possibilities:

Case 1: Minors that contain all of row*, the $(i-1)$st row and the ith row will be 0 because row* is a linear combinantion of the ith row and the $(i-1)$th row.

Case 2: Minors that contain row* and the ith row (or the $(i-1)$st row and row*) are also nonnegative. This is because we can rewrite row* as a linear combination of the $(i-1)$st and the ith row and "decompose" the minors into a two specified minors. A schematic is given below. We use R^* and R_i, repsectively, to represent the segments of row* and the ith row contained in the minor, and A_1 and A_2 for the remaining entries in the minor.

$$det(\begin{bmatrix} A_1 \\ R^* \\ R_i \\ A_2 \end{bmatrix}) = det(\begin{bmatrix} A_1 \\ c_2 R_i \\ R_i \\ A_2 \end{bmatrix}) + det(\begin{bmatrix} A_1 \\ c_1 R_{i-1} \\ R_i \\ A_2 \end{bmatrix}) = 0 + c_1 det(\begin{bmatrix} A_1 \\ R_{i-1} \\ R_i \\ A_2 \end{bmatrix}) \geq 0$$

, where $det(\begin{bmatrix} A_1 \\ R_{i-1} \\ R_i \\ A_2 \end{bmatrix})$ is a given nonnegative minor. A similar argument applies to minors containing the $(i-1)$st row and row*.

Case 3: Minors that only contain elements of row* are also nonegative. The argument is similar to that in Case 2. We can decompose the minor into two nonnegative minors:

$$det(\begin{bmatrix} A_1 \\ R^* \\ A_2 \end{bmatrix}) = det(\begin{bmatrix} A_1 \\ c_1 R_{i-1} + c_2 R_i \\ A_2 \end{bmatrix}) = c_1 det(\begin{bmatrix} A_1 \\ R_{i-1} \\ A_2 \end{bmatrix}) + c_2 det(\begin{bmatrix} A_1 \\ R_i \\ A_2 \end{bmatrix}) \geq 0$$

, with $det(\begin{bmatrix} A_1 \\ R_{i-1} \\ A_2 \end{bmatrix})$ and $det(\begin{bmatrix} A_1 \\ R_i \\ A_2 \end{bmatrix})$ nonnegative.

This covers all possible cases when $detA = det \begin{bmatrix} a_{i,1} & a_{i+1,1} \\ a_{i,n} & a_{i+1,n} \end{bmatrix} > 0$. We now discuss cases in which the determinant is 0 and show that a TN line insertion is still possible as long as the specified entries respect the NS principle. There are four cases to consider:

Case 1: $a_{i-1,1} = a_{i-1,n} = 0$ $(a_{i,1} = a_{i,n} = 0)$

The NS principle dictates that there are only two possible scenarios:

1. The $(i-1)$st row is all 0 (The ith row is all 0). We can simply remove the zero row and complete the insertion with the new row above (below).

2. Assuming there are positive entries between $a_{i-1,1}$ and $a_{i-1,n}$, the NS principle requires all entries below $a_{i-1,1}$ and entries above $a_{i-1,n}$ to be zero (Entries below $a_{i,1}$ and entries above $a_{i,n}$). Notice that this implies $a_{i,1}$ and a_1^* have to be zero (a_n^* and $a_{i-1,n}$ have to be zero). We can then complete row* as a positive multiple of the ith row (a positive multiple of the $(i-1)$st row).

Case 2: $a_{i-1,1} = a_{i,1} = 0$ $(a_{i-1,n} = a_{i,n} = 0)$

In this case a_1^* (a_n^*) has to be 0 because of the NS principle. We can still complete the ith row as a linear combination of the ith row and the $(i-1)$th row by treating a_n^* as a weighted sum of $a_{i-1,n}$ and $a_{i,n}$ (a_1^* as a weighted sum of $a_{i-1,1}$ and $a_{i,1}$).

Case 3: $\begin{bmatrix} a_{i,1} \\ a_{i,n} \end{bmatrix}$ is a positive multiple of $\begin{bmatrix} a_{i-1,1} \\ a_{i-1,n} \end{bmatrix}$: $\begin{bmatrix} a_{i,1} \\ a_{i,n} \end{bmatrix} = k \begin{bmatrix} a_{i-1,1} \\ a_{i-1,n} \end{bmatrix}$

In this case, $\begin{bmatrix} a_1^* \\ a_n^* \end{bmatrix}$ must also be a positive multiple of $\begin{bmatrix} a_{i-1,1} \\ a_{i-1,n} \end{bmatrix}$ by nonnegativity. The two-by-two minors of the submatrix

$$\begin{bmatrix} a_{i-1,1} & a_{i-1,n} \\ a_1^* & a_n^* \\ c_1 a_{i-1,1} & c_1 a_{i-1,n} \end{bmatrix}$$

implies $a_{i-1,1}a_n^* - a_{i-1,n}a_1^* \geq 0$, and $k(a_{i-1,n}a_1^* - a_{i-1,1}a_n^*) \geq 0$, which can only be satisfied if $\begin{bmatrix} a_1^* \\ a_n^* \end{bmatrix}$ is also a positive multiple of $\begin{bmatrix} a_{i-1,1} \\ a_{i-1,n} \end{bmatrix}$. Thus we can complete row* as a positive multiple of the $(i-1)$st row.

Case 4: $a_{i-1,1} = a_{i-1,n} = a_{i,1} = a_{i,n} = 0$

These conditions imply that $a*_1 = a*_n = 0$. We can simply complete the line with all zeros.

Having considered all possible cases, we proved that a line insertion is possible between two consecutive rows. A line insertion between two consecutive columns can be proven by considering A^T. □

Corollary 4.3. *Given any m-by-n TN matrix, it is possible to append a line with two extremal boundary conditions on the border of the matrix.*

Proof. We can use the theorem proved above to show that it is always possible to append a line outside the first (last) row or column. We give a proof for appending a line outside the first row or the last row. Suppose we are trying to append a line outside the first row in a TN matrix. We denote the two extremal conditions as a_1^*, the initial entry, and a_n^*, the last entry. Assuming $a_{1,1} \neq 0$ and $a_{1,n} \neq 0$, we add one more row above the first row, as shown in the display below. The new row will be a copy of the old first row plus a positive perturbation δ. Following the scheme, the first entry will be equal $a_{1,1} + \delta$, $\delta > 0$. The perturbation is chosen to make the determinant of the matrix $\begin{bmatrix} a_{1,1} + \delta & a_{1,n} \\ a_1^* & a_n^* \end{bmatrix}$ positive, which requires $\delta > \frac{a_1^* a_{1,n} - a_n^* a_{1,1}}{a_n^*}$.

Because the original matrix is TN, by Lemma 2.1, the resulting matrix is still TN. We have now transformed the problem to a line insertion problem in the second row. The fact that the determinant $A = \begin{bmatrix} a_{1,1}+\delta & a_{1,n} \\ a_{1,1} & a_{1,n} \end{bmatrix}$ is positive implies that we could find a pair of positive coefficients as in the second row insertion.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & \dots & \dots \\ a_{m,1} & \dots & a_{m,n} \\ a_{m+1,1} & \dots & a_{m+1,n} \end{bmatrix} \to \begin{bmatrix} a_{1,1}+\delta & \dots & a_{1,n} \\ a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & \dots & \dots \\ a_{m,1} & \dots & a_{m,n} \\ a_{m+1,1} & \dots & a_{m+1,n} \end{bmatrix} \to \begin{bmatrix} a_{1,1}+\delta & \dots & a_{1,n} \\ a_1^* & ? & a^*n \\ a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & \dots & \dots \\ a_{m,1} & \dots & a_{m,n} \\ a_{m+1,1} & \dots & a_{m+1,n} \end{bmatrix}$$

It is possible for a_n^* to be 0, which means we cannot find a proper δ. Howver, in that case, we can omit all the work above and simply complete all unspecified entries (to the left of a_n^*) with zeros without violating total nonnegativity and the NS principle.

The line insertion in the last row can be transformed in a similar way: first add a copy of the last row below the last row and then perturb the last entry of the new row by a positive δ such that $\begin{bmatrix} a_1^* & a_n^* \\ a_{m,1} & a_{m,n}+\delta \end{bmatrix}$ is positive, which requires $\delta > \frac{a_1^* a_{m,n} - a_n^* a_{m,1}}{a_1^*}$ (If a_1^* is 0, we can simply complete all unspecified entries, all to the right of a_1^*, with all zeros, without violating nonnegativity and the NS principle). Again, this will yield a new TN matrix and we have transformed the line insertion in the last row to a line insertion in the second to last row.

We are now only left with cases in which either the first entry or the last entry of the first (final) row is 0. The NS principle implies that either the first entry or the last entry of the line to be inserted is also 0. We can then complete the line as a positive multiple of the first (final) row.Appending a line outside the first and last columns can be proven by considering A^T. □

4.3 TN Completability of the Border Pattern under Northeast-Southwest Principle

With Theorem 3.1, we can show that there exists a TN completion for an m-by-n partial TN matrix with the border pattern.

Theorem 4.4. *The m-by-n border pattern, $m, n \geq 3$ is TN completable if it respects the Northeast-Southwest (NS) principle*

Proof. Let's first assume there are no zeros on the border, in which case we can complete the pattern by viewing the completion of each row as a TN line insertion with two extremal boundary conditions. The completion of the second row is equivalent to inserting the second row into the submatrix consisting of the first and the last rows; the completion of the third row is equivalent to insert the third row into the submatrix consisting of the first, the second and and the last rows and so forth. Upon completing the $(n-1)$st row, we have a TN completion of the original partial matrix.

Now let's consider cases in which one or more zeros occur on the border. Again, the pattern is TN completable as long as it respects the NS principle.

Case 1: $a_{1,1} = 0$ $(a_{m,n} = 0)$

The NS principle prescribes either entries in the first row (northeast) or the first column (southwest) are all zero (entries in the last row (southwest) or the first column (northeast) are all zero). We can first attempt to complete the submatrix excluding the zero row/-column. Then the problem is transformed into line insertions with one specified entries. We can apply the same strategy as in the proof of theorem 3.1 by completing one row as a positive multiple of the previous row. After we complete the submatrix, we can add the zero row/column back and we have a TN completion.

Case 2: $a_{i,1} = a_{i,n} = 0$, $i \subset \{2, ..., m-1\}$ or $a_{1,j} = a_{m,j} = 0$, $j \subset \{2, ..., n-1\}$ In this case we can first complete the submatrix excluding the rows indexed by i and columns indexed by j and insert them back at the same row- and column- indexes as zero rows and columns.

Case 3: $a_{i,1} = 0$, $i \geq r$, $r \in \{2, ..., m\}$ or $a_{j,n} = 0$, $j \leq s$, $s \in \{1, ..., m-1\}$ or $a_{1,k}$, $k \geq t$, $t \in \{2, ..., n\}$ or $a_{n,h}$, $h \leq u$, $u \in \{1, ..., n-1\}$

The case 3 is describing the following patterns and their transposes, as well as any combination of them:

$$\begin{bmatrix} x & x & \dots & x & x \\ x & ? & \dots & ? & x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ? & \dots & ? & x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x & \dots & x & x \end{bmatrix} \begin{bmatrix} x & x & \dots & x & 0 \\ x & ? & \dots & ? & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x & ? & \dots & ? & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x & x & \dots & x & x \end{bmatrix}$$

The way to complete is as follows: after taking care of lines in which the first entry and the last entry are both 0 (Case 2), we can then complete the pattern by repeatedly applying TN line insertions with two extremal boundary conditions. The line insertions are always possible because the pattern respects the NS principle. □

Chapter 5

TP Completability of the Border Pattern

5.1 The 3-by-n Case

The argument for TP completion can be proved by applying Theorem 2.1. We repeatedly specify one entry at a time. The order with which we complete the entries is arbitrary.

Theorem 5.1. *Every 3-by-n (n-by-3), n$\geq$ 3, border pattern is TP-completable.*

Proof. Although unspecified entries may be completed sequentially in any order, we complete from left to right for simplicity. Starting from position (2,2), the TP completability of the (2,2) entry is implied by Theorem 2.1 as the entry resides in a 3-by-3 matrix consisting of columns 1,2,n and all three rows. For position (2,3), which is shown in the figure below, its TP completability is implied as it resides in a 3-by-4 matrix consisting of columns 1,2,3 and n (entry $a^*_{2,2}$ is choen). In general, suppose we want to complete position (2,k), $2 \leq k \leq n-1$. It's TP completability is guaranteed by the theorem as the entry resides in the matrix consisting all columns i, $i = 1, 2, ..., k$ and the last column. Notice that because all unspecified entries sit on the same row, there is no interaction between the entry about to be specified and entries in columns that still have unspecified entries.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a*_{2,2} & ? & \dots & ? & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n-1} & a_{3,n} \end{bmatrix}$$

Example of completing position (3,3)

A comment on this proof is that although the TP completability of unspecified entries is implied by the pattern itself, values of the entries specified earlier will limit the range of which values are chosen to complete remaining unspecified entries. A "smart" completion of first several unspecfied entries will yield a larger range of possible values for the unspecified entries later. □

5.2 The m-by-n Case

Here we use Theorem 2.2 stated in Section 2 to give a new theorem. With Theorem 2.2, we can complete a row with two extremal boundary conditions that appear in the second or penultimate row.

Theorem 5.2. *Given any TP matrix, it is possible to insert a line between the second and third rows or between the penultimate row and the last row with two extremal boundary conditions as long as the specified entries make the corresponding partial matrix partial TP.*

Proof. The proof for a partial TP matrix with size 3-by-n (inserting at the second row), is given above.

For a partial TP matrix with size m-by-n, $m, n \geq 4$, we can apply Theorem 2.2 sequentially. The strategy is to complete one unspecified entry each time and repeat until completion. For the second row, we complete one entry each time from right to left, and for the penultimate row, from left to right. Since the completion for the second row and that for the penultimate row are symmetric, we only prove the case of the second row: Again, the strategy is to complete one unspecified entry each time and repeat similar steps until we get a completion. A diagram of the completion strategy is shown below. Observe that the unspecified $(2, n-1)$ entry is positioned at the $(2, 2)$ entry of the submatrix consisting of rows 1 to n and columns 1, n-1 and n. Theorem 2.2 implies the TP completability of the $(2, n-1)$ entry with respect to the submatrix and hence a value for the $(2, n-1)$ entry can be chosne to keep the whole matrix partial TP. Notice that because all entries to the left of the $(2, n-1)$ (except $a_{2,1}$) entry are unspecified, the $(2, n-1)$ entry does not interact with entries in columns other than 1, $n-1$ and n when being completed at current step. This fact reduces the number of minors to be taken care of at each step. Such simplification is possible because minors that contain the $(2, n-1)$ entry and entries in columns other than 1, $n-1$ and n have at least one more entry to be unspecified later.

Following the direction from right to left, next entry to be specified is $(2, n-2)$ entry. It's TP completability is implied by Theorem 2.2, as it is at the $(2, 2)$ position of the submatrix composed of rows 1 to n and columns 1, $n-2$, $n-1$ and n.

In general, following the correct direction, the $(2, n-k)$ entry, $k \in (1, n-2)$, can be viewed as the single unspecified entry residing at the (2,2) position of the submatrix that contains rows 1 to n and columns 1, $n-k+1$,...,n. Its TP completability with respect to the submatrix is implied by Theorem 2.2 and hence it may be chosen to keep the whole matrix partial TP. The non-interference argument above for the $(2, n-1)$ entry is also applied here.

Repeating same steps, our last entry to be specified is the $(2, 2)$ entry, with all entries $(2, k)$, $k = 3, ..., n-1$ have been specified. Again TP completability of the $(2, 2)$ entry is guaranteed by Theorem 2.2.Upon completing the $(2, 2)$ entry, we complete all entries on the line and get a TP completion.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & ? & ? & \dots & \leftarrow & a_{2,n}* & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & \dots & a_{3,n-1} & a_{3,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & \dots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Figure: Example of second line insertion with two extremal entries.

□

With this theorem, we can show that there exists a TP completion for an m-by-n partial TP matrix with the border pattern.

Theorem 5.3. *The m-by-n border pattern, $m, n \geq 3$ is TP completable.*

Proof. The second row is TP completable from the 3-by-n case. The third row is positioned at the penultimate row of the submatrix consisting of row 1,2,3 and n and thus is TP completable by Theorem 4.2 (graph below). Notice that the unspecified entries in the third row will not interact with specified entries in rows i, $i = 3, ..., n-1$. This is because in every two-by-two minor consisting of entries in the third row and specified entries in rows below except the last row, there will always be one more unspecified entry to be determined later. Like in the TN case, the values we choose for the third row will affect the range of possible values from which we choose for unspecified entries in rows below. However, the TP completability will always be guaranteed by Theorem 4.2.

We can complete rows below using the similar argument. Suppose we try to complete row i, $4 \leq i \leq (n-1)$. The row is positioned at the penultimate row in the submatrix consisting of rows $1, 2, ..., i$ and n and its TP completability is implied by Theorem 4.2. Repeatedly applying the arguemtn until the $(n-1)$th row, we will have a TP completion for the border pattern.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a*_{2,2} & a^*_{2,3} & \dots & a^*_{2,n-1} & a_{2,n} \\ a_{3,1} & \rightarrow & \rightarrow & \dots & \rightarrow & a_{3,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n-1} & a_{n,n} \end{bmatrix} \tag{5.1}$$

Example of completing the third row

□

Having seen that a TN line insertion with two boundary conditions is possible at any row index, and a TP line insertion with two boundary conditions is poosible at the second and penultimate row, we conjecture the following:

Conjecture 5.4. *Given any m-by-n TP matrix, it is possible to insert a ilne between any two consecutive rows or columns with two extremal boundary conditions as long as the specified entries make the corresponding partial matrix partial TP.*

5.3 Extension: Block in Border

Further questions about patterns with specified border may be answered with Theorem 2.2. In this section, we discuss which patterns of specified positions, by no means exhaustive, may occur "inside" a pattern with fully specified border such that completability is still assured.

The first case that can be shown to be TP completable is the pattern in which a block is inside the border.

Definition 5.5. A partial m-by-n, $m, n \geq 4$, TP matrix has *the border-with-block pattern* if all entries on its border and entries surrounded by row i, row i+s, column j and column j+t, $i \in [2, m-1]$, $j \in [2, n-1]$, $s \in [0, m-1-i]$ and $t \in [0, n-1-j]$, are specified.

This notion can also be described as follows: A partial m-by-n, $m, n \geq 4$, TP matrix has *the border-with-block pattern* if besides the specified entries on the border of a partial TP matrix, there is one and only one rectangular block of specified entries inside the border. The block is allowed to be contiguous with the border.
An example of the pattern is shown below.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & ? & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & ? & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & ? & \cdots & a_{i,j} & a_{i,j+1} & \cdots & a_{i,j+k} & \cdots & ? & a_{i,n} \\ \cdots & \cdots & \cdots & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,j+k} & \cdots & \cdots & a_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{i+l,j} & a_{i+l,j+1} & \cdots & a_{i+l,j+k} & \cdots & \cdots & a_{i+l,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Figure: A block inside the border pattern

Theorem 5.6. *If a partial m-by-n TP matrix follows the border-with-block pattern, it is TP completable.*

Proof. If $min\{m, n\} = 3$, the claim can be proven following the same technique used in the proof of Theorem 2.2.
For cases with $min\{m, n\} \geq 4$, the strategy of completion is as follows:
We first complete the rows that contain the block. We begin with entries to the left of the block and, then, entries to the right of the block. Whether we start from entries to the

right or entries to the left is arbitrary and does not affect our proof of TP completability. To complete entries to the left, we complete one column each time from right to left and, when inside the column, one entry each time from bottom to top. Each time we try to complete an entry at position (i^*, j^*), $i^* \in [i, i+l]$ $j^* \in [2, j-1]$, it is positioned at the (2,2) position in the submatrix that includes rows 1, i*,...,i+l and m, and columns, 1, j*,j*+1,...,j+k and n. Thus its TP completability is implied by Theorem 2.2. The entries to the right can be completed using the same argument by completing one column each time from left to right and, when inside the column, one entry each time from top to bottom.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & ? & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & ? & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & ? & \cdots & a_{i,j} & a_{i,j+1} & \cdots & a_{i,j+k} & \downarrow & ? & a_{i,n} \\ \cdots & \cdots & \uparrow & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,j+k} & \downarrow & \cdots & a_{i+1,n} \\ \cdots & \cdots & \uparrow & \cdots & \cdots & \cdots & \cdots & \downarrow & \cdots & \cdots \\ \cdots & \cdots & \uparrow & a_{i+l,j} & a_{i+l,j+1} & \cdots & a_{i+l,j+k} & \cdots & \cdots & a_{i+l,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{m,n-1} & a_{m,n} \end{bmatrix}$$

Figure: Completion of rows that contain the block

After we complete the rows that contain the block, we can complete the rest of the unspecified entries one row each time by applying Theorem 4.2. □

Chapter 6

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Appendix A

Some Matlab Code

In this section, we document some Matlab codes that are used for this research.

GenerateTP.m: This program generates TP matrix with integer entries using results of bidiagnalization.

```
function D=GenerateTP(n,q)
D=diag(randi([1,q],n,1));
k=nchoosek(n,2);
l=randi([1,q],k,1);
u=randi([1,q],k,1);
temp=eye(n-1);
left_multiple=zeros(n);
right_multiple=zeros(n);
left_multiple(1:n-1,2:n)=temp;
right_multiple(2:n,1:n-1)=temp;
index=1;
for i=n-1:-1:1
    L=zeros(n);
    L(n,n-1)=1;
    for j=1:i
        D=(eye(n)+L*l(index))*D;
        L=left_multiple*L*right_multiple;
        index=index+1;
    end
```

```
end
index=1;
for k=n-1:-1:1
    U=zeros(n);
    U(n-1,n)=1;
    for m=1:k
        D=D*(eye(n)+U*u(index));
        U=left_multiple*U*right_multiple;
        index=index+1;
    end
end
end
```

NormalizeTP.m: This program normalizes the first row and first column of a TP matrix to 1.

```
function output=NormalizeTP(matrix)
    output=matrix;
    dimension=size(output);
    rows=dimension(1);
    cols=dimension(2);
    for i=1:rows
        output(i,:)=output(i,:)/output(i,1);
    end
    for j=1:cols
        output(:,j)=output(:,j)/output(1,j);
    end
end
```

CheckTP.m: This program checks whether an input matrix is TP or not.

```
function output=CheckTP(ma)
   output=1;
   [row,column]=size(ma);
   size_max=min(row,column);
   row_list=1:row;
   column_list=1:column;
   if min(min(ma))<=0
       fprintf('This is not a TP matrix.');
       return;
   end
   for i=2:size_max
```

```
        row_list_temp=transpose(nchoosek(row_list,i));
        column_list_temp=transpose(nchoosek(column_list,i));
        for j=row_list_temp
            for k=column_list_temp
                if det(ma(j,k))<=0
                    fprintf('This is not a TP matrix.');
                    output=0 ;
                    return;
                end
            end
        end
    end
    fprintf('This is a TP matrix');
    output=1 ;
    return;
end
```

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