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Drazin inversion in the von Neumann algebra generated by two orthogonal projections

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A B S T R A C T

Criteria for Drazin and Moore–Penrose invertibility of operators in the von Neumann algebra generated by two orthogonal projections are established and explicit representations for the corresponding inverses are given. The results are illustrated by several examples that have recently been considered in the literature.

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1. Main results

This note concerns operators lying in the von Neumann or W^*-algebra $W^*(P, Q)$ generated by two orthogonal projections P and Q. In [7], one of the authors explored ranges, null spaces, and related characteristics of operators in $W^*(P, Q)$ and established in particular a criterion for Moore–Penrose invertibility. For several special operators in the algebra, Drazin invertibility was recently studied in [1] (and in a more general context also in [2]), and paper [1] actually prompted us to write the present note. Our aim is to show that a criterion for Drazin invertibility in $W^*(P, Q)$ and an explicit representation of the Drazin inverse can be easily derived from the corresponding results for Moore–Penrose inverses and that many of the results of [1] pertaining to various specific operators can be deduced very comfortably from a single general theorem.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, a *Moore–Penrose* inverse is an operator $X \in \mathcal{B}(\mathcal{H})$ such that

$$XAX = X, \qquad AXA = A, \qquad (AX)^* = AX, \qquad (XA)^* = XA. \tag{1}$$

Such X exists if and only if the range of A is closed, in which case it is defined uniquely. The standard notation for the Moore–Penrose inverse of A is $A^\dagger$. Due to (1), $AA^\dagger$ and $A^\dagger A$ are the orthogonal projections onto the range of A and of A^*, respectively.

The *Drazin inverse*, on the other hand, exists by definition if and only if both sequences $\operatorname{Im} A^j$ and $\operatorname{Ker} A^j$ stabilize. In this case there is a smallest non-negative integer k for which $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+1}$ and $\operatorname{Im} A^k = \operatorname{Im} A^{k+1}$, and the Drazin inverse of A, denoted by A^D, is the uniquely determined operator $X \in \mathcal{B}(\mathcal{H})$ satisfying

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA. \tag{2}$$

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Obviously, for invertible operators A both the Moore–Penrose inverse and the Drazin inverse coincide with the usual inverse A^{-1}. Equalities (2) then hold with $k = 0$.

Now let $P \in \mathcal{B}(\mathcal{H})$ and $Q \in \mathcal{B}(H)$ be two orthogonal projections and denote by $W^*(P, Q)$ the smallest von Neumann subalgebra of $\mathcal{B}(H)$ that contains P, Q, and the identity operator I. Let L and N denote the ranges of P and Q, respectively. We denote by P_M the orthogonal projection of $\mathcal{H}$ onto a closed subspace M and may therefore also write $P = P_L$ and $Q = P_N$. The structure of $W^*(P, Q)$ was described in [4] on the basis of the pioneering papers [3] and [5]. Namely, $W^*(P, Q)$ consists of all operators of the form

$$A = (\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00}) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}, \tag{3}$$

where, in notation slightly different from [7],

$$(\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00}) = \alpha_{11} I_{M_{11}} \oplus \alpha_{10} I_{M_{10}} \oplus \alpha_{01} I_{M_{01}} \oplus \alpha_{00} I_{M_{00}}$$

with

$$M_{11} = L \cap N, \qquad M_{10} = L \cap N^\perp, \qquad M_{01} = L^\perp \cap N, \qquad M_{00} = L^\perp \cap N^\perp,$$

H is the compression of $I - P_N$ to the subspace $M_0 = L \ominus (M_{11} \oplus M_{10})$, the operator R performs a unitary equivalence of H with the compression of P_N to $M_1 = L^\perp \ominus (M_{01} \oplus M_{00})$ (the existence of such unitary equivalence is a non-trivial fact, lying at the heart of the "two projections theory"), $\alpha_{ij} \in \mathbf{C}$, and φ_{ij} are Borel-measurable and essentially bounded functions on the spectrum $\sigma(H)$ of H. The null sets here and in what follows are always in the sense of the spectral type of H, that is, sets mapped to $\{0\}$ by the spectral measure of H.

Of course, the first orthogonal sum in (3) is limited to the terms (if any) with $\dim M_{ij} > 0$ and the last term is present if and only if $\dim M_0 > 0$. Observe also that $\sigma(H) \subseteq [0, 1]$ and that the choice of M_0 precludes $0, 1$ from lying in the point spectrum of H.

Since unitary multiples have no effect on the generalized invertibility of the operators in question, we may without loss of generality identify the subspaces M_0 and M_1 via the unitary mapping $R : M_1 \to M_0$. Consequently, representation (3) simplifies to

$$(\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00}) \oplus \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix}. \tag{4}$$

For the generating projections themselves, representation (4) will then look as

$$P = (1, 1, 0, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

$$Q = (1, 0, 1, 0) \oplus \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ \sqrt{H(I - H)} & H \end{pmatrix}. \tag{6}$$

In what follows, we let

$$\Phi_A = \begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix}.$$

Then of course the last term in (4) simply becomes $\Phi_A(H)$. In particular,

$$\Phi_P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Phi_Q(t) = \begin{pmatrix} 1 - t & \sqrt{t(1 - t)} \\ \sqrt{t(1 - t)} & t \end{pmatrix}.$$

We also put

$$\omega_A = \det \Phi_A = \varphi_{00} \varphi_{11} - \varphi_{01} \varphi_{10}, \qquad \varphi_A = |\varphi_{00}|^2 + |\varphi_{01}|^2 + |\varphi_{10}|^2 + |\varphi_{11}|^2,$$

and denote by $\Delta_r(A)$ the set of all $t \in \sigma(H)$ such that the rank of $\Phi_A(t)$ equals r ($r = 0, 1, 2$).

Theorem 1 of [7] contains, among other things, a necessary and sufficient condition for $\operatorname{Im} A$ to be closed (and thus for $A^\dagger$ to exist). It reads as follows.

Theorem 1.1. *Let $A \in W^*(P, Q)$. Then the range of A is closed if and only if the functions ω_A and φ_A are separated from zero on $\Delta_2(A)$ and $\Delta_1(A)$, respectively.*

If the range of A is closed, the Moore–Penrose inverse $A^\dagger$ can be expressed in terms of H as follows. For $r = 0, 1, 2$, denote by $M^{(r)}$ the spectral subspace of $\mathcal{H}$ corresponding to the part $\Delta_r(A)$ of $\sigma(H)$ and let H_r be the restriction of H to $M^{(r)}$. In [7] it was shown that

$$A^\dagger = (\alpha_{11}^\dagger, \alpha_{10}^\dagger, \alpha_{01}^\dagger, \alpha_{00}^\dagger) \oplus 0_{M^{(0)} \oplus M^{(0)}} \oplus \begin{pmatrix} \varphi_A(H_1) & 0 \\ 0 & \varphi_A(H_1) \end{pmatrix}^{-1} \Phi_A(H_1)^* \oplus \Phi_A(H_2)^{-1}, \tag{7}$$

where $\alpha^\dagger = 1/\alpha$ if $\alpha \neq 0$ and $0^\dagger = 0$. Note that $\Phi(H_2)$ is invertible whenever ω_A is bounded away from zero on Δ_2.

To state the result for the existence of A^D, we need to stratify the set $\Delta_1(A)$ further. Namely, let

$$\Delta_{10}(A) = \left\{ t \in \Delta_1(A): \operatorname{tr} \Phi_A(t) = 0 \right\}, \qquad \Delta_{11}(A) = \Delta_1(A) \setminus \Delta_{10}(A).$$

Theorem 1.2. *An operator $A \in W^*(P, Q)$ is Drazin invertible if and only if the functions ω_A and $\operatorname{tr} \Phi_A$ are separated from zero on $\Delta_2(A)$ and $\Delta_{11}(A)$, respectively.*

Proof. *Necessity.* It is well known that conditions (2) imply that the range of $B := A^k$ is closed. Indeed, we have $\operatorname{Im} B = \operatorname{Im} B^2$ and $\operatorname{Ker} B = \operatorname{Ker} B^2$ and hence B is Drazin invertible with $k = 1$, which guarantees a $Z \in \mathcal{B}(\mathcal{H})$ such that $B^2 Z = B$ and $BZ = ZB$. It follows that if $y_n = Bx_n \to y$, then $y_n = B^2 Z x_n = BZ Bx_n = BZ y_n$ and thus $y = BZy \in \operatorname{Im} B$. Since $\operatorname{Im} A^j = \operatorname{Im} A^k$ for $j \geqslant k$, the range of A^j is closed for all $j \geqslant k$. Applying Theorem 1.1 to A^j, we conclude that ω_{A^j} and φ_{A^j} must be separated from zero on $\Delta_2(A^j)$ and $\Delta_1(A^j)$, respectively ($j \geqslant k$). It remains to observe that

$$\omega_{A^j} = (\omega_A)^j, \qquad \varphi_{A^j} = |\operatorname{tr} \Phi_A|^{2(j-1)} \varphi_A \quad \text{on } \Delta_1(A)$$

while

$$\Delta_2(A^j) = \Delta_2(A) \quad \text{for } j = 1, 2, \dots \quad \text{and} \quad \Delta_1(A^j) = \Delta_{11}(A) \quad \text{for } j = 2, 3, \dots.$$

Sufficiency. In addition to the subspaces $M^{(r)}$ and operators H_r introduced above for $r = 0, 1, 2$, we denote by $M^{(r)}$ the spectral subspace of $\mathcal{H}$ corresponding to the part $\Delta_r(A)$ of $\sigma(H)$ and by H_r the restriction of H to $M^{(r)}$ for $r = 10$ and $r = 11$. Then (4) can be rewritten as

$$A = (\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00}) \oplus 0_{M^{(0)} \oplus M^{(0)}} \oplus \Phi_A(H_{10}) \oplus \Phi_A(H_{11}) \oplus \Phi_A(H_2).$$

The operator $\Phi_A(H_2)$ is invertible (due to the condition on ω_A), the operator $\Phi_A(H_{10})$ is nilpotent of degree two, and

$$\left(\Phi_A(H_{11}) \right)^2 = \begin{pmatrix} (\operatorname{tr} \Phi_A)(H_{11}) & 0 \\ 0 & (\operatorname{tr} \Phi_A)(H_{11}) \end{pmatrix} \Phi_A(H_{11}),$$

the first term on the right-hand side being invertible because of the condition on $\operatorname{tr} \Phi_A$. It can be checked directly that the Drazin inverse of A is then given by the formula

$$A^D = \left(\alpha_{11}^\dagger, \alpha_{10}^\dagger, \alpha_{01}^\dagger, \alpha_{00}^\dagger \right) \oplus 0_{M^{(0)} \oplus M^{(0)}} \oplus 0_{M^{(10)} \oplus M^{(10)}}$$
$$\oplus \begin{pmatrix} (\operatorname{tr} \Phi_A)(H_{11}) & 0 \\ 0 & (\operatorname{tr} \Phi_A)(H_{11}) \end{pmatrix}^{-2} \Phi_A(H_{11}) \oplus \Phi_A(H_2)^{-1}, \tag{8}$$

which completes the proof. □

Observe that $M^{(10)} \oplus M^{(10)}$ is a reducing subspace for A and therefore for $A^\dagger$ as well. Since $A|_{M^{(10)} \oplus M^{(10)}}$ is non-zero (unless $M^{(10)} = \{0\}$), the restriction of $A^\dagger$ onto $M^{(10)} \oplus M^{(10)}$ also is non-zero. Comparing this to (8) we conclude that A^D and $A^\dagger$ may coincide only if $M^{(10)} = \{0\}$ (that is, $\Delta_1(A) = \Delta_{11}(A)$).

Suppose now that this condition holds (and consequently $H_1 = H_{11}$). Another glance at (7) and (8) then reveals that $A^D = A^\dagger$ if and only if for all $t \in \Delta_1$ the matrix

$$\Phi_A(t) =: \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \tag{9}$$

has the property

$$\frac{1}{(a+d)^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}. \tag{10}$$

Direct computations show that, under the condition $ad = bc$ (which holds since $t \in \Delta_1(A)$), (10) is equivalent to

$$\arg a = \arg d, \qquad |b| = |c|. \tag{11}$$

On the other hand, normality of the matrix (9) is equivalent to

$$\arg b + \arg c = 2 \arg(a - d), \qquad |b| = |c|. \tag{12}$$

But (once again, under the condition $ad = bc$) (11) and (12) are equivalent.

Consequently, $A^D = A^\dagger$ if and only if $M^{(10)} = \{0\}$ and the matrix $\Phi_A(t)$ is normal for $t \in \Delta_1(A)$.

Note also that $k = 0$ (that is, the operator A is invertible and therefore $A^\dagger = A^D = A^{-1}$) if and only if $\alpha_{ij} \neq 0$ whenever $\dim M_{ij} > 0$ and $M^{(0)} = M^{(1)} = \{0\}$. On the other hand, $k = 2$ if and only if $M^{(10)} \neq \{0\}$. In all other cases when A^D exists, it does so with $k = 1$.

We now consider the particular situation when A is actually a polynomial in P and Q. Direct computations (see e.g. formula (2.2) in [8]) show that φ_{00} and φ_{11} in representation (4) are also polynomials, while φ_{10} and φ_{01} are polynomials times $\sqrt{t(1-t)}$ (explicit formulas for these polynomials are available in [8] but they are not important for our purposes). Consequently, the functions ω_A and $\text{tr}\,\Phi_A$ are in this setting polynomials as well, and therefore either vanish identically or have only isolated roots. Theorem 1.2 may therefore be simplified as follows.

Theorem 1.3. *Let A be a polynomial in P and Q. Then A is Drazin invertible if and only if one of the following three situations takes place:*

(i) ω_A *does not vanish at the limit points of* $\sigma(H)$,
(ii) $\omega_A \equiv 0$, *and* $\text{tr}\,\Phi_A$ *does not vanish at the limit points of* $\sigma(H)$,
(iii) $\omega_A \equiv 0$, $\text{tr}\,\Phi_A \equiv 0$.

Proof. If the polynomial ω_A is not identically zero, the set $\Delta_2(A)$ is $\sigma(H)$ with zeros of ω_A (if any) deleted. So, for ω_A to be separated from zero on $\Delta_2(A)$ it is necessary and sufficient that these zeros are isolated points of $\sigma(H)$. On the other hand, $\Delta_1(A)$ (and therefore $\Delta_{11}(A)$) in this case consists of isolated points only, so that the condition on $\text{tr}\,\Phi_A$ from Theorem 1.2 holds automatically. Thus, in this setting condition (i) is necessary and sufficient for A to be Drazin invertible.

Let now $\omega_A \equiv 0$. Then $\Delta_2(A) = \emptyset$, so that the condition on ω_A from Theorem 1.2 holds again automatically. If in addition $\text{tr}\,\Phi_A$ is not identically zero, then $\Delta_{11}(A)$ differs from $\sigma(H)$ by at most finitely many points, and $\text{tr}\,\Phi_A$ is separated from zero on $\Delta_{11}(A)$ if and only if its zeros are not limit points of $\sigma(H)$. This is exactly condition (ii).

Finally, if ω_A and $\text{tr}\,\Phi_A$ are both identically equal to zero, then the sets $\Delta_2(A)$ and $\Delta_{11}(A)$ are void. Conditions of Theorem 1.2 then hold vacuously, so that A is Drazin invertible. $\square$

Observe that in cases (i) and (ii) of Theorem 1.3 the range of A is closed, so that its Moore–Penrose inverse also exists. On the other hand, case (iii) corresponds to a nilpotent A, and then $\text{Im}\,A$ may or may not be closed.

2. Examples

The purpose of this section is to demonstrate how Theorems 1.1 and 1.2 and the explicit representations (7) and (8) work in concrete situations, in particular in the cases studied in [1] and [2].

Let us first consider $P + Q$. From (5) and (6) we infer that

$$P + Q = (2, 1, 1, 0) \oplus \begin{pmatrix} 2I - H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & H \end{pmatrix}. \tag{13}$$

Theorem 2.1. *The following are equivalent*: (i) $P + Q$ *is Drazin invertible,* (ii) $P + Q$ *is Moore–Penrose invertible,* (iii) $M_0 = \{0\}$ *or* $M_0 \neq \{0\}$ *and H is invertible. If $M_0 = \{0\}$, then*

$$(P + Q)^D = (P + Q)^\dagger = \left(\frac{1}{2}, 1, 1, 0\right)$$

and if $M_0 \neq \{0\}$ and H is invertible, then $(P + Q)^D$ and $(P + Q)^\dagger$ are equal to

$$\left(\frac{1}{2}, 1, 1, 0\right) \oplus \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix} \begin{pmatrix} H & -\sqrt{H(I-H)} \\ -\sqrt{H(I-H)} & 2I - H \end{pmatrix}. \tag{14}$$

Proof. If $M_0 = \{0\}$, then the matrix in (13) is actually absent, so that $P + Q = (2, 1, 1, 0)$, which implies that $(P + Q)^D$ and $(P + Q)^\dagger$ exist and are just $(1/2, 1, 1, 0)$. So assume $M_0 \neq \{0\}$. We then have

$$\Phi_{P+Q}(t) = \begin{pmatrix} 2 - t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix},$$

whence $\omega_{P+Q}(t) = t$, $\varphi_{P+Q}(t) = 4 - 2t$, $\text{tr}\,\Phi_{P+Q}(t) = 2$. It follows that

$$\Delta_0 = \emptyset, \qquad \Delta_{10} = \emptyset, \qquad \Delta_{11} = \Delta_1 = \{0\}, \qquad \Delta_2 = \sigma(H) \setminus \{0\}.$$

Since $M^{(10)}$ is absent (or may be taken to be $\{0\}$) and $\Phi_{P+Q}(t)$ is normal for $t \in \Delta_1$, we conclude that $P + Q$ is Drazin invertible if and only if it is Moore–Penrose invertible and that the two inverses coincide. Theorem 1.2 shows that $P + Q$ is Drazin invertible exactly if $\sigma(H) \subseteq \{0\} \cup [\varepsilon, 1)$ for some $\varepsilon > 0$, which is equivalent to the invertibility of H because 0 is known to be not an eigenvalue of the Hermitian operator H. If H is invertible, then $H_2 = H$ and equality (8) yields the asserted formula for $(P + Q)^D = (P + Q)^\dagger$. $\square$

Theorem 2.1 gives the Drazin inverse in terms of H. In paper [6], the authors raised the problem of expressing $(A + B)^D$ via A, B, A^D, B^D in the case where A and B are arbitrary matrices. We here deal with orthogonal projections P and Q. If A

is a projection (not necessarily orthogonal), then (2) is obviously true with $X = A$ and $k = 1$, so that $A^D = A$. Thus, in our setting the problem of [6] amounts to finding a formula for $(P + Q)^D$ in terms of only P and Q. From (8) it follows that the Drazin inverse of every Drazin invertible operator in $W^*(P, Q)$ belongs also to $W^*(P, Q)$. This shows that the formula we are looking for must exist.

The following theorem provides us with such a formula. In connection with this theorem notice that if K and M are closed subspaces of $\mathcal{H}$, then $P_{K \cap M}$ can be expressed in terms of P_K and P_M as the strong limit

$$P_{K \cap M} = \underset{n \to \infty}{\text{s-lim}} \, (P_K P_M)^n = \text{s-lim} \, P_K P_M P_K P_M P_K P_M \cdots.$$

This formula goes back to von Neumann [9] and is frequently called the method of alternating projections or von Neumann's algorithm. Incidentally, the formula follows easily from (5) and (6), applied to $K = L$, $M = N$. Indeed,

$$P_{K \cap M} = (1, 0, 0, 0) \oplus 0_{M_0 \oplus M_0}$$

while

$$(P_K P_M)^n = (1, 0, 0, 0) \oplus \begin{pmatrix} (I - H)^{n-1} & 0 \\ 0 & (I - H)^{n-1} \end{pmatrix} \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ 0 & 0 \end{pmatrix},$$

and since the Hermitian operator $I - H$ has its spectrum in $[0, 1]$ and does not have 1 as its eigenvalue, its powers converge strongly to zero.

In our context $P_L = P$ and $P_N = Q$ and hence

$$P_{L \cap N} = \underset{n \to \infty}{\text{s-lim}} \, (PQ)^n, \qquad P_{L^\perp \cap N^\perp} = \underset{n \to \infty}{\text{s-lim}} \, \big((I - P)(I - Q)\big)^n.$$

We put

$$S = P(I - Q)P + (I - P)Q(I - P) = P + Q - PQ - QP,$$

$$T = P_{L \cap N} + P_{L^\perp \cap N^\perp} + S.$$

Theorem 2.2. *The operator $P + Q$ is Drazin invertible if and only if T is invertible, in which case*

$$(P + Q)^D = \frac{1}{2} P_{L \cap N} - 2 P_{L^\perp \cap N^\perp} + T^{-1}(2I - P - Q).$$

Proof. Suppose first that $M_0 \neq \{0\}$. From (5) and (6) we see that

$$P(I - Q)P = (0, 1, 0, 0) \oplus \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix},$$

$$(I - P)Q(I - P) = (0, 0, 1, 0) \oplus \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix},$$

which gives

$$S = (0, 1, 1, 0) \oplus \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}.$$

Since $P_{L \cap N} = (1, 0, 0, 0)$ and $P_{L^\perp \cap N^\perp} = (0, 0, 0, 1)$, we obtain that

$$T = (1, 1, 1, 1) \oplus \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}.$$

Clearly, T is invertible if and only if so is H. Thus, by Theorem 2.1, $P + Q$ is Drazin invertible if and only if T is invertible. In that case

$$T^{-1} = (1, 1, 1, 1) \oplus \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix}.$$

A straightforward computation yields

$$2I - P - Q = (0, 1, 1, 2) \oplus \begin{pmatrix} H & -\sqrt{H(I - H)} \\ -\sqrt{H(I - H)} & 2I - H \end{pmatrix}.$$

Combining the last two formulas we see that $T^{-1}(2I - P - Q)$ equals

$$(0, 1, 1, 2) \oplus \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix} \begin{pmatrix} H & -\sqrt{H(I - H)} \\ -\sqrt{H(I - H)} & 2I - H \end{pmatrix},$$

which after addition of $(1/2)P_{L\cap N} - 2P_{L^{\perp}\cap N^{\perp}} = (1/2, 0, 0, -2)$ becomes (14). This completes the proof under the assumption that $M_0 \neq \{0\}$. If $M_0 = \{0\}$, then $T = (1, 1, 1, 1)$ is the identity operator and the theorem is equivalent to saying that $P + Q$ is Drazin invertible with

$$(P + Q)^D = \frac{1}{2}P_{L\cap N} - 2P_{L^{\perp}\cap N^{\perp}} + 2I - P - Q = \left(\frac{1}{2}, 1, 1, 0\right).$$

But this is immediate from Theorem 2.1. □

The following theorem is a generalization of Theorem 2.2. The proof is similar to the proofs of the previous two theorems and is therefore omitted.

Theorem 2.3. *Let $a, b \in \mathbf{C}$. The operator $aP + bQ$ is Drazin invertible if and only if it is Moore–Penrose invertible, which is in turn equivalent to the invertibility of the operator T. If T is invertible, then*

$$(P - Q)^D = (P - Q)^{\dagger} = T^{-1}(P - Q),$$

while if $ab \neq 0$, $a + b \neq 0$, then both $(aP + bQ)^D$ and $(aP + bQ)^{\dagger}$ are equal to

$$\frac{1}{a+b}P_{L\cap N} - \left(\frac{1}{a} + \frac{1}{b}\right)P_{L^{\perp}\cap N^{\perp}} + \frac{1}{ab}T^{-1}\big(a(I - P) + b(I - Q)\big).$$

We now turn to the products PQ and PQP. Representations (5) and (6) give

$$PQ = (1, 0, 0, 0) \oplus \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ 0 & 0 \end{pmatrix}, \tag{15}$$

$$PQP = (1, 0, 0, 0) \oplus \begin{pmatrix} I - H & 0 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

Theorem 2.4. *For the operators PQ and PQP, both Drazin and Moore–Penrose invertibility are equivalent to the condition that either $M_0 = \{0\}$ or $M_0 \neq \{0\}$ and $I - H$ is invertible. If $M_0 = \{0\}$, then*

$$(PQ)^D = (PQ)^{\dagger} = (PQP)^D = (PQP)^{\dagger} = (1, 0, 0, 0),$$

and if $M_0 \neq \{0\}$ and $I - H$ is invertible, then

$$(PQ)^D = (1, 0, 0, 0) \oplus \begin{pmatrix} (I - H)^{-2} & 0 \\ 0 & (I - H)^{-2} \end{pmatrix} \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ 0 & 0 \end{pmatrix},$$

$$(PQ)^{\dagger} = (1, 0, 0, 0) \oplus \begin{pmatrix} (I - H)^{-1} & 0 \\ 0 & (I - H)^{-1} \end{pmatrix} \begin{pmatrix} I - H & 0 \\ \sqrt{H(I - H)} & 0 \end{pmatrix},$$

$$(PQP)^D = (PQP)^{\dagger} = (1, 0, 0, 0) \oplus \begin{pmatrix} (I - H)^{-2} & 0 \\ 0 & (I - H)^{-2} \end{pmatrix} \begin{pmatrix} I - H & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. The case $M_0 = \{0\}$ is trivial. So let $M_0 \neq \{0\}$. We have

$$\Phi_{PQ}(t) = \begin{pmatrix} 1 - t & \sqrt{t(1 - t)} \\ 0 & 0 \end{pmatrix},$$

and hence $\omega_{PQ}(t) = 0$, $\varphi_{PQ}(t) = 1 - t$, $\operatorname{tr}\Phi_{PQ}(t) = 1 - t$, $\Delta_0 = \{1\}$, $\Delta_{10} = \emptyset$, $\Delta_{11} = \sigma(H) \setminus \{1\}$, $\Delta_2 = \emptyset$. From Theorem 1.2 we therefore obtain that PQ is Drazin invertible if and only if $1 - t$ is separated from zero on $\sigma(H) \setminus \{1\}$, which happens if and only if $\sigma(H) \subseteq (0, 1 - \varepsilon] \cup \{1\}$ for some $\varepsilon > 0$. As 1 is not an eigenvalue of H, this is equivalent to the invertibility of $I - H$. In the same way we deduce from Theorem 1.1 that PQ is Moore–Penrose invertible exactly if $I - H$ is invertible. In the case at hand, $H_{11} = H$. The representations of the Drazin and Moore–Penrose inverses of PQ are therefore immediate from (7) and (8). The operator PQP can be tackled equally. □

Let now

$$U = PQP + (I - P)(I - Q)(I - P), \qquad V = P_{L\cap N^{\perp}} + P_{L^{\perp}\cap N} + U$$

and recall that $P_{L\cap N^{\perp}}$ and $P_{L^{\perp}\cap N}$ are the strong limits of

$$P(I - Q)P(I - Q)P(I - Q)\cdots \quad \text{and} \quad (I - P)Q(I - P)Q(I - P)Q\cdots,$$

respectively.

Theorem 2.5. *For each of the operators PQ and PQP, both Drazin and Moore–Penrose invertibility are equivalent to the invertibility of V. In case V is invertible,*

$$(PQ)^D = V^{-2}PQ, \qquad (PQ)^\dagger = V^{-1}QP, \qquad (PQP)^D = (PQP)^\dagger = V^{-2}PQP.$$

Proof. Since

$$U = (1, 0, 0, 1) \oplus \begin{pmatrix} I - H & 0 \\ 0 & I - H \end{pmatrix}, \qquad V = (1, 1, 1, 1) \oplus \begin{pmatrix} I - H & 0 \\ 0 & I - H \end{pmatrix},$$

this follows from Theorem 2.4 in conjunction with (15) and (16). $\quad\square$

In analogy to Theorem 2.5 one can prove the following.

Theorem 2.6. *Let A be one of the operators PQ, PQP, $PQPQ$, $PQPQP$, ... Then A is Drazin invertible if and only if it is Moore–Penrose invertible, and this is in turn the case if and only if V is invertible. If V is invertible, then for every natural number $m \geqslant 1$,*

$$\big((PQ)^m\big)^D = V^{-m-1}PQ, \qquad \big((PQ)^m\big)^\dagger = V^{-m}QP, \qquad \big((PQ)^mP\big)^D = \big((PQ)^mP\big)^\dagger = V^{-m-1}PQP.$$

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