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Maps preserving product $XY - YX^*$ on factor von Neumann algebras[☆]

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ABSTRACT

Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. For $A, B \in \mathcal{A}$, define by $[A, B]_* = AB - BA^*$ the new product of A and B. In this paper, we prove that a nonlinear bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A, B]_*) = [\Phi(A), \Phi(B)]_*$ for all $A, B \in \mathcal{A}$ if and only if Φ is a $*$-ring isomorphism. In particular, if the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism.

1. Introduction

As a kind of new products in a $*$-ring, the operation $XY - YX^*$ was discussed in [6]. This product $XY - YX^*$ is found playing a more and more important role in some research topics, and its study has recently attracted many authors' attention. This product was extensively studied because, by the fundamental theorem of Šemrl in [6], maps of the form $T \mapsto TA - AT^*$ naturally arise in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [7,8]). Šemrl in [9] proved every Jordan $*$-derivation $J : \mathcal{B}(H) \to \mathcal{B}(H)$ (satisfying $J(T^2) = TJ(T) + J(T)T^*$)

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is of the form $J(T) = TA - AT^*$, where $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H. Motivated by the work of Šemrl and the theory of rings (and algebras) equipped with a Lie product $[T, S] = TS - ST$ or a Jordan product $T \circ S = TS + ST$, Molnár recently in [5] initiated the systematic study of this new product, and studied the relation between subspaces and ideals of $\mathcal{B}(H)$. Where he showed that if a subspace $\mathcal{N}$ of $\mathcal{B}(H)$ satisfies $AB - BA^* \in \mathcal{N}$ for $A \in \mathcal{B}(H)$ and $B \in \mathcal{N}$, then $\mathcal{N}$ is an ideal; and also, if the dimension of H is an odd natural number, then $\mathcal{N} = \mathcal{B}(H)$. In addition, he proved that if H is of dimension greater than 1 and $\mathcal{N} \subseteq \mathcal{B}(H)$ is an ideal, then $\mathrm{span}\{AB - BA^* | A \in \mathcal{N}, B \in \mathcal{B}(H)\} = \mathrm{span}\{BA - AB^* | A \in \mathcal{N}, B \in \mathcal{B}(H)\} = \mathcal{N}$. In [2], Brešar and Fošner generalized Molnár's results to rings with involution in different ways, and studied the relationship between (ordinary) ideals of a ring R and left and right ideals of R with respect to the product $AB - BA^*$. Their approach is entirely algebraic and is completely different from one used by Molnár, and it is based on discovering certain identities that connect the product $AB - BA^*$ with the initial, associative product.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-rings. For $A, B \in \mathcal{A}$, denote by $[A, B]_* = AB - BA^*$ the new product of A and B. A map $\phi : \mathcal{A} \to \mathcal{B}$ is called new products preserving if $\phi([A, B]_*) = [\phi(A), \phi(B)]_*$ for all $A, B \in \mathcal{A}$. In [1], the authors studied the bijective map preserving this new product on $\mathcal{B}(H)$, where H is a complex Hilbert space of dimension greater than 2. They showed that such maps are in fact $*$-automorphisms or conjugate $*$-automorphisms. This result shows that, in some sense, the new product $AB - BA^*$ structure is determine enough the $*$-algebraic structure of $\mathcal{B}(H)$. In this paper, we will discuss such a problem on more general factor von Neumann algebras. We prove that such a bijective map on factor von Neumann algebras must be a $*$-additive isomorphism (see Main Theorem). In particular, if the factor is of type I, then $*$-isomorphism is spatial, which generalized the main result in [1] to any complex Hilbert space case (see Corollary 1). We mention here the method used in [1] is not completely fit for general von Neumann algebras since the notion of finite-rank is meaningless in general von Neumann algebras.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Recall that a factor is a von Neumann algebra whose center only contains the scalar operators. An algebra $\mathcal{R}$ is called prime if $A\mathcal{R}B = \{0\}$ implies that $A = 0$ or $B = 0$. It is well known that every factor von Neumann algebra is a prime algebra.

Our main result is as follows:

Main Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective map. Then Φ satisfies $\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$* if and only if *Φ is a $*$-ring isomorphism.*

2. The proof of Main Theorem

In this section, we will complete the proof of the main theorem by proving several claims. The following results will be used many times in the proof of theorem.

Lemma 1. *Let $\mathcal{A}$ be a factor von Neumann algebra, and $A \in \mathcal{A}$. Then $AB = BA^*$ for every $B \in \mathcal{A}$ implies that $A \in \mathbb{R}I$.*

Proof. In fact, take $B = I$, then $A = A^*$, and therefore, $AB = BA$ for every $B \in \mathcal{A}$, hence A belongs to the center of $\mathcal{A}$. Note that $\mathcal{A}$ is a factor, it follows that $A \in \mathbb{R}I$. $\square$

Lemma 2. *Let $\mathcal{A}$ be a factor von Neumann algebra, and $B \in \mathcal{A}$. Then $AB = BA^*$ for every $A \in \mathcal{A}$ implies that $B = 0$.*

Proof. It follows that $AB = BA$ for every Hermitian element A, and hence $AB = BA$ for every $A \in \mathcal{A}$ since $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$, where $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ are Hermitian. So there exists a scalar $\alpha \in \mathbb{C}$ such that $B = \alpha I$. Taking $A \in \mathcal{A}$ such that $A \neq A^*$, one has $\alpha(A - A^*) = 0$, and consequently, $\alpha = 0$ and $B = 0$. $\square$

Proof of Main Theorem

Claim 1. $\Phi(0) = 0$.

For any $A \in \mathcal{A}$, we have $\Phi(A)\Phi(0) - \Phi(0)\Phi(A)^* = \Phi(0)$. It follows from the surjectivity of Φ that there exists $A \in \mathcal{A}$ such that $\Phi(A) = iI$ (where, i is the imaginary number unit), so $2i\Phi(0) = \Phi(0)$, and hence $\Phi(0) = 0$.

Claim 2. *$\Phi(\mathbb{R}I) = \mathbb{R}I$, $\Phi(\mathbb{C}I) = \mathbb{C}I$ and Φ preserves Hermitian elements in both directions.*

Claim 1 and the injectivity of Φ imply that

$$AB = BA^* \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)^* \quad \text{for all } A, B \in \mathcal{A}. \tag{2.1}$$

Let $\alpha \in \mathbb{R}$ be arbitrary. Then the equality $\alpha I \cdot A = A \cdot (\alpha I)^*$ $(\forall A \in \mathcal{A})$ implies that $\Phi(\alpha I)\Phi(A) = \Phi(A)\Phi(\alpha I)^*$. Now the surjectivity of Φ ensures that

$$\Phi(\alpha I)S = S\Phi(\alpha I)^* \quad \text{for all } S \in \mathcal{B}.$$

Lemma 1 implies that $\Phi(\alpha I) \in \mathbb{R}I$. A similar discussion implies that $\Phi(A) \in \mathbb{R}I \Rightarrow A \in \mathbb{R}I$. So $\Phi(\mathbb{R}I) = \mathbb{R}I$.

For any Hermitian element $A \in \mathcal{A}$ (that is, $A^* = A$), Eq. (2.1) implies that $\Phi(A)\Phi(I) = \Phi(I)\Phi(A)^*$, and hence it follows from $\Phi(I) \in \mathbb{R}I$ and $\Phi(I) \neq 0$ that $\Phi(A) = \Phi(A)^*$. Conversely, assume that $\Phi(A)$ is Hermitian. Then, it follows from $\Phi(\mathbb{R}I) = \mathbb{R}I$ and Eq. (2.1) again that A is Hermitian. So Φ preserves Hermitian elements in both directions.

Let $\alpha \in \mathbb{C}$ be arbitrary. Then, for every Hermitian element $A \in \mathcal{A}$, the equality $A \cdot \alpha I = (\alpha I) \cdot A^*$ implies that $\Phi(A)\Phi(\alpha I) = \Phi(\alpha I)\Phi(A)^*$. Since Φ preserves Hermitian elements in both directions, it follows from the surjectivity of Φ that $S\Phi(\alpha I) = \Phi(\alpha I)S$ for every Hermitian element $S \in \mathcal{B}$, and therefore, $T\Phi(\alpha I) = \Phi(\alpha I)T$ for all $T \in \mathcal{B}$ since $T = S_1 + iS_2$ with S_1 and S_2 being Hermitian, so $\Phi(\alpha I) \in \mathbb{C}I$. A similar discussion implies that $\Phi(A) \in \mathbb{C}I \Rightarrow A \in \mathbb{C}I$. So $\Phi(\mathbb{C}I) = \mathbb{C}I$.

Claim 3. *$\Phi(iA) = i\Phi(A)$ $(\forall A \in \mathcal{A})$ or $\Phi(iA) = -i\Phi(A)$ $(\forall A \in \mathcal{A})$ and Φ preserves projections in both directions.*

Applying Claim 2, we have $\Phi\left(\pm\frac{1}{2}iI\right) \in (\mathbb{C} \setminus \mathbb{R})I$ and $\Phi\left(\pm\frac{1}{2}I\right) \in \mathbb{R}I$. It follows from $\frac{1}{2}iI = \left[-\frac{1}{2}iI, -\frac{1}{2}I\right]_*$ that

$$\Phi\left(\frac{1}{2}iI\right) = 2\Phi\left(-\frac{1}{2}I\right)\Phi\left(-\frac{1}{2}iI\right), \tag{2.2}$$

Also the equality $-\frac{1}{2}I = \left[\frac{1}{2}iI, \frac{1}{2}iI\right]_*$ implies that

$$\Phi\left(-\frac{1}{2}I\right) = 2\Phi\left(\frac{1}{2}iI\right)^2 \tag{2.3}$$

and $-\frac{1}{2}I = \left[-\frac{1}{2}iI, -\frac{1}{2}iI\right]_*$ ensures that

$$\Phi\left(-\frac{1}{2}I\right) = 2\Phi\left(-\frac{1}{2}iI\right)^2. \tag{2.4}$$

Now Eqs. (2.2)–(2.4) ensure that $\Phi\left(-\frac{1}{2}I\right) = -\frac{1}{2}I$, and

$$\Phi\left(\frac{1}{2}iI\right) = \pm\frac{1}{2}iI. \tag{2.5}$$

So, for every $A \in \mathcal{A}$, we have

$$\Phi(iA) = \Phi\left(\left[\frac{1}{2}iI, A\right]_*\right) = \left[\Phi\left(\frac{1}{2}iI\right), \Phi(A)\right]_* = \left(\Phi\left(\frac{1}{2}iI\right) - \Phi\left(\frac{1}{2}iI\right)^*\right)\Phi(A),$$

which, together with Eq. (2.5), implies that

$$\Phi(iA) = i\Phi(A)\ (\forall A \in \mathcal{A}) \text{ or } \Phi(iA) = -i\Phi(A)(\forall A \in \mathcal{A}).$$

For every projection $P \in \mathcal{A}$, since $2iP = iPI - I(iP)^*$, we have $\Phi(2iP) = \pm 2i\Phi(P)$, and hence,

$$\pm 2i\Phi(P) = \Phi(2iP) = \Phi([iP, P]_*) = [\Phi(iP), \Phi(P)]_* = \pm 2i\Phi(P)^2,$$

so $\Phi(P)^2 = \Phi(P)$. That is, $\Phi(P)$ is a projection. Conversely, assume that $\Phi(P)$ is a projection. Since Φ^{-1} has the same property as Φ has, a similar discussion implies that P is a projection. Hence Φ preserves projections in both directions.

Claim 4. *Let $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ denote respectively the set of all projections in $\mathcal{A}$ and $\mathcal{B}$, then $\Phi : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ preserves the order and orthogonality in both directions.*

Let $P, R \in \mathcal{P}(\mathcal{A})$ be arbitrary and $PR = RP = 0$. That is, P and R are orthogonal projections. Then it follows from Claims 1 and 3 that

$$0 = \Phi([iP, R]_*) = [\Phi(iP), \Phi(R)]_* = \pm i(\Phi(P)\Phi(R) + \Phi(R)\Phi(P)),$$

and consequently, $\Phi(P)\Phi(R) + \Phi(R)\Phi(P) = 0$. Note that $\Phi(P)$ and $\Phi(R)$ are projections, so $\Phi(P)\Phi(R) = \Phi(R)\Phi(P) = 0$. Conversely, if $\Phi(P)$ and $\Phi(R)$ are orthogonal projections in $\mathcal{B}$, then a similar discussion implies that P and R are orthogonal projections. Hence $\Phi : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ preserves the orthogonality in both directions.

For any $P, R \in \mathcal{P}(\mathcal{A})$ with $P \leqslant R$, that is, $PR = RP = P$. By Claim 3,

$$\pm 2i\Phi(P) = \Phi(2iP) = \Phi([iP, R]_*) = [\Phi(iP), \Phi(R)]_* = \pm i(\Phi(P)\Phi(R) + \Phi(R)\Phi(P)),$$

and therefore, $2\Phi(P) = \Phi(P)\Phi(R) + \Phi(R)\Phi(P)$. So $\Phi(P) = \Phi(P)\Phi(R) = \Phi(R)\Phi(P)$. That is, $\Phi(P) \leqslant \Phi(R)$. Let P, $R \in \mathcal{P}(\mathcal{A})$ such that $\Phi(P) \leqslant \Phi(R)$, a similar discussion is applied to Φ^{-1}, we get that $P \leqslant R$, and hence, $\Phi : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ preserves the order in both directions.

Claim 5. *Let $A \in \mathcal{A}$ be an Hermitian element and $\lambda \in \mathbb{R}$. Then*

$$\Phi(A) + \Phi(\lambda I - A) \in \mathbb{R}I.$$

Let $A \in \mathcal{A}$ be an Hermitian element and $\lambda \in \mathbb{R}$. For every Hermitian element $X \in \mathcal{A}$, since $[A, X]_* = [X, \lambda I - A]_*$, one has

$$[\Phi(A), \Phi(X)]_* = [\Phi(X), \Phi(\lambda I - A)]_*,$$

which, together with Claim 2, implies that, for every Hermitian element $X \in \mathcal{A}$,

$$(\Phi(A) + \Phi(\lambda I - A))\Phi(X) = \Phi(X)(\Phi(A) + \Phi(\lambda I - A)).$$

It follows from Claim 2 again that $\Phi(A) + \Phi(\lambda I - A)$ commutes with every Hermitian element in $\mathcal{B}$, and hence, commutes with every element in $\mathcal{B}$, so $\Phi(A) + \Phi(\lambda I - A) \in \mathbb{C}I$. Note that $\Phi(A) + \Phi(\lambda I - A)$ is Hermitian, it follows that $\Phi(A) + \Phi(\lambda I - A) \in \mathbb{R}I$.

Choose an arbitrary nontrivial projection P_1 in $\mathcal{A}$ and let $P_2 = I - P_1$. Then, Claims 2 and 3 ensure that there exist nontrivial projections Q_i $(i = 1, 2)$ such that $\Phi(P_i) = Q_i$. By Claim 5, $Q_1 + Q_2 = I$. Let $i, j = 1, 2$, write $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ and $\mathcal{B}_{ij} = Q_i\mathcal{B}Q_j$, then

$$\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij} \quad \text{and} \quad \mathcal{B} = \sum_{i,j=1}^{2} \mathcal{B}_{ij}.$$

Claim 6. $\Phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ $(i \neq j)$.

Let $i \neq j$ and $X \in \mathcal{A}_{ij}$ be arbitrary. Since $X = [P_i, X]_*$, we have

$$\Phi(X) = [\Phi(P_i), \Phi(X)]_* = Q_i\Phi(X) - \Phi(X)Q_i.$$

It follows that $\Phi(X)Q_i = 0$ and $\Phi(X) = Q_i\Phi(X)$, and hence $Q_j\Phi(X) = 0$. So

$$\Phi(X) = \sum_{i,j=1}^{2} Q_i\Phi(X)Q_j = Q_i\Phi(X)Q_j \in \mathcal{B}_{ij}.$$

That is, $\Phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$. Since Φ^{-1} has the same property as Φ has, we have $\mathcal{B}_{ij} \subseteq \Phi(\mathcal{A}_{ij})$ Therefore, $\Phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$.

Claim 7. $\Phi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$ $(i = 1, 2)$.

Let $j \neq i$. Set $Q \in \mathcal{B}_{jj}$ be an arbitrary projection. Then $Q_i + Q \in \mathcal{B}$ is a projection and $Q_i + Q \geqslant Q_i$. By Claims 3 and 4, there exists a projection $P_Q \in \mathcal{A}$ with $P_Q \geqslant P_i$ such that $\Phi(P_Q) = Q_i + Q$. Since, for every $A \in \mathcal{A}_{ii}$, we have $[P_Q, A]_* = 0$, it follows that $[Q_i + Q, \Phi(A)]_* = 0$, which, together with $[Q_i, \Phi(A)]_* = 0$, implies that

$$\Phi(A)Q = Q\Phi(A) \text{ for every projection } Q \in \mathcal{B}_{jj}. \tag{2.6}$$

Taking $Q = Q_j$ in Eq. (2.6) and multiplying Q_i respectively from the left side and the right side of Eq. (2.6), one obtains that

$$Q_i\Phi(A)Q_j = 0 \quad \text{and} \quad Q_j\Phi(A)Q_i = 0. \tag{2.7}$$

Note that $\mathcal{B}_{jj}$ is a factor von Neumann algebra, and a von Neumann algebra is generalized by its projections if and only if it has no infinite dimensional abelian summand (see, for example, [4]). It follows from Eq. (2.6) that $Q_j\Phi(A)Q_j \in \mathbb{C}Q_j$, which, together with Eq. (2.7), implies that, for every $A \in \mathcal{A}_{ii}$, $\Phi(A) \in \mathcal{B}_{ii} + \mathbb{C}Q_j$.
For every $A_i \in \mathcal{A}_{ii}$, define a function $f_i : \mathcal{A}_{ii} \to \mathbb{C}$ as follows:

$$f_i(A_i)Q_j = Q_j\Phi(A_i)Q_j, \quad \text{where } j \neq i.$$

Then $\Phi(A_i) = Q_i\Phi(A_i)Q_i + f_i(A_i)Q_j$. Take a nonzero element $X \in \mathcal{A}_{ij}$ $(i \neq j)$. Then it follows from $[X, A_i]_* = 0$ that $[\Phi(X), \Phi(A_i)]_* = 0$, that is,

$$\Phi(X)(Q_i\Phi(A_i)Q_i + f_i(A_i)Q_j) = (Q_i\Phi(A_i)Q_i + f_i(A_i)Q_j)\Phi(X)^*.$$

By Claim 6, $\Phi(X) \in \mathcal{B}_{ij}$. Multiplying Q_j from the right side of the above expression, one has $f_i(A_i)\Phi(X) = 0$. Note that $\Phi(X) \neq 0$, we have $f_i(A_i) = 0$ for every $A_i \in \mathcal{A}_{ii}$. So $f_i(\cdot) \equiv 0$, and therefore, $\Phi(\mathcal{A}_{ii}) \subseteq \mathcal{B}_{ii}$. The same discussion is applied to Φ^{-1}, the inverse inclusion relation can be similarly proved. Hence $\Phi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$.

Claim 8. *For $i, j = 1, 2$, let $A_{ij} \in \mathcal{A}_{ij}$, then*

$$\begin{aligned}
\Phi(A_{ii} + A_{ij}) &= \Phi(A_{ii}) + \Phi(A_{ij}), \quad i \neq j,\\
\Phi(A_{ii} + A_{ji}) &= \Phi(A_{ii}) + \Phi(A_{ji}), \quad i \neq j,\\
\Phi(A_{ii} + A_{jj}) &= \Phi(A_{ii}) + \Phi(A_{jj}), \quad i \neq j,\\
\Phi(A_{ij} + A_{ji}) &= \Phi(A_{ij}) + \Phi(A_{ji}), \quad i \neq j.
\end{aligned}$$

Let $\Phi(T) = \Phi(A_{ii}) + \Phi(A_{ij})$. Then, for every $X_{jj} \in \mathcal{A}_{jj}$, it follows from Claims 6 and 7 that

$$\begin{aligned}
\Phi([X_{jj}, T]_*) &= [\Phi(X_{jj}), \Phi(T)]_*\\
&= [\Phi(X_{jj}), \Phi(A_{ii}) + \Phi(A_{ij})]_*\\
&= [\Phi(X_{jj}), \Phi(A_{ij})]_* = \Phi([X_{jj}, A_{ij}]_*).
\end{aligned}$$

This, together with the injectivity of Φ, implies that

$$X_{jj}(T-A_{ij}) = (T-A_{ij})X_{jj}^* \text{ for every } X_{jj} \in \mathcal{A}_{jj}. \tag{2.8}$$

Multiplying P_i from the right side of Eq. (2.8), one has $X_{jj}(T-A_{ij})P_i = 0$ for every $X_{jj} \in \mathcal{A}_{jj}$. That is, $P_jXP_j(T-A_{ij})P_i = 0$ for every $X \in \mathcal{A}$. Note that $\mathcal{A}$ is prime, so $P_jTP_i = 0$. Multiplying P_i from the left side of Eq. (2.8), similarly, one gets that $P_iTP_j = A_{ij}$. It follows from Eq. (2.8) again and Lemma 2 that $P_jTP_j = 0$.

On the other hand, for every $X_{ii} \in \mathcal{A}_{ii}$, by Claims 6 and 7, one has

$$\begin{aligned}\Phi([T, X_{ii}]_*) &= [\Phi(T), \Phi(X_{ii})]_* \\ &= [\Phi(A_{ii}) + \Phi(A_{ij}), \Phi(X_{ii})]_* \\ &= [\Phi(A_{ii}), \Phi(X_{ii})]_* = \Phi([A_{ii}, X_{ii}]_*).\end{aligned}$$

So $(T-A_{ii})X_{ii} = X_{ii}(T-A_{ii})^*$ for every $X_{ii} \in \mathcal{A}_{ii}$, and hence, there exists a real number α_T such that $P_iTP_i = A_{ii} + \alpha_T P_i$. Therefore $T = \sum_{i,j=1}^{2} P_iTP_j = A_{ii} + A_{ij} + \alpha_T P_i$ and

$$\Phi(A_{ii} + A_{ij} + \alpha_T P_i) = \Phi(A_{ii}) + \Phi(A_{ij}).$$

For every $X_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$, it follows from the above expression that there exists a real number α such that

$$\begin{aligned}\Phi(A_{ii}X_{ij} - X_{ij}A_{ij}^* + \alpha_T X_{ij}) &= \Phi([A_{ii} + A_{ij} + \alpha_T P_i, X_{ij}]_*) \\ &= [\Phi(A_{ii} + A_{ij} + \alpha_T P_i), \Phi(X_{ij})]_* \\ &= [\Phi(A_{ii}) + \Phi(A_{ij}), \Phi(X_{ij})]_* \\ &= \Phi([A_{ii}, X_{ij}]_*) + \Phi([A_{ij}, X_{ij}]_*) \\ &= \Phi(A_{ii}X_{ij}) + \Phi(-X_{ij}A_{ij}^*) \\ &= \Phi(A_{ii}X_{ij} - X_{ij}A_{ij}^* + \alpha P_i).\end{aligned}$$

Thus $\alpha_T X_{ij} = \alpha P_i$, and hence $\alpha_T = 0$. So

$$\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{2.9}$$

For every $T_{jj} \in \mathcal{A}_{jj}$, it follows from $[T_{jj}, A_{ii} + A_{ji}]_* = [T_{jj}, A_{ji}]_*$ that

$$[\Phi(T_{jj}), \Phi(A_{ii} + A_{ji}) - \Phi(A_{ji})]_* = 0.$$

By Claim 7, we have, for every $S_{jj} \in \mathcal{B}_{jj}$,

$$S_{jj}(\Phi(A_{ii} + A_{ji}) - \Phi(A_{ji})) = (\Phi(A_{ii} + A_{ji}) - \Phi(A_{ji}))S_{jj}^*.$$

A similar discussion just as Eq. (2.8) implies that

$$Q_j\Phi(A_{ii} + A_{ji})Q_i = \Phi(A_{ji}), \quad Q_i\Phi(A_{ii} + A_{ji})Q_j = 0 \quad \text{and} \quad Q_j\Phi(A_{ii} + A_{ji})Q_j = 0.$$

By Claim 7, there exists $B_{ii} \in \mathcal{A}_{ii}$ such that

$$Q_i\Phi(A_{ii} + A_{ji})Q_i = \Phi(B_{ii}).$$

So

$$\Phi(A_{ii} + A_{ji}) = \Phi(B_{ii}) + \Phi(A_{ji}).$$

For every $T_{ji} \in \mathcal{A}_{ji}$ $(j \neq i)$, the above expression and Eq. (2.9) imply that

$$\begin{aligned}\Phi(-T_{ji}A_{ii}^*) + \Phi(-T_{ji}A_{ji}^*) &= \Phi(-T_{ji}A_{ii}^* - T_{ji}A_{ji}^*) = \Phi([A_{ii} + A_{ji}, T_{ji}]_*) \\ &= [\Phi(A_{ii} + A_{ji}), \Phi(T_{ji})]_* \\ &= [\Phi(B_{ii}) + \Phi(A_{ji}), \Phi(T_{ji})]_* \\ &= \Phi(-T_{ji}B_{ii}^*) + \Phi(-T_{ji}A_{ji}^*).\end{aligned}$$

So $T_{ji}(A_{ii}^* - B_{ii}^*) = 0$ for every $T_{ji} \in \mathcal{A}_{ji}$, and hence, $B_{ii} = A_{ii}$. Hence

$$\Phi(A_{ii} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ji}).$$

Let $\Phi(T) = \Phi(A_{11}) + \Phi(A_{22})$. For every $T_{11} \in \mathcal{A}_{11}$, we have

$$\Phi([T_{11}, T]_*) = [\Phi(T_{11}), \Phi(T)]_* = [\Phi(T_{11}), \Phi(A_{11}) + \Phi(A_{22})]_* = \Phi([T_{11}, A_{11}]_*).$$

This implies that $T_{11}(T - A_{11}) = (T - A_{11})T_{11}^*$ for every $T_{11} \in \mathcal{A}_{11}$, and therefore $P_1TP_2 = 0, P_2TP_1 = 0$ and $P_1TP_1 = A_{11}$. Similarly, one can prove that $P_2TP_2 = A_{22}$. So

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}). \tag{2.10}$$

For every $T_{ii} \in \mathcal{A}_{ii}$, we have always $[A_{ij} + A_{ji}, T_{ii}]_* = [A_{ji}, T_{ii}]_*$ $(i \neq j)$. It follows that, for every $S_{ii} \in \mathcal{B}_{ii}$,

$$(\Phi(A_{ij} + A_{ji}) - \Phi(A_{ji}))S_{ii} = S_{ii}(\Phi(A_{ij} + A_{ji}) - \Phi(A_{ji}))^*,$$

so $Q_j\Phi(A_{ij} + A_{ji})Q_i = \Phi(A_{ji})$ and there exists $\xi \in \mathbb{R}$ such that $Q_i\Phi(A_{ij} + A_{ji})Q_i = \xi Q_i$. Similarly, it follows from $[A_{ij} + A_{ji}, T_{jj}]_* = [A_{ij}, T_{jj}]_*$ $(\forall T_{jj} \in \mathcal{A}_{jj})$ that $Q_i\Phi(A_{ij} + A_{ji})Q_j = \Phi(A_{ij})$ and $Q_j\Phi(A_{ij} + A_{ji})Q_j = \mu Q_j$ for some $\mu \in \mathbb{R}$. Thus

$$\Phi(A_{ij} + A_{ji}) = \Phi(A_{ij}) + \Phi(A_{ji}) + \xi Q_i + \mu Q_j. \tag{2.11}$$

Take a nonzero element $T_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$, it follows from Eqs. (2.10) and (2.11) that

$$\begin{aligned}
\Phi(A_{ji}T_{ij}) + \Phi(-T_{ij}A_{ij}^*) &= \Phi(A_{ji}T_{ij} - T_{ij}A_{ij}^*) \\
&= \Phi([A_{ij} + A_{ji}, T_{ij}]_*) \\
&= [\Phi(A_{ij} + A_{ji}), \Phi(T_{ij})]_* \\
&= [\Phi(A_{ij}) + \Phi(A_{ji}) + \xi Q_i + \mu Q_j, \Phi(T_{ij})]_* \\
&= \Phi(A_{ji}T_{ij}) + \Phi(-T_{ij}A_{ij}^*) + (\xi - \mu)\Phi(T_{ij}),
\end{aligned}$$

and hence $(\xi - \mu)\Phi(T_{ij}) = 0$, so $\xi = \mu$.

For all $T_{ii} \in \mathcal{A}_{ii}$, it follows from Eq. (2.11) again that there exist $\eta \in \mathbb{R}$ such that

$$\begin{aligned}
&\Phi(T_{ii}A_{ij}) + \Phi(-A_{ji}T_{ii}^*) + \eta Q_i + \eta Q_j \\
&= \Phi(T_{ii}A_{ij} - A_{ji}T_{ii}^*) \\
&= \Phi([T_{ii}, A_{ij} + A_{ji}]_*) \\
&= [\Phi(T_{ii}), \Phi(A_{ij}) + \Phi(A_{ji}) + \xi Q_i + \xi Q_j]_* \\
&= \Phi(T_{ii}A_{ij}) + \Phi(-A_{ji}T_{ii}^*) + \xi(\Phi(T_{ii}) - \Phi(T_{ii})^*).
\end{aligned}$$

It follows that

$$\eta Q_i + \eta Q_j = \xi(\Phi(T_{ii}) - \Phi(T_{ii})^*) \quad \text{for all } T_{ii} \in \mathcal{A}_{ii}.$$

Multiply Q_j in the above expression, then $\eta Q_j = 0$, and hence $\eta = 0$ and

$$0 = \xi(\Phi(T_{ii}) - \Phi(T_{ii})^*) \quad \text{for all } T_{ii} \in \mathcal{A}_{ii},$$

which implies that $\xi = 0$. Therefore

$$\Phi(A_{ij} + A_{ji}) = \Phi(A_{ij}) + \Phi(A_{ji}).$$

Claim 9. *Let $A_{ij} \in \mathcal{A}_{ij}$ $(i, j = 1, 2)$. Then $\Phi\left(\sum_{i=1}^2 A_{ij}\right) = \sum_{i=1}^2 \Phi(A_{ij})$.*

For every $T_{ii} \in \mathcal{A}_{ii}$, since $[A_{ii} + A_{ij} + A_{ji}, T_{ii}]_* = [A_{ii} + A_{ji}, T_{ii}]_*$ $(i \neq j)$, it follows from Claim 7 that, for every $S_{ii} \in \mathcal{B}_{ii}$,

$$(\Phi(A_{ii} + A_{ij} + A_{ji}) - \Phi(A_{ii} + A_{ji}))S_{ii} = S_{ii}(\Phi(A_{ii} + A_{ij} + A_{ji}) - \Phi(A_{ii} + A_{ji}))^*.$$

Hence Claims 6–8 and Lemma 1 imply that $Q_j\Phi(A_{ii}+A_{ij}+A_{ji})Q_i=\Phi(A_{ji})$ and

$$Q_i\Phi(A_{ii}+A_{ij}+A_{ji})Q_i=\Phi(A_{ii})+\alpha Q_i \quad \text{for some } \alpha\in\mathbb{R}.$$

For every $T_{jj}\in\mathcal{A}_{jj}$, it follows from $[A_{ii}+A_{ij}+A_{ji},T_{jj}]_*=[A_{ij},T_{jj}]_*$ $(i\neq j)$ and Claim 7 that, for every $S_{jj}\in\mathcal{B}_{jj}$,

$$(\Phi(A_{ii}+A_{ij}+A_{ji})-\Phi(A_{ij}))S_{jj}=S_{jj}(\Phi(A_{ii}+A_{ij}+A_{ji})-\Phi(A_{ij}))^*.$$

So $Q_i\Phi(A_{ii}+A_{ij}+A_{ji})Q_j=\Phi(A_{ij})$ and there exists $\beta\in\mathbb{R}$ such that $Q_j\Phi(A_{ii}+A_{ij}+A_{ji})Q_j=\beta Q_j$. Thus

$$\Phi(A_{ii}+A_{ij}+A_{ji})=\Phi(A_{ii})+\alpha Q_i+\Phi(A_{ij})+\Phi(A_{ji})+\beta Q_j. \tag{2.12}$$

For every $T_{ii}\in\mathcal{A}_{ii}$, it follows from Eq. (2.12) that there exist $\gamma,\delta\in\mathbb{R}$ such that

$$\begin{aligned}
&\Phi(T_{ii}A_{ii}-A_{ii}T_{ii}^*)+\Phi(T_{ii}A_{ij})+\Phi(-A_{ji}T_{ii}^*)+\gamma Q_i+\delta Q_j\\
&=\Phi(T_{ii}A_{ii}+T_{ii}A_{ij}-A_{ii}T_{ii}^*-A_{ji}T_{ii}^*)\\
&=\Phi([T_{ii},A_{ii}+A_{ij}+A_{ji}]_*)\\
&=[\Phi(T_{ii}),\Phi(A_{ii})+\alpha Q_i+\Phi(A_{ij})+\Phi(A_{ji})+\beta Q_j]_*\\
&=\Phi(T_{ii}A_{ii}-A_{ii}T_{ii}^*)+\Phi(T_{ii}A_{ij})+\Phi(-A_{ji}T_{ii}^*)+\alpha(\Phi(T_{ii})-\Phi(T_{ii})^*).
\end{aligned}$$

So $\alpha=\gamma=\delta=0$. On the other hand, for every $T_{jj}\in\mathcal{A}_{jj}$, it follows from Eq. (2.12) and Claim 8 that

$$\begin{aligned}
\Phi(T_{jj}A_{ji})+\Phi(-A_{ij}T_{jj}^*)&=\Phi(T_{jj}A_{ji}-A_{ij}T_{jj}^*)\\
&=\Phi([T_{jj},A_{ii}+A_{ij}+A_{ji}]_*)\\
&=[\Phi(T_{jj}),\Phi(A_{ii})+\Phi(A_{ij})+\Phi(A_{ji})+\beta Q_j]_*\\
&=\Phi(T_{jj}A_{ji})+\Phi(-A_{ij}T_{jj}^*)+\beta(\Phi(T_{jj})-\Phi(T_{jj})^*),
\end{aligned}$$

and hence $\beta=0$. So

$$\Phi(A_{ii}+A_{ij}+A_{ji})=\Phi(A_{ii})+\Phi(A_{ij})+\Phi(A_{ji}). \tag{2.13}$$

For every $T_{11}\in\mathcal{A}_{11}$, we have $[T_{11},A_{11}+A_{12}+A_{21}+A_{22}]_*=[T_{11},A_{11}+A_{12}+A_{21}]_*$. So, for every $S_{11}\in\mathcal{B}_{11}$, it follows that

$$\begin{aligned}
&S_{11}(\Phi(A_{11}+A_{12}+A_{21}+A_{22})-\Phi(A_{11}+A_{12}+A_{21}))\\
&=(\Phi(A_{11}+A_{12}+A_{21}+A_{22})-\Phi(A_{11}+A_{12}+A_{21}))S_{11}^*.
\end{aligned}$$

This, together with Eq. (2.13), ensures that

$$\begin{aligned}
P_1\Phi(A_{11}+A_{12}+A_{21}+A_{22})P_1&=\Phi(A_{11}),\\
P_1\Phi(A_{11}+A_{12}+A_{21}+A_{22})P_2&=\Phi(A_{12}),\\
P_2\Phi(A_{11}+A_{12}+A_{21}+A_{22})P_1&=\Phi(A_{21}).
\end{aligned}$$

By Claim 7, there exists $C_{22}\in\mathcal{A}_{22}$ such that

$$P_2\Phi(A_{11}+A_{12}+A_{21}+A_{22})P_2=\Phi(C_{22}).$$

Thus

$$\Phi(A_{11}+A_{12}+A_{21}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(C_{22}).$$

For every $T_{22}\in\mathcal{A}_{22}$, it follows from the above expression and Eq. (2.13) that

$$\begin{aligned}
&\Phi(T_{22}A_{21})+\Phi(-A_{12}T_{22}^*)+\Phi(T_{22}A_{22}-A_{22}T_{22}^*)\\
&=\Phi(T_{22}A_{21}-A_{12}T_{22}^*+T_{22}A_{22}-A_{22}T_{22}^*)\\
&=\Phi([T_{22},A_{11}+A_{12}+A_{21}+A_{22}]_*)
\end{aligned}$$

$$= [\Phi(T_{22}), \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(C_{22})]_*$$
$$= \Phi(T_{22}A_{21}) + \Phi(-A_{12}T_{22}^*) + \Phi(T_{22}C_{22} - C_{22}T_{22}^*).$$

So $T_{22}(C_{22} - A_{22}) = (C_{22} - A_{22})T_{22}^*$ for every $T_{22} \in \mathcal{A}_{22}$, and hence, $C_{22} = A_{22}$ and the claim holds.

Claim 10. *Let $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ and $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$. Then*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}) \text{ and } \Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

It follows from $[P_i + A_{ij}, P_j + B_{ij}]_* = A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*$, Claim 8 and Eq. (2.13) that

$$\begin{aligned}
&\Phi(A_{ij}) + \Phi(B_{ij}) - \Phi(A_{ij})^* - \Phi(B_{ij})\Phi(A_{ij})^* \\
&= [Q_i + \Phi(A_{ij}), Q_j + \Phi(B_{ij})]_* \\
&= [\Phi(P_i + A_{ij}), \Phi(P_j + B_{ij})]_* \\
&= \Phi([P_i + A_{ij}, P_j + B_{ij}]_*) \\
&= \Phi(A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*) \\
&= \Phi(A_{ij} + B_{ij}) + \Phi(-A_{ij}^*) + \Phi(-B_{ij}A_{ij}^*).
\end{aligned}$$

Multiplying respectively Q_i and Q_j from the left side and right side of the above expression, one has

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Let $T_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$ be arbitrary. Applying the above expression, we have

$$\begin{aligned}
\Phi(A_{ii} + B_{ii})\Phi(T_{ij}) &= [\Phi(A_{ii} + B_{ii}), \Phi(T_{ij})]_* \\
&= \Phi([A_{ii} + B_{ii}, T_{ij}]_*) = \Phi(A_{ii}T_{ij} + B_{ii}T_{ij}) \\
&= \Phi(A_{ii}T_{ij}) + \Phi(B_{ii}T_{ij}) \\
&= \Phi([A_{ii}, T_{ij}]_*) + \Phi([B_{ii}, T_{ij}]_*) \\
&= [\Phi(A_{ii}), \Phi(T_{ij})]_* + [\Phi(B_{ii}), \Phi(T_{ij})]_* \\
&= (\Phi(A_{ii}) + \Phi(B_{ii}))\Phi(T_{ij}).
\end{aligned}$$

Claim 6 implies that $(\Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}))S_{ij} = 0$ for all $S_{ij} \in \mathcal{B}_{ij}$, and hence $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$.

Claim 11. *Let $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ and $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$. Then*

$$\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}), \ \Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji}),$$
$$\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij}), \ \Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj}).$$

Let $X \in \mathcal{A}_{ij}$ $(i \neq j)$ be arbitrary. Then $\Phi(A_{ii}X) = \Phi([A_{ii}, X]_*) = \Phi(A_{ii})\Phi(X)$, and hence,

$$\Phi(A_{ii}B_{ii})\Phi(X) = \Phi(A_{ii}B_{ii}X) = \Phi(A_{ii})\Phi(B_{ii}X) = \Phi(A_{ii})\Phi(B_{ii})\Phi(X)$$

for all $X \in \mathcal{A}_{ij}$. Now Claim 6 implies that

$$\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}). \tag{2.14}$$

From Claim 6, it follows that

$$\Phi(A_{ij}B_{ji}) = \Phi([A_{ij}, B_{ji}]_*) = [\Phi(A_{ij}), \Phi(B_{ji})]_* = \Phi(A_{ij})\Phi(B_{ji}).$$

Thus, for every $T_{ji} \in \mathcal{A}_{ji}$ $(j \neq i)$, the above expression and Eq. (2.14) imply that

$$\Phi(A_{ii}B_{ij})\Phi(T_{ji}) = \Phi(A_{ii}B_{ij}T_{ji}) = \Phi(A_{ii})\Phi(B_{ij}T_{ji}) = \Phi(A_{ii})\Phi(B_{ij})\Phi(T_{ji}),$$

and therefore,

$$\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij}).$$

Similarly, for every $T_{ji} \in \mathcal{A}_{ji}$ ($j \neq i$),

$$\Phi(T_{ji})\Phi(A_{ij}B_{jj}) = \Phi(T_{ji}A_{ij}B_{jj}) = \Phi(T_{ji}A_{ij})\Phi(B_{jj}) = \Phi(T_{ji})\Phi(A_{ij})\Phi(B_{jj}).$$

So

$$\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj}).$$

Claim 12. *$\Phi : \mathcal{A} \to \mathcal{B}$ is a $*$-ring isomorphism.*

Claims 9 and 10 imply that Φ is additive. Next, we prove that Φ is multiplicative. For any $A, B \in \mathcal{A}$, write $A = \sum_{i=1}^{2} A_{ij}$ and $B = \sum_{i=1}^{2} B_{ij}$. Then

$$\begin{aligned} AB = & (A_{11}B_{11} + A_{12}B_{21}) + (A_{11}B_{12} + A_{12}B_{22}) \\ & + (A_{21}B_{11} + A_{22}B_{21}) + (A_{21}B_{12} + A_{22}B_{22}). \end{aligned}$$

It follows from Claims 9, 10 and 11 that $\Phi(AB) = \Phi(A)\Phi(B)$.

For every $A \in \mathcal{A}$, we have $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$, where $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ are self-adjoint. It follows from $\Phi(iA) = \pm i\Phi(A)$ $(\forall A \in \mathcal{A})$ that $\Phi(A^*) = \Phi(A)^*$ for every $A \in \mathcal{A}$, so Φ is a $*$-ring isomorphism.

The following corollary generalized the result in [1], where the author assume that the Hilbert space is at least of dimension 3.

Corollary 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be type I factor von Neumann algebras acting on complex Hilbert spaces H and K, respectively. Then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*$ if and only if there exists a unitary or conjugate unitary operator $U : H \to K$ such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}$.*

Proof. Since every ring isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is spatial, the result follows from Main theorem. $\square$

Corollary 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras acting on complex Hilbert spaces H and K, respectively. Assume that $\mathcal{A}$ and $\mathcal{B}$ are finite factors. Then a linear bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*$ if and only if there exists a unitary operator $U : H \to K$ such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}$.*

Proof. The result follows from Main Theorem and [3, Proposition 10, pp. 304], which states that every isomorphism between finite factors is spatial. $\square$

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