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Mathematical Studies of Optimal Economic Growth Model with Monetary Policy

Xiang Liu

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Mathematical Studies of Optimal Economic Growth Model with Monetary Policy

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelors of Science in Mathematics from The College of William and Mary

by

Xiang Liu

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December 12nd, 2017
Mathematical Studies of Optimal Economic Growth Model with Monetary Policy

Xiang Liu
Abstract

In this paper, efforts will be made to study an extended Neoclassic economic growth model derived from Solow-Swan Model and Ramsey-Cass-Koopsman Model. Some growth models (e.g. Solow-Swan Model) attempt to explain long-run economic growth by looking at capital accumulation, labor or population growth, and increases in productivity, while our derived model tends to look at growth from individual household and how their choice of saving, consumption and money holdings would affect the overall economic capital accumulation over a long period of time.

First an optimal control model is set up, and a system of differential equations and algebraic equations is derived from the optimal solutions which is an extension of the existing model. Secondly, the equilibrium points and their stability of both models are studied by calculating the determinant of their respective Jacobian matrix. Last but not least, model numerical simulations are performed for both models using Matlab, in hopes that it will give us a better understanding of the system and a clearer pattern for the behaviors of each model.
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Chapter 1

Introduction

1.1 On the Shoulders of Giants

Let us start with the Solow-Swan model, where it all began. The Solow-Swan growth model is a famous neoclassical economic model trying to capture long-run economic growth [9]. It attempts to explain the growth of an economy by looking at capital accumulation, labor or population growth, and increases in productivity (or technological progress).

Mathematically speaking, the Solow-Swan model is a nonlinear system consisting a single ordinary differential equation that models the evolution of the per capita stock of capital:

\[
\dot{k}(t) = s \cdot f(k(t)) - \delta \cdot k(t). \tag{1.1}
\]

In (1.1), \( k(t) \) stands for capital accumulation and \( f(k) \) represents an explicit production function. This relation can be effortlessly explained: the total change of an individual’s capital accumulation at year \( t \) is the value of the goods he/she retained at the end of the year (how much the individual produced multiplying by his/her saving rate \( s \)) deducts the depreciation of his/her previous capital holdings. The required investment function can be thought of as a depreciation function (what is the least amount of savings required to avoid having a negative value for capital). Potential output will eventually reach an
equilibrium due to law of diminishing marginal returns (see Figure 1.1).

From Figure 1.1, we can see that output and investment holdings are increasing as capital increases but at a slowing rate, which is known as the diminishing marginal returns of capital. The economy will eventually reach an equilibrium state as the growth rates of potential output and savings growth approach zero. One thing worth noticing here is that $s$ is an exogenous variable in this model, or put in other words, the Solow-Swan model depicts a fixed saving rate $s$. This suggests that individual household would not be able to choose their own level of consumption. In real life however, this is hardly the case. Households usually have great flexibility in choosing their own level of saving/consumption/money holdings.

Various extensions have been made to this model due to its particularly attractive mathematical characteristics. In 1965, David Cass [3] and Tjalling Koopmans [5] integrated Frank Ramsey’s analysis of consumer optimization [7], thereby endogenizing the savings rate, to create what is now known as the Ramsey-Cass-Koopmans (RCK) model. RCK model is aiming at maximizing levels of consumption over successive generations [4].
The first key equation for the model resembles the one describing the Solow-Swan Model:

$$k'(t) = f(k(t)) - (n + \delta) \cdot k(t) - c(t).$$  \hfill (1.2)

Here $n$ stands for population growth and $c$ is the previous exogenous, now endogenous (saving) consumption variable. The second important equation in the RCK model is the solution to the social planner’s problem of maximizing a social welfare function:

$$W = \int_{0}^{\infty} e^{-rt} U(c(t)) dt.$$

The goal here is to maximize the value of (1.3) over successive generations, hence the integration bound from 0 to $\infty$. We will take a closer look into this welfare function $W$ in the following section. As of now, all we need to know is that the current utility function $U$ only has consumption as variable, and it is within our interest to study a similar model but with money holdings also added into the utility function.

![Figure 1.2: Ramsey-Cass-Koopmans Model Phase Diagram](source: [1, p322]}

3
Figure 1.2 depicts the phase diagram of the RCK model, [1, p322]. The graph shows that there exists a unique level of \( k_{gold} \) that maximizes consumption per capita. Given this direction of movements, it is clear that there exists a unique stable path such that all orbits away from this stable path diverge, and eventually reach zero consumption or zero capital stock as shown in the figure.

In 1967, Argentine economist Miguel Sidrauski [8] further modified the RCK Model into a Money In the Utility (MIU) Model by adding money as an variable into the utility function, assuming that money yields direct utility.

![Figure 1.3: Phase Diagram of the Sidrauski Model](source)

Figure 1.3 illustrates the phase diagram of the MIU model proposed by Sidrauski [8]. The heavy arrows represent a dynamic saddle path solution of the economy in which all the constraints present in the model are satisfied. It is a stable path of the dynamic system. The light arrows represent dynamic paths which are ruled out by the transversality condition. However, his remarks on the equilibrium growth path are inconclusive, in the sense that he did not prove his claims nor provide a numerical simulation of the model. It is within our interests today to take on this problem and attempt to generate more insights and implications for the model, for future references and studies in the field.
1.2 Summary of Our Goals

We will further study the existing Sidrauski Model by attempting to numerically solve for its equilibrium point. Meanwhile we will prove whether or not our MIU system has an unstable saddle type equilibrium, as suggested by Sidrauski, through various linear algebra and differential equation techniques. Finally, we will simulate our model and compare it with the existing RCK model using the mathematical software Matlab.
Chapter 2

Mathematical Model

2.1 Introduce Parameters and Variables

First of all, the following table contains all the variables that we will use throughout this thesis. Any additional variables or terms will be explained as we encounter them.

<table>
<thead>
<tr>
<th>variable</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$c(t)$</td>
<td>consumption</td>
</tr>
<tr>
<td>$m(t)$</td>
<td>real money holding</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>gross money holding</td>
</tr>
<tr>
<td>$k(t)$</td>
<td>capital</td>
</tr>
<tr>
<td>$a(t)$</td>
<td>wealth</td>
</tr>
</tbody>
</table>

Table 2.1: Variables in the model
<table>
<thead>
<tr>
<th>parameter</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v)</td>
<td>government transfer</td>
</tr>
<tr>
<td>(\pi)</td>
<td>inflation rate</td>
</tr>
<tr>
<td>(n)</td>
<td>population change rate</td>
</tr>
<tr>
<td>(\delta)</td>
<td>depreciation rate</td>
</tr>
<tr>
<td>(\rho)</td>
<td>rate of time preference</td>
</tr>
<tr>
<td>(\theta)</td>
<td>utility function parameter</td>
</tr>
<tr>
<td>(\tau)</td>
<td>reciprocal of (1 - \theta)</td>
</tr>
<tr>
<td>(g)</td>
<td>production function parameter</td>
</tr>
</tbody>
</table>

Table 2.2: Parameters in the model

Note: variables may be written in shortened form in, for example \(c(t)\), \(m(t)\), \(k(t)\), \(a(t)\) might be written as simply \(c\), \(m\), \(k\), and \(a\) for simplification purpose.

### 2.2 Utility and Production Functions

First of all, let us consider what our utility function should look like. A typical household utility function would take the form of \(U = U(c(t), z(t))\), where \(c(t)\) represents the consumption per capita at time \(t\), and \(z(t)\) represents the flow of services one can obtain with money holding at time \(t\).

The price of each unit of services or goods is \(\frac{1}{P}\), and when multiplied with with \(M\), units of nominal money holdings, it gives us the units of services \(z(t)\) we could purchase with our current money holding. Then dividing it by population \(N\), we can establish the relation \(z(t) = m(t)\):

\[
z(t) = \frac{M}{P \times N} = m(t).
\]

The basic unit in our monetary economy is each household, which has utility function in the form of \(U = U(c(t), m(t))\). It is possible for the marginal utility of money holdings
to be negative, meaning that there exists some $m > 0$ such that $U(c(t), m(t)) \leq 0$ for all $m > \bar{m}$. Another assumption we have here is that the utility function should be strictly increasing and concave with continuous first and second derivatives [8, 11]:

$$\frac{\partial U}{\partial c} > 0, \quad \frac{\partial U}{\partial m} > 0, \quad \frac{\partial^2 U}{\partial c^2} < 0, \quad \frac{\partial^2 U}{\partial m^2} < 0, \quad \text{for } m > 0 \text{ and } c > 0. \quad (2.1)$$

It is fairly easy to grasp the concept behind this condition: the diminishing marginal return of utility. As an individual increases his/her consumption of a product, there will be a point where the individual will start to experience a decline in the marginal utility from consuming each additional unit of that product. In our case this product would be the combination of money holdings and consumption.

There are two types of commonly seen utility functions that can satisfy (2.1). First is a Cobb-Douglas utility function:

$$U(c, m) = \frac{1}{\theta} c^\theta m^\theta, \quad 0 < \theta < 1, \quad (2.2)$$

and the second one is a utility function that satisfies the law of Constant Elasticity of Substitution:

$$U(c, m) = \frac{1}{\theta} c^\theta + \frac{g}{\theta} m^\theta, \quad 0 < \theta < 1, \quad g > 0. \quad (2.3)$$

After testing both utility functions, we decided to use (2.3) in the following chapters because it yields better results, and it is also a popular choice for other MIU research papers [2].

Similarly, we also have preliminary conditions for choice of production function, [1, p35]:

$$f(0) = 0, \quad f'(k) > 0, \quad f''(k) < 0, \quad k > 0. \quad (2.4)$$

Two options we have for the production function are as follows:

$$f(k) = \frac{zk}{a_0k + a_1}, \quad z, a_0, a_1 > 0, \quad (2.5)$$

or

$$f(k) = zk^\theta \quad 0 < \theta < 1. \quad (2.6)$$
For now, we will not choose a specific production function but instead consider a
generalized system for a general production function $f(k)$. However, in the last chapter
of this thesis, a specific production function will be chosen in order to numerically solve
the system.

2.3 Optimal Control Model

Now back to model building. For each household in the economy, its goal would be to
maximize total lifetime utility by choosing time paths for consumption and real money
holdings, described in the following wealth equation

$$W = \int_0^\infty e^{-pt}U(c(t), m(t))dt,$$

(2.7)
given that

$$a(t) = k(t) + m(t).$$

(2.8)

Here $a(t)$ stands for wealth, which equals to the sum of capital wealth and money holdings,
at time $t$.

At the same time, a family’s disposable income equals to sum of real consumption $c(t)$
and gross real saving $s(t)$. A family’s disposable income is also the sum of output $f(k)$
and net transfer that the economic unit receives from the government $v(t)$. Hence the
following relation holds:

$$f(k(t)) + v(t) = c(t) + s(t).$$

(2.9)

Savings, on the other hand, is composed of gross capital accumulation $i(t)$ and gross
money holdings $x(t)$:

$$s(t) = i(t) + x(t).$$

(2.10)

Gross capital accumulation at year $t$ equals to the sum of the change in capital stock and
the after-depreciation value of the stock $\delta k(t)$. We also need to distribute current year
capital to adjust for our newborn population, and the additional capital needed would be
$nk(t)$. Let $\delta$ stand for the instantaneous rate of depreciation of capital and $n$ stand for the instantaneous growth rate of population $N$. Then we have

$$i(t) = k'(t) + (\delta + n)k(t).$$ \hspace{1cm} (2.11)

Without loss of generality, we can assume that the gross accumulation for real cash would be similar (with $\pi$ being the expected inflation rate):

$$x(t) = m'(t) + (\pi + n)m(t).$$ \hspace{1cm} (2.12)

Substituting (2.12) and (2.11) into (2.10), and then into (2.9), we have the overall flow of constraint for one economic unit:

$$f(k(t)) + v - c(t) - (\delta + n)k(t) - (\pi + n)m(t) - m'(t) - k'(t) = 0.$$ \hspace{1cm} (2.13)

Differentiating (2.8) with respect to time, we get $a'(t) = k'(t) + m'(t)$. Substituting it into (2.13) gives us

$$a'(t) = f(a(t)) + v - c(t) - (\delta + n)a(t) - (\pi + n)m(t).$$ \hspace{1cm} (2.14)

Substituting $k(t) = a(t) - m(t)$ into (2.14), we finally obtain

$$a'(t) = f(a(t) - m(t)) + v - (\delta + n)a(t) + (\delta - \pi)m(t) - c(t).$$ \hspace{1cm} (2.15)

Equation (2.15) is now our working constraint for stock and flow when calculating the maximization of the lifelong utility of an economic unit.

Now we have an optimal control problem

$$\max_{c(t), m(t)} \int_0^\infty e^{-\rho t} U(c(t), m(t))dt$$ \hspace{1cm} (2.16)

subject to the constraint

$$a'(t) = f(a(t) - m(t)) + v - (\delta + n)a(t) + (\delta - \pi)m(t) - c(t).$$ \hspace{1cm} (2.17)
Our current value Hamiltonian equation for (2.16) and (2.17) is

\[ H(c, m, a, \lambda) = U(c, m) + \lambda[(f(a - m) + v - (\delta + n)a + (\delta - \pi)m - c)]. \] (2.18)

From [1, Theorem 7.14] the Maximum Principle for Discounted Infinite-Horizon Problems, the optimal path for \((c(t), m(t), a(t))\) should satisfy the following Euler-Lagrange equation:

\[ \frac{\partial H}{\partial c}(c(t), m(t), a(t), \lambda(t)) = 0, \] (2.19)
\[ \frac{\partial H}{\partial m}(c(t), m(t), a(t), \lambda(t)) = 0, \] (2.20)
\[ \rho \lambda(t) - \lambda' = \frac{\partial H}{\partial a}(c(t), m(t), a(t), \lambda(t)), \] (2.21)
\[ a'(t) = \frac{\partial H}{\partial \lambda}(c(t), m(t), a(t), \lambda(t)), \] (2.22)

with a transversality condition

\[ \lim_{t \to \infty} e^{-\rho t} \lambda(t)a(t) = 0. \]

If we organize and substitute this system with known relationships, we can derive the following system

\[ \frac{\partial U}{\partial c} - \lambda = 0, \] (2.23)
\[ \frac{\partial U}{\partial m} + \lambda(-f'(a(t) - m(t)) + \delta - \pi) = 0, \] (2.24)
\[ \rho \lambda(t) - \lambda'(t) = \lambda(t)(f'(a(t) - m(t)) - (\delta + n)), \] (2.25)
\[ a'(t) = f(a - m) + v - (\delta + n)a + (\delta - \pi)m - c. \] (2.26)

The system (2.23)-(2.26) consists of two algebraic equations and two differential equations in variables \(c(t), m(t), a(t)\) and \(\lambda(t)\). Now our strategy is to eliminate \(\lambda(t)\) from the system and obtain a system of variables \(c(t), m(t)\) and \(a(t)\) only.

Now we use (2.3) as our utility function. From our choice of utility function, we calculated that

\[ \frac{\partial U}{\partial m} = gm^{\theta-1}, \quad \frac{\partial U}{\partial c} = c^{\theta-1}. \] (2.27)
Then from Equation (2.23) and (2.27), we obtain $c^{\theta-1} = \lambda$. Taking logarithm and differentiating, we get

$$(\theta - 1) \frac{c'}{c} = \frac{\lambda'}{\lambda}.$$  

If we combine this result with Equation (2.25), we get the expression for $c'$:

$$c'(t) = \frac{c}{1 - \theta} (f'(a - m) - (\rho + \delta + n)). \quad (2.28)$$

On the other hand, substituting Equation (2.27) into (2.24), we get

$$gm^{\theta-1} + c^{\theta-1}(f'(a - m) + \delta - \pi) = 0,$$

which can be reorganized into

$$\left( \frac{m}{c} \right)^{\theta-1} = \frac{1}{g} ((\pi - \delta) + f'(a - m)). \quad (2.29)$$

Using (2.28) and (2.29) to replace (2.23), (2.24) and (2.25), we now have a system of two differential equations and one algebraic equation. This constitutes our master system which describes the optimal growth problem for any general production function $f(k)$ and utility function $U(c, m) = \frac{1}{\theta} c^\theta + \frac{g}{\theta} m^\theta$:

$$\begin{cases}
a'(t) = f(a(t) - m(t)) + v - (\delta + n)a(t) + (\delta - \pi)m(t) - c(t), \\
c'(t) = \tau c(t)(f'(a(t) - m(t)) - (\rho + \delta + n)), \\
0 = c(t) - m(t)(\frac{\pi - \delta + f'(a(t) - m(t))}{g})^\tau, \text{ where } \tau = \frac{1}{1 - \theta}. \end{cases} \quad (2.30)$$

Now let us pause a bit and think about how this model that we are building differs from the existing widely-acknowledged models. In Introduction to Modern Economic Growth by Daron Acemoglu [1, P322], a simpler RCK model was studied. Its system of equations takes the following form:

$$\begin{cases}k'(t) = f(k(t)) + v - (\delta + n)k(t) - c(t), \\
c'(t) = \tau c(t)(f'(k(t)) - (\rho + \delta + n)). \end{cases} \quad (2.31)$$

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This neoclassical growth model (2.31) resembles our model (2.30) in every way except that it did not take real money holding into the equation, and therefore lacking one equation for variable $m(t)$. Our fundamental constraint for wealth: $a(t) = m(t) + k(t)$ suggests that wealth equals the sum of real money holdings and capital accumulation, while in constraint (2.31), is simply $a(t) = k(t)$.

**Note 1:** $\tau$, or $\frac{1}{1-\theta}$ reflects the curvature of the utility function; its inverse is known as the (intertemporal) elasticity of substitution and indicates how much the representative agent wishes to smooth consumption over time. It is assumed that this elasticity is a positive constant.

**Note 2:** In our further study of the model, we would assume a government transfer of zero. Mathematically speak, addition of a constant would not make much impact on the model, especially in the later simulated graphs. The addition of a constant would only shift the graph up or down a bit. Therefore for simplification purpose, $v(t)$ will be omitted in the remaining chapters.
Chapter 3

Equilibrium Points and Stabilities

3.1 System Equilibriums

In this chapter, we will explore equilibrium points of both the RCK model (2.31) and the Money in Utility model (2.30) and prove their stabilities.

3.1.1 Equilibrium Points of Capital-Consumption Model

Now let us take a moment to calculate the general equilibrium solution for an existing 2D model (2.31), that is without money in the utility function. By setting $k'(t)$ and $c'(t)$ to zero, we obtain the following positive equilibrium. We will make use of this result in the next chapter.

$$\begin{align*}
    k^* &= (f')^{-1}(\rho + \delta + n), \\
    c^* &= f(k^*) - (\delta + n)k^*.
\end{align*}$$

(3.1)

The point $(k^*, c^*)$ is one of the three equilibrium points that we saw in Figure 1.3. The other two are the origin $(0, 0)$, and the point $(k_*, 0)$ which occurs when consumption is equal to zero. In the next section, we will prove the stabilities of all these equilibrium points with respect to (2.31).
3.1.2 Equilibrium Points of the MIU Model

Now we can use the same technique to find equilibrium points of the MIU model. To refresh a bit, our system is as follows:

\[
\begin{align*}
    a'(t) &= f(a(t) - m(t)) - (\delta + n)a(t) + (\delta - \pi)m(t) - c(t), \\
    c'(t) &= \tau c(t)(f'(a(t) - m(t)) - (\rho + \delta + n)), \\
    0 &= c(t) - m(t)(\frac{\pi - \delta + f'(a(t) - m(t))}{g})^\tau, \quad \text{where } \tau = \frac{1}{1 - \theta},
\end{align*}
\]  

(3.2)

By setting expressions for \(a', c'\) from (3.2) to zero, we have the equation for equilibrium \((a, c, m)\):

\[
\begin{align*}
    0 &= f(a - m) - (\delta + n)a + (\delta - \pi)m - c, \\
    0 &= \tau c(f'(a - m) - (\rho + \delta + n)), \\
    0 &= c - m(\frac{\pi - \delta + f'(a - m)}{g})^\tau, \quad \text{where } \tau = \frac{1}{1 - \theta}.
\end{align*}
\]  

(3.3)

We discuss all possible equilibrium points in the following cases.

**Case 1:** \(c = 0\) and \(m = 0\).

Then from the first equation of (3.3), we have \(f(a) - (\delta + n)a = 0\). If \(a = 0\), we would have \((0, 0, 0)\) as an equilibrium. If \(a \neq 0\), then \(f(a) = (\delta + n)a\) has a unique solution \(a_\ast > 0\) as in the Solow-Swan model. So we have a unique equilibrium \((a_\ast, 0, 0)\) in this form.

**Case 2:** \(c = 0\) and \(m \neq 0\).

From the third equation of (3.3), we must have \(f'(a - m) = \delta - \pi\), therefore \(a - m = (f')^{-1}(\delta - \pi) = M\). Note that from (2.4), \(f' > 0\) so it is an invertible function. Now from the first equation of (3.3), we know that \((\delta + n)a - (\delta - \pi)m = f(M)\). Then the system of two linear equations

\[
a - m = M, \quad (\delta + n)a - (\delta - \pi)m = f(M),
\]

(3.4)

has a unique solution \((\tilde{a}, \tilde{m})\) which is positive. Hence we have our third equilibrium \((\tilde{a}, 0, \tilde{m})\).
Case 3: \( c \neq 0 \).

In this case, again let us denote \( a - m \) as \( M \). Then from the second equation of (3.3), we get that \( f'(M) = \rho + \delta + n \). Therefore, \( M = (f')^{-1}(\rho + \delta + n) \). From the first and third equations of (3.3) we obtain that

\[
0 = f(M) - (\delta + n)(m + M) + (\delta - \pi)m - \left(\frac{f'(M)}{g}\right)\tau m. \tag{3.5}
\]

After organizing the equation (3.5), we get an expressions for \( m^* \), along with \( a^* \) and \( c^* \):

\[
\begin{aligned}
m^* &= \frac{f(M) - (\delta + n)M}{n + \pi + \left(\frac{f'(M)}{g}\right)\tau}, \\
a^* &= m^* + M, \quad \text{where } M = (f')^{-1}(\rho + \delta + n), \\
c^* &= \left(\frac{\pi + \rho + n}{g}\right)\tau m^*.
\end{aligned} \tag{3.6}
\]

This equilibrium \((a^*, c^*, m^*)\) is strictly positive, and it applies to any production function \( f(k) \) and our choice of utility function (2.3).

### 3.1.3 Conclusion

Altogether we have found three equilibrium points for the RCK model (2.31): \((k^*, c^*), (k_*, 0), (0, 0)\) and four equilibrium points for the MIU system (3.2): \((a^*, c^*, m^*), (\bar{a}, 0, \bar{m}), (a_*, 0, 0), (0, 0, 0)\). In the next section, we will study their stabilities one by one before moving on to simulate both models in Matlab.
3.2 Equilibrium Stability of the Existing 2D Model

According to Acemoglu in his book Introduction to Modern Economics [1], the three equilibrium points \((k^*, c^*), (k_*, 0), (0, 0)\) of (2.31) should be a saddle point, a nodal sink and a nodal source respectively. Here we will validate this claim and later compare this model with the MIU model (3.2).

3.2.1 Equilibrium Stability of \((k^*, c^*)\)

Recall our RCK model (2.31) with \(v = 0\):

\[
\begin{align*}
  k'(t) &= f(k(t)) - (\delta + n)k(t) - c(t), \\
  c'(t) &= c(t)\tau(f'(k(t)) - (\rho + \delta + n)).
\end{align*}
\]

At this equilibrium state, we have both equations equal to zero and thus \(f'(k) = (\rho + \delta + n)\).

The corresponding Jacobian matrix for \((k^*, c^*)\) is

\[
J_{(k^*, c^*)} = \begin{pmatrix}
  f'(k^*) - (n + \delta) & -1 \\
  c\tau f''(k^*) & \tau(f'(k^*) - (\rho + \delta + n))
\end{pmatrix} = \begin{pmatrix}
  \rho & -1 \\
  c\tau f''(k^*) & 0
\end{pmatrix}.
\]

It is clear that the determinant of this Jacobian matrix is negative, since \(c\tau f''(k^*)\) is negative, given that \(f''(k^*) < 0\). Therefore, \((k^*, c^*)\) is an unstable saddle point.

3.2.2 Equilibrium Stability of \((0, 0)\)

We will use the same techniques to find the Jacobian matrix for this point:

\[
J_{(0, 0)} = \begin{pmatrix}
  f'(0) - (n + \delta) & -1 \\
  0 & \tau(f'(0) - (\rho + n + \delta))
\end{pmatrix}.
\]

For a Jacobian matrix of this form, its eigenvalues are the diagonal entries. To prove that they are both positive, we need to take a closer look at the phase diagram in Figure 1.2. It is clear that \(0 < k^* < k_*\) from the figure. From the given condition in (2.4), we
know that production function $f(k)$ has a negative second derivative, thus the following equation holds:

$$f'(0) > f'(k^*) > f'(k_*) .$$  (3.8)

From our earlier exploration of the stability of $(k^*, c^*)$, we get $f'(k^*) = \rho + n + \delta$. Therefore given (3.8), we have $f'(0) > \rho + n + \delta$ and $f'(0) > n + \delta$. Now we just proved that the two diagonal entries of $J_{(0,0)}$ are positive, and $(0,0)$ is therefore a nodal source with two positive eigenvalues.

### 3.2.3 Equilibrium Stability of $(k_*, 0)$

If we substitute $c^* = 0$ into (3.7), we would have the following Jacobian matrix:

$$J_{(k_*, 0)} = \begin{pmatrix}
  f'(k_*) - (n + \delta) & -1 \\
  0 & \tau(f'(k_*) - (\rho + n + \delta))
\end{pmatrix}.
$$

For a Jacobian matrix of this form, the eigenvalues would simply be the diagonal entries: $\lambda_1 = f'(k_*) - (n + \delta)$ and $\lambda_2 = \tau(f'(k_*) - (\rho + n + \delta))$.

If we substitute $(k_*, 0)$ into (3.7), we get $f(k_*) - (\delta + n)k_* = 0$ and hence $\frac{f(k_*)}{k_*} = \delta + n$. We claim that if $f(k)$ satisfies (2.4), then $\frac{f(k)}{k} > f'(k)$ for $k > 0$. Indeed we define $F(k) = f(k) - f'(k)k$, then $F'(k) = f'(k) - f''(k)k = -kf''(k) > 0$, therefore $F(k) > F(0) = 0$, which gives us $f(k) - f'(k)k > 0$ or $\frac{f(k)}{k} > f'(k)$. This implies that $f'(k_*) - (n + \delta) < 0$ and $\tau(f'(k_*) - (\rho + n + \delta)) < 0$. Now we can conclude that with both eigenvalues being negative, $(k_*, 0)$ is a nodal sink.

So far we have validated the stabilities of the three equilibrium points of the RCK model and it’s time to move on to our MIU model.
3.3 Equilibrium Stabilities for Our 3D Model

3.3.1 Stability of MIU Model

We already have a generic system of solutions for our master system, and our next step is to derive a system we can work with, since the third equation is our current generic system is not a differential equation.

In the book, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering* by Steven H Strogatz, a set of clear and thorough guidelines of techniques was given for transforming our current system.[10]

Now that we have explicitly solved for equilibrium, we can use linear approximation to get a relatively accurate measure for its generic state. Assume that

\[ (a(t), c(t), m(t)) \approx (a^*, c^*, m^*) + (\phi(t), \psi(t), \eta(t)). \]  

Then \((\phi(t), \psi(t), \eta(t))\) satisfies the linearized equation:

\[
\begin{aligned}
\dot{\phi} &= f'(a^* - m^*)(\phi - \eta) - (n + \delta)\phi - \psi + (\delta - \pi)\eta, \\
\dot{\psi} &= \tau(f''(a^* - m^*)c^*(\phi - \eta) + f'(a^* - m^*)\psi - (\delta + \rho + n)\psi), \\
0 &= \psi - \eta\left(\frac{\pi - \delta + f'(a^* - m^*)}{g}\right)^{\tau - 1} \frac{f''(a^* - m^*)}{g}(\phi - \eta).
\end{aligned}
\]  

(3.10)

Now we will try to prove that this system has a saddle point equilibrium by analyzing the preceding linearized equation. However, our current system does not have three differential equations, instead, the third equation is an algebraic equation.

Our current Jacobian matrix is in the form

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial m} \\
\frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial m} \\
\frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial c} & \frac{\partial f_3}{\partial m}
\end{pmatrix}
\]

Here \(f_1, f_2, f_3\) are the functions on the right hand side of (2.30).
From (3.10), we have:

\[
J = \begin{pmatrix}
  f'(a - m) - (\delta + n) & -1 & -f'(a - m) + (\delta - \pi) \\
  \tau cf''(a - m) & \tau(f'(a - m) - (\rho + \delta + n)) & -\tau cf''(a - m) \\
  -m\tau(\pi - \delta + f'(a - m))g^{-1}f''(a - m) & 1 & J_{33}
\end{pmatrix}.
\]

with

\[
J_{33} = -\left(\frac{\pi - \delta + f'(a - m)}{g}\right)^\tau + m\tau\left(\frac{\pi - \delta + f'(a - m)}{g}\right)^{\tau-1}f''(a - m).
\]

We will transform this system into a two dimensional system through the following steps. First, let us set up the system for finding eigenvalues, and denote each derivate by \(J_{ij}\), where \(i\) denotes the row number and \(j\) denotes the column number.

Now setting the solution of (3.10) to \(e^{\lambda t}(A, B, C)^T\), we get

\[
\begin{pmatrix}
  J_{11} & J_{12} & J_{13} \\
  J_{21} & J_{22} & J_{23} \\
  J_{31} & J_{32} & J_{33}
\end{pmatrix}
\begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix} = \lambda
\begin{pmatrix}
  A \\
  B \\
  0
\end{pmatrix}.
\]

(3.11)

Then the third row equation of (3.11) implies that

\[
J_{31}A + J_{32}B + J_{33}C = 0,
\]

\[
C = -\frac{1}{J_{33}}(J_{31}A + J_{32}B).
\]

Now we can substitute the expression for \(C\) into the first two equations of (3.11) and calculate its eigenvalues. The reduced system becomes:

\[
\begin{cases}
(J_{11} - \frac{J_{13}J_{31}}{J_{33}})A + (J_{12} - \frac{J_{13}J_{32}}{J_{33}})B = \lambda A, \\
(J_{21} - \frac{J_{23}J_{31}}{J_{33}})A + (J_{22} - \frac{J_{23}J_{32}}{J_{33}})B = \lambda B.
\end{cases}
\]

(3.12)

Therefore \(\lambda\) is the eigenvalue of the 2 \(\times\) 2 matrix of the following form:

\[
\tilde{J} = \begin{pmatrix}
  J_{11} - \frac{J_{13}J_{31}}{J_{33}} & J_{12} - \frac{J_{13}J_{32}}{J_{33}} \\
  J_{21} - \frac{J_{23}J_{31}}{J_{33}} & J_{22} - \frac{J_{23}J_{32}}{J_{33}}
\end{pmatrix}.
\]

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Now all we need to do to find individual stability for each equilibrium point is to plug
in specific values into our first and second Jacobian matrix to determine the sign of its
eigenvalues.

### 3.3.2 Equilibrium Stability of \((a^*, m^*, c^*)\)

By equaling \(a', c'\) to zero and plugging in the expression \(f'(a^* - m^*) = \rho + \delta + n\), we get
the following Jacobian matrix:

\[
J_{(a^*, c^*, m^*)} = \begin{pmatrix}
\rho & -1 & -\rho - \pi - n \\
\tau cf''(a^* - m^*) & 0 & -\tau cf''(a^* - m^*) \\
-m^*\tau\left(\frac{\pi - \delta + f'(a^* - m^*)}{g}\right)^{\tau-1}f''(a^* - m^*) & 1 & J_{33}
\end{pmatrix},
\]

with

\[
J_{33} = \left(-\frac{\pi + \rho + n}{g}\right)^{\tau} + m^*\tau\left(\frac{\pi + \rho + n}{g}\right)^{\tau-1}f''(a^* - m^*).
\]

To simplify further calculation, we will substitute some repetitive expressions with
shorter notation. Let \(\frac{\pi + \rho + n}{g} = Q\) and \(\tau f''(a^* - m^*) = P\). Now, the Jacobian matrix
for \((a^*, c^*, m^*)\) can be expressed as

\[
J_{(a^*, c^*, m^*)} = \begin{pmatrix}
\rho & -1 & -gQ \\
cP & 0 & -cP \\
-mP & g & \frac{mP}{g}Q^{\tau-1} - Q^\tau
\end{pmatrix},
\]

Our eigenvalue Jacobian matrix is

\[
\tilde{J}_{(a^*, c^*, m^*)} = \begin{pmatrix}
\frac{\rho mP}{g}Q^{\tau-1} - \rho Q^\tau - mPQ^\tau & -\frac{mP}{g}Q^{\tau-1} + Q^\tau + gQ \\
-cPQ^\tau & cP
\end{pmatrix}.
\]

It turns out that after some gruesome calculations, the determinant of the preceding
Jacobian matrix can be simplified to the following expression:

\[
DetJ_{(a^*, c^*, m^*)} = cPQ^{\tau-1}\left(\frac{mP}{g} - Q\right)(\rho - gQ - Q^\tau).
\]

(3.13)
We know that \( Q > 0 \) and \( P < 0 \) since the second derivative of our production function is negative by assumption (2.4). If we substitute \( Q = \frac{\pi + \rho + n}{g} \) into (3.13), we will see that \( \rho - gQ - Q^r = -\pi - n - Q^r < 0 \) and \( \frac{mP}{g} - Q < 0 \). This shows that regardless of the form of the production function \( f(k) \), this determinant will always be negative, and thus it ensures a saddle point for the system at \((a^*, c^*, m^*)\).

### 3.3.3 Equilibrium Stability of \((0, 0, 0)\)

If \( a = 0 \), \( c = 0 \) and \( m = 0 \), the Jacobian matrix for system is as follows:

\[
J_{(0,0,0)} = \begin{pmatrix}
    f'(0) - (\delta + n) & -1 & -f'(0) + (\delta - \pi) \\
    0 & \tau(f'(0) - (\rho + \delta + n)) & 0 \\
    0 & 1 & -\frac{(\pi - \delta + f'(0))}{g}
\end{pmatrix},
\]

And \( \tilde{J} \) is:

\[
\tilde{J}_{(0,0,0)} = \begin{pmatrix}
    f'(0) - (\delta + n) & -1 - \frac{f'(0) - (\delta - \pi)}{(\pi - \delta + f'(0))} \\
    0 & \tau(f'(0) - (\delta + \rho + n))
\end{pmatrix}.
\]

The eigenvalues of \( \tilde{J}_{(0,0,0)} \) are the two diagonal entries:

\[
\lambda_1 = f'(0) - (\delta + n), \quad \lambda_2 = \tau(f'(0) - (\delta + \rho + n)).
\]

Now remember that from the previous equilibrium state, we have \( f'(a^* - m^*) = \delta + \rho + n \). Since \( a^* - m^* \) will always be non-negative (\( k \) would never be negative), and our production function \( f \) has negative second derivative, we can safely say that \( f'(0) > f'(a^* - m^*) \) and therefore \( f'(0) > \delta + \rho + n \). This provides us the proof that both of the eigenvalues for this matrix are positive, hence \((0, 0, 0)\) is a nodal source.
3.3.4 Equilibrium Stability of \((a_*, 0, 0)\)

Now let us consider our third equilibrium point. Using the same technique, let us plug in \(a_*, c_* = 0, m_* = 0\) into our Jacobian matrix:

\[
J_{a_*,0,0} = \begin{pmatrix}
    f'(a_*) - (\delta + n) & -1 & -f'(a_*) + (\delta - \pi) \\
    0 & \tau(f'(a_*) - (\rho + \delta + n)) & 0 \\
    0 & 1 & -\left(\frac{\pi - \delta + f'(a_*)}{g}\right)\tau
\end{pmatrix},
\]

and our \(\bar{J}\) is

\[
\bar{J}_{(a_*,0,0)} = \begin{pmatrix}
    f'(a_*) - (\delta + n) & -1 & -f'(0) - (\delta - \pi) \\
    0 & \tau(f'(a_*) - (\delta + \rho + n)) & 0 \\
    \frac{\pi - \delta + f'(a_*)}{g}\tau & 0 & \tau(f'(a_*) - (\delta + \rho + n))
\end{pmatrix}.
\]

Similar to how we proved the stability of \((0,0,0)\), the two \(\lambda\) values for this system would still be the two elements on the diagonal:

\[
\lambda_1 = f'(a_*) - (\delta + n), \quad \lambda_2 = \tau(f'(a_*) - (\delta + \rho + n)).
\]

We still have the expression \(f'(a^* - m^*) = \delta + \rho + n\). Since \(a_*\) will always be bigger than \(a^* - m^*\) and our production function \(f\) has negative second derivative, we can show that \(f'(a_*) < f'(a^* - m^*)\) and therefore \(f'(a_*) < \delta + \rho + n\). This would show that \(\lambda_2\) is negative.

If we substitute \((a_*, 0, 0)\) into the first equation of our master system (2.30), we have \(\frac{f(a_*)}{a_*} = \delta + n\). Then we can use exactly the same technique we used to show that \((k_*, 0)\) has negative eigenvalues. By borrowing the already proved fact: \(\frac{f(k)}{k} > f'(k)\) and substituting \(k\) with our variable \(a^*\): \(\frac{f(a_*)}{a_*} > f'(a_*)\), we would eventually have \(\delta + n > f'(a_*)\). Hence \(\lambda_1\) is also negative. With both eigenvalues being negative, \((a_*,0,0)\) is a nodal sink.
3.3.5 Equilibrium Stability of \((\tilde{a}, 0, \tilde{m})\)

If \(c = 0\), the Jacobian matrix for the this equilibrium is

\[
J(\tilde{a}, 0, \tilde{m}) = \begin{pmatrix}
0 & -1 & 0 \\
0 & \tau(-\rho + \delta + n) & 0 \\
-m\tau\left(\frac{\pi - \delta + f'(a - m)}{g}\right) & 1 & J_{33}
\end{pmatrix},
\]

with

\[
J_{33} = -\left(\frac{\pi - \delta + f'(a - m)}{g}\right)^\tau + m\tau\left(\frac{\pi - \delta + f'(a - m)}{g}\right)^{\tau - 1}f''(a - m)
\]

and

\[
\tilde{J}(\tilde{a}, 0, \tilde{m}) = \begin{pmatrix}
0 & -1 \\
0 & \tau(-\delta + \rho + n)
\end{pmatrix}.
\]

The two eigenvalues of this \(\tilde{J}\) would be \(\lambda_1 = 0\) and \(\lambda_2 = -\tau(\delta + \rho + n) < 0\). This gives us a degenerate nodal sink, since one eigenvalue is zero, and the other is negative.

3.3.6 Conclusion

From the stability proved above and the phase diagram of (3.2), we reach the following conclusion:

**Proposition 3.1.** If the utility function is defined as (2.3), and the production function satisfies (2.4), for any \(a(0) > 0\), there exists a unique \(c(0) > 0\) and \(m(0)\) satisfying

\[
c(0) - m(0)\left(\frac{\pi - \delta + f'(a(0) - m(0))}{g}\right)^\tau = 0,
\]

such that the solution \((a(t), c(t), m(t))\) of (3.2) with initial condition \((a(0), c(0), m(0))\) satisfies

\[
\lim_{t \to \infty} (a(t), c(t), m(t)) = (a^*, c^*, m^*)
\]

with the numerical values of \((a^*, c^*, m^*)\) are given by (3.6).
Chapter 4

Numerical Analysis using Matlab

We made it! In this last chapter, we finally get to simulate our model in the amazing mathematical software Matlab. We will simulate both the RCK model (2.31) and the MIU model (2.30) to compare and study their behaviors.

Most solving functions provided by Matlab require a system of expressions in differential equation form, which is not our case for the MIU model. Conveniently enough, Matlab has this useful function ode15s that allows us to numerically solve for solutions of a system of Differential Algebraic Equations. Differential algebraic equations are a type of differential equation system where one or more derivatives of the dependent variables are not present in the equations. Variables that appear in the equations without their derivative are called algebraic. For example, in our case, we do not have a differential equation for $m$, therefore the third equation is our “algebraic” equation and $a'$, $c'$ make up the “differential” part.

To refresh our memory, here are the two models that we will work with:

\[
\begin{align*}
  k'(t) &= f(k(t)) + v - (\delta + n)k(t) - c(t), \\
  c'(t) &= \tau c(t)(f'(k(t)) - (\rho + \delta + n)).
\end{align*}
\] (4.1)

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\[
\begin{aligned}
\begin{cases}
    a'(t) &= f(a(t) - m(t)) + v - (\delta + n)a(t) + (\delta - \pi)m(t) - c(t), \\
    c'(t) &= \tau c(t)(f'(a(t) - m(t)) - (\rho + \delta + n)), \\
    0 &= c(t) - m(t)\left(\frac{\pi - \delta + f'(a(t) - m(t))}{g}\right)^\tau,
\end{cases}
\end{aligned}
\] 

where \( \tau = \frac{1}{1 - \theta}. \)
4.1 Simulation of RCK Model

Let us start with the 2D model (4.1) and compare our result with the existing phase
diagram of the system, Figure 1.2. Fortunately there is already a tool which can help us
plot the solution path and find equilibrium points without going through much trouble:
the PPlane plug-in gadget written by John C. Polking from Rice University. The trickiest
part of the model simulation might be to find the right parameters for the model, especially
in our case of 6 parameters. Here is a table of proposed values for the parameters and a
phase diagram for the overall RCK model using this set of parameters:

<table>
<thead>
<tr>
<th>parameter</th>
<th>$z$</th>
<th>$\delta$</th>
<th>$\tau$</th>
<th>$\rho$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>5</td>
<td>0.1</td>
<td>1.1</td>
<td>0.02</td>
<td>3</td>
<td>1</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters Values for Ramsey-Cass-Koopsman Model

We can see that the phase diagram of our simulated model resembles Figure 1.2 (from
[1, p322]). The three equilibrium points have the following properties:
<table>
<thead>
<tr>
<th>equilibrium point</th>
<th>eigenvalues</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>(4.89, 5.357)</td>
<td>Nodal source</td>
</tr>
<tr>
<td>(1.734, 1.207)</td>
<td>(0.419, -0.399)</td>
<td>Saddle point</td>
</tr>
<tr>
<td>(14.818, 0)</td>
<td>(-0.108, -0.140)</td>
<td>Nodal sink</td>
</tr>
</tbody>
</table>

Table 4.2: Numerical Values and Stabilities of Equilibrium Points (2D)

After trying different initial values for $k$ and $c$, we were able to simulate the path converging to each of the equilibrium points.

Figure 4.2: Simulation for Orbit I in Figure 4.1.

Initial condition $k(0) = 0.05$, $c(0) = 0.1$; capital function $k(t)$ becomes negative for large value of $t$. 
Figure 4.3: Simulation for Orbit II, Shooting to \((k^*, c^*)\).

Initial condition \(k(0) = 0.05, c(0) = 0.044\); system converges to \((k^*, c^*)\) for large \(t\).
Figure 4.4: Simulation for Orbit III, Shooting to \((k^*, c^*)\).

Initial condition \(k(0) = 2.55, c(0) = 1.509\). System converges to \((k^*, c^*)\) for large \(t\).
Figure 4.5: Simulation for Orbit IV, Shooting to \((k^*, 0)\).

Initial condition \(k(0) = 0.05, c(0) = 0.01\). System converges to \((k^*, 0)\) for large \(t\).
4.2 Simulation of MIU Model

Similarly, let us now take a look at the 3D model (4.2) and compare its solution to the RCK model (4.1). We used same set of parameters but with some additions of new parameters:

<table>
<thead>
<tr>
<th>param.</th>
<th>$z$</th>
<th>$\delta$</th>
<th>$\tau$</th>
<th>$\rho$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$n$</th>
<th>$g$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>5</td>
<td>0.1</td>
<td>1.1</td>
<td>0.02</td>
<td>3</td>
<td>1</td>
<td>0.01</td>
<td>1</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 4.3: Parameters Values for 3D Model

The four equilibrium points have the following properties:

<table>
<thead>
<tr>
<th>equilibrium point</th>
<th>eigenvalues</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(11.617, 0.614, 9.883)</td>
<td>(0.0974, −0.120)</td>
<td>Saddle point</td>
</tr>
<tr>
<td>(14.818, 0, 0)</td>
<td>(−0.074, −0.104)</td>
<td>Nodal sink</td>
</tr>
<tr>
<td>(0, 0, 0)</td>
<td>(4.890, 5.357)</td>
<td>Nodal source</td>
</tr>
<tr>
<td>(22.5, 0, 19.5)</td>
<td>(0, −0.143)</td>
<td>Degenerate sink</td>
</tr>
</tbody>
</table>

Table 4.4: Numerical Solutions and Stabilities of Equilibrium Points (3D)

Note that the production function we used here is (2.5). (2.6) would yield similar results.

Now we need to set initial values for $a$, $c$, and $m$ accordingly, aiming at simulating different equilibriums. After a lot of tuning, we finally obtain the followings illustration figures.
Figure 4.6: Simulation for Equilibrium $(0, 0, 0)$.

Initial condition $a(0) = 0.4$, $c(0) = 0.19$, $m(0) = 0.21$; consumption function shoots off and wealth becomes negative for large value of $t$. 
Figure 4.7: Simulation for Equilibrium $(a_*, 0, 0)$.

Initial condition $a(0) = 0.4$, $c(0) = 0.005$, $m(0) = 0.25$; system converges to $(a_*, 0, 0)$. 
Figure 4.8: Simulation for Equilibrium \((a^*, c^*, m^*)\).

Initial condition \(a(0) = 0.4, c(0) = 0.098, m(0) = 0.22\); system converges to \((a^*, c^*, m^*)\).
Figure 4.9: Simulation for Equilibrium $(\tilde{a}, 0, \tilde{m})$.

Initial condition $a(0) = 0.4$, $c(0) = 0.35$, $m(0) = 0.01$; system converges to $(\tilde{a}, 0, \tilde{m})$. 
Chapter 5

Conclusions

Through reproduction and validation of the Ramsay-Cass-Koopsman Model, we successfully simulated its multiple equilibrium solution paths and proved their stabilities. To conclude, there exist three equilibrium points for a typical RCK economy: \((0, 0), (k^*, c^*), (k^*, 0)\):

- \((0, 0)\) is a nodal source with two positive eigenvalues;
- \((k^*, c^*)\) is a saddle point with one positive and one negative eigenvalue;
- \((k^*, 0)\) is nodal sink with two negative eigenvalues.

With this model kept in mind, we proceeded to examine a Money In Utility Model (MIU) model which can be derived from this RCK model by endogenizing real money holding as variable \(m\). Along with the relation \(a(t) - m(t) = k(t)\), this will give us a system of three variables \(a(t), c(t),\) and \(m(t)\). After conducting similar calculations for system equilibrium points and stabilities, we obtained four equilibrium points, similar to that of the two dimensional model:

- \((0, 0, 0)\) is a nodal source with two positive eigenvalues;
- \((a^*, c^*, m^*)\) is a saddle point with one positive, one negative eigenvalue;
- \((a^*, 0, 0)\) is nodal sink with two negative eigenvalues.
- \((\tilde{a}, 0, \tilde{m})\) is a degenerate nodal sink with a zero and a negative eigenvalue.
In Miguel Sidrauski’s paper *Rational Choice and Patterns of Growth in a Monetary Economy*, he focused on deriving the system without giving too much details on the proof for the stabilities of its three different equilibrium points. In this paper, we found an additional equilibrium point, even though its behavior might not be as determined as other three, which might be the reason why Sidrauski left out this equilibrium point in his paper. We made efforts to mathematically prove the stabilities of the system after solving for our variables of interest explicitly. Hopefully our efforts can be of future use to other scholars in the area.

Through modeling a monetary economy, we have a better understanding of the question such as how should we model the demand for money? How does real economy differs from the Solow-Swan economy that give rise to a positive value for money? However, the research we conducted is from a mathematical perspective; in order to fully understand and make use of the MIU model, systematic empirical research needs to be conducted in order to link the quantities in the theoretical model to measurable data and furthermore proven useful.

### 5.1 Acknowledgements

This research is supported by William and Mary Charles Center Honors Fellowship for both of my summer research and senior honors thesis. Thank you to the best advisor I could have possibly had: Professor Junping Shi, for your passion in academic research, and your patient guidance for me on this project from beginning to end. Thank you to Professor Rex Kincaid and Professor Carlisle Moody who willingly served on my Honors Committee and provided insightful and valuable advices.
Appendices
Appendix A

Matlab Code

A.1 Explicitly Solving Equilibrium Points in RCK Model

Note: We used the `pplane8` program written by John C. Polking of Rice University to solve for the two dimensional RCK model. The code itself is too long to be included. For further interest in the program, please reach out to me at xliu12@email.wm.edu.

A.2 Explicitly Solving Equilibrium Points in MIU Model

```matlab
% old solution system:
z = 5;
v = 0;
delta = 0.1;
n = 0.01;
rho = 0.02;
```
tau = 1.1;
a0 = 3;
a1 = 1;
g = 1;
ir = 0.05;

%(0 ,0 ,0) and its eigenvalues
f_0 = z/a1;
lambda_01 = f_0 - (delta+n);
lambda_02 = tau*(f_0 - (rho+delta+n));

% (a ,c ,m)
    k = -a1/a0 + 1/(a0)*((z*a1)/(delta+n+rho))ˆ0.5;
m = (((z*k)/(a0*k+a1))+v-delta*k-n*k)/(n+ir+((ir+rho+n)/(g))ˆtau);
a = k+m;
% eigenvalues for (a , c , m)
c = m*((ir+rho+n)/g)ˆtau;
Q = (ir + rho + n)/g;
f_two = -2*z*a1*a0*(a0*(a-m)+a1)ˆ(-3);
P = tau*f_two;
J11 = (rho*m*P*Q^((tau-1)/g)-rho*Q^tau-m*P*Q^tau;
J12 = -m*P*Q^((tau-1)/g) + Q^tau + g*Q;
J21 = -c*P*Q^tau;
J22 = c*P;
A = [J11 J12; J21 J22];
eigenval = eig(A);
d = det(A);
\( (a, 0, 0) \)
\[
a_1 = \frac{(z-(\delta+n)a_1)}{(\delta+n)a_0);
\]

\% eigenvalues for \((a, 0, 0)\)
\[
fd_a = \frac{z(z-(\delta+n)a_1)}{(\delta+n)a_0/(a_0(z-(\delta+n)a_1))\ldots
/(\delta+n)a_0+a_1)^2; \\
lambda_{a1} = fd_a - (\delta+n); \\
lambda_{a2} = \tau(a - (\rho+\delta+n));
\]

\% \((a, 0, m)\)
\[
syms a m
\]
eqn1 = \(ir - \delta + z a/(a_0(a-m)+a_1)^2 = 0; \)
eqn2 = \(z(a-m)/(a_0(a-m)+a_1) - (\delta+n)a + (\delta-ir)*m == 0; \)
sol = solve([eqn1, eqn2], [a, m]);
aSol = sol.a;
mSol = sol.m;

\% eigenvalues for \((a, 0, m)\)
\[
lambda_1 = 0; \\
lambda_2 = -\tau(-ir - \rho - n);
\]

A.3 Numerical Simulation of RCK Model

function TwoD_model
% parameters
z=5;
v=0;
delta=0.1;
n=0.01;
rho=0.02;
tau=1.1;
a0=3;
a1=1;

% intervals of time which to run the model
tspan=[0 0.5];
% initial condition
% simulation for (0,0)
y0=[0.05 0.1];

[T,Y]=ode45(@f,tspan,y0);

plot(T,Y(:,1),'r',T,Y(:,2),'b','linewidth',2);
xlabel('Time');
legend('capital', 'consumption')

function dy=f(t,x)
dy = zeros(2,1);
dy(1)=z*x(1)/(a0*x(1)+a1)+v *(n+delta)*x(1)-x(2);
dy(2)=x(2)*tau*(z*a1/(a0*x(1)+a1)^2 -(rho + delta + n));
end
end
A.4 Numerical Simulation of MIU Model

```matlab
function ThreeD_model
M = [1 0 0
     0 1 0
     0 0 0];

% intervals of time which to run the model
tspan = [0 300];

% initial condition
x0 = [0.4 0.098 0.22];

options = odeset('Mass',M,'RelTol',1e-4,'AbsTol',[5 5 5]);
[t,x] = ode15s(@f,tspan,x0,options);

figure(1)
plot(t,x(:,1),t,x(:,2),t,x(:,3),'--b','linewidth',1.5)
legend('wealth','consumption','money holdings')
xlabel('Time');

function out = f(t,x)
z=5;
v=0;
delta=0.1;
n=0.01;
rho=0.02;
tau=1.1;
```

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\begin{align*}
a0 &= 3; \\
a1 &= 1; \\
g &= 1; \\
ir &= 0.05; \\
\text{out} &= \left[ \frac{z(x(1) - x(3))}{(a0(x(1) - x(3)) + a1)} + v - (\delta + n)x(1) + \ldots \right. \\
&\quad \left. \left( \delta - \text{ir} \right)x(3) - x(1) \right] \\
&\quad x(2) \tau \left( \frac{z \alpha l}{(a0(x(1) - x(3)) + a1)^2} - (\rho + \delta + n) \right); \\
&\quad -(x(2)) + x(3) \left( \frac{(\text{ir} - \delta) + (z \alpha l)}{(a0(x(1) - x(3)) + a1)^2} \right) \left( \frac{g}{\tau} \right); \\
\end{align*}
Bibliography


