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SPECTRUM, NUMERICAL RANGE AND DAVIS-WIELANDT SHELL OF A NORMAL OPERATOR

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Abstract. We denote the numerical range of the normal operator $T$ by $W(T)$. A characterization is given to the points in $W(T)$ that lie on the boundary. The collection of such boundary points together with the interior of the convex hull of the spectrum of $T$ will then be the set $W(T)$. Moreover, it is shown that such boundary points reveal a lot of information about the normal operator. For instance, such a boundary point always associates with an invariant (reducing) subspace of the normal operator. It follows that a normal operator acting on a separable Hilbert space cannot have a closed strictly convex set as its numerical range. Similar results are obtained for the Davis-Wielandt shell of a normal operator. One can deduce additional information of the normal operator by studying the boundary of its Davis-Wielandt shell. Further extension of the result to the joint numerical range of commuting operators is discussed.

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1. Introduction. Let $B(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. We identify $B(\mathcal{H})$ with the algebra $M_n$ of $n \times n$ complex matrices if $\mathcal{H}$ has dimension $n$. The numerical range of $T \in B(\mathcal{H})$ is defined by

$$W(T) = \{ (Tx, x) : x \in \mathcal{H}, \langle x, x \rangle = 1 \},$$

which is useful for studying operators; see [5–7]. In particular, the geometrical properties of $W(T)$ often provide useful information about the algebraic and analytic properties of $T$. For instance, $W(T) = \{ \mu \}$ if and only if $T = \mu I$; $W(T) \subseteq \mathbb{R}$ if and only if $T = T^*$; $W(T)$ has no interior point if and only if there are $a, b \in \mathbb{C}$ with $a \neq 0$ such that $aT + bI$ is self-adjoint. Moreover, there are nice connections between $W(T)$ and the spectrum $\sigma(T)$ of $T$. For example, the closure of $W(T)$, denoted by $\text{cl}(W(T))$, always contains $\sigma(T)$. If $T$ is normal, then $\text{cl}(W(T)) = \text{conv} \sigma(T)$, where $\text{conv} S$ denotes the convex hull of the set $S$. Hence, $\text{cl}(W(T))$ is completely determined by $\sigma(T)$ for a normal operator $T$. However, one can easily find examples of normal operators $A$ and $B$ with the same spectrum such that $W(A) \neq W(B)$.

Example 1.1. Let $A = \text{diag}(1, 1/2, 1/3, \ldots)$, $B = \text{diag}(0, 1, 1/2, 1/3, \ldots)$ be two diagonal operators acting on $\ell_2$. Then, $W(A) = \{0, 1\} \neq [0, 1] = W(B)$ and $\sigma(A) = \sigma(B) = \{1/n : n \geq 1\} \cup \{0\}$. 
For two normal operators $A$ and $B$ with the same spectrum, we have $\text{cl}(W(A)) = \text{conv} \sigma(A) = \text{conv} \sigma(B) = \text{cl}(W(B))$. Thus, $W(A)$ and $W(B)$ can differ only by their boundaries $\partial W(A)$ and $\partial W(B)$. Hence, to describe the numerical range of a normal operator $T$, it suffices to determine which boundary points of $W(T)$ actually belong to $W(T)$. In this paper, a characterization is given to such boundary points. Moreover, we show that a point in $W(T) \cap \partial W(T)$ always leads to an orthogonal decomposition of the Hilbert space, and a corresponding decomposition of the operator $T$. It follows that a normal operator acting on a separable Hilbert space cannot have a closed strictly convex set as its numerical range. On the contrary, the numerical range of a non-normal operator cannot be detected by the geometrical features of $W(T)$; see [7, Theorem 1.3.6].

Motivated by theoretical study and applications, researchers considered different generalizations of the numerical range; see for example [5, 6] and [7, Chapter 1]. One of these generalizations is the Davis-Wielandt shell of $T \in \mathcal{B}(\mathcal{H})$ defined by

$$DW(T) = \{(\langle Tx, x \rangle, \langle Tx, Tx \rangle) : x \in \mathcal{H}, \langle x, x \rangle = 1\};$$

see [3, 4, 10]. Evidently, the projection of the set $DW(T)$ on the first co-ordinate is the classical numerical range. So, $DW(T)$ captures more information about the operator $T$. For a normal operator $T \in \mathcal{B}(\mathcal{H})$, it is known that the closure of $DW(T)$ is the set

$$\text{conv}\{\langle \lambda, |\lambda|^2 \rangle : \lambda \in \sigma(T)\};$$

see, for example [9, Theorem 2.1]. Thus, the interior of $DW(T)$ can be easily determined. However, the points in $DW(T)$ that lie on its boundary are not so well understood. We characterize such points and show that they lead to direct sum decomposition of $T$ that cannot be detected by the geometrical features of $W(T)$. Inspired by some comments of the referee on an early version of this paper, we include a discussion of the extension of our results to the joint numerical range of commuting operators.

In the following discussion, we denote $\text{cl}(S)$ and $\partial S$ as the closure and the boundary of a set $S$, respectively. Moreover, we use $\text{int}(S)$ to denote the relative interior of $S$. For instance, if $\text{cl}(S)$ is a line segment in $\mathbb{C}$, then $\text{int}(S)$ will be the line segment obtained from $\text{cl}(S)$ by removing its end points, although $S$ has no interior points in $\mathbb{C}$. For $T \in \mathcal{B}(\mathcal{H})$, the point spectrum of $T \in \mathcal{B}(\mathcal{H})$ is denoted by $\sigma_p(T)$.


**Theorem 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then, $\mu \in W(T)$ is a boundary point if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, with $\mu \in W(T_1) \subseteq L$, for a straight line $L$ and $W(T_2) \cap L = \emptyset$.

**Proof.** Let $\mu \in W(T)$ be a boundary point of $W(T)$. We may replace $T$ by $aT + bI$ so that $\mu = 0$ and $\text{Re } v \leq 0$ for all $v \in W(T)$. Let $T = H + iG$, where $H$ and $G$ are self-adjoint. Since $W(H) = \{\text{Re } v : v \in W(T)\}$, we see that $\langle Hx, x \rangle \leq 0$ for any unit vector $x \in \mathcal{H}$. Thus, $H$ is negative semidefinite. Let $\mathcal{H}_1$ be the kernel of $H$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp$. Then, $H = 0_{\mathcal{H}_1} \oplus H_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Since $HG = GH$, we see that $G = G_1 \oplus G_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Thus, $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Since $T_1 = iG_1$, $W(T_1) \subseteq i\mathbb{R}$, and $T_2 = H_2 + iG_2$ such that $W(H_2) \subseteq (-\infty, 0)$, it follows that $W(T_2) \cap i\mathbb{R} = \emptyset$. 


Using the fact that $W(T_1 \oplus T_2) = \text{conv}\{W(T_1) \cup W(T_2)\}$ (see, for example, [7, 1.2.10]), one can verify the converse. □

In Theorem 2.1, $W(T_1)$ may be a point or a line segment containing none, one or all of its end points; $W(T_2)$ may be an open set, a closed set, or neither.

Example 2.2. We have $0 \in W(T) \cap \partial W(T)$ if $T = T_1 \oplus T_2 \in B(\ell_2 \oplus \ell_2)$ for any choices of the following $T_1$ and $T_2$:

- $T_1 = 0$ so that $W(T_1) = \{0\}$,
- $T_1 = i(-I \oplus I)$ so that $W(T_1) = \{i\mu : \mu \in [-1, 1]\}$, or
- $T_1 = i([\text{diag}(1/2, 2/3, 3/4, \ldots) \oplus \text{diag}(-1/2, -2/3, -3/4, \ldots)]$ so that $W(T_1) = \{i\mu : \mu \in (-1, 1)\}$;
- $T_2 = \text{diag}(e^{2\pi i/3}, e^{4\pi i/3}, -1/2)$ so that $W(T_2) = \text{conv}\sigma(T_2)$,
- $T_2 = e^{2\pi i/3}D \oplus e^{4\pi i/3}D \oplus (D - I)$ with $D = \text{diag}(2/3, 3/4, 4/5, \ldots)$ so that $W(T_2) = \text{int}(\text{conv}\sigma(T_2)) = \text{int}(\text{conv}\{e^{2\pi i/3}, e^{4\pi i/3}, 0\})$, or
- $T_2 = \text{diag}(e^{2\pi i/3}, e^{4\pi i/3}) \oplus -\text{diag}(1/3, 1/4, 1/5, \ldots)$ so that $W(T_2) = \text{int}(\{e^{2\pi i/3}, e^{4\pi i/3}, 0\}) \cup \text{conv}\{e^{2\pi i/3}, e^{4\pi i/3}\}$.

In connection to Theorem 2.1 and the above example, we give a detailed analysis of an operator $A$ such that $W(A)$ is a subset of a straight line in $\mathbb{C}$ in the following. In particular, we give a description of $W(A)$ in terms of $\sigma(A)$ and $\sigma_p(A)$ and determine the algebraic structure of $A$. Note that the following proposition is valid for a general operator $A$.

Proposition 2.3. Let $A \in B(H)$ be such that $W(A) \subseteq L$, where $L$ is a straight line in $\mathbb{C}$. Then,

$$W(A) = \text{int}(\text{conv}\sigma(A)) \cup \sigma_p(A)$$

and one of the following holds:

(a) $A = \mu I$ and $W(A) = \{\mu\} \subseteq L$.

(b) There are $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\text{cl}(W(A)) = a[-1, 1] + b \subseteq a\mathbb{R} + b$. In such case, an end point $\mu$ of the line segment $a[-1, 1] + b$ belongs to $W(A)$ if and only if $\mu \in \sigma_p(A)$.

Proof. Suppose $W(A)$ is a subset of a line $L$ in $\mathbb{C}$. Note that $W(A) = \{\mu\}$ if and only if $A = \mu I$. Assume that it is not the case. Then, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\text{cl}(W(A)) = a[-1, 1] + b \subseteq a\mathbb{R} + b$. Thus, $A = aS + bI$ such that $S = S^*$ with $\text{cl}(W(S)) \subseteq [-1, 1]$. In particular, $\|S\| = 1$.

If the end point $a + b$ of $\text{cl}(W(A))$ belongs to $W(A)$, then $1 \in W(S)$. So, there is a unit vector $x \in H$ such that

$$1 = \langle Sx, x \rangle \leq \|Sx\|\|x\| \leq \|S\| \leq 1.$$

By the equality case of the Cauchy-Schwartz inequality, $Sx = x$, and thus $Ax = (a + b)x$. Thus, $a + b \in \sigma_p(A)$. Conversely, if $a + b \in \sigma_p(A)$, then $a + b \in W(A)$. Similarly, $-a + b \in W(A)$ if and only if $-a + b \in \sigma_p(A)$. □

The following corollary is immediate.

Corollary 2.4. Suppose $A \in B(H)$ is normal and $\mu \in W(A)$ is a boundary point. Then, there is a straight line $L$ in $\mathbb{C}$ such that $W(A) \cap L = \{\mu\}$ if and only if $H$ admits an orthogonal decomposition $H_1 \oplus H_2$ and $A = \mu I_{H_1} \oplus A_2 \in B(H_1 \oplus H_2)$ with $\mu \notin W(A_2)$. 

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We present another example to illustrate our results and show that the set \( W(T) \cap \partial W(T) \) cannot be determined by (and does not determine) \( \sigma(T) \) and \( \sigma_p(T) \) in general. The following corollary is useful for presenting the example:

**Corollary 2.5.** Suppose \( A = d_1 I_{H_1} \oplus d_2 I_{H_2} \oplus \cdots \in \mathcal{B}(H) \) such that \( H \) is an orthogonal sum of the closed subspaces \( H_1, H_2, \ldots \). Then,

\[
W(A) = \text{conv} \{ d_n : n \geq 1 \}.
\]

**Proof.** The result follows from the inclusions

\[
\text{int}(W(A)) \subseteq \text{conv} \{ d_n : n \geq 1 \} \subseteq W(A) \subseteq \text{cl}(W(A)) = \text{cl}(\text{conv} \{ d_n : n \geq 1 \})
\]

and the description of \( \partial(W(A)) \cap W(A) \) in Theorem 2.1. \( \square \)

We are now ready to present the promised example. In particular, we construct normal operators \( A, B, C \in \mathcal{B}(H) \) so that \( \text{cl}(W(A)) = \text{cl}(W(B)) = \text{cl}(W(C)) \); \( A \) and \( C \) have different spectra and point spectra but \( \partial W(A) \cap W(A) = \partial W(C) \cap W(C) \); \( B \) and \( C \) have the same spectrum and point spectrum but \( \partial W(B) \cap W(B) \neq \partial W(C) \cap W(C) \).

**Example 2.6.** Let \( \{ r_n : n \geq 1 \} \) be a countable dense subset of the open interval \( (0, 1) \) and \( \{ d_n : n \geq 1 \} \) a countable dense subset of the interior of \( \text{conv} \{ 0, 1, i \} \). Let \( A = [i] \oplus \text{diag}(r_1, r_2, \ldots), B = [i] \oplus \text{diag}(d_1, d_2, \ldots) \) and \( C = B \oplus M \), where \( M \) is the multiplication operator on \( L_2([0, 1]) \) defined by \( M(f)(t) = tf(t) \) for \( t \in [0, 1] \). Then,

\[
\text{cl}(W(A)) = \text{cl}(W(B)) = \text{cl}(W(C)) = \text{conv} \{ 0, 1, i \}.
\]

Using Theorem 2.1, we have \( \partial W(B) \cap W(B) = \{ i \} \) and

\[
\partial W(A) \cap W(A) = \{ i \} \cup (0, 1) = \partial W(C) \cap W(C)
\]

so that

\[
\partial W(A) \cap W(A) = \partial W(C) \cap W(C) \neq \partial W(B) \cap W(B).
\]

It is easy to check that

\[
\sigma_p(A) = \{ i \} \cup \{ r_n : n \geq 1 \}, \quad \sigma_p(B) = \sigma_p(C) = \{ i \} \cup \{ d_n : n \geq 1 \}, \quad \sigma(A) = \{ i \} \cup [0, 1] \quad \text{and} \quad \sigma(B) = \sigma(C) = \text{conv} \{ 0, 1, i \}.
\]

**Corollary 2.7.** Suppose \( \dim H \) is infinite and \( A \in \mathcal{B}(H) \) is normal. If \( A \) is not unitarily reducible, then \( W(A) = \text{int}(W(A)) \). In other words, \( W(A) \) is a non-empty open set in \( \mathbb{C} \) or \( W(A) \) is a non-degenerate line segment without end points.

Suppose \( S \) is a closed, bounded and convex subset of \( \mathbb{C} \), with non-empty interior. We say that \( S \) is strictly convex if \( \partial S \) equals the set \( \text{Ext}(S) \) of extreme points of \( S \).

**Corollary 2.8.** Let \( A \in \mathcal{B}(H) \) be normal and \( E = W(A) \cap \text{Ext}(\text{cl}(W(A))) \) be uncountable. Then, \( H \) is nonseparable and every point in \( E \) is an eigenvalue of \( A \). In particular, if \( W(A) = \text{cl}(W(A)) \) is strictly convex with non-empty interior, then \( H \) is nonseparable and every boundary point of \( W(A) \) is an eigenvalue.
**Corollary 2.9.** Let $S$ be a bounded and convex subset of $\mathbb{C}$. Then, there exist a separable Hilbert space $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ such that $S = W(A)$ if and only if $S \cap \text{Ext}(\text{cl}(S))$ is countable.

**Proof.** Suppose $S$ is a bounded convex set such that $S \cap \text{Ext}(\text{cl}(S))$ is countable. Let $A = \text{diag}(d_1, d_2, \ldots)$ such that $\{d_n : n \geq 1\}$ is the union of $S \cap \text{Ext}(\text{cl}(S))$ and a countable dense set of the relative interior of $S$, then $W(A) = S$. The converse follows from Corollary 2.8.

### 3. Davis-Wielandt Shells.

In this section, we characterize $DW(T) \cap \partial DW(T)$ for normal $T \in \mathcal{B}(\mathcal{H})$. In our discussion, we always identify $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$.

**Theorem 3.1.** Suppose $T \in \mathcal{B}(\mathcal{H})$ is a normal operator. Then, $DW(T)$ and $\text{conv}\{ (\xi, |\xi|^2) : \xi \in \sigma(A) \}$ have the same interior. A point $(\mu, r) \in DW(T)$ is a boundary point if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ with $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $(\mu, r) \in DW(T_1) \subseteq \mathcal{P}$ for a plane $\mathcal{P}$ in $\mathbb{C} \times \mathbb{R}$ and $DW(T_2) \cap \mathcal{P} = \emptyset$.

**Proof.** Let $T = H + iG$ be such that $H = H^*$ and $G = G^*$. Then, $DW(T)$ can be identified with the joint numerical range

$$W(H, G, T^*T) = \{(\langle Hx, x \rangle, \langle Gx, x \rangle, \langle T^*Tx, x \rangle) : x \in \mathcal{H}, \langle x, x \rangle = 1\} \subseteq \mathbb{R}^3.$$  

Let $x \in \mathcal{B}(\mathcal{H})$ be a unit vector such that

$$(\mu_1, \mu_2, r) = (\langle Hx, x \rangle, \langle Gx, x \rangle, \langle T^*Tx, x \rangle)$$

is a boundary point of $W(H, G, T^*T)$. Let $\mathcal{P}$ be a support plane of $DW(T)$ passing through $(\mu_1, \mu_2, r)$. Then, there are real numbers $a, b, c, d$ such that

$$av_1 + bv_2 + c\tilde{r} - d \leq a\mu_1 + b\mu_2 + cr - d = 0$$

for all $(v_1, v_2, \tilde{r}) \in W(H, G, T^*T)$. As a result, the operator $\tilde{T} = aH + cT^*T - dI$ is negative semidefinite with a non-zero kernel. Let $\mathcal{H}_1$ be the kernel of $\tilde{T}$. Then, $\mathcal{H}_1 \oplus \mathcal{H}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $(\langle \tilde{T}_2y, y \rangle) < 0$ for any unit vector $y$. Note that $T$ commutes with $H, G$. It follows that $H = H_1 \oplus H_2$ and $G = G_1 \oplus G_2$ acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$ so that $T^*T = T_1^*T_1 \oplus T_2^*T_2$ for $T_1 = H_1 + iG_1$ and $T_2 = H_2 + iG_2$. Clearly, $W(H_1, G_1, T_1^*T_1) \subseteq \mathcal{P}$ and $W(H_2, G_2, T_2^*T_2)$ are contained in one of the half space determined by $\mathcal{P}$. Identifying $DW(T_j) = W(H_j, G_j, T_j^*T_j)$ for $j = 1, 2$, we get the desired conclusion on $DW(T)$.

It is easy to verify the sufficiency of the theorem.

By Theorem 3.1, the study of points in $DW(T) \cap \partial DW(T)$ for a normal operator $T$ reduces to the study of points in $DW(T_1)$ such that $DW(T_1)$ is a subset of a plane in $\mathbb{C} \times \mathbb{R}$. In the following, we give a detailed analysis of an operator $A$ for which $DW(A)$ is a subset of a plane in $\mathbb{C} \times \mathbb{R}$. In particular, we give a description of $DW(A)$ in terms of $\sigma(A)$ and $\sigma_p(A)$.

Note that $DW(A) \subseteq \text{conv}\mathcal{P}$ for any $A \in \mathcal{B}(\mathcal{H})$, where

$$\mathcal{P} = \{(\xi, |\xi|^2) : \xi \in \mathbb{C}\}$$

(1)
is a paraboloid. Also, observe that if \( A, A' \in \mathcal{B}(\mathcal{H}) \) with \( A' = \alpha A + \beta I \), where \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \), then

\[
DW(A') = \{ (\alpha \mu + \beta, |\alpha|^2 r + 2\text{Re}(\alpha \bar{\beta} \mu) + |\beta|^2) : (\mu, r) \in DW(A) \}.
\]

So, \( DW(A') \) is the image of \( DW(A) \) under a real bijective affine transform. Clearly, there is also a one-to-one correspondence between \( \sigma_p(A') \) and \( \sigma_p(A) \). Moreover, the affine transform will establish a one-to-one correspondence between the boundary points of \( DW(A') \) and those of \( DW(A) \). Hence, replacing \( A \) by \( A' \) will not affect the hypothesis and conclusion of the results in the following discussion.

**Theorem 3.2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be normal. Then, \( DW(A) \) is a subset of a plane in \( \mathbb{C} \times \mathbb{R} \) if and only if one of the following holds:

(a) \( \mathcal{H} = \mu I \) so that \( DW(A) = \{ (\mu, |\mu|^2) \} \) is a singleton.

(b) \( \mathcal{H} \) has a closed subspace \( \mathcal{H}_1 \) such that \( A = \mu_1 I_{\mathcal{H}_1} \oplus \mu_2 I_{\mathcal{H}_2} = B(\mathcal{H}_1 \oplus \mathcal{H}_2^\perp) \) and

\[
DW(A) = \operatorname{conv} \{ (|\mu_1|^2), (|\mu_2|^2) \}.
\]

(c) \( \sigma(A) \) has more than two elements and there are \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \) such that \( \alpha A + \beta I \) is a self-adjoint operator and \( DW(A) \) is contained in a plane parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \).

(d) \( \sigma(A) \) has more than two elements and there are \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \) such that \( \alpha A + \beta I \) is a unitary operator and \( DW(A) \) is contained in a plane parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \).

In all the cases (a) – (d) we have

\[
DW(A) = \operatorname{int} \left( \operatorname{conv} \{ (\mu, |\mu|^2) : \mu \in \sigma(A) \} \right) \cup \operatorname{conv} \{ (\xi, |\xi|^2) : \xi \in \sigma_p(A) \}.
\]

**Proof.** Suppose (a) – (c) hold. Then,

\[
DW(A) \subseteq \operatorname{cl}(DW(A)) = \operatorname{conv} \{ (\mu, \mu^2) : \mu \in \sigma(A) \}
\]

is a subset of a plane in \( \mathbb{C} \times \mathbb{R} \) parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \). Suppose (d) holds. Then, the operator \( A' = \alpha A + \beta I \) satisfies \( \|A' x\| = 1 \) for all unit vectors \( x \in \mathcal{H} \). Thus, \( DW(A') \) is a subset of a plane parallel to the complex plane in \( \mathbb{C} \times \mathbb{R} \).

Since \( \alpha \neq 0 \) and \( \sigma(A') = \sigma(\alpha A + \beta I) \) has at least three elements not in a line, it follows from (2) that \( DW(A) \) is a subset of a plane not parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \).

Suppose \( DW(A) \) is a subset of a line or \( DW(A) \) is a subset of a plane parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \). Then, the projection of \( DW(A) \) to the first co-ordinate will be \( W(A) \) and is a subset of a straight line in \( \mathbb{C} \). Then there exist \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \) such that \( \alpha A + \beta I \) is self-adjoint. It follows that (a), (b) or (c) holds depending on \( \sigma(A) \) has one, two or more elements.

Now, suppose \( DW(A) \) is not a subset of a line, and \( DW(A) \subseteq P \), where \( P \) is not parallel to the line \( \{ (0, s) : s \in \mathbb{R} \} \) in \( \mathbb{C} \times \mathbb{R} \). Then there exist \( b, c \) and \( d \in \mathbb{R} \) such that for all \( (\mu_1 + i\mu_2, r) \in DW(A) \), we have

\[
r + 2(b\mu_1 + c\mu_2) = d.
\]

Since \( r \geq \mu_1^2 + \mu_2^2 \), we have

\[
d + (b^2 + c^2) = (r - (\mu_1^2 + \mu_2^2)) + (b + \mu_1)^2 + (c + \mu_2)^2 \geq 0.
\]
If $d + (b^2 + c^2) = 0$, then $DW(A')$ consists of one point $(-b - ic, b^2 + c^2)$ so that $A'$ is a scalar operator, which is a contradiction. Hence, $d + (b^2 + c^2) > 0$. Let $\alpha = 1/\sqrt{d + (b^2 + c^2)}$ and $\beta = (b + ic)/\sqrt{d + (b^2 + c^2)}$. Then for every $(\mu_1 + i\mu_2, r) \in DW(A)$, we have

$$|\alpha|^2 r + 2\text{Re}(\alpha \beta \mu) + |\beta|^2 = \frac{1}{d + (b^2 + c^2)} \left( r + 2(b\mu_1 + c\mu_2) + b^2 + c^2 \right) = 1.$$ 

Therefore, for $A' = \alpha A + \beta I$, we have

$$DW(A') \subseteq \{(\xi, 1) : \xi \in \mathbb{C} \} = P',$$

that is, $\|A'x\|^2 = 1$ for all unit vector $x \in \mathcal{H}_1$. Since $A$ is normal and so is $A'$, it follows that $A'$ is unitary.

Finally, we consider the equality

$$DW(A) = \text{int} (\text{conv} \{(\mu, |\mu|^2) : \mu \in \sigma(A)\}) \cup \text{conv} \{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.$$ 

Clearly, the equality is valid if (a) or (b) holds. The “$\supseteq$” inclusion is clear. To prove the reverse inclusion, we establish the following:

**Claim.** If

$$(\mu, r) \in DW(A) \setminus \text{int} (\text{cl} (DW(A))),$$

then

$$(\mu, r) \in \text{conv} \{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.$$ 

Suppose (c) holds. We may replace $A$ by $\alpha A + \beta I$ and assume that $A$ is self-adjoint. Then

$$DW(A) \subseteq \text{conv} \{(\mu, |\mu|^2) : \mu \in \sigma(A)\}$$

is a convex lamina in $\mathbb{R} \times [0, \infty)$. If $c$ and $d$ are the maximum and minimum of $\sigma(A)$, then the upper edge of the lamina equals $\text{conv} \{(c, |c|^2), (d, |d|^2)\}$. The points on this set may or may not lie in $DW(A)$ depending on whether $c, d \in \sigma_p(A)$. Similarly, we have to examine the lower edges or boundary curve of the lamina.

To establish the claim in this case, let $x \in \mathcal{H}$ be a unit vector such that $((Ax, x), \|Ax\|^2) = (\mu, r) \notin \text{int} (\text{cl} (DW(A)))$. If $r = \mu^2$, then by the Cauchy-Schwartz inequality, $Ax = \mu x$ and hence $\mu \in \sigma_p(A)$. Suppose $r \neq \mu^2$. Let $L$ be a support line of $DW(A)$ passing through $(\mu, r)$ and suppose $L$ intersects the parabola $P = \{(s, s^2) : s \in \mathbb{R}\}$ at $(\mu_1, |\mu_1|^2)$ and $(\mu_2, |\mu_2|^2)$. Clearly, $\mu_1, \mu_2, \mu$ are all distinct. We may replace $A$ by $A - (\mu_1 + \mu_2)I/2$ and assume that $\mu_1 + \mu_2 = 0$. We may further assume that $|\mu_1| = 1$. Otherwise, replace $A$ by $A/|\mu_1|$. Thus, we may assume that $L = \{(\xi, 1) : \xi \in \mathbb{R}\}$ is an upper edge or a lower edge of the convex lamina $DW(A)$ with $(\mu, r) = (\mu, 1) \in L$.

Consequently, $L$ is either the maximum or the minimum of $\sigma(A^*A)$.

Let $\mathcal{H}_0$ be the kernel of $A^*A - I$. Since $((Ax, x), \|Ax\|^2) = (\mu, 1)$, we see that $x \in \mathcal{H}_0$. Since $A$ is self-adjoint, we can further decompose $\mathcal{H}_0$ into the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$, which are the kernel of $A - I$ and $A + I$, respectively. Note that neither $\mathcal{H}_1$ nor $\mathcal{H}_2$ can be a zero space, otherwise, we cannot have $x \in \mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that
\langle Ax, x \rangle = \mu. Thus, \( A \) can be written as \( I_{H_1} \oplus -I_{H_2} \oplus A_0 \in B(H_1 \oplus H_2 \oplus H_0^+) \). Then

\[
(\mu, r) \in DW(I_{H_1} \oplus -I_{H_2}) = \text{conv } \{(1, 1), (-1, 1)\} \subseteq \text{conv } \{(|\xi|, |\xi|^2) : \xi \in \sigma_p(A)\}.
\]

Finally, suppose (d) holds. We may replace \( A \) by \( \alpha A + \beta I \) and assume that \( A \) is unitary. Hence, \( DW(A) \subseteq \{(\mu, 1) : \mu \in W(A)\} \), \( W(A) \) is a subset of the closed unit disk and \( \sigma(A) \) is a subset of the unit circle in \( \mathbb{C} \). Suppose \( (\mu, r) \notin \text{int} (\text{cl} (DW(A))) \). Then, there is a supporting line \( L \) on \( W(A) \) passing through \( \mu \). By Theorem 2.1, \( A = A_1 \oplus A_2 \), with \( \mu \in W(A_1) \). Note that \( DW(A_1) \subseteq DW(A) \subseteq \{(v, 1) : v \in W(A)\} \). Thus, \( DW(A_1) \) is a subset of a line segment passing through \( (\mu, 1) \). From the result in (b), we see that \( (\mu, 1) \in \text{conv } \{(v, 1) : v \in \sigma_p(A_1)\} \subseteq \text{conv } \{(|\xi|, |\xi|^2) : \xi \in \sigma_p(A)\} \).

Similar to Corollary 2.5, we have the following corollary for the Davis-Wielandt shell:

**Corollary 3.3.** Suppose \( A = d_1 I_{H_1} \oplus d_2 I_{H_2} \oplus \cdots \in B(H) \) such that \( H \) is an orthogonal sum of the closed subspaces \( H_1, H_2, \ldots \). Then

\[
DW(A) = \text{conv } \{(d_n, |d_n|^2) : n \geq 1\}.
\]

We can use the operators in Example 2.6 to illustrate our results on Davis-Wielandt shells.

**Example 3.4.** Let \( A, B, C \) be defined as in Example 2.6. Then,

\[
\partial DW(A) \cap DW(A) = \left( \bigcup_{n \geq 1} \text{conv } \{(i, 1), (r_n, r_n^2)\} \right) \cup \text{conv } \{(r_n, r_n^2) : n \geq 1\},
\]

\[
\partial DW(B) \cap DW(B) = \{(i, 1)\} \cup \{(d_n, d_n^2) : n \geq 1\},
\]

and

\[
\partial DW(C) \cap DW(C) = \{(i, 1)\} \cup \{(d_n, d_n^2) : n \geq 1\} \cup \{(\mu, r) : \mu \in (0, 1), \mu^2 < r < \mu\}.
\]

By Corollary 3.3, we have

\[
DW(X) = \text{conv } \{(\mu, |\mu|^2) : \mu \in \sigma_p(X)\}
\]

for \( X = A, B, C \), and

\[
DW(C) = \text{conv } \{DW(B) \cup DW(M)\} = \text{conv } \{(\mu, |\mu|^2) : \mu \in \sigma_p(C)\} \cup \{(\mu, r) : \mu \in (0, 1), \mu^2 < r < \mu\}.
\]

Recall that \( \partial W(A) \cap W(A) = \partial W(C) \cap W(C) = \{i\} \cup (0, 1) \). It is clear that the boundary structure of \( DW(A) \) can provide more information of \( A \) than \( W(A) \). In particular, we have

\[
\sigma(A) = \{\mu \in \mathbb{C} : (\mu, |\mu|^2) \in \partial DW(A)\}
\]
and

$$\sigma_p(A) = \{ \mu \in \mathbb{C} : (\mu, |\mu|^2) \in DW(A) \}.$$ 

Note that the analog of Corollary 2.9 does not hold for the Davis-Wielandt shell. In particular, the operator C in the above example acts on a separable Hilbert space and (DW(C)) has uncountably many extreme point lying in DW(C).

4. Joint numerical ranges. Inspired by the comments of the referee on an early version of the paper, we see that our results on the numerical range and the Davis-Wielandt shell can be further extended to the joint numerical range $W(A_1, \ldots, A_m)$ of mutually commuting operators $A_1, \ldots, A_m \in B(\mathcal{H})$ defined as the set of $(a_1, \ldots, a_m) \in \mathbb{C}^m$ with

$$a_j = \langle A_j x, x \rangle \quad \text{for } j = 1, \ldots, m,$$

for some unit vector $x \in \mathcal{H}$; see [2, 8, 11] and references there in. While $W(A)$ and $DW(A)$ are useful for studying an operator $A$, the joint numerical range $W(A_1, \ldots, A_m)$ is useful in studying the joint behavior of the operators $A_1, \ldots, A_m$. Suppose $A_j = H_j + iG_j$ for $j = 1, \ldots, m$, then $W(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ can be identified with $W(H_1, G_1, \ldots, H_m, G_m) \subseteq \mathbb{R}^{2m}$. So, we can focus on the joint numerical ranges of self-adjoint operators $A_1, \ldots, A_m \in B(\mathcal{H})$. Define the joint approximate point spectrum $\sigma_\pi(A_1, \ldots, A_m)$ to be the set of points $(a_1, \ldots, a_m)$ such that $\sum_{j=1}^m \| (A_j - a_j I) x_n \| \to 0$ for a sequence $\{x_n\}$ of unit vector in $\mathcal{H}$. It is known that

$$\text{cl} (W(A_1, \ldots, A_m)) = \text{conv} \sigma_\pi(A_1, \ldots, A_m)$$

if $A_1, \ldots, A_m \in B(\mathcal{H})$ are mutually commuting self-adjoint operators; see [1, Corollary 36.11] and [11].

Suppose $B_1, \ldots, B_m \in B(\mathcal{H})$ are mutually commuting self-adjoint operators. If the real linear span of $iH_1, B_1, \ldots, B_m$ has dimension $k \leq m$, then $W(B_1, \ldots, B_m)$ is a subset of a $(k-1)$-dimensional hyperplane in $\mathbb{R}^m$, that is,

$$W(B_1, \ldots, B_m) \subseteq (b_1, \ldots, b_m) + V$$

for a $(k-1)$-dimensional subspace $V$ of $\mathbb{R}^m$. We can extend Theorem 2.1 and Theorem 3.1 (and their proofs) to the following:

**Theorem 4.1.** Suppose $A_1, \ldots, A_m \in B(\mathcal{H})$ are mutually commuting self-adjoint operators. Then $(a_1, \ldots, a_m) \in W(A_1, \ldots, A_m) \cap \partial W(A_1, \ldots, A_m)$ if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $A_j = B_j \oplus C_j$ for $j = 1, \ldots, m$ such that $(a_1, \ldots, a_m) \in W(B_1, \ldots, B_m) \subseteq P$ for a hyperplane in $\mathbb{R}^m$ and $W(C_1, \ldots, C_m) \cap P = \emptyset$.

Similar to the study in Sections 2 and 3, one may analyze the geometric structure of $W(B_1, \ldots, B_m)$ in connection to the algebraic structure of $B_1, \ldots, B_m$ in Theorem 4.1. If the boundary point $(a_1, \ldots, a_m)$ of $W(A_1, \ldots, A_m)$ lies in the relative interior of $W(B_1, \ldots, B_m)$, then not much can be said. Otherwise, we can apply the theorem again to further decompose $B_j$ into the direct sum of two operators for $j = 1, \ldots, m$. If this procedure can be repeated until we have $(a_1, \ldots, a_m) \in W(\tilde{B}_1, \ldots, \tilde{B}_m)$ so that $W(\tilde{B}_1, \ldots, \tilde{B}_m)$ lies on a hyperplane of dimension 0 or 1, then we can apply Theorem 3.2 to conclude that each $\tilde{B}_j$ is a scalar operator, or $\tilde{B}_j = \mu_j I \oplus \nu_j F$ with $a_j \in (\mu_j, \nu_j)$.
for all \( j = 1, \ldots, m \). Of course, in the latter case, \((a_1, \ldots, a_m)\) is again in the relative interior of \(W(\tilde{B}_1, \ldots, \tilde{B}_m)\). Summarizing the above discussion, we have the following:

**Proposition 4.2.** Under the hypotheses of Theorem 4.1. If \((a_1, \ldots, a_m) \in W(A_1, \ldots, A_m)\) is a boundary point, then \(B_1, \ldots, B_m\) can be chosen so that one of the following holds:

(a) \((a_1, \ldots, a_m)\) is in the relative interior of \(W(B_1, \ldots, B_m)\).

(b) \(B_j = a_jI\) for \( j = 1, \ldots, m \). This case holds if and only if \((a_1, \ldots, a_m)\) is an extreme point in \(W(A_1, \ldots, A_m)\).

Statement (b) of the above theorem is the main theorem in [8]. Similar to Corollary 2.9, we have the following:

**Corollary 4.3.** Let \( S \) be a bounded and convex subset of \( \mathbb{R}^m \). Then there exist a separable Hilbert space \( \mathcal{H} \) and mutually commuting self-adjoint operators \( A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H}) \) such that \( S = W(A_1, \ldots, A_m) \) if and only if \( S \cap \text{Ext(cl}(S)) \) is countable.

Note that one may sometimes use the joint numerical range to study \( DW(A) \) as in our proof of Theorem 3.1. But one cannot just treat \( DW(A) \) as a special case of the joint numerical range. For instance, one can extend Corollary 2.9 to the joint numerical range (Corollary 4.3) but not to the Davis-Wielandt shell (as noted at the end of Section 3). In this connection, it would be interesting to characterize those bounded convex sets in \( \mathbb{R}^3 \) that can be realized as \( DW(A) \) for a normal operator \( A \) acting on a separable Hilbert space.

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**REFERENCES**