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The joint essential numerical range of operators: convexity and related results

by

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Abstract. Let $W(A)$ and $W_e(A)$ be the joint numerical range and the joint essential numerical range of an $m$-tuple of self-adjoint operators $A = (A_1, \ldots, A_m)$ acting on an infinite-dimensional Hilbert space. It is shown that $W_e(A)$ is always convex and admits many equivalent formulations. In particular, for any fixed $i \in \{1, \ldots, m\}$, $W_e(A)$ can be obtained as the intersection of all sets of the form

$$\text{cl}(W(A_1, \ldots, A_{i+1}, A_i + F, A_{i+1}, \ldots, A_m)),$$

where $F = F^*$ has finite rank. Moreover, the closure $\text{cl}(W(A))$ of $W(A)$ is always star-shaped with the elements in $W_e(A)$ as star centers. Although $\text{cl}(W(A))$ is usually not convex, an analog of the separation theorem is obtained, namely, for any element $d \notin \text{cl}(W(A))$, there is a linear functional $f$ such that $f(d) > \sup\{f(a) : a \in \text{cl}(W(\tilde{A})))\}$, where $\tilde{A}$ is obtained from $A$ by perturbing one of the components $A_i$ by a finite rank self-adjoint operator. Other results on $W(A)$ and $W_e(A)$ extending those on a single operator are obtained.

1. Introduction. Let $B(H)$ denote the algebra of bounded linear operators acting on a complex Hilbert space $H$. The numerical range of $A \in B(H)$ is defined as

$$W(A) = \{\langle Ax, x \rangle : x \in H, \langle x, x \rangle = 1\},$$

which is useful in studying operators; see [10, 11, 22, 24] and [25, Chapter 1]. Let $S(H)$ denote the set of self-adjoint operators in $B(H)$. Since every $A \in B(H)$ admits a decomposition $A = A_1 + iA_2$ with $A_1, A_2 \in S(H)$, we can identify $W(A)$ with

$$\{(\langle A_1 x, x \rangle, \langle A_2 x, x \rangle) : x \in H, \langle x, x \rangle = 1\} \subseteq \mathbb{R}^2.$$

This leads to the joint numerical range of $A = (A_1, \ldots, A_m) \in S(H)^m$,

$$W(A) = \{(\langle A_1 x, x \rangle, \ldots, \langle A_m x, x \rangle) : x \in H, \langle x, x \rangle = 1\} \subseteq \mathbb{R}^m,$$
which has been studied by many researchers in order to understand the joint behavior of several operators \(A_1, \ldots, A_m\). One may see [1, 5, 12, 14, 15, 16, 19, 23, 28, 31, 33, 35] and their references for the background and many applications of the joint numerical range.

Let \(\mathcal{F}(\mathcal{H})\) and \(\mathcal{K}(\mathcal{H})\) be the sets of finite rank and compact operators in \(\mathcal{B}(\mathcal{H})\). In the study of finite rank or compact perturbations of operators, researchers consider the **joint essential numerical range** of \(A \in \mathcal{S}(\mathcal{H})^m\) defined by

\[
W_e(A) = \bigcap \{\text{cl}(W(A + K)) : K = (K_1, \ldots, K_m) \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}.
\]

Here \(\text{cl}(S)\) denotes the closure of the set \(S\). For \(m = 2\), \(W_e(A)\) can be identified with the **essential numerical range** of \(A = A_1 + iA_2 \in \mathcal{B}(\mathcal{H})\), defined by

\[
W_e(A) = \bigcap \{\text{cl}(W(A + K)) : K \in \mathcal{K}(\mathcal{H})\}.
\]

One may see [2, 3, 6, 7, 13, 18, 20, 21, 26, 27, 30, 32, 36, 37] and their references for the background and many interesting results on \(W_e(A)\) and \(W_e(A)\).

In theoretical studies as well as applications, it is desirable to deal with \(A\) such that \(W(A)\) or \(\text{cl}(W(A))\) is convex. For example, if \(\text{cl}(W(A))\) is convex, one can apply the separation theorem to show that \(0 \notin \text{cl}(W(A))\) if and only if there exist \(r > 0\) and \(c = (c_1, \ldots, c_m) \in \mathbb{R}^m\) such that \((\sum_{i=1}^m c_i A_i) > rI_{\mathcal{H}}\). Unfortunately, \(\text{cl}(W(A))\) is not always convex. Here are some results concerning the convexity of \(W(A)\) and \(\text{cl}(W(A))\), and related to \(W_e(A)\) (for example, see [5, 10, 11, 36, 21, 29, 31] and their references).

1. **(P1)** [31] \(W(A_1, \ldots, A_m)\) is convex if
   (a) \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 3, or
   (b) \(\dim \mathcal{H} \geq 3\) and \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 4.

2. **(P2)** [31] For any \(A_1, A_2, A_3 \in \mathcal{S}(\mathcal{H})\) such that \(\text{span}\{I, A_1, A_2, A_3\}\) has dimension 4, there is always an \(A_4 \in \mathcal{S}(\mathcal{H})\) for which \(W(A_1, \ldots, A_4)\) is not convex.

3. **(P3)** [31] If \(m \geq 4\) then there exists \(A \in \mathcal{S}(\mathcal{H})^m\) such that \(W(A)\) is non-convex.

4. **(P4)** For any positive integer \(m\) and any \(A \in \mathcal{S}(\mathcal{H})^m\), \(W_e(A)\) is a compact set contained in \(W(A)\). If \(\text{span}\{I, A_1, \ldots, A_m\}\) has dimension at most 4, then \(W_e(A)\) is convex.

5. **(P5)** [36] For \(S \subseteq \mathbb{R}^m\), let \(\text{Ext}(S)\) be the set of all points in \(S\) that do not lie in the open line segment joining two distinct points in \(S\). Then \(\text{Ext}(\text{cl}(W(A))) \subseteq \text{Ext}(W(A)) \cup \text{Ext}(W_e(A))\).

We remark that (P1)–(P3) also hold if we replace \(W(A)\) by \(\text{cl}(W(A))\). In view of (P2) and (P3), if \(m > 3\), then for \(A \in \mathcal{S}(\mathcal{H})^m\) and \(K \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\) the set \(\text{cl}(W(A + K))\) is usually non-convex. Since \(W_e(A)\)
is the intersection of non-convex sets, one does not expect the set $W_e(A)$ to be convex. This might be the reason why the convexity of $W_e(A)$ is seldom discussed for $m > 3$. In fact, some researchers have studied different geometrical properties of $W_e(A)$ under the assumption that $W_e(A)$ is convex, and some have examined $W_e(A)$ for different classes of operators without discussing their convexity; for example, see [6, 26, 27, 30, 32].

In this paper, we prove the rather unexpected result that $W_e(A)$ is always convex. Moreover, it is shown that the closure $\text{cl}(W(A))$ of $W(A)$ is always star-shaped with the elements in $W_e(A)$ as star centers. Many results relating $W_e(A)$ and $W(A)$ are also obtained. Our paper is organized as follows.

In Section 2, we extend the results of [21] by establishing several equivalent formulations of the essential joint numerical range for $A \in S(\mathcal{H})^m$. One key obstacle for such an extension is the fact that $W(A)$ may not be convex. To get around this problem, we show that $\text{cl}(W(A))$ is star-shaped. The star-shapedness of $\text{cl}(W(A))$ and the conditions equivalent to membership in $W_e(A)$, given in Section 2, lead to our main result that $W_e(A)$ is convex and its elements are star centers of the set $\text{cl}(W(A))$, which is presented in Section 3. With the convexity theorem, we obtain additional descriptions of $W_e(A)$ in Section 4 in terms of the perturbations of one of the components of $A$, and also in terms of linear combinations of the components of $A$. For example, we show that $W_e(A_1, \ldots, A_m)$ is equal to the sets

$$\bigcap \{\text{cl}(W(A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m) : F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H})\}$$

and

$$\{(a_1, \ldots, a_m) : \sum_{j=1}^m c_j a_j \in W_e\left(\sum_{j=1}^m c_j A_j\right) \text{ for all } (c_1, \ldots, c_m) \in \Omega\},$$

where $\Omega = \{(c_1, \ldots, c_m) \in \mathbb{R}^m : \sum_{j=1}^m c_j^2 = 1\}$. Also, we obtain an analog of the separation theorem for the not necessarily convex set $\text{cl}(W(A))$, namely, for any element $d \notin \text{cl}(W(A))$, there is a linear functional $f$ such that $f(d) > \sup\{f(a) : a \in \text{cl}(W(\hat{A}))\}$, where $\hat{A}$ is obtained from $A$ by perturbing one of the components $A_j$ by a finite rank self-adjoint operator. In Section 5, we present additional results on $W(A)$ and $W_e(A)$. For instance, $W_e(A) = \text{cl}(W(A))$ if and only if the extreme points of $W(A)$ are contained in $W_e(A)$; the convex hull of $\text{cl}(W(A))$ can always be realized as the joint essential numerical range of $(\hat{A}_1, \ldots, \hat{A}_m)$ for linear operators $\hat{A}_1, \ldots, \hat{A}_m$ acting on a separable Hilbert space.

In our discussion, we always assume that $\mathcal{H}$ is infinite-dimensional. For any vector $x \in \mathcal{H}$ and $A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m$, we will use the notation

$$\langle Ax, x \rangle = (\langle A_1 x, x \rangle, \ldots, \langle A_m x, x \rangle).$$
Furthermore, $\mathbb{R}^m$ will be used to denote the inner product space of $1 \times m$ real vectors with the usual inner product $\langle x, y \rangle$.

2. Equivalent conditions for $W_e(A)$. Following [21, Theorem 5.1] and its corollary on a single operator $A \in \mathcal{B}(\mathcal{H})$, we obtain several conditions equivalent to membership in $W_e(A)$.

**Theorem 2.1.** Let $A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. The following conditions are equivalent for a real vector $a = (a_1, \ldots, a_m)$:

1. $a \in W_e(A) = \bigcap \{ \text{cl}(W(A + K)) : K \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \}$. 
2. $a \in \bigcap \{ \text{cl}(W(A + F)) : F \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \}$. 
3. There is an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of vectors such that $\lim_{n \to \infty} \langle Ax_n, x_n \rangle = a$.
4. There is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of unit vectors converging weakly to $0$ in $\mathcal{H}$ such that $\lim_{n \to \infty} \langle Ax_n, x_n \rangle = a$.
5. There is an infinite-dimensional projection $P \in \mathcal{S}(\mathcal{H})$ such that $P(A_j - a_j I)P \in \mathcal{K}(\mathcal{H})$ for $j = 1, \ldots, k$.

Most of the argument in [21] can be applied here except for one crucial step, where the convexity of $W(A)$ for $m = 2$ is needed. Since $W(A)$ may not be convex for $m > 3$, we need the following auxiliary result to overcome the obstacle. As a byproduct, it shows that $\text{cl}(W(A))$ is star-shaped.

**Theorem 2.2.** Let $A$ satisfy the hypothesis of Theorem 2.1, and let $W_3(A)$ be the set of real vectors $a$ satisfying condition (3) of Theorem 2.1. Then $W_3(A)$ is non-empty and closed. Moreover, each element $a \in W_3(A)$ is a star center of $\text{cl}(W(A))$, i.e., for any $b \in \text{cl}(W(A))$ we have $(1 - t)a + tb \in \text{cl}(W(A))$ for all $0 \leq t \leq 1$.

**Proof.** To prove that $W_3(A)$ is non-empty, let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal sequence of vectors in $\mathcal{H}$. Then the sequence $\{\langle Ax_n, x_n \rangle\}_{n=1}^{\infty}$ is bounded. By choosing a subsequence if necessary, we can assume that $\langle Ax_n, x_n \rangle$ converges. Hence, $W_3(A)$ is non-empty.

Next, we show that $W_3(A)$ is closed. Suppose $a \in \text{cl}(W_3(A))$. Then for each $n \geq 1$, there exists an orthonormal sequence $\{x_k^n\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} \langle Ax_k^n, x_k^n \rangle = a^n \in \mathbb{R}^m$ and $\lim_{n \to \infty} a^n = a$.

Let $\delta_n = 1/(4n^2)$. By going to subsequences if necessary, we may assume that $\|\langle Ax_k^n, x_k^n \rangle - a\| < \delta_n$ for all $n, k$. We may also assume that $\|A_1\|^2 + \cdots + \|A_m\|^2 \leq 1$. Then $\|\langle Ax, y \rangle\| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{H}$.
Choose \( x_1 = x_1^1 \). Then \( \| (Ax_1, x_1) - a \| < 1 \). Suppose we have chosen \( \{x_1, \ldots, x_n\} \) orthonormal with \( \| (Ax_k, x_k) - a \| < 1/k \) for \( 1 \leq k \leq n \). Then choose \( N \) such that for all \( 1 \leq k \leq n \),

\[
\| (x_k, x_k^{n+1}) \|, \| (Ax_k, x_k^{n+1}) \| < \delta_{n+1}.
\]

Let \( y = x_N^{n+1} - \sum_{k=1}^n (x_N^{n+1}, x_k)x_k \). Then

\[
\| y - x_N^{n+1} \| \leq n \delta_{n+1}, \quad \text{so} \quad 1 - n \delta_{n+1} \leq \| y \| \leq 1 + n \delta_{n+1}.
\]

Therefore,

\[
\| (Ay, y) - a \| \leq \| (A(y - x_N^{n+1}), y) \| + \| (Ax_N^{n+1}, y - x_N^{n+1}) \| + \| (Ax_N^{n+1}, x_N^{n+1}) - a \|
\]

\[
\leq \| y - x_N^{n+1} \| (\| y \| + \| x_N^{n+1} \|) + \delta_{n+1} \leq (2n + 2) \delta_{n+1}.
\]

Let \( x_{n+1} = y/\| y \| \). Then

\[
\| x_{n+1} - y \| = |1 - \| y \| | \leq n \delta_{n+1}.
\]

Hence, \( \{x_1, \ldots, x_n, x_{n+1}\} \) is an orthonormal set and

\[
\| (Ax_{n+1}, x_{n+1}) - a \| \leq \| y - x_{n+1} \| (\| y \| + \| x_{n+1} \|) + (2n + 2) \delta_{n+1}
\]

\[
\leq (4n + 3) \delta_{n+1} < 1/(n + 1).
\]

To prove the last assertion, let \( a \in W_3(A) \) and \( b \in \text{cl}(W(A)) \). Suppose \( \{x_n\} \) is an orthonormal sequence in \( \mathcal{H} \) such that \( (Ax_n, x_n) \to a \). For \( 0 \leq t \leq 1 \), we are going to show that \( (1 - t)a + tb \in \text{cl}(W(A)) \). Given \( \varepsilon > 0 \), let \( y \) be a unit vector in \( \mathcal{H} \) such that \( \| (Ay, y) - b \| < \varepsilon \). Choose \( n \) such that \( \| (Ax_n, x_n) - a \| < \varepsilon \) and \( \| (Ay, x_n) \| < \varepsilon \). Choose \( \theta \in \mathbb{R} \) such that \( (e^{i\theta}y, x_n) \) is imaginary. Let \( z = \sqrt{t} e^{i\theta} y + \sqrt{1-t} x_n \). Then

\[
(z, z) = t(y, y) + (1 - t)(x_n, x_n) + 2\sqrt{t(1-t)}(\langle e^{i\theta}y, x_n \rangle + \langle x_n, e^{i\theta}y \rangle) = 1
\]

and

\[
\| (Az, z) - ((1 - t)a + tb) \| \leq (1-t)\| (Ax_n, x_n) - a \| + t\| (Ay, y) - b \|
\]

\[
+ \sqrt{t} \sqrt{1-t} \| (e^{i\theta}Ay, x_n) + (Ax_n, e^{i\theta}y) \| \leq 2 \varepsilon.
\]

Therefore, \( (1 - t)a + tb \in \text{cl}(W(A)) \).

The referee indicated that \( W_3(A) \) is clearly closed, and a short proof is possible. We include a detailed proof for the sake of completeness and easy reference.

**Proof of Theorem 2.1.** For \( j = 2, 3, 4, 5 \), let \( W_j(A) \) be the set of \( a \) satisfying condition \( (j) \). Clearly, we have

\[
W_5(A) \subseteq W_3(A) \subseteq W_4(A) \subseteq W_2(A) \subseteq W_2(A).
\]

Suppose \( a \in W_2(A) \). We are going to show that \( a \in W_5(A) \). Without loss of generality, we may assume \( a = 0 \).
Since $0 \in W_2(A) \subseteq \text{cl}(W(A))$, there exists a unit vector $x_1 \in H$ such that $\|\langle Ax_1, x_1 \rangle\| < 1/2$. Suppose we have an orthonormal set $\{x_1, \ldots, x_n\}$ such that $\|\langle Ax_n, x_n \rangle\| < 1/2^n$. Let $Q$ be the orthogonal projection of $H$ onto the subspace $S$ spanned by $x_1, \ldots, x_n$ and let
\[
B = ((I - Q)A_1(I - Q)|_{S^\perp}, \ldots, (I - Q)A_m(I - Q)|_{S^\perp}).
\]
Let $b = (b_1, \ldots, b_m) \in W_3(B)$ and $bI_S = (b_1I_S, \ldots, b_mI_S)$. Then for $Q = I - Q$, we have
\[
bI_S \oplus B = (b_1Q + \overline{Q}A_1\overline{Q}, \ldots, b_mQ + \overline{Q}A_m\overline{Q}) = A + F
\]
for some $F \in \mathcal{F}(H)^m \cap S(H)^m$. Therefore, $0 \in \text{cl}(W(bI_S \oplus B))$. Hence, there exists a unit vector $x \in H$ such that $\|\langle (A + F)x, x \rangle\| < 1/2^{n+2}$. Let $x = y + z$, where $y \in S$ and $z \in S^\perp$. Then $\|y\|^2 + \|z\|^2 = \|x\|^2 = 1$. If $z = 0$, then $\langle (A + F)x, x \rangle = b \in W_3(B) \subseteq \text{cl}(W(B))$. If $z \neq 0$, then by Theorem 2.2, we have
\[
\langle (A + F)x, x \rangle = \|y\|^2b + \|z\|^2\langle B(z/\|z\|), z/\|z\| \rangle \in \text{cl}(W(B)).
\]
So there exists a unit vector $x_{n+1} \in S^\perp$ such that
\[
\|\langle (A + F)x, x \rangle - \langle Bx_{n+1}, x_{n+1} \rangle\| < \frac{1}{2^{n+2}},
\]
and hence
\[
\|\langle Ax_{n+1}, x_{n+1} \rangle\| = \|\langle Bx_{n+1}, x_{n+1} \rangle\| < \frac{1}{2^{n+1}},
\]
because $\langle Fx_{n+1}, x_{n+1} \rangle = 0$. Inductively, we can choose an orthonormal sequence $\{x_n\}_{n=1}^\infty$ such that
\[
(1) \quad \|\langle Ax_n, x_n \rangle\| < \frac{1}{2^n} \quad \text{for all } n \geq 1.
\]
Let $n_1 = 1$. For every $1 \leq i \leq m$, we have
\[
\sum_{n=1}^\infty |\langle A_ix_n, x_n \rangle|^2 \leq \|A_ix_n\|^2 \quad \text{and} \quad \sum_{n=1}^\infty |\langle A_ix_n, x_{n_1} \rangle|^2 \leq \|A_i^*x_{n_1}\|^2.
\]
Hence, there exists $n_2 > n_1$ such that
\[
\sum_{n=n_2}^\infty |\langle A_ix_n, x_n \rangle|^2 < \frac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^\infty |\langle A_ix_n, x_{n_1} \rangle|^2 < \frac{1}{2}
\]
for all $1 \leq i \leq m$. Repeating this procedure, we get a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that for all $1 \leq i \leq m$, we have
\[
(2) \quad \sum_{n=n_{k+1}}^\infty |\langle A_ix_{n_k}, x_n \rangle|^2 < \frac{1}{2^k} \quad \text{and} \quad \sum_{n=n_{k+1}}^\infty |\langle A_ix_n, x_{n_k} \rangle|^2 < \frac{1}{2^k}.
\]
Formulas (1) and (2) imply that

\[
\sum_{k,l=1}^{\infty} |\langle A_i x_{n_k}, x_{n_l} \rangle|^2 < \infty.
\]

Let \( P \) be the orthogonal projection onto the subspace spanned by \( \{x_{n_k}\}_{k=1}^{\infty} \).
Then it follows from (3) that \( PA_i P \) is compact for all \( 1 \leq i \leq m \).

\[ \textbf{3. Convexity and star-shapedness} \]

\textbf{Theorem 3.1.} Let \( A \in \mathcal{S}(\mathcal{H})^m \). Then \( W_e(A) \) is a compact convex subset of \( \text{cl}(W(A)) \). Moreover, each element in \( W_e(A) \) is a star center of the star-shaped set \( \text{cl}(W(A)) \).

\textbf{Proof.} Because \( W_e(A) \) is the intersection of compact sets, it is compact. To prove the convexity, let \( a, b \in W_e(A) \) and \( 0 \leq t \leq 1 \). Then for every \( F \in \mathcal{F}(\mathcal{H})^m \cap S(\mathcal{H})^m \), we have \( a \in W_e(A) = W_e(A + F) \) and \( b \in W_e(A) \subseteq \text{cl}(W(A + F)) \). So, by Theorem 2.2, we have \( ta + (1 - t)b \in \text{cl}(W(A + F)) \). Hence,

\[
ta + (1 - t)b \in \bigcap \{\text{cl}(W(A + F)) : F \in \mathcal{F}(\mathcal{H})^m \cap S(\mathcal{H})^m \} = W_e(A).
\]

By Theorems 2.1 and 2.2, we have the last assertion.

Note that \( W_e(A) \cap W(A) \) may be empty. For example, if

\[
A = \text{diag}(1, 1/2, 1/3, \ldots)
\]

acts on \( \ell^2 \), then \( W_e(A) = \{0\} \) and \( W(A) = (0, 1] \). One may wonder whether a point \( a \in W_e(A) \cap W(A) \) is a star center of \( W(A) \). This is not true, as shown by the example below. Moreover, the example shows that for \( m \geq 4 \) there exists \( A \in S(\mathcal{H})^m \) such that \( \text{cl}(W(A)) \) is convex whereas \( W(A) \) is not. Of course, this is impossible for \( m \leq 3 \) as \( W(A) \) is always convex.

\textbf{Example 3.2.} Consider \( \mathcal{H} = \ell^2 \) with canonical basis \( \{e_n : n \geq 1\} \). Let \( A = (A_1, \ldots, A_4) \) with

\[
A_1 = \text{diag}(1, 0, 1/3, 1/4, \ldots), \quad A_2 = \text{diag}(1, 0) \oplus 0,
\]

\[
A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0, \quad A_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0.
\]

Then \( (1, 1, 0, 0) \in W(A) \) and \( (0, 0, 0, 0) \in W(A) \cap W_e(A) \), but \( (1/2, 1/2, 0, 0) \notin W(A) \). Hence, \( W(A) \) is not convex. However, \( \text{cl}(W(A)) \) is convex.

\textbf{Proof.} Note that \( (1, 1, 0, 0) = \langle Ae_1, e_1 \rangle \in W(A) \) and

\[
(0, 0, 0, 0) = \langle Ae_2, e_2 \rangle = \lim_{n \to \infty} \langle Ae_n, e_n \rangle \in W(A) \cap W_e(A).
\]
To show that \((1/2,1/2,0,0) \notin W(A)\), consider a unit vector \(x = \sum x_je_j\) such that \(\sum_{n=1}^{\infty} |x_n|^2 = 1\). If \(\langle A_1x, x \rangle = \langle A_2x, x \rangle = 1/2\), then
\[
|x_1|^2 + \sum_{n=3}^{\infty} |x_n|^2/n = |x_1|^2 = 1/2.
\]
Thus, \(x_n = 0\) for all \(n \geq 3\) and \(|x_1|^2 = |x_2|^2 = 1/2\). It then follows that \((\langle A_3x, x \rangle, \langle A_4x, x \rangle) \neq (0,0)\). This proves that \((1/2,1/2,0,0) \notin W(A)\).

Hence, \((0,0,0,0) \in W_e(A) \cap W(A)\) is not a star center of \(W(A)\), and \(W(A)\) is not convex.

To see that \(\text{cl}(W(A))\) is convex, note that \(0 \in W_e(A)\). Thus, by Theorem 3.1, for every \(b \in \text{cl}(W(A))\) we have \(t0 + (1-t)b \in \text{cl}(W(A))\) for any \(t \in [0,1]\).

Let \(B = (B_1, B_2, B_3, B_4)\), where
\[B_1 = \text{diag}(0,1,0), \quad B_2 = \text{diag}(0,1,0),\]
\[B_3 = [0] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_4 = [0] \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\]
and \(C = (C_1, C_2, C_3, C_4)\), where \(C_1 = \text{diag}(1/3,1/4,\ldots) \oplus [0], \quad C_2 = C_3 = C_4 = \text{diag}(0,0,\ldots) \oplus [0]\). Then it is easy to verify that
\[W(B) = \{(r,r,s,t) \in \mathbb{R}^4 : 4(r-1/2)^2 + s^2 + t^2 \leq 1\}\]
and
\[W(C) = \{(c,0,0,0) : c \in [0,1/3]\}\]
are both compact and convex. Hence, \(W(B \oplus C) = \text{conv}(W(B) \cup W(C))\) is compact and convex and
\[W(A) \subseteq W(B \oplus C) \Rightarrow \text{cl}(W(A)) \subseteq W(B \oplus C).\]

On the other hand, \(B \oplus C = [0] \oplus A \oplus [0]\). Therefore,
\[W(B \oplus C) = \{t0 + (1-t)b : b \in W(A)\} \subseteq \text{cl}(W(A)).\]
So, \(\text{cl}(W(A)) = W(B \oplus C)\) is convex. 

4. Other descriptions of \(W_e(A)\). For \(c = (c_1,\ldots,c_m) \in \mathbb{R}^m\) and \(A = (A_1,\ldots,A_m) \in S(\mathcal{H})^m\), let \(c \cdot A = \sum_{i=1}^{m} c_i A_i\). Using the convexity of \(W_e(A)\), we obtain additional conditions equivalent to membership in \(W_e(A)\) in terms of \(c \cdot A \in S(\mathcal{H})\) so that the joint behavior of \(A_1,\ldots,A_m\) can be understood from their linear combinations. For \(A \in S(\mathcal{H})\) and a positive integer \(k\), let
\[\lambda_k(A) = \inf\{\max \sigma(A + F) : F \in S(\mathcal{H}) \text{ with } \text{rank}(F) < k\}.\]

**Theorem 4.1.** Let \(A \in S(\mathcal{H})^m\) and \(a = (a_1,\ldots,a_m) \in \mathbb{R}^m\). Then \(a \in W_e(A)\) if and only if any one (and hence all) of the following conditions holds:
we have the following analog of the separation theorem for a convex set.

(1) For every \( c \in \mathbb{R}^m \), \( c \cdot a \in W_e(c \cdot A) \).

(2) For every \( c \in \mathbb{R}^m \), \( c \cdot a \in \bigcap \{ \text{cl}(W(c \cdot A + F)) : F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \} \).

(3) For every \( c \in \mathbb{R}^m \), there is an orthonormal sequence \( \{ x_n \}_{n=1}^\infty \subset \mathcal{H} \) such that

\[
\lim_{n \to \infty} \langle c \cdot Ax_n, x_n \rangle = c \cdot a.
\]

(4) For every \( c \in \mathbb{R}^m \), there is a sequence \( \{ x_n \}_{n=1}^\infty \subset \mathcal{H} \) of unit vectors such that \( \{ x_n \}_{n=1}^\infty \) converges weakly to \( 0 \) in \( \mathcal{H} \) and

\[
\lim_{n \to \infty} \langle c \cdot Ax_n, x_n \rangle = c \cdot a.
\]

(5) For every \( c \in \mathbb{R}^m \), there is an infinite-dimensional projection \( P \in S(\mathcal{H}) \) such that \( P(c \cdot A - c \cdot aI)P \in \mathcal{K}(\mathcal{H}) \).

(6) For every \( c \in \mathbb{R}^m \) and \( k \geq 1 \), \( \lambda_k(c \cdot A - c \cdot aI) \geq 0 \).

Proof. By the convexity of \( W_e(A) \), we can apply the separation theorem to Theorem 2.1 to show that \( a \in W_e(A) \) if and only if any one of the conditions (1) to (5) holds.

To prove the equivalence of condition (6), suppose \( a \in \mathbb{R}^m \). Without loss of generality, we may assume that \( a = 0 \). Suppose \( 0 \) satisfies condition (6). Then for every \( c \in \mathbb{R}^m \) and \( F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \) with \( \text{rank}(F) = k \), we have

\[
\lambda_1(c \cdot A + F) \geq \lambda_{k+1}(c \cdot A) \geq 0 \quad \text{and} \quad \lambda_1(-(c \cdot A + F)) \geq \lambda_{k+1}(-c \cdot A) \geq 0.
\]

Hence, \( c \cdot 0 = 0 \in \text{cl}(W(c \cdot A + F)) \). Therefore, condition (2) is satisfied.

Conversely, if \( 0 \) does not satisfy condition (6), then there exist \( c \in \mathbb{R}^m \) and \( k \geq 1 \) such that \( \lambda_k(c \cdot A) < 0 \). Thus there exists \( F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \) such that \( c \cdot A + F < 0 \) and \( 0 \) does not satisfy condition (2).

Let \( A \in S(\mathcal{H})^m \). Although the set \( \text{cl}(W(A)) \) may not be convex if \( m \geq 4 \), we have the following analog of the separation theorem for a convex set.

Theorem 4.2. Let \( A = (A_1, \ldots, A_m) \in S(\mathcal{H})^m \) and \( d = (d_1, \ldots, d_m) \in \mathbb{R}^m \). Then \( d \notin W_e(A) \) if and only if any one (and hence all) of the following conditions holds:

(a) There exists \( K \in \mathcal{K}(\mathcal{H})^m \cap S(\mathcal{H})^m \) such that \( d \notin \text{cl}(W(A + K)) \).

(b) There exists \( F \in \mathcal{F}(\mathcal{H})^m \cap S(\mathcal{H})^m \) with \( d \notin \text{conv}(\text{cl}(W(A + F))) \).

(c) There exist \( F \in \mathcal{F}(\mathcal{H}) \cap S(\mathcal{H}) \), \( r > 0 \) and \( c = (c_1, \ldots, c_m) \in \mathbb{R}^m \) such that

\[
\left( \sum_{i=1}^m c_i(A_i - d_i I) \right) + F > rI_{\mathcal{H}}.
\]

Proof. For simplicity, replace \( (A_1, \ldots, A_m) \) by \( (A_1 - d_1 I, \ldots, A_m - d_m I) \) and assume that \( d = (0, \ldots, 0) \).
(c)⇒(b). If (c) holds, we may perturb \((c_1, \ldots, c_m)\) so that \(c_j \neq 0\) for all \(j \in \{1, \ldots, m\}\) and condition (4) still holds true. In particular, \(c_1 \neq 0\). Then let \(F = (F/c_1, 0, \ldots, 0)\). We have \(c \cdot a > r > 0\) for all \(a \in W(A + F)\). Therefore, \(0 \notin \text{conv}(\text{cl}(W(A + F)))\).

Clearly, we have (b)⇒(a), which implies that \(0 \notin W_e(A)\).

Finally, suppose \(0 \notin W_e(A)\). Then by Theorem 4.1(2), there exist a real vector \(c = (c_1, \ldots, c_m)\) and \(F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})\) such that \(0 = c \cdot 0 \notin \text{cl}(W(c \cdot A + F))\). Since \(\text{cl}(W(c \cdot A + F))\) is a closed subinterval \([s, t]\) of \(\mathbb{R}\), we may assume that \(0 < s \leq t\). Let \(r = s/2\). Then \((\sum_{i=1}^{m} c_i A_i) + F > r I_\mathcal{H}\). Hence, (c) holds.

Let \(\Omega = \{c \in \mathbb{R}^m : \langle c, c \rangle = 1\}\). By Theorem 4.2, we have the following result showing that \(W_e(A)\) can be expressed as the intersection of half-spaces.

**Corollary 4.3.** Let \(A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m\). Then
\[
W_e(A) = \bigcap_{c \in \Omega} \{d \in \mathbb{R}^m : \langle c, d \rangle \leq \max W_e(c \cdot A)\}
= \{d \in \mathbb{R}^m : \langle c, d \rangle \in W_e(c \cdot A) \text{ for all } c \in \Omega\}.
\]

For \(A \in \mathcal{B}(\mathcal{H})\), let \(\sigma_e(A) = \bigcap \{\sigma(A + K) : K \in \mathcal{K}(\mathcal{H})\}\) be the essential spectrum of \(A\). Then for \(A \in \mathcal{S}(\mathcal{H})\), we have
\[
W_e(A) = \text{conv} \sigma_e(A).
\]
Thus, one may replace \(\max W_e(c \cdot A)\) by \(\max \sigma_e(c \cdot A)\) in Corollary 4.3.

**Corollary 4.4.** Let \(A = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m\). If \(d \notin \text{cl}(W(A))\), then for any \(i \in \{1, \ldots, m\}\) there exists \(F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})\) such that \(d \notin \text{conv}(\text{cl}(W(A)))\), where \(A = (A_1, \ldots, A_{i-1}, A_i + F, A_{i+1}, \ldots, A_m)\).

**Proof.** If \(d \notin \text{cl}(W(A))\), then \(d \notin W_e(A)\). The result readily follows from the arguments in the last paragraph in the proof of Theorem 4.2. 

It follows from Theorem 2.1 that the intersection of the non-convex sets \(\text{cl}(W(A + K))\), which equals \(W_e(A)\), is a convex set. By Theorem 4.2 and Corollary 4.4, we see that one can replace \(\text{cl}(W(A + K))\) by its convex hull in the intersection to obtain the same convex set \(W_e(A)\). It is known that for any \(B = (B_1, \ldots, B_m) \in \mathcal{B}(\mathcal{H})^m\),
\[
\text{conv}(\text{cl}(W(B))) = \{(f(B_1), \ldots, f(B_m)) : f \in \Xi\},
\]
where \(\Xi\) is the set of linear functionals \(f\) on \(\mathcal{B}(\mathcal{H})\) satisfying \(1 = f(I) = \max \{f(X) : X \in \mathcal{B}(\mathcal{H}), \|X\| \leq 1\}\) (for example, see [10, 11]). So, it is easier to determine \(\text{conv}(\text{cl}(W(A + K)))\) than \(\text{cl}(W(A + K))\). In fact, we have the following.
**Corollary 4.5.** Let $A \in S(H)^m$ and $i \in \{1, \ldots, m\}$. Then

$$W_e(A) = \bigcap \{\text{cl}(W(A + F)) : F \in \{0\}^{i-1} \times (F(H) \cap S(H)) \times \{0\}^{m-i}\}$$

$$= \bigcap \{\text{conv}(\text{cl}(W(A + F))) : F \in \{0\}^{i-1} \times (F(H) \cap S(H)) \times \{0\}^{m-i}\}.$$  

**Proof.** Let $F \in \{0\}^{i-1} \times (F(H) \cap S(H)) \times \{0\}^{m-i}$. Clearly,

$$W_e(A) \subseteq \text{cl}(W(A + F)) \subseteq \text{conv}(\text{cl}(W(A + F))).$$

So, we may take the intersection of the second and third sets over all $F \in \{0\}^{i-1} \times (F(H) \cap S(H)) \times \{0\}^{m-i}$, and get an inclusion involving the three sets in the corollary. Finally, if $d \notin W_e(A)$, then $d$ will not belong to the third set by Corollary 4.4. So, the third set is a subset of $W_e(A)$. Hence, the three sets in the corollary are equal. □

**5. Additional results.** The following result shows that $W_e(A)$ is unchanged under certain operations on $A$.

**Theorem 5.1.** Let $A = (A_1, \ldots, A_m) \in S(H)^m$.

(a) Suppose $H_1$ is a closed subspace of $H$ such that $H_1^\perp$ is finite-dimensional. If $X : H_1 \to H$ is such that $X^*X = I_{H_1}$, then

$$W_e(A) = W_e(X^*A_1X, \ldots, X^*A_mX).$$

(b) For each $j \in \{1, \ldots, m\}$, suppose $P_j : H \to H$ is an orthogonal projection such that $I - P_j$ has finite rank. Then

$$W_e(A) = W_e(P_1A_1P_1, \ldots, P_mA_mP_m).$$

**Proof.** Use Theorem 2.1. □

We will establish some additional relationships between the sets $W_e(A)$ and $W(A)$. The next theorem generalizes the results of [29] and [14].

**Theorem 5.2.** Let $A \in S(H)^m$. Then $W_e(A) = \text{cl}(W(A))$ if and only if $\text{Ext}(W(A)) \subseteq W_e(A)$.

**Proof.** If $W_e(A) = \text{cl}(W(A))$, then

$$\text{Ext}(W(A)) \subseteq W(A) \subseteq W_e(A).$$

Conversely, if $\text{Ext}(W(A)) \subseteq W_e(A)$, then by (P5),

$$\text{Ext}(\text{cl}(W(A))) \subseteq W_e(A).$$

Hence,

$$\text{cl}(W(A)) \subseteq \text{conv}(\text{Ext}(\text{cl}(W(A)))) \subseteq \text{conv}(W_e(A)) = W_e(A).$$

Since $W_e(A) \subseteq \text{cl}(W(A))$, we have $W_e(A) = \text{cl}(W(A))$. □
For \( k \geq 1 \), let \( I_k \) denote the \( k \times k \) identity matrix. Then for \( A = (A_1, \ldots, A_m) \in S(H)^m \), we have
\[
A \otimes I_k = (A_1 \otimes I_k, \ldots, A_m \otimes I_k) \in S(\overline{H} \oplus \cdots \oplus H)^m.
\]

Similarly, let \( I_\infty \) denote the identity operator acting on \( \ell_2 \). Then for \( A = (A_1, \ldots, A_m) \in S(H)^m \), we have
\[
A \otimes I_\infty = (A_1 \otimes I_\infty, \ldots, A_m \otimes I_\infty) \in S(\overline{H} \oplus H \oplus \cdots)^m.
\]

**Theorem 5.3.** Let \( A = (A_1, \ldots, A_m) \in S(H)^m \). Then for any positive integer \( k > \sqrt{m} - 1 \),
\[
W(A \otimes I_k) = \text{conv}(W(A)).
\]
Moreover,
\[
W_e(A \otimes I_\infty) = \text{cl}(\text{conv}(W(A))).
\]

**Proof.** Suppose \( k > \sqrt{m} - 1 \). By the result in [34], every \( a \in \text{conv}(W(A)) \) can be written as \( a = \sum_{j=1}^k t_j \langle Ax_j, x_j \rangle \) for some unit vectors \( x_1, \ldots, x_k \in H \). Thus, for \( x = (\sqrt{t_1} x_1, \ldots, \sqrt{t_k} x_k) \in H \oplus \cdots \oplus H \), we have \( \langle A \otimes I_k x, x \rangle = a \). Conversely, if \( a = \langle A \otimes I_k x, x \rangle \in W(A \otimes I_k) \), one can decompose the unit vector \( x \) into \( k \) parts \( y_1, \ldots, y_k \) according to the structure of \( H \otimes I_k \).

Then
\[
a = \sum_{j=1}^k \|y_j\|^2 \langle Ay_j/\|y_j\|, y_j/\|y_j\| \rangle \in \text{conv}(W(A)).
\]

If \( a \in \text{cl}(\text{conv}(W(A))) \), then there is a sequence \( \{x_n\} \) of unit vectors in \( H \) such that \( \langle Ax_n, x_n \rangle \to a \). Let
\[
\tilde{x}_n = \left(0, \ldots, 0, x_n, 0, \ldots\right) \in \overline{H} \oplus \overline{H} \oplus \cdots.
\]
Then \( \{\tilde{x}_n\} \) is an orthonormal sequence in \( H \oplus H \oplus \cdots \) and \( \langle A \otimes I_\infty \tilde{x}_n, \tilde{x}_n \rangle \to a \). Therefore, \( a \in W_e(A \otimes I_\infty) \).

Since
\[
W_e(A \otimes I_\infty) \subseteq \text{cl}(W(A \otimes I_\infty)) = \text{cl} \left( \bigcup_{k=1}^\infty W(A \otimes I_k) \right) \subseteq \text{cl}(\text{conv}(W(A))),
\]
we get the reverse inclusion. \( \blacksquare \)

**Corollary 5.4.** Let \( S \) be a compact convex subset of \( \mathbb{R}^m \). Then there are \( A, \tilde{A} \in S(H)^m \) with \( H = \ell^2 \) such that \( W(A) \) is convex and
\[
W(A) \subseteq S = \text{cl}(W(A)) = W_e(\tilde{A}).
\]

**Proof.** For \( j = 1, \ldots, m \), let \( A_j = \text{diag}(a_{1j}, a_{2j}, \ldots) \) act on \( \ell^2 \) with the standard canonical basis \( \{e_n : n \geq 1\} \) and be such that \( \{(a_{i1}, \ldots, a_{im}) : \)
$i \geq 1$} is a dense subset of $S$. Then for $A = (A_1, \ldots, A_m)$ the set

$$W(A) = \text{conv}\{(a_{i1}, \ldots, a_{im}) : i \geq 1\}$$

is convex, and $\tilde{A} = A \otimes I_\infty$ satisfies the assertion by Theorem 5.3.

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