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Potential Stability of Matrix Sign Patterns

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Potential Stability of Matrix Sign Patterns

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelors of Science in Mathematics from The College of William and Mary

by

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Potential Stability of Matrix Sign Patterns

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Abstract

The topic of matrix stability is very important for determining the stability of solutions to systems of differential equations. We examine several problems in the field of matrix stability, including minimal conditions for a $7 \times 7$ matrix sign pattern to be potentially stable, and applications of sign patterns to the study of Turing instability in the $3 \times 3$ case. Furthermore, some of our work serves as a model for a new method of approaching similar problems in the future.
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Chapter 1

Introduction

1.1 Motivation

Stable matrices appear in a variety of applied areas of mathematics, namely the study of systems of differential equations. There is a well-defined method of studying the long term behavior of such a system, by calculating the eigenvalues of its associated matrix representation, however it is desirable to determine the stability of a square matrix from its property.

By studying matrix stability from the perspective of matrix sign patterns, we can gain some insight into the structure of potentially stable matrices with a great deal of generality. This makes the study of sign pattern stability a valuable tool for approaching the problem.

One particular area of interest for stable matrices is defined by Alan Turing in his 1952 paper, The Chemical Basis of Morphogenesis [1]. This paper deals with a problem in mathematical biology, the study of certain reaction-diffusion systems, by examining the properties of a class of equations of two matrices $A - tP$, where $A$ is stable, $P$ is positive, and $t$ is a scaling parameter. This paper gave rise to the concept of Turing instability, a concept which is still widely studied today. It is possible to apply the study of potentially stable sign patterns to gain some general as well as some specific insights into the possible structures of a Turing-unstable system.
1.2 Basic Concepts

For a system of linear differential equations

\[
x' = \begin{bmatrix} x_1' \\ \vdots \\ x'_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \cdot x,
\]

the long term behavior of the system depends entirely on the eigenvalues of the matrix \(A\). The equilibrium \(x = 0\) is asymptotically stable for the system (1.1) if all eigenvalues of \(A\) have negative real part. Thus, in order to better understand the stability of a general system, we should examine which restrictions can be placed on the structure of \(A\) in order to allow for negative eigenvalues.

Let \(M_n\) be the set of all \(n \times n\) matrices with real-valued entries. A matrix \(A \in M_n\) is said to be stable if, for each of its eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\), \(\text{Re}(\lambda_i) < 0\). A system which is modeled by such a matrix \(A\) has stable equilibria, and given small perturbations of its initial conditions the system will return to these equilibrium points.

Let \(M_n\) be the set of all \(n \times n\) matrices with real-valued entries. A matrix \(A \in M_n\) is said to be stable if, for each of its eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\), \(\text{Re}(\lambda_i) < 0\). A system which is modeled by such a matrix \(A\) has stable equilibria, and given small perturbations of its initial conditions the system will return to these equilibrium points.

We define the sign pattern of a matrix \(A = [a_{ij}]\) to be an \(n \times n\) matrix \(S(A) = [s_{ij}]\) such that, for \(i, j \in \{1, \ldots, n\}\), \(s_{ij} = 0\) when \(a_{ij} = 0\), \(s_{ij} = -\) when \(a_{ij} < 0\), and \(s_{ij} = +\) when \(a_{ij} > 0\). If some matrix \(A \in M_n\) is found to be stable, then the sign pattern \(S(A)\) is said to be potentially stable, or PS for short. In the case where \(A \in M_n\) is an upper triangular, lower triangular, or diagonal matrix, or when \(A\) is permutationally similar to such a matrix, the problem becomes trivial due to the ease of calculating the eigenvalues of these matrices. Therefore we restrict our examination to irreducible matrices, or the matrices \(A \in M_n\) such that there does not exist a permutational matrix \(P\) such that

\[
PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}, A_{11} \in M_k, A_{22} \in M_{n-k}, 0 < k < n.
\]

Let the minimum number of nonzero entries required for an \(n \times n\) irreducible sign pattern to be potentially stable be denoted by \(m_n\). Then the following result has been proved in [2].
Theorem 1.1.

\[
\begin{aligned}
    m_n &= 2n - 1, & \text{if } n = 2, 3, \\
    m_n &= 2n - 2, & \text{if } n = 4, 5, \\
    m_n &= 2n - 3, & \text{if } n = 6, \\
    m_n &\leq 2n - (\lfloor \frac{n}{3} \rfloor + 1), & n \geq 7.
\end{aligned}
\]

Hence the value of \( m_n \) for \( n = 2, 3, 4, 5, 6 \) was determined in Theorem 1.1, as well as an upper bound for \( m_n \) for any \( n \geq 7 \).

For a stable matrix \( A \), if there is a positive diagonal matrix \( P \) such that the matrix \( A - tP \) is unstable for some positive \( t > 0 \), then \( A \) is said to exhibit Turing instability. We consider the minimal number of nonzero entries that such a matrix \( A \) must have in order for it to exhibit Turing instability. We define this number for any \( n \times n \) matrix \( A \) to be \( S_n \). Note that, trivially, \( S_n \geq m_n \) for any \( n \) since \( A \) is assumed to be stable.

In his original paper on the subject, Turing proved the following [1]:

Theorem 1.2. \( S_2 = 4 \).

While this result was not explicitly stated in [1], it is well known to the mathematical biologists studying pattern formation.

1.3 Main Results

In this thesis we prove the following two theorems which improve on the previous results in Theorems 1.1 and 1.2:

Theorem 1.3. \( m_7 = 2(7) - 3 = 11 \).

Theorem 1.4. \( S_3 = 6 \).

Note that Theorem 1.3 shows that the upper bound in Theorem 1.1 is indeed also the lower bound for the case of \( 7 \times 7 \) matrices. So the exact value of \( m_n \) now is known for \( 2 \leq n \leq 7 \), but it is still not known for \( n \geq 8 \).
In a 2014 paper by Raspovic et.al. [3], it was claimed that in order for a $3 \times 3$ matrix to exhibit Turing instability, it must have at least 6 nonzero entries. But the claim was not proved in the paper. Theorem 1.4 provides the justification for that claim.
Chapter 2

Preliminaries

2.1 Digraphs

We define the *digraph* of a matrix $A = (a_{ij})$ to be a directed graph containing the vertex set \{1, \ldots, n\}, and for each $i, j \in \{1, \ldots, n\}$, if $a_{ij} \neq 0$ there exists an edge from vertex $i$ to vertex $j$. For a digraph, we define a *path* as an ordered set of edges such that, for some vertices $i, j, l \in \{1, \ldots, n\}$, if the $m^{th}$ edge in the set is defined by $(i, j)$, then the $(m + 1)^{th}$ edge is defined by $(j, l)$. We define the *length* of a path as the number of edges in the path. If for each pair of vertices $p$ and $q$, such that $p \in \{1, \ldots, n\}$, $q \in \{1, \ldots, n\} \setminus \{p\}$, in a given digraph there exists a path which begins at $p$ and ends at $q$, we say that the digraph is *strongly connected*. It is the case that for any $A \in M_n$, $A$ is irreducible if and only if the digraph of $A$ is strongly connected [4]. We define a *cycle* to be a path which begins and ends at the same point, and which only intersects itself at this point. We refer to a cycle of length 1 as a *loop*. Also note that a permutation similarity which swaps the $i^{th}$ and $j^{th}$ rows/columns of $A$ is reflected in the digraph of $A$ by swapping the labels of the $i^{th}$ and $j^{th}$ vertices of the digraph.

The *circumference* of a digraph $G$ is defined as the length of the longest cycle present within the graph. We write this as $\text{circ}(G)$. Note that as the circumference decreases, the minimum number of edges needed to be strongly connected increases.
2.2 Minors

The following lemma is from well-known elementary algebra and it is useful for better defining the properties of the characteristic polynomial of a stable matrix.

**Lemma 2.1.** Let \( p(x) = x^2 + cx + d \) with \( c, d \in \mathbb{R} \). Then \( p \) has roots \( \lambda_1, \lambda_2 \) with \( \text{Re}(\lambda_1) < 0 \) and \( \text{Re}(\lambda_2) < 0 \) if and only if \( c > 0 \) and \( d > 0 \).

An \( m \times m \) principal submatrix of \( A \) is a matrix \( B = [b_{ij}] \in M_m \), \( 1 \leq m \leq n \) such that \( b_{ij} = a_{v_i v_j} \) for some \( v_1, \ldots, v_m \in \{1, \ldots, n\} \) with \( v_1 < \ldots < v_m \). A principal minor of \( A \) is defined as the determinant of some principal submatrix \( B = [b_{ij}] \) of \( A \). We denote the \( m \times m \) principal minor of \( A \) indexed by \( v_1 < \ldots < v_m \in \{1, \ldots, n\} \) as \( M(A)_{v_1,\ldots,v_m} \). For example,

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \implies M(A)_{23} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(A)_{13} = \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

There exists a direct relationship between the minors of a matrix and its eigenvalues. The sum of all \( k \times k \) principal minors of a matrix \( A \) is equal to the sum of all products of unique combinations of \( k \) eigenvalues of \( A \). That is,

\[
E_k = \sum_{1 \leq v_1 < \ldots < v_k \leq n} M(A)_{v_1,\ldots,v_k} = \sum_{1 \leq u_1 < \ldots < u_k \leq n} \lambda_{u_1} \cdots \lambda_{u_k}.
\]

(2.1)

Furthermore, the coefficient of \( t^{n-k} \) in the characteristic polynomial \( P_A(t) = \det(tI - A) \) of the \( A \) is equal to \((-1)^k E_k\). Due to the relationship between the minors and the eigenvalues of a matrix, we have the following lemma, which is well known:

**Lemma 2.2.** If \( A \in M_n \) is stable, then the following are true:

1. For all \( k = 1, \ldots, n \), the sign of the sum \( E_k \) of the \( k \times k \) minors of \( A \) is \((-1)^k\).

2. The characteristic polynomial of \( A \),

\[
P_A(t) = \det(tI - A) = \sum_{k=0}^{n} (-1)^k E_k t^{n-k},
\]

has all positive coefficients.
Note that the above lemma gives us a necessary condition for a given matrix $A$ to be stable. This condition will be very important in our work. If a given sign pattern can be realized by a real valued matrix $A$ which meets the condition that the sign of the sum of the $k \times k$ minors of $A$ is $(-1)^k$, then we say that this sign pattern has \textit{correct minors}. If for some $k$, the sum of $k \times k$ minors is equal to zero, then that sign pattern cannot be PS, as this would imply that either some of its eigenvalues are positive and some are negative, or that at least one of the eigenvalues is equal to zero.

The condition on the coefficients of $P_A(t)$ is necessary for the stability of $A$, however it is not sufficient. For example if $A$ is some real-valued matrix which has characteristic polynomial

$$P_A(t) = t^3 + 0.8t^2 + 0.81t + 1.01 = (t + 1)(t - 0.1 + i)(t - 0.1 - i),$$

then $P_A(t)$ has all positive coefficients, but $A$ has eigenvalues $\lambda = 0.1 \pm i$ which have strictly positive real parts, and so $A$ is not stable. In section 2.4, we will introduce a necessary and sufficient condition for the stability of $A$.

### 2.3 Digraph Cycles

There exists a direct relationship between the minors of a matrix and the cycles present in its digraph. If two or more cycles do not share any vertices, then we say that they are \textit{independent}. If the digraph of a sign pattern contains a cycle made up of $k$ edges, then this implies that at least one of its $k \times k$ minors is not equal to zero. Additionally, if there exist independent cycles of length $a_1, \ldots, a_m$, then this implies that, if $\sum_{i=1}^m a_i \leq n$, at least one of its $(\sum_{i=1}^m a_i) \times (\sum_{i=1}^m a_i)$ minors is not equal to zero. Below are examples of a digraph with correct minors, and one without:
Therefore, if a given digraph has independent cycles whose lengths add up to one of $1, \ldots, n$, then we can assign signs to the entries of the corresponding matrix such that it has correct minors.

### 2.4 Routh-Hurwitz Stability Criterion

The following result, proven by A. Hurwitz [5], provides a necessary and sufficient condition for a given polynomial to have roots with strictly negative real parts. However, due to the general nature of the polynomials we are examining, as well as the increasing complexity of the Hurwitz condition as dimensions increase, we will develop a separate condition which we use in our examination of $7 \times 7$ matrices.

**Lemma 2.3.** Suppose that $f$ is a degree-$n$ polynomial of the form $f(z) = \sum_{k=0}^{n} c_k z^{n-k}$ where $c_k \in \mathbb{R}$ and $c_0 = 1$. Then all the zeros of $f(z)$ have negative real parts if and only if the leading $k \times k$ principal minors $\Delta_k$ is positive for the following $n \times n$ matrix:

$$H_n = \begin{pmatrix}
    c_1 & c_3 & c_5 & \cdots & \cdots \\
    1 & c_2 & c_4 & \cdots & \cdots \\
    c_1 & c_3 & c_5 & \cdots & \cdots \\
    1 & c_2 & c_4 & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}. \tag{2.2}$$

Note that the matrix $H_n$ has the property that the odd indexed rows have all the odd coefficients $c_1, c_3, c_5, \ldots$ listed from the $(1, 1), (3, 3), (5, 5)$ positions; while the even indexed rows have all the even coefficients $1, c_2, c_4, \ldots$ listed from $(2, 1), (4, 3), (6, 5)$ positions.
In particular for \( n = 2, 3 \), Lemma 2.3 implies the following conditions for stability of \( A \):

1. \((n = 2)\) \( H_2 = \begin{pmatrix} c_1 & c_3 \\ 1 & c_2 \end{pmatrix} \), \( \Delta_1 = c_1 > 0, \Delta_2 = c_1c_2 > c_3 \). That is, \( c_1, c_2 > 0 \).

2. \((n = 3)\) \( H_3 = \begin{pmatrix} c_1 & c_3 & 0 \\ 1 & c_2 & 0 \\ 0 & c_1 & c_3 \end{pmatrix} \), \( \Delta_1 = c_1 > 0, \Delta_2 = c_1c_2 - c_3 > 0, \Delta_3 = c_3(c_1c_2 - c_3) > 0 \).

That is, \( c_1, c_2, c_3 > 0 \) and \( c_1c_2 > c_3 \).

Note, we can directly apply the Hurwitz condition to our examination of stable matrices by replacing the entries \( c_k \) with \((-1)^kE_k\), as in the characteristic polynomial \( P_A(t) \) of a matrix \( A \in M_n \), giving us the matrix:

\[
H_n = \begin{pmatrix}
-E_1 & -E_3 & -E_5 & \cdots & \cdots \\
1 & E_2 & E_4 & \cdots & \cdots \\
 & -E_1 & -E_3 & -E_5 & \cdots \\
1 & E_2 & E_4 & \cdots & \cdots \\
 & & \cdots & \cdots & \cdots 
\end{pmatrix}.
\] (2.3)
Chapter 3

Minimal Number of Nonzero Entries

In this chapter, we will detail the proof of Theorem 1.3.

3.1 Methods

For a $7 \times 7$ irreducible sign pattern, by Theorem 1.1 we have that the minimum number of nonzero entries needed to be potentially stable is

$$m_7 \leq 2(7) - (\lfloor 7/3 \rfloor + 1) = 14 - (2 + 1) = 11.$$  

Here we prove that the minimum for a $7 \times 7$ sign pattern is strictly equal to 11. In order to prove this minimum, we need only show that there cannot exist a potentially stable $7 \times 7$ sign pattern with only 10 nonzero entries. Note that if there were a potentially stable $7 \times 7$ sign pattern with fewer than 10 nonzero entries, then we would similarly be able to construct a potentially stable pattern with 10 nonzero entries by adding additional nonzero entries to an existing potentially stable pattern. Thus it is sufficient to prove that no potentially stable pattern with only 10 nonzero entries exists.

In order to prove that no such sign pattern exists, we first construct a list of all digraphs with 7 vertices and 10 edges which allow for correct minors (as defined by the relationship between cycles in the graph and the minors of the associated matrix in section 2.2). Once we have
constructed this list of digraphs, we will construct the associated set of nonequivalent matrix sign patterns which have correct minors. Finally we utilize a variant of Routh-Hurwitz stability criterion to show that none of these candidate sign patterns have a stable realization. From this we will conclude that the minimum number of nonzero entries must be equal to 11.

3.2 Candidate Digraphs

In this section we construct all candidate digraphs with 7 vertices and 10 edges which allow correct minors for a stable matrix. In order to better organize this list, we classify the graphs based on their circumference (the maximum length of cycle in the graph).

Case 1: $\text{circ}(G) = 7$.

In this case there must be at least one loop, and either there is one additional loop or there is a 2-cycle (see the minimum configuration below).

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) [circle,fill,inner sep=2pt] {};
  \node (v2) at (1,0) [circle,fill,inner sep=2pt] {};
  \node (v3) at (2,0) [circle,fill,inner sep=2pt] {};
  \node (v4) at (3,0) [circle,fill,inner sep=2pt] {};
  \node (v5) at (4,0) [circle,fill,inner sep=2pt] {};
  \node (v6) at (5,0) [circle,fill,inner sep=2pt] {};
  \node (v7) at (6,0) [circle,fill,inner sep=2pt] {};
  \node (v8) at (7,0) [circle,fill,inner sep=2pt] {};

  \draw [->] (v1) to (v2);
  \draw [->] (v2) to (v3);
  \draw [->] (v3) to (v4);
  \draw [->] (v4) to (v5);
  \draw [->] (v5) to (v6);
  \draw [->] (v6) to (v7);
  \draw [->] (v7) to (v8);
  \draw [->] (v8) to (v1);
\end{tikzpicture}
\end{center}

Case 1.1: There are at least two loops. Then 9 edges have been utilized. Suppose another edge is added to create a $k$-cycle where $k < 7$. The possible sizes of nonzero minors are $1, 2, k, k + 1, k + 2, 7$ (possibly less if the $k$-cycle intersects any of the two loops.) Thus, there is at least one $3 < r < 7$ such that the $r \times r$ minor of the adjacency matrix is zero. Therefore $G$ is not potentially stable.

Case 1.2: There is exactly one loop and a 2-cycle of two adjacent (numbering-wise) vertices. This utilizes 9 edges. Suppose the remaining edge is contained in a $k$-cycle, where $2 \leq k < 7$. If the 2-cycle and the loop have a vertex in common, then the possible sizes of nonzero minors are $1, 2, k, k + 1, k + 2, 7$, so we miss at least one minor, and therefore $G$ is not potentially stable. Similarly, if the $k$-cycle has a vertex in common with either the loop or the 2-cycle, we get a non-PS adjacency matrix. Thus, the 2-cycle, loop and $k$-cycle must be pairwise disjoint. In this
case, the possible sizes of nonzero minors are $1, 2, 3, k, k + 1, k + 2, k + 3, 7$. Thus, $k = 4$ or $k = 3$.

In this case we have the candidate graphs as shown in Figure 3.1.1, Figure 3.1.2, Figure 3.1.3 and Figure 3.1.4.

**Case 1.3:** There is exactly one loop and a 2-cycle of two non-adjacent (numbering-wise) vertices. Suppose these two additional edges create a $k$-cycle and an $r$-cycle and nonzero minors of sizes $1, 2, 3, k, r, r + 1, 7$. Thus $(k, r) = (4, 5)$.

In this case we have the candidate graph Figure 3.1.5.

**Case 2:** $circ(G) = 6$.

In this case there must be at least one loop. Either there is a loop on the vertex that does not belong to the 6-cycle or there is none (see the two possible configurations below.) Two of the three edges must be utilized to make sure that the graph is strongly connected. That is, one edge must be coming from the lone vertex and one must be going to the lone vertex.

**Case 2.1:** There is a loop in the lone vertex (say $v_1$) and another loop in another vertex. So far, we can guarantee nonzero minors of size $1, 2, 6, 7$. For the two remaining edges, one must be outgoing from $v_1$ and one must be incoming from $v_1$. If these two edges form a $k$-cycle (which intersects a loop and the 6-cycle), then we get nonzero minors of size $k$ and $k + 1$ and nothing else. Thus $G$ will not be potentially stable.

**Case 2.2:** There is a loop in the lone vertex and no loop in any other vertex. Suppose the outgoing and incoming edge to the lone vertex form a $k$-cycle (which intersects the loop and the 6-cycle), with $k < 7$. Then minors of size $1, k, 6, 7$, are nonzero. Suppose the remaining edge gives rise to another cycle of length $1 < r < 7$ (this means it must necessarily intersect the 6-cycle). This may give rise to nonzero minors of size $r, r + 1$, and $r + k$ (less if the $r$-cycle also
intersects with the loop or the $k$-cycle). Thus, the $r$-cycle must not intersect with the $k$-cycle and \{\(k, r, r + 1, r + k\)\} = \{2, 3, 4, 5\}. There is no choice but for $k = 2$ and $r = 3$.

Thus, in this case, we have the candidate graphs as in Figure 3.1.6 and Figure 3.1.7.

**Case 2.3:** There is no loop in the lone vertex. Hence, there is at least one loop intersecting the 6-cycle. Suppose the incoming and outgoing edges to the lone vertex forms a $k$-cycle, where $1 < k < 7$. At this point, 9 edges have been accounted for and nonzero minors of sizes 1, 6, $k, k + 1$. Suppose the last edge forms an $r$-cycle, where $r < 7$. By assumption, this $r$-cycle must intersect the 6-cycle. If this $r$-cycle intersects the loop or the $k$-cycle, then there will be a zero minor and thus, the adjacency matrix cannot be PS. If the $r$-cycle, loop, and the $k$-cycle are pairwise disjoint, then we get additional nonzero minors of size $r, r + 1, r + k, r + k + 1$.

We want \(\{2, 3, 4, 5, 7\} \subseteq \{k, k + 1, r, r + 1, r + k, r + k + 1\}\). Thus, either \((r, k) = (2, 4)\) or \((r, k) = (4, 2)\).

Thus, in this case, we have the candidate graphs: Figure 3.1.8 and Figure 3.1.9.

**Case 3:** $\text{circ}(G) = 5$.

In this case a 5-cycle uses 5 edges, and another edge must form a loop. At least three out of the four remaining edges must be used to ensure strong connectedness of the graph. Either there is a loop intersecting the 5-cycle or there is none (see the two possible configurations below.)

![Diagram of two possible configurations](image)

**Case 3.1:** There is a loop intersecting the 5-cycle (the one we choose at the beginning, there may be more than one 5-cycle). Either there is an edge between the two remaining vertices or there is none.

**Subcase 3.1.1:** Suppose there is no edge connecting the two vertices (let’s call them $v_1$ and $v_2$). Then, two of the four remaining edges should connect $v_1$ to vertices in the 5-cycle to form a $k$-cycle, where $k \leq 5$. Similarly, the remaining two edges must connect $v_2$ to vertices in
the 5-cycle to form an r-cycle, where $k \leq 5$. Assuming the $k$-cycle, $r$-cycle, and the loop are disjoint, then we have nonzero minors of size $1, 5, k, r, k+1, r+1, k+r, k+r+1$. (Note that if they are not pairwise disjoint, there will be at least one minor size that will be missing.) Thus \( \{2, 3, 4, 6, 7\} \in \{k, r, k+1, r+1, k+r, k+r+1\} \). Thus \((k, r) = (2, 4)\) or \((k, r) = (4, 2)\). Thus, we have the candidate graph in Figure 3.1.10.

**Subcase 3.1.2:** Suppose $v_1$ and $v_2$ form a 2-cycle and $v_2$ is not adjacent to any vertex in the 5-cycle. So far, we have accounted for 8 edges and nonzero minors of sizes $1, 2, 3, 5$. The two remaining edges must be incoming and outgoing from $v_1$ to make a strongly connected graph. Say these two remaining edges form a $k$-cycle, where $k \leq 5$. This adds nonzero minors of size $k$ and $k+1$, which is not enough to make a potentially stable adjacency matrix.

**Subcase 3.1.3:** Suppose $v_1$ and $v_2$ are part of a $k$-cycle, with $2 < k \leq 5$. So far, we have accounted for at least 9 edges and nonzero minors of sizes $1, 2, 3, 5$. To get a nonzero $2 \times 2$ minor, either there must be another loop or there is a 2-cycle. Adding a loop can only guarantee at least two more nonzero minor sizes. Thus, there must be a 2-cycle in the graph. If the 2-cycle is disjoint from the 5-cycle (that is, $v_1$ and $v_2$ form the 2-cycle), we only get nonzero minors of size $\{1, 2, 3, 5, k, k+1\} \neq \{1, 2, 3, 4, 5, 6, 7\}$. Hence the two vertices in the 2-cycle must be part of the 5-cycle. In this case, we get nonzero minors, $1, 2, 3, k, k+1, k+2, k+3, 5$. Thus $k = 4$.

Thus, we have the candidate graph in Figure 3.1.11.

**Case 3.2:** There is no loop intersecting the 5-cycle. Again, either there is an edge connecting the remaining two vertices $v_1$ and $v_2$ or there is none.

**Subcase 3.2.1:** There is no edge connecting the remaining two vertices $v_1$ and $v_2$. Then, two of the four remaining edges should connect $v_1$ to vertices in the 5-cycle to form a $k$-cycle, where $k \leq 5$. Similarly, the remaining two edges must connect $v_2$ to vertices in the 5-cycle to form an $r$-cycle, where $k \leq 5$. Assuming the $k$-cycle, $r$-cycle and the loop are disjoint, then we have nonzero minors of size $1, 5, 6, k, r, r+1, r+k$. Note that there is no choice of $2 \leq k, r \leq 5$ that will give a complete set of nonzero minors. Thus, this case will not give a PS adjacency matrix.
Subcase 3.2.2: Suppose \( v_1 \) and \( v_2 \) form a 2-cycle and one of \( v_1 \) or \( v_2 \) is not adjacent to any vertex in the 5-cycle. So far, we have accounted for 8 edges and nonzero minors of sizes 1, 2, 5, 6, 7. The two remaining edges must be incoming and outgoing from \( v_1, v_2 \) to make a strongly connected graph. Say these two remaining edges form a \( k \)-cycle, where \( k \leq 5 \). This adds nonzero minors of size \( k \) and possibly (if the \( k \)-cycle does not contain the loop) \( k + 1 \). Thus, we have the graph in Figure 3.1.12.

Subcase 3.2.3: Suppose \( v_1 \) and \( v_2 \) are part of a \( k \)-cycle, with \( 2 < k \leq 5 \). So far, we have accounted for at least 9 edges and nonzero minors of sizes 1, 5, 6, \( k \), where \( k \geq 3 \). To get a nonzero \( 2 \times 2 \) minor, there should be another loop or a 2-cycle. Adding a loop can only guarantee at most two more sizes of nonzero minors. If there is a 2-cycle between \( v_1 \) and \( v_2 \), we get additional nonzero minor sizes 2, 7, which is not enough for the graph to be PS. If the 2-cycle is disjoint with the \( k \)-cycle, we get additional nonzero minor sizes 2, 3, \( k + 2 \). This is still not enough to get a PS matrix.

Case 4: \( \text{circ}(G) = 4 \).

Let \( v_1, v_2, v_3, v_4 \) form a 4-cycle. For each of \( v_5, v_6, v_7 \), there must be incoming edges \{5, \text{in}\}, \{6, \text{in}\}, \{7, \text{in}\} and outgoing edges \{5, \text{out}\}, \{6, \text{out}\}, \{7, \text{out}\}. Note that the sets \{\{5, \text{in}\}, \{6, \text{in}\}, \{7, \text{in}\}\}, and \{\{5, \text{out}\}, \{6, \text{out}\}, \{7, \text{out}\}\} must have at least 1 element in common since we still have to account for the loop. We can list all possible nonequivalent strongly connected graphs with fewer than 9 edges and maximum cycle length 4 as follows:

For the top left and two middle graphs, adding a loop will give an adjacency matrix that has zero
determinant. For the top rightmost graph, a loop that is disjoint from the 2-cycle and 4-cycle must be added to get all nonzero minors. For the lower left graph, a loop must be added so that the loop, a 4-cycle, and a 2-cycle are all disjoint. Finally, for the lower right graph, a disjoint loop and 2-cycle must be added in order to get nonzero minors.

Thus, we have the following candidate graphs: Figure 3.1.13, Figure 3.1.14, and Figure 3.1.15.

**Case 5:** \( \text{circ}(G) = 3 \).

If the circumference of the graph is 3, then we have at least one 3-cycle, so there are 4 additional vertices to connect to one another, as well as the existing 3-cycle. If one of the 4 vertices is connected by a 2-cycle to the existing 3-cycle, then there must be at least one incoming and one outgoing edge connecting the 3 remaining vertices to the other 4, and at least two additional edges connecting between the 3 remaining vertices. If two of the 4 vertices is connected to the existing 3-cycle by another 3-cycle, then there must be at least 3 additional edges connecting the remaining 2 vertices to each other and back to the existing 3-cycles. So in any case, the graph must have at least 9 edges in order to be strongly connected. Let \( v_1, v_2 \) and \( v_3 \) form a 3-cycle. Since the graph needs at least 9 edges in order to be strongly connected, it can have at most 1 loop, giving a total of \( 9 + 1 = 10 \) edges. Then the graph must have a 2-cycle as well.

**Case 5.1:** Suppose the 2-cycle shares an edge with the 3-cycle \((v_1v_2v_3)\). Then, between the 2-cycle, the 3-cycle, and the loop, we have used 5 of the 10 available edges. So there are 5 edges remaining with which to connect the vertices \( v_4, v_5, v_6 \) and \( v_7 \). Each of these vertices requires at least one incoming edge and one outgoing edge. Since \( \text{circ}(G) = 3 \), it would require at least 3 edges in order to connect any two of the remaining vertices to the original 3-cycle. From that point, it would require at least an additional 3 edges in order to connect the remaining two vertices. However there are only 5 edges available, and thus the 2-cycle cannot share an edge with the 3-cycle.

**Case 5.2:** Suppose the 2-cycle does not share an edge with the 3-cycle \((v_1v_2v_3)\), say the 2-cycle is \((v_4v_5)\) without loss of generality. Then, between the 2-cycle, the 3-cycle, and the loop, we
have used 6 of the 10 available edges. So there are 4 edges remaining with which to connect the vertices $v_6$ and $v_7$ with the cycle $(v_1v_2v_3)$ and the cycle $(v_4v_5)$. Since the circumference of the graph is 4, it would require at least 3 edges in order to connect $v_6$ and $v_7$ to either the 3-cycle or the 2-cycle. Then there is at most 1 edge remaining, which is insufficient to connect the remaining separated cycles. Therefore the 2-cycle cannot be separate from the 3-cycle, and so there are no digraphs with 7 vertices and a circumference of 3 which have correct minors.

**Case 6:** $\text{circ}(G) = 2$.

If the circumference of the graph is equal to 2, then each vertex must be connected to at least one other vertex by 2 edges, allowing for some vertices to be connected to multiple edges. Then, allowing for repeated vertices, there must be at least 6 pairs of connections between the 7 vertices, giving a minimum of $6 \times 2 = 12$ edges for the graph to be strongly connected. Thus, since we are limiting ourselves to 10 edges, the circumference cannot equal 2.

Summarizing the above discussion, we reach the main result in this section.

**Proposition 3.1.** Suppose that $(V,E)$ is a strongly connected digraph with 7 vertices and 10 edges which has all non-zero minors. Then $(V,E)$ is equivalent to one of digraphs in Figure 3.1.

### 3.3 Stability Calculations

We now convert the graphs from Figure 3.1 into properly signed matrices, and show that none of them can be realized by a stable matrix. First however, we prove the following lemma.

**Lemma 3.2.** Let $A$ be a $7 \times 7$ real-valued matrix with the characteristic polynomial

$$P_A(t) = t^7 + c_1t^6 + c_2t^5 + c_3t^4 + c_4t^3 + c_5t^2 + c_6t + c_7. \quad (3.1)$$

If $A$ is stable, then all of the following inequalities must hold:

1. $c_2c_4 - c_6 > 0,$
Figure 3.1: List of potential digraphs with 7 vertices and 10 edges.
2. $c_1 c_2 - c_3 > 0$,

3. $c_1 c_6 - c_7 > 0$,

4. $c_2 c_5 - c_7 > 0$.

*Proof.* By Lemma 2.1, a matrix $A$ is stable if and only if there exist $a_1, a_2, a_3, a_4, b_1, b_2, b_3 > 0$ such that

$$P_A(t) = (t^2 + b_1 t + a_1)(t^2 + b_2 t + a_2)(t^2 + b_3 t + a_3)(t + a_4). \quad (3.2)$$

Comparing the coefficients of (3.1) and (3.2), we have:

$$c_1 = a_4 + b_1 + b_2 + b_3,$$
$$c_2 = a_1 + a_2 + a_3 + a_4(b_1 + b_2 + b_3) + b_1 b_2 + b_1 b_3 + b_2 b_3,$$
$$c_3 = a_1 a_4 + a_2 a_4 + a_3 a_4 + a_2 b_1 + a_3 b_1 + a_1 b_2 + a_3 b_2 + a_4 b_1 b_2 + a_1 b_3 + a_2 b_3 + a_4 b_1 b_3 + b_1 b_2 b_3,$$
$$c_4 = a_1 a_2 + a_1 a_3 + a_2 a_3 + a_2 a_4 b_1 + a_3 a_4 b_1 + a_1 a_4 b_2 + a_3 a_4 b_2 + a_3 b_1 b_2 + a_1 a_4 b_3 + a_2 a_4 b_3 + a_2 b_1 b_3 + a_1 b_2 b_3 + a_4 b_1 b_2 b_3,$$
$$c_5 = a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 + a_2 a_3 b_1 + a_1 a_3 b_2 + a_3 a_4 b_1 + a_1 a_2 b_3 + a_2 a_4 b_3 + a_1 a_4 b_3,$$
$$c_6 = a_1 a_2 a_3 + a_2 a_3 a_4 b_1 + a_1 a_3 a_4 b_2 + a_1 a_2 a_4 b_3,$$
$$c_7 = a_1 a_2 a_3 a_4.$$

We can verify that for each case of $c_i c_j - c_{i+j}$ listed above, $c_i c_j - c_{i+j}$ can be expressed as a sum of products of $a_i$ and $b_j$'s, hence $c_i c_j - c_{i+j} > 0$ as all $a_i$ and $b_j$ are positive. \hfill \Box

Now we use Lemma 3.2 to exclude all the 15 digraphs (or equivalently sign patterns) in Figure 3.1 to be potentially stable. This will complete the proof of Theorem 1.3.
Here we have: $c_1 = a_{11}$, $c_2 = a_{23}a_{32}$, and $c_3 = a_{11}a_{23}a_{32} + a_{45}a_{56}a_{64}$. So $c_1c_2 - c_3 = -a_{45}a_{56}a_{64} < 0$. Thus this sign pattern is not potentially stable by Lemma 3.2(2).

Here we have: $c_1 = a_{11}$, $c_2 = a_{23}a_{32}$, $c_3 = a_{11}a_{23}a_{32} + a_{56}a_{67}a_{75}$. So $c_1c_2 - c_3 = -a_{56}a_{67}a_{75} < 0$. Thus this sign pattern is not potentially stable by Lemma 3.2(2).

Here we have: $c_2 = a_{23}a_{32}$, $c_4 = a_{45}a_{56}a_{67}a_{74}$, $c_6 = a_{23}a_{32}a_{45}a_{56}a_{67}a_{74}$. So $c_2c_4 - c_6 = 0$. Note that while both positive and negative values of $a_{71}$ allow for correct minors, the value
of $a_{71}$ does not appear in our contradiction, and thus the contradiction holds regardless of the value of $a_{71}$. Thus this sign pattern is not potentially stable by Lemma 3.2(1).

$$\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{34} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -a_{43} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{45} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{71} & 0 & 0 & 0 & 0 & -a_{75} & 0
\end{bmatrix}$$

Here we have: $c_1 = a_{11}, c_2 = a_{34}a_{43}, c_3 = a_{11}a_{34}a_{43} + a_{56}a_{67}a_{75}$. So $c_1c_2 - c_3 = -a_{56}a_{67}a_{75} < 0$. Thus this sign pattern is not potentially stable by Lemma 3.2(2).

$$\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{34} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{43} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{45} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{71} & 0 & -a_{73} & 0 & 0 & 0 & 0
\end{bmatrix}$$

Here we have: $c_1 = a_{11}, c_2 = a_{37}a_{73}, c_3 = a_{11}a_{37}a_{73}$. So $c_1c_2 - c_3 = 0$. Thus this sign pattern is not potentially stable by Lemma 3.2(2).

$$\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
-a_{21} & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{34} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{45} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{53} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{67} \\
0 & -a_{72} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Here we have: \( c_1 = a_{11}, \ c_6 = a_{23}a_{43}a_{45}a_{56}a_{67}a_{72}, \ c_7 = a_{11}a_{23}a_{43}a_{45}a_{56}a_{67}a_{72}. \) So \( c_1c_6 - c_7 = 0. \) Thus this sign pattern is not potentially stable by Lemma 3.2(3).

\[
\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
-a_{21} & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{56} & 0 \\
0 & 0 & 0 & -a_{64} & 0 & 0 & a_{67} \\
0 & -a_{72} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Here we have: \( c_1 = a_{11}, \ c_6 = a_{23}a_{43}a_{45}a_{56}a_{67}a_{72}, \ c_7 = a_{11}a_{23}a_{43}a_{45}a_{56}a_{67}a_{72}. \) So \( c_1c_6 - c_7 = 0. \) Thus this sign pattern is not potentially stable by Lemma 3.2(3).

\[
\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\
0 & -a_{52} & 0 & 0 & 0 & a_{56} & 0 \\
-a_{61} & 0 & 0 & 0 & 0 & 0 & a_{67} \\
0 & 0 & 0 & 0 & 0 & -a_{76} & 0
\end{bmatrix}
\]

Here we have: \( c_2 = a_{67}a_{76}, \ c_4 = a_{23}a_{34}a_{45}a_{52}, \ c_6 = a_{67}a_{76}a_{23}a_{34}a_{45}a_{52}. \) So \( c_2c_4 - c_6 = 0. \) Thus this sign pattern is not potentially stable by Lemma 3.2(2).

\[
\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & -a_{32} & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{56} & 0 \\
\pm a_{61} & 0 & 0 & 0 & 0 & 0 & a_{67} \\
0 & 0 & 0 & -a_{74} & 0 & 0 & 0
\end{bmatrix}
\]
Here we have: \( c_2 = a_{23}a_{32}, \ c_5 = a_{11}a_{45}a_{56}a_{67}a_{74}, \ c_7 = a_{23}a_{32}a_{11}a_{45}a_{56}a_{67}a_{74}. \) So \( c_2c_5 - c_7 = 0. \) Note that while both positive and negative values of \( a_{61} \) allow for correct minors, the value of \( a_{61} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{61}. \) Thus this sign pattern is not potentially stable by Lemma 3.2(4).

\[
\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{45} & 0 & a_{47} \\
\pm a_{51} & 0 & 0 & 0 & 0 & a_{56} & 0 \\
0 & 0 & 0 & 0 & -a_{65} & 0 & 0 \\
0 & -a_{72} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Here we have: \( c_2 = a_{56}a_{65}, \ c_4 = a_{23}a_{34}a_{47}a_{72}, \ c_6 = a_{56}a_{65}a_{23}a_{34}a_{47}a_{72}. \) So \( c_2c_4 - c_6 = 0. \) Note that while both positive and negative values of \( a_{51} \) allow for correct minors, the value of \( a_{51} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{51}. \) Thus this sign pattern is not potentially stable by Lemma 3.2(2).

\[
\begin{bmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\
0 & -a_{32} & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\
\pm a_{51} & 0 & 0 & 0 & 0 & a_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{67} \\
0 & 0 & 0 & -a_{74} & 0 & 0 & 0
\end{bmatrix}
\]

Here we have: \( c_2 = a_{23}a_{32}, \ c_4 = a_{45}a_{56}a_{67}a_{74}, \ c_6 = a_{23}a_{32}a_{45}a_{56}a_{67}a_{74}. \) So \( c_2c_4 - c_6 = 0. \) Note that while both positive and negative values of \( a_{51} \) allow for correct minors, the value of \( a_{51} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{51}. \) Thus this sign pattern is not potentially stable by Lemma 3.2(2).
Here we have: 

\[ c_2 = a_{12} a_{21}, \quad c_5 = a_{34} a_{45} a_{56} a_{67} a_{73}, \quad c_7 = a_{12} a_{21} a_{34} a_{45} a_{56} a_{67} a_{73}. \]

So 

\[ c_2 c_5 - c_7 = 0. \]

Thus this sign pattern is not potentially stable by Lemma 3.2(4).

Here we have: 

\[ c_2 = a_{27} a_{72}, \quad c_4 = a_{34} a_{45} a_{56} a_{63}, \quad c_6 = a_{27} a_{72} a_{34} a_{45} a_{56} a_{63}. \]

So 

\[ c_2 c_4 - c_6 = 0. \]

Note that while both positive and negative values of \( a_{31} \) allow for correct minors, the value of \( a_{31} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{31} \). Thus this sign pattern is not potentially stable by Lemma 3.2(2).
Here we have: \( c_2 = a_{45}a_{54}, \ c_5 = a_{11}a_{23}a_{36}a_{67}a_{72}, \ c_7 = a_{45}a_{54}a_{11}a_{23}a_{36}a_{67}a_{72}. \) So \( c_2c_5 - c_7 = 0. \) Note that while both positive and negative values of \( a_{41} \) allow for correct minors, the value of \( a_{41} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{41}. \) Thus this sign pattern is not potentially stable by Lemma 3.2(4).

\[
\begin{pmatrix}
-a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & 0 & a_{25} & 0 & 0 \\
0 & 0 & 0 & a_{34} & 0 & 0 & 0 \\
\pm a_{41} & 0 & -a_{43} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{67} \\
0 & -a_{72} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Here we have: \( c_2 = a_{43}a_{34}, \ c_5 = a_{11}a_{25}a_{56}a_{67}a_{72}, \ c_7 = a_{43}a_{34}a_{11}a_{25}a_{56}a_{67}a_{72}. \) So \( c_2c_5 - c_7 = 0. \) Note that while both positive and negative values of \( a_{41} \) allow for correct minors, the value of \( a_{41} \) does not appear in our contradiction, and thus the contradiction holds regardless of the value of \( a_{41}. \) Thus this sign pattern is not potentially stable by Lemma 3.2(4).
Chapter 4

Turing Instability of a $3 \times 3$ Matrix

In this chapter we prove Theorem 1.4.

4.1 Methods

We prove that for a $3 \times 3$ matrix $A$ to exhibit Turing instability (there exists a positive diagonal matrix $P$, such that $A - tP$ is unstable for some $t > 0$), it must have at least 6 nonzero entries. However, in order for an irreducible matrix of the same dimensions to be stable it must have at least 5 nonzero entries. In order to prove the statement regarding Turing instability, we should consider each $3 \times 3$ potentially stable sign pattern containing only 5 nonzero entries and attempt to show that no matrix with one of these patterns can exhibit Turing instability.

We approach the problem in a general way by examining the potential for Turing instability in each $3 \times 3$ potentially stable sign pattern with only 5 nonzero entries. By showing that any realization of such a sign pattern cannot exhibit Turing instability, and by providing an example of a $3 \times 3$ matrix with 6 nonzero entries which does exhibit Turing instability, we will prove the theorem.
4.2 Candidate Digraphs

In order for a $3 \times 3$ matrix to be irreducible, its digraph must either contain (a) two connected 2-cycles, or (b) one 3-cycle. Then, in order for such a matrix to be potentially stable, in both cases a loop is required, then in case (a) you must have either a 2-cycle which can intersect the loop or be separate from it, or an additional loop, and in case (b) you can have the loop either on the end of one of the 2-cycles, or at the intersection of the two 2-cycles. Thus, we have the list of digraphs in Figure 4.1 to consider. Note that the 5th graph does not contain a $3 \times 3$ minor. Because of this, its determinant will always be 0, and thus we only need to consider the first 4 graphs.
Figure 4.1: List of potential digraphs with 3 vertices and 5 edges.
We now input the correct signs into the above patterns in order to allow for the sums of minors to have correct signs.

1.a. \[
\begin{bmatrix}
- & + & 0 \\
 0 & 0 & + \\
- & - & 0 \\
\end{bmatrix}
\]

1.b. \[
\begin{bmatrix}
- & + & 0 \\
 0 & 0 & + \\
+ & - & 0 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
- & + & 0 \\
- & 0 & + \\
- & 0 & 0 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
- & + & 0 \\
0 & - & + \\
- & 0 & 0 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
- & + & 0 \\
- & 0 & + \\
0 & - & 0 \\
\end{bmatrix}
\]

By the Routh-Hurwitz condition (see section 2.4), it can be shown that matrix 1.a. is not potentially stable, but the other sign patterns are potentially stable. Therefore in order to prove the statement regarding Turing instability for the \(3 \times 3\) case we need only show that the bottom 4 sign patterns can not exhibit Turing instability.
4.3 Stability Calculations

Case 1:

\[ A = \begin{bmatrix}
-a_{11} & a_{12} & 0 \\
0 & 0 & a_{23} \\
a_{31} & -a_{32} & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
p_1 & 0 & 0 \\
0 & p_2 & 0 \\
0 & 0 & p_3
\end{bmatrix}, \quad a_{ij} > 0, p_k > 0. \]

\[ A - tP = \begin{bmatrix}
-a_{11} - tp_1 & a_{12} & 0 \\
0 & -tp_2 & a_{23} \\
a_{31} & -a_{32} & -tp_3
\end{bmatrix}. \]

Then for any \( t \geq 0, \)

\[ c_1(t) = t(p_1 + p_2 + p_3) + (a_{11}) > 0, \]

\[ c_2(t) = t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3) + (a_{23}a_{32}) > 0, \]

\[ c_3(t) = t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{23}a_{32}p_1) + (a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31}) > 0, \]

\[ c_1(t)c_2(t) - c_3(t) = (t(p_1 + p_2 + p_3) + a_{11})(t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3) + (a_{23}a_{32})) \\
- (t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{23}a_{32}p_1) + (a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31})) > 0. \]

Therefore \( A - tP \) is stable for all \( t \geq 0. \)
Case 2:

\[ A = \begin{bmatrix}
-a_{11} & a_{12} & 0 \\
-a_{21} & 0 & a_{23} \\
-a_{31} & 0 & 0 \\
\end{bmatrix}, \quad P = \begin{bmatrix}
p_1 & 0 & 0 \\
0 & p_2 & 0 \\
0 & 0 & p_3 \\
\end{bmatrix}, \quad a_{ij} > 0, p_k > 0. \]

\[ A - tP = \begin{bmatrix}
-a_{11} - tp_1 & a_{12} & 0 \\
-a_{21} - tp_2 & 0 & a_{23} \\
-a_{31} & 0 & -tp_3 \\
\end{bmatrix}. \]

Then for any \( t \geq 0, \)

\[ c_1(t) = t(p_1 + p_2 + p_3) + (a_{11}) > 0, \]

\[ c_2(t) = t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3 + a_{12}a_{21}) > 0, \]

\[ c_3(t) = t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{12}a_{21}p_3) + (a_{12}a_{23}a_{31}) > 0, \]

\[ c_1(t)c_2(t) - c_3(t) = (t(p_1 + p_2 + p_3) + a_{11}) (t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3 + a_{12}a_{21})) - (t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{12}a_{21}p_3) + (a_{12}a_{23}a_{31})) > 0. \]

Therefore \( A - tP \) is stable for all \( t \geq 0. \)
Case 3:

\[
A = \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & -a_{22} & a_{23} \\ -a_{31} & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad a_{ij} > 0, p_k > 0.
\]

\[
A - tP = \begin{bmatrix} -a_{11} - tp_1 & a_{12} & 0 \\ 0 & -a_{22} - tp_2 & a_{23} \\ -a_{31} & 0 & -tp_3 \end{bmatrix}.
\]

\[
c_1(t) = t(p_1 + p_2 + p_3) + (a_{11} + a_{22}) > 0,
\]

\[
c_2(t) = t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{22}p_1 + a_{11}p_3 + a_{22}p_3) + (a_{11}a_{22}) > 0,
\]

\[
c_3(t) = t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3 + a_{22}p_1p_3) + t(a_{11}a_{22}p_3) + (a_{12}a_{23}a_{31}) > 0,
\]

\[
c_1(t)c_2(t) - c_3(t) = (t(p_1 + p_2 + p_3) + (a_{11} + a_{22}))(t^2(p_1p_2 + p_1p_3 + p_2p_3)
+ t(a_{11}p_2 + a_{22}p_1 + a_{11}p_3 + a_{22}p_3) + (a_{11}a_{22}))
- (t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3 + a_{22}p_1p_3) + t(a_{11}a_{22}p_3) + (a_{12}a_{23}a_{31})) > 0.
\]

Therefore \(A - tP\) is stable for all \(t \geq 0\).
Case 4:

\[
A = \begin{bmatrix}
-a_{11} & a_{12} & 0 \\
-a_{21} & 0 & a_{23} \\
0 & -a_{32} & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
p_1 & 0 & 0 \\
0 & p_2 & 0 \\
0 & 0 & p_3
\end{bmatrix}, \quad a_{ij} > 0, p_k > 0.
\]

\[
A - tP = \begin{bmatrix}
-a_{11} - tp_1 & a_{12} & 0 \\
-a_{21} & -tp_2 & a_{23} \\
0 & -a_{32} & -tp_3
\end{bmatrix}.
\]

\[
c_1(t) = t(p_1 + p_2 + p_3) + (a_{11}) > 0,
\]

\[
c_2(t) = t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3) + (a_{12}a_{21} + a_{23}a_{32}) > 0,
\]

\[
c_3(t) = t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{12}a_{21}p_3 + a_{23}a_{32}p_1) + (a_{11}a_{23}a_{32}) > 0,
\]

\[
c_1(t)c_2(t) - c_3(t) = (t(p_1 + p_2 + p_3) + (a_{11}))
\]

\[
(t^2(p_1p_2 + p_1p_3 + p_2p_3) + t(a_{11}p_2 + a_{11}p_3) + (a_{12}a_{21} + a_{23}a_{32}))
\]

\[
- (t^3(p_1p_2p_3) + t^2(a_{11}p_2p_3) + t(a_{12}a_{21}p_3 + a_{23}a_{32}p_1) + (a_{11}a_{23}a_{32})) > 0.
\]

Therefore \( A - tP \) is stable for all \( t \geq 0 \).

From the above 4 cases, since \( A - tP \) is stable for all \( t \geq 0 \) for each of the irreducible sign patterns shown above, there is no \( 3 \times 3 \) irreducible, stable matrix with only 5 entries which can exhibit Turing instability for some positive diagonal matrix \( P \). This completes the proof of Theorem 1.4.
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Bibliography


