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Strongly Real Conjugacy Classes in Unitary Groups over Fields of Even Characteristic

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Strongly real conjugacy classes of unitary groups over fields of even characteristic

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Abstract

An element g of a group G is called strongly real if there is an s in G such that $s^2 = 1$ and $sgs^{-1} = g^{-1}$. It is a fact that if g in G is strongly real, then every element in its conjugacy class is strongly real. Thus we can classify each conjugacy class as strongly real or not strongly real. Gates, Singh, and Vinroot have classified the strongly real conjugacy classes of $\mathrm{U}(n, q^2)$ in the case that q is odd. Vinroot and Schaeffer Fry have classified some of the conjugacy classes of $\mathrm{U}(n, q^2)$ where q is even. We conjecture the full classification, and under that conjecture provide a generating function for the number of unipotent strongly real conjugacy classes in $\mathrm{U}(n, q^2)$. We also give some computational results.

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Chapter 1

Introduction

1.1 Summary of Results

In the first chapter we introduce the prerequisite concepts needed to understand the question that this paper works on: Which conjugacy classes of $\mathrm{U}(n, q^2)$ are strongly real when q is a power of 2. In particular we define conjugacy classes and strongly reality in the second section, and the finite unitary group in the third section. In addition we give basic results associated with these, some of which we use later.

In the second chapter, first section, we develop two famous canonical forms, the Jordan canonical form and rational canonical form. These allow us to specify conjugacy classes by (in effect) listing the elementary divisors of a member, and answer some question related to the topic of this paper (which conjugacy classes are strongly real in $\mathrm{GL}(n, q)$ and which conjugacy classes are real in $\mathrm{U}(n, q^2)$) in the second section.

In the third and final chapter, first section, we present the classification of conjugacy classes of $\mathrm{U}(n, q^2)$ when q is odd. In the second section we layout which conjugacy classes were previously known to be (or not be) strongly real in $\mathrm{U}(n, q^2)$ when q is a power of 2. In the third section we present a conjugacy class recently determined to not be strongly real, and then give a partial classification of the strongly real classes in $\mathrm{U}(n, q^2)$. Next we show that if one assumes that the (unipotent) classes corresponding to $(5^r, 4^l, 2^k, 1^m)$ and $(3^r, 2^k)$, r odd, are not strongly real, then the partial classification becomes complete. Finally, under the conjectured classification, we give a generating function for the number of strongly real unipotent classes in $\mathrm{U}(n, q^2)$, q a power of 2. In the last section we list some computational results that lend support to the above assumption. The appendix provides the GAP code that was used.

1.2 Conjugacy Classes

Throughout this section G is an arbitrary group unless stated otherwise.

Definition 1.1. Let $a, b \in G$.

- a is said to be *conjugate* to b if there is an h in G for which $hah^{-1} = b$. In this case we say h conjugates a to b.
- An element s of G is said to be an *involution* if $s^2 = 1$.
- a is called *real* if there is an s in G that conjugates a to a^{-1}.
- a is called *strongly real* if there is an involution s that conjugates a to a^{-1}.

It follows directly that the relation on G given by $a \sim b$ iff a is conjugate to b, is an equivalence relation. The corresponding equivalence classes of $\sim$ are called the conjugacy classes of G and are identified by naming a member within them. Furthermore,

Theorem 1.2. *If x in G is (strongly) real, then so is every element of its conjugacy class.*

Proof. Suppose x is real and let y be an element of its conjugacy class. This means $y \sim x$ so that there is a g for which $gyg^{-1} = x$. Also, since x is real there is an h for which $hxh^{-1} = x^{-1}$. Substituting the first of these equations into the second gives $hgyg^{-1}h^{-1} = (gyg^{-1})^{-1} = (g^{-1})^{-1}y^{-1}g^{-1} = gy^{-1}g^{-1}$. Multiplying on the left by g^{-1} and on the right by g gives $g^{-1}hgyg^{-1}h^{-1}g = (g^{-1}hg)y(g^{-1}hg)^{-1} = y^{-1}$ so that y is conjugate to y^{-1} and hence real. Now instead suppose x is strongly real with h an involution conjugating x to x^{-1}. Then, as before, we will have $(g^{-1}hg)y(g^{-1}hg)^{-1} = y^{-1}$ so that $g^{-1}hg$ conjugates y to y^{-1}. Since $g^{-1}hg$ is an involution ($(g^{-1}hg)^2 = g^{-1}hgg^{-1}hg = g^{-1}h(gg^{-1})hg = g^{-1}h1hg = g^{-1}h^2g = g^{-1}g = 1$), y is strongly real. Since y is arbitrary, every element of the conjugacy class of x is (strongly) real when x is (strongly) real. $\square$

Because of this, we refer to a conjugacy class itself as (strongly) real if one (so that all) of its members are (strongly) real. The study of conjugacy classes can be motivated by the fact that the equivalence classes under matrix similarity in $\mathrm{GL}(n, \mathbb{C})$ are also the conjugacy classes of $\mathrm{GL}(n, \mathbb{C})$, and similar matrices are known to share many properties [6]. Also, there are some nice things that are equivalent to being real or strongly real: An element g of finite group G is real iff for every complex representation ρ of G, $\mathrm{Tr}(\rho(g))$ is real [7, Problem 2.11]. This is why such elements are called real. Also,

Proposition 1.3. *An element is strongly real iff it is a product of two involutions.*

Proof. Let x be strongly real so that there is an involution $h \in G$ for which $h^{-1}xh = x^{-1}$. Since $h^2 = 1$, $h^{-1} = h$ and we have $hxh = x^{-1}$. Inverting both sides gives $hx^{-1}h = x$. Since hx^{-1} is an involution ($hx^{-1}hx^{-1} = (hx^{-1}h)x^{-1} = xx^{-1} = 1$) and h is an involution, x is a product of two involutions. Now let x be equal to hs where h and s are involutions. Then $x^{-1} = s^{-1}h^{-1} = sh = 1sh = h^{-1}hsh = h^{-1}(hs)h = h^{-1}xh$, where h is an involution. Hence, x is strongly real. $\square$

It should be clear from definition that all strongly real elements are real. Thus, partitioning into strongly real classes refines the partition of G into real classes.

All the elements of a group can be strongly real as is the case with S_n [3, Proposition 1.2] (which implies that all the elements of a group can be real), and

Proposition 1.4. *There are groups with none of their nonidentity elements real. In particular, all Abelian groups of odd order have only their identity element being real.*

Proof. Note that the identity of any group is strongly real and thus real. This is because 1 is an involution which conjugates 1 to its inverse (1). Let G be an Abelian group of odd order (an example is $\mathbb{Z}_9$) and suppose g is a real nonidentity element of G. Then, there is an h in G for which $hgh^{-1} = g^{-1}$. Since G is Abelian, $hgh^{-1} = hh^{-1}g = g$ so $g = g^{-1}$. Multiplying on both sides by g gives $g^2 = 1$, so since g is not the identity, it has order 2. Lagrange's Theorem implies that the order of any element divides the order of the group, so 2 divides $|G|$. But $|G|$ is odd, so we have a contradiction. Hence, G has no nonidentity real elements. □

1.3 U(n, q^2) and U(V)

Now that we've defined strongly real conjugacy classes, we work towards defining U(V), the unitary group. First, GL(V) is defined as the group of invertible linear transformations from an n-dimensional F-vector space to itself. After choosing a basis, these transformations correspond to the group of invertible matrices in $M_n(F)$, denoted GL(n, F), regardless of which F-vector space we use. Recall that every finite field has its size a prime power. Also, for any prime power p^n, there is a unique field of size p^n denoted $\mathbb{F}_{p^n}$. In the case that $F = \mathbb{F}_{p^m} = \mathbb{F}_q$, GL$(n, F)$ will be denoted by GL(n, q). Now, we will define U(n, q^2) as the matrix group $\{g \in GL(n, \mathbb{F}_{q^2}) | g^* g := \overline{g^t} g = 1\}$ where conjugation will be entrywise and the conjugate of $a \in \mathbb{F}_{q^2}$ is a^q (the Frobenius automorphism applied to a). Note that this is a natural analogue to the set of unitary matrices with complex value entries, since the Frobenius automorphism is like complex conjugation in that it is an order two field automorphism. In fact, this is really the only natural analogue to unitary matrices in $M_n(\mathbb{C})$, since

Proposition 1.5. *The Frobenius automorphism is the unique order two automorphism on $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$.*

Proof. First recall the following fact from Galois Theory: If K is the splitting field over F of a separable polynomial $f(x)$, then K is Galois over F (meaning $|\mathrm{Aut}(E/F)| = [E : F]$). We know that $\mathbb{F}_{q^2}$ is the splitting field of the separable $x^{q^2} - x$ and so also is a degree two (a.k.a quadratic) extension of $\mathbb{F}_q$. The above fact then gives that the Automorphism group of F_{q^2} over F_q is of size 2. But we are already aware of two automorphisms in this group: the identity and the Frobenius automorphism. The identity is of order 1 and the Frobenius automorphism is of order 2, so the proposition follows. $\square$

This definition of $\mathrm{U}(n, q^2)$ is not usually the most useful for our computations. For our purposes, we will define a collection of groups that are all isomorphic to $\mathrm{U}(n, q^2)$. Then, since isomorphism induces a bijection between conjugacy classes, we can work within the collection we will define.

A matrix J in $M_n(\mathbb{F}_{q^2})$ is said to be Hermitian when $J^* = J$. This is analogous to the definition of Hermitian matrices in $M_n(\mathbb{C})$. Now, note that our definition of $\mathrm{U}(n, q^2)$ may be written as

$$\{g \in \mathrm{GL}(n, q^2) | g^* 1 g := \overline{g^t} 1 g = 1\}.$$

and 1 is a Hermitian matrix. Replacing 1 by any other invertible Hermitian matrix J, will give a group isomorphic to $\mathrm{U}(n, q^2)$ [3, Proposition 2.11] , which we now discuss further. We will call this new group the finite unitary matrix group defined by the Hermitian matrix J. One reason we care about these groups is because they are the matrix groups we get upon choosing bases for V in $\mathrm{U}(V)$.

Definition 1.6. A sesquilinear form on V, an n dimensional F-vector space, relative to an order two automorphism represented by conjugation, is a function $B : V \times V \to F$ for which

- $B(u+v, w) = B(u, w) + B(v, w)$
- $B(u, v+w) = B(u, v) + B(u, w)$
- $B(au, v) = aB(u, v)$ and $B(u, av) = \overline{a}B(u, v)$

for all $u, v, w \in V$ and $a \in F$. If B also satisfies

- $B(u, v) = \overline{B(v, u)}$ for all $u, v \in V$

then B is called a Hermitian form.

For any Hermitian form B and basis for V, $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$, there is a matrix $\hat{B}_{\mathcal{B}}$, $\hat{B}$ for short, given by $\hat{B}_{ij} = B(v_i, v_j)$ for which $B(u, v) = [u]^t_{\mathcal{B}}\hat{B}[v]_{\mathcal{B}}$ for all $u, v \in V$ [5, Chapter 10]. Such matrices have the property that $\hat{B}^* = \hat{B}$ (since $B(v_i, v_j) = \overline{B(v_j, v_I)}$ for all i and j) and are thus Hermitian.

If $\mathcal{C} = \{w_1, w_2, \ldots, w_n\}$ is another basis for V and D is a matrix such that $w_j = \sum_i D_{ij}v_i$, then $\hat{B}_{\mathcal{C}} = D^t\hat{B}_{\mathcal{B}}\overline{D}$ and so $\det \hat{B}_{\mathcal{C}} = \det D^t \det \hat{B}_{\mathcal{B}} \det \overline{D} = \det D \det \hat{B}_{\mathcal{B}}\overline{\det D}$. Since D is a change of basis matrix, it is invertible and so $\det \hat{B}_{\mathcal{C}}$ is nonzero for a basis $\mathcal{C}$ if and only if $\det \hat{B}_{\mathcal{B}}$ is nonzero for *every* basis $\mathcal{B}$ of V. This motivates us to define a non-degenerate Hermitian form as a Hermitian B for which $\det \hat{B}_{\mathcal{B}} \neq 0$ for some basis $\mathcal{B}$ of V. Equivalently, a non-degenerate Hermitian form is one whose associated Hermitian matrices are invertible.

Definition 1.7. Let E/F be a field extension and V be an E-vector space. Then V together with with a non-degenerate Hermitian form H is called a unitary space over E/F. The space of invertible transformations that are isometries relative to H is a group called the unitary group relative to H. It is denoted $\mathrm{U}(V)$.

After picking a basis, we see that $\mathrm{U}(V)$ is isomorphic to $\{T \in \mathrm{GL}(n, E) | T^t \hat{H} \overline{T} = \hat{H}\}$. In the case that this is a finite group, picking a different invertible Hermitian $\hat{H}$ gives a group isomorphic to this one [5, Corollary 10.4]. We see that $\mathrm{U}(V)$ is independent (up to isomorphism) of the choice of non-degenerate Hermitian form.

Now we will derive the correspondence between partitions of n and conjugacy classes of $\mathrm{U}(n, q^2)$. Then, we will present results classifying the conjugacy classes of $\mathrm{U}(n, q^2)$.

Chapter 2

Partitions and Conjugacy Classes

An element of $M_n(\mathbb{F}_{q^2})$, g, is called unipotent if $(g-1)^m = 0$ for some m. Notice that if g is unipotent, then so is every element of its conjugacy class. To see this, let $g = hxh^{-1}$ be such that $(g-1)^m = 0$. Then $(x-1)^m = (h^{-1}gh - h^{-1}h)^m = (h^{-1}(g-1)h)^m = h^{-1}(g-1)hh^{-1}(g-1)hh^{-1}\cdots(g-1)h = h^{-1}(g-1)^m h = 0$. Hence we call a conjugacy class unipotent when one, so that all, of its members are unipotent. The unipotent conjugacy classes of $\mathrm{U}(n, q^2)$, q even, are parameterized by the partitions of n. We derive this parameterization.

2.1 The Rational and Jordan Canonical Forms

All of the results in this section are proved or given as exercises in [1, Chapter 10] and [1, Chapter 12].

Definition 2.1. Let R be a ring. A *left R-module* is a set M together with a binary operation $+$ on M and an action of R on M such that M is an abelian group with respect to $+$ and for any $r, s \in R$ and $m, m' \in M$, $(r+s)m = rm + sm$, $r(m+m') = rm + rm'$, and $(rs)m = r(sm)$. If R has an identity 1, we also require that $1m = m$.

Note the similarity of the definition of a R-module to an F-vector space. You might expect then that any F- vector space, V, is also a (left) F-module . This is true because any field is a ring, any vector space is an abelian group with respect to addition, addition of field elements distributes over a vector, the action by the field (scalar multiplication) distributes over a sum of vectors, and $(ab)v = a(bv)$ and $1v = v$ for any $a, b \in F$ and $v \in V$. Hence, modules are a generalization of vector spaces. Also note that we can take M to be R. The definition of a ring then implies that R is a left R-module. This is analogous to how any field is a vector space over itself.

Now, some rules of arithmetic in left R-modules:

Lemma 2.2. *Let M be a left R-module where R is a ring with 1. If $r \in R$ and $m \in M$ then*

- $0m = 0$
- $r(-m) = -(rm) = (-r)m$
- $r0 = 0.$

Proof. $m + 0m = 1m + 0m = (1+0)m = 1m = m$. So we have $m + 0m = m = m + 0$. Cancelling m on the left gives $0m = 0$. Next, since $(-1)m + m = (-1)m + 1m = (-1+1)m = 0m$, the above gives that $(-1)m + m = 0$ so that. $(-1)m = -m$. Now from the above, $r(-m) = r((-1)m) = (r(-1))m = (-r)m$. Also, since $(-r)m + rm = ((-r) + r)m = 0m = 0$, $(-r)m = -(rm)$. Finally, $r0 + m = r(-m + m) + 1m = r(-m) + rm + 1m = (-r)m + rm + 1m = ((-r) + r + 1)m = 1m = m$ for all $m \in M$ so $r0 = 0$. $\square$

The following is an important class of R-modules that we will use to get important results on representatives of matrix conjugacy classes.

Theorem 2.3. *Let F be a field, R be the polynomial ring $F[x]$, V, an F-vector space, and T be a linear transformation from V to V. Then V, with action by $F[x]$ (with respect to T) is given by $p(x)(v) = p(T)(v) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0)(T)(v) = (a_n T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I)(v)$ (where I is the identity transformation and T^n is defined by, $T^1 = T$, $T^n = T \circ T^{n-1}$), is a left $F[x]$-module.*

Proof. As a vector space, V is an Abelian group with respect its addition. The definition above gives an action since the space of linear transformations between F-vector spaces V and W is itself an F-vector space, and a composition of linear transformations is a linear transformation. The necessary properties in the definition follow directly. $\square$

$F[x]$-modules are natural to consider once we notice that considering the action by the subring F (actually, the constant polynomials of $F[x]$) of $F[x]$, gives V as an F-vector space. In other words, since V is an F-vector space, we know how F acts on V. The definition above then gives a way to extend the action by F to an action by $F[x]$.

In order to state the needed results, we must present several more definitions

Definition 2.4. Let R be a ring and M be an R-module.

- A subgroup N of M which is closed under the action by R is called an *R-submodule of M*. Formally, it a subgroup N of M such that $rn \in N$ for any $r \in R$ and $n \in N$.

- $m \in M$ is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements is denoted $\mathrm{Tor}(M)$. A module is *torsion free* if $\mathrm{Tor}(M) = \{0\}$ and is a *torsion module* if $\mathrm{Tor}(M) = M$.

- The *annihilator* of a submodule N in R is the set of $r \in R$ for which $rn = 0$ for any $n \in N$. It is denoted $\mathrm{Ann}_R(N)$

Proposition 2.5. *If R is an integral domain and M is an R-module, then Tor(M) is a submodule of M. In this case, Tor(M) is called the torsion submodule of M. The converse can be true; there is ring S that is not an integral domain and an Abelian group N for which N is an S-module and Tor(N) is not a submodule of N. Also, the annihilator of an R-submodule is a two-sided ideal of R.*

Proof. Suppose R is an integral domain and let $m, n \in \text{Tor}(M)$. Since $m, n \in \text{Tor}(M)$, there are nonzero $r, s \in R$ for which $rm = sn = 0$. Since R is an integral domain, it has no zero divisors, and so $rs \neq 0$. Also, being an integral domain, R is commutative, so $rs = sr$. Compute $rs(m - n) = (rs)m + (rs)(-n) = (sr)m + (rs)(-n)$. Using the lemma, this is $s(rm) - (rs)(n) = s0 - r(sn) = 0 - r0 = 0 - 0 = 0$. Since $rs \neq 0$ and $rs(m - n) = 0$, $m - n \in \text{Tor}(M)$. It follows that $\text{Tor}(M)$ is a subgroup of M. Now let $r \in R$ be nonzero and $n \in \text{Tor}(M)$. By the definition of $\text{Tor}(M)$ there is a nonzero $s \in R$ for which $sn = 0$. Since R is an integral domain, $rs = sr \neq 0$ and so $s(rn) = (sr)n = (rs)n = r(sn) = r0 = 0$. If r were 0, then $s(rn) = s0 = 0$. Either way, $rn \in \text{Tor}(M)$ and by definition, $\text{Tor}(M)$ is a submodule of M. The converse can be true since $\mathbb{Z}_6$ is not an integral domain and the $\mathbb{Z}_6$-module, $M = \mathbb{Z}_6$, has torsion elements 2 and 3 ($2, 3 \neq 0$, $2(3) = 3(2) = 0$) whose sum is 5 which is not a torsion element (so that $\text{Tor}(M)$ is not a subgroup of M). Now, to show that the annihilator of a submodule N is an ideal: Suppose $r, s \in \text{Ann}_R(N)$. Then for any $n \in N$, $(r - s)n = (r + (-s))n = rn + (-s)n = 0 - (sn) = 0 - 0 = 0$, where we have used the fact that $s, r \in \text{Ann}_R(N)$. Also, if $r \in R$ (not necessarily in $\text{Ann}_R(N)$) , $(rs)n = r(sn) = r0 = 0$ so we have that $\text{Ann}_R(N)$ is a subring of R and that $\text{Ann}_R(N)$ is closed under left multiplication by elements of R. It is similarly closed under right multiplication by elements of R: If $r \in R$ then $rn \in N$ since N is a submodule and so $(sr)n = s(rn) = 0$. By definition, $\text{Ann}_R(N)$ is an ideal of R. □

The following proposition is straightforward and analogous to how the set of linear combinations of vectors in a vector space is a subspace.

Proposition 2.6. *Let R be a ring with 1 and M be an R-module. If A is a subset of M, then $RA := \{r_1a_1 + r_2a_2 + \cdots + r_ka_k | r_i \in R$ and $a_i \in A$ for all $i \in [k]$ for some $k \in \mathbb{N}\}$ is a submodule of M.*

Definition 2.7. RA above is called the *submodule of M generated by A*. A submodule N of M is said to be *finitely generated by A* if there is a finite subset A of M for which $N = RA$. A submodule N of M is said to be *cyclic* if it is generated by a singleton. A module M is *free* (analogous to a free group) if it is generated by A and its elements may uniquely be written as finite R-linear combinations of elements from A. Such an A is called a *basis* for M. Here we will say that the *rank* of M is the maximal number of R-linearly independent elements of M where linear independence is what you would expect.

Another proposition analogous to one for vector spaces is the following.

Proposition 2.8. *Let M_1, $M_2, \ldots, M_k$ be R-modules. Then the external direct sum of $M_1, \ldots, M_k$ defined as $M_1 \times M_2 \times \cdots \times M_k$, with addition and action by R defined componentwise, is an R-module denoted by $M_1 \oplus M_2 \oplus \cdots \oplus M_k$,*

Definition 2.9. Let M and N be R-modules. A mapping ϕ from M to N that is a group homomorphism for which $\phi(rm) = r\phi(m)$ for all $r \in R$ and $m \in M$ is called an R-module homomorphism. A R-module isomorphism is a bijective R-module homomorphism. If there is an R-module isomorphism between M and N, then we say M is isomorphic to N and write $M \cong N$. The kernel and image of ϕ are what one would expect. The set of all R-module homomorphisms from M to N is denoted $\mathrm{Hom}_R(M, N)$.

There are isomorphism theorems for R-modules that are similar (with similar proofs) to those for groups and rings:

Theorem 2.10. *If M and N are R-modules and $\phi : M \to N$ is a R-module homomorphism, then ker ϕ is a submodule of M and $M/ker\ \phi \cong \phi(M)$.*

Recall that a principal ideal domain R is an integral domain in which every ideal is of the form (r) for some $r \in R$. From now on suppose R is a principal ideal domain, or PID for short.

Let C be a cyclic R-module so that there is some $x \in C$ for which every element in C is of the form rx where $r \in R$. The existence of such an x suggests defining the R-module homomorphism $\phi : R \to C$ by $\phi(r) = rx$ (it is easy to check that this is a homomorphism). The above theorem then gives that $R/\ker \phi \cong C$. The above theorem also says that $\ker \phi$ is a submodule of R, which implies that it is an ideal of R. Then, since R is a P.I.D., $\ker \phi = (a)$ for some $a \in R$. But the kernel consists of exactly those elements of R for which right multiplying by x gives 0. Thus, letting $rx \in C$ and $s \in (a)$, we see that $s(rx) = (sr)x = (rs)x = r(sx) = r0 = 0$ so that $s \in \mathrm{Ann}_R(C)$. If $s \in \mathrm{Ann}_R(C)$ then for any $r \in R$, $0 = s(rx) = (sr)x$. In particular $sx = s(1x) = 0$ so that $s \in \ker \phi = (a)$. We see that $(a) = \mathrm{Ann}_R(C)$ and conclude that all cyclic submodules C of R are of the form $R/(a)$ where $(a) = \mathrm{Ann}_R(C)$.

We are now finally in a position to state the fundamental results that will greatly simplify the problem of classifying the conjugacy classes of $\mathrm{U}(n, q^2)$. I've presented enough that one should be able to read the proof in Dummit and Foote [1, Section 12.1].

Theorem 2.11. *If R is a P.I.D. and M is a finitely generated R-module then*

- *M is isomorphic to the direct sum of finitely many cyclic modules. In particular, $M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$ for some integer $r \geq 0$ and nonzero a_i that are not units and satisfy $a_1 | a_2 | \cdots | a_k$. This is called the invariant factor decomposition.*

- *M is torsion free iff M is free.*

- *$Tor(M) \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$ so that M is a torsion module iff $r = 0$. If $r = 0$ then $Ann_R(M) = (a_k)$.*

Note that the decomposition given above is guaranteed to be an R-module since the (a_i) are annihilators of submodules of M so that they are ideals of R (Proposition 2.5). Hence, each $R/(a_i)$ is a subring of R (and hence an R-module), and by Proposition 2.8 the external direct sum of these with R^r is an R-module.

Since every PID, R, is a UFD (unique factorization domain), every nonzero a in R can be written as $up_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ where each p_i is prime and u is a unit. Then, since each p_i is prime, $(p_i^{\alpha_i}) + (p_j^{\alpha_j}) = (gcd(p_i^{\alpha_i}, p_j^{\alpha_j})) = (1) = R$ for any $i \neq j$. Also, since $\bigcap_{i=1}^{s}(p_i^{\alpha_i}) = (lcm(p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_s^{\alpha_s})) = (a)$, the Chinese Remainder Theorem gives that $R/(a) \cong R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots, R/(p_s^{\alpha_s})$. Applying this procedure to each cyclic module in the above decomposition gives:

Theorem 2.12. *If R is a PID and M is a finitely generated R-module, then, $M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_s^{\alpha_s})$ for some whole numbers r and s.*

Dummit and Foote proves that these decompositions are unique. The prime powers in the above decomposition are called the elementary divisors of M.

Let V be an F-vector space of dimension n and T be a linear transformation on V. Then, there exists a set (a basis) $\mathcal{B}$ of size n such that V is the set of F-linear combinations of elements from $\mathcal{B}$. Equivalently, V is a finitely generated F-module. Then, since F can be identified with a subset of $F[t]$ that acts identically to F on V, the set of F-linear combinations from $\mathcal{B}$, V, is contained in the set of $F[t]$-linear combinations from $\mathcal{B}$. Hence, V is a finitely generated $F[t]$-module. Recall that since F is a field, $F[t]$ is a Euclidean domain and that all Euclidean domains are PIDs. Hence $F[t]$ is a PID, and so the above theorems hold for V. If in the decomposition of V given in Theorem 2.11, $r \neq 0$, then V would contain a submodule (here the same as a subspace) isomorphic to $F[t]$ ($V \cong F[t]^r \oplus F[t]/(a_1) \oplus F[t]/(a_2) \oplus \cdots \oplus F[t]/(a_k) \supset F[t] \oplus \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\}$). But $F[t]$ is of infinite dimension as an F-vector space, and so no subspace of V can be isomorphic to it (since such a subspace would have infinite dimension). We have a contradiction. Hence $r = 0$ in the decomposition above and by Theorem 2.11, V is a torsion module.

Now, for any $a_nt^n + \cdots + a_0 = f(t) \in F[t]$, since $F(t)$ is an F-vector space, $F[t]/(f(t))$ is an F-vector space ($(f(t))$ is a one-dimensional subspace). Then, since each member of $F[t]/(f(t))$ can be represented by a representative's remainder upon division by $f(t)$, $\{1 + (f(t)), t + (f(t)), \ldots, t^{n-1} + (f(t))\}$ is a basis for $F[t]/(f(t))$ (There is no nontrivial linear combination of these that is 0 because there is no way to combine them to get a term with nonzero nth degree term. Moreover, for any remainder r, r is a polynomial with degree at most $n - 1$ and so can be written as a linear combination of these.) Let $a(t) = t^k + b_{k-1}t^{k-1} + \cdots b_0$. Then the linear transformation on $F[t]/(a(t))$ given by $g(t) + (a(t)) \to tg(t) + (a(t))$ has matrix representation relative to the above basis. It is

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & -b_0 \\ 1 & 0 & \cdots & \cdots & \cdots & -b_1 \\ 0 & 1 & \cdots & \cdots & \cdots & -b_2 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & -b_{k-1} \end{pmatrix}.$$

By convention, blank entires are 0. Any matrix of this form is called the companion matrix of $a(t) = t^k + b_{k-1}t^{k-1} + \cdots + b_0$, and is denoted $\mathcal{C}_{a(t)}$. Since $T(v) = tv$ for any $v \in V$, and $V \cong F[t]/(a_1(t)) \oplus F[t]/(a_2(t)) \oplus \cdots \oplus F[t]/(a_k(t))$, where all the a_i can be made unique by taking them to be monic, the matrix of T relative to the basis of vectors in V corresponding (under the isomorphism of the decomposition) to

$\{(a_i(t)+t^j)e_i \mid i \in [k] \text{ and } j \in [\deg(a_i(t))]\}$ is

$$\begin{pmatrix} \mathcal{C}_{a_1(t)} & & & \\ & \mathcal{C}_{a_2(t)} & & \\ & & \ddots & \\ & & & \mathcal{C}_{a_k(t)} \end{pmatrix}.$$

This is called the rational canonical form of the linear transformation T. It is unique up the order of factors when the a_i are required to be monic.

Since matrix similarity corresponds to a change of bases, if two matrices have the same Rational Canonical Form, then they are similar and vice versa. Since invertibility is preserved under similarity and the equivalence classes under similarity are exactly the conjugacy classes in $\mathrm{GL}(n,q)$, we may represent conjugacy classes in $\mathrm{GL}(n,q)$ by matrices in Rational Canonical Form.

The Jordan Canonical Form is derived in [1, Section 12.3] using the elementary divisor decomposition similar to how we have used the invariant factor decomposition above to obtain the rational canonical form. The Jordan Canonical Form is used to prove many results about matrices with complex entries in Matrix Analysis by Horn and Johnson. An alternative way of getting these canonical forms is presented in Linear Algebra by Friedberg, Insel, and Spence [2, Chapter 7] . Here is the Jordan Canonical Form:

Definition 2.13. A Jordan block of size k corresponding to Λ is a k-by-k matrix with Λ down the main diagonal and 1 down the superdiagonal. It has different notations but here will be denoted here as $J_k(\Lambda)$.

Theorem 2.14. *If V is a finite dimensional F-vector space and T is a linear transformation on V such that F contains all the eigenvalues of T, then there is a basis in which the matrix representation of T is $J_{\Lambda_1} \oplus J_{\Lambda_2} \oplus \cdots \oplus J_{\Lambda_k}$, where $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$*

are the eigenvalues of T and each J_{Λ_i} is a direct sum of Jordan blocks corresponding to Λ_i.

2.2 Partitions and Conjugacy Classes

Let $g \in \mathrm{GL}(n,q)$. For any irreducible $f \in \mathbb{F}_q[t]$ we may consider the elementary divisors of g that are powers of f, namely $f^{\lambda_1}, f^{\lambda_2}, \ldots, f^{\lambda_k}$ (note that some or all of the exponents here can be equal). We define $\lambda^{(f)}$ as $(\lambda_1, \lambda_2 \ldots, \lambda_k)$. Then it follows from the elementary divisor decomposition that $\sum_f \deg(f)|\lambda^{(f)}| = n$ and $g \sim \prod_f \mathcal{C}_{\lambda^{(f)},f}$ where $\mathcal{C}_{\lambda^{(f)},f} = \mathcal{C}_{f^{\lambda_1}} \times \mathcal{C}_{f^{\lambda_2}} \times \cdots \times \mathcal{C}_{f^{\lambda_k}}$ and $\lambda^{(f)} = (\lambda_1, \lambda_2 \ldots, \lambda_k)$. Consider $f = a_d x^d + \cdots + a_0$, an irreducible polynomial over $\mathbb{F}_q$. There is a minimal n for which $\mathbb{F}_{q^n}$ contains all of f's roots (all finite field extensions are of this form and $\overline{\mathbb{F}}_q = \bigcup_{n\geq 1} \mathbb{F}_{q^n}$, see [1, Section 14.3]). Thus, some $\mathbb{F}_{q^n}$ is the splitting field of f and generalizing Proposition 1.5 we see that the roots of f are α, α^q, α^{q^2}, $\ldots$, $\alpha^{q^{d-1}}$ for some α. Hence, over $\overline{\mathbb{F}}_q$ every $\mathbb{F}_q$ irreducible polynomial breaks into linear factors. Using this and the Jordan Canonical Form we can see that $g \sim \prod_f J_{\lambda^{(f)},f}$ in GL(n, $\overline{\mathbb{F}}_q$) where $\lambda^{(f)} = (\lambda_1, \ldots, \lambda_k)$, $J_{\lambda^{(f)},f} = J_{\lambda_1,f} \times \cdots \times J_{\lambda_k,f}$, and $J_{\lambda_i,f}$ is a product of Jordan blocks corresponding to the roots of f^{λ_i}.

Looking at g in Jordan Canonical Form, we can see that g^{-1} is conjugate to the Jordan canonical matrix of g with each diagonal entry replaced with its inverse. Thus, the elementary divisors of g^{-1} correspond to the elementary divisors of g in the following way. If f that factors to $(x-\alpha)(x-\alpha^q)\cdots(x-\alpha^{q^{d-1}})$ over $\overline{\mathbb{F}}_q$ is an elementary divisor of g, then $\overline{f}$ that factors to $(x-\alpha^{-1})(x-\alpha^{-q})\cdots(x-\alpha^{-q^{d-1}})$ over $\overline{\mathbb{F}}_q$ is an elementary divisor of g^{-1}. It follows that g is real iff

$$g \sim [\prod_{f\neq\overline{f}} (\mathcal{C}_{\lambda^{(f)},f} \times \mathcal{C}_{\lambda^{(\overline{f})},\overline{f}})] \times [\prod_{f=\overline{f}} \mathcal{C}_{\lambda^{(f)},f}]$$

.

The realness of conjugacy classes in $\mathrm{U}(n,q^2)$ is similar, except instead of associating $f=(x-\alpha)(x-\alpha^q)\cdots(x-\alpha^{q^{d-1}})$ with $(x-\alpha^{-1})(x-\alpha^{-q})\cdots(x-\alpha^{-q^{d-1}})$ we associate it with $(x-\alpha)(x-\alpha^{-q})(x-\alpha^{q^2})\cdots(x-\alpha^{(-q)^{d-1}})$. It is shown in [3] that an element g is strongly real iff its unipotent part (matrix with elementary divisors the elementary divisors of g of the form $(t-1)^k$) is strongly real. When g is unipotent and q is even, its elementary divisors are all of the form $(t-1)^k$ and since conjugacy is determined by elementary divisors, the conjugacy classes of g correspond to partitions $(k_1^{m_{k_1}}, k_2^{m_{k_2}}, \ldots, k_i^{m_{k_i}})$ where each $(t-1)^{k_i}$ occurs m_{k_i} times as an elementary divisor of g.

From here on some familiarity with partitions is needed. A partition is written as $(k_1^{m_{k_1}}, k_2^{m_{k_2}}, \ldots, k_i^{m_{k_i}})$ with each $m_i \geq 1$ where the bases are the parts and each superscripted number is the number of times that that part appears, called its multiplicity. If a number is not a part, we say that it has multiplicity 0. Also, we write the parts in decreasing order. For example, $(3^1, 2^2)$ or $(3, 2^2)$ is a partition of 7 since 7 can be written as a sum of one 3 and two 2s. 1 has multiplicity 0 here.

When a conjugacy class corresponds to the partition $(k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 2^{m_2}, 1^{m_1})$ (remember that this means that its elements have $(t-1)^k$ as an elementary divisor m_k times, $(t-1)^{k-1}$ as an elementary divisor m_{k-1} times, and so on) where we allow ourselves to write in numbers with 0 multiplicity (except require that $m_k > 0$), giving them a superscript of 0 for convenience, we say that that class is of type $(k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 2^{m_2}, 1^{m_1})$. Since any element of a fixed conjugacy class corresponds to the same partition, by "parts of the element g" I mean the parts of the partition corresponding to g. I will also say that a partition is strongly real if elements of its type are strongly real.

Chapter 3

Classification of Conjugacy Classes

3.1 Odd Characteristic

The abbreviation sr will often be used for strongly real henceforth.

The following lemma is true regardless of the characteristic of the field. It is how most of the general results are derived. It comes from [4, Proposition 4.1]:

Lemma 3.1. *Suppose the unipotent class of type* $\mu = (k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 2^{m_2}, 1^{m_1})$ *is s.r.. Then the following are strongly real partitions:*

- $((k-2)^{m_k}, (k-3)^{m_{k-1}}, \ldots, 2^{m_4}, 1^{m_1+m_3})$ *when* $m_2 > 0$.
- $((k-2)^{m_k}, (k-3)^{m_{k-1}}, \ldots, l^{m_{l+2}}, (l-1)^{m_{l-1}+m_{l+1}}, (l-2)^{m_{l-2}+m_l}, (l-3)^{m_{l-3}}, (l-4)^{m_{l-4}}, \ldots, 1^{m_1})$ *when* l *for which* $3 \leq l < k$ *has positive multiplicity.*
- $((k-1)^{m_{k-1}}, (k-2)^{m_{k-2}+m_k}, (k-3)^{m_{k-3}}, (k-4)^{m_{k-4}}, \ldots, 1^{m_1})$ *when* k *is the largest part of* μ.

This lets one show that $(k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 2^{m_2}, 1^{m_1})$ is *not* strongly real by showing that one of the types above with a smaller largest part is not strongly real.

A basic result that is very useful in piecing together particular cases into general results is:

Proposition 3.2. *Let G_i be a group for all $i \in [n]$. An element $x = (x_1, x_2, \ldots, x_n) \in \prod_{i=1}^n G_i$ is strongly real if and only if each x_i is strongly real in G_i.*

The idea behind this is that if s_i are the conjugating involutions of g_i then take their direct sum for a conjugating involution of the product of g_i and vice versa

Since a direct sum of unipotent elements is unipotent, this allows us to combine strongly real partitions to get new strongly real partitions.

Example: We will show (3,1) for even characteristic is strongly real. (1) for even characteristic is also strongly real so it follows that (3,1,1) is strongly real.

The classification of strongly real conjugacy classes of $\mathrm{U}(n, q^2)$ for q odd was completed in [4]. Here it is:

Theorem 3.3. *A real element g of $U(n, q^2)$ is strongly real if and only if every elementary divisor of the form $(t \pm 1)^{2m}$ of g has even multiplicity.*

In the same article, a generating function for $T_{n,q}$, the number of strongly real classes in $\mathrm{U}(n, q^2)$ where $n = 0$ corresponds to the group $\{1\}$ was provided:

$$\sum_{n=0}^{\infty} T_{n,q} z^n = \prod_{k=1}^{\infty} \frac{(1 + qz^{2k-1})^2}{1 - qz^{2k}}$$

3.2 Even Characteristic, Some Past Results

The following results come from [8]. As you might expect from the title of the section, we let q be a power of 2 throughout.

Proposition 3.4. *A unipotent element of type $(5, 1^r)$ in $U(5 + r, q^2)$ is not strongly real.*

Proof. Consider the group of unitary matrices relative to $J = N_5 \oplus I_r$ where N_i is the

i by i matrix with an all ones antidiagonal and zeroes elsewhere. Let

$$g = \begin{pmatrix} 1 & a & b & 0 & c & 0 \\ 0 & 1 & \overline{a} & \overline{b} & 0 & 0 \\ 0 & 0 & 1 & a & b & 0 \\ 0 & 0 & 0 & 1 & \overline{a} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r \end{pmatrix}$$

where the 0s in the right column are 1 by r all zero matrices and $a \neq 0$, $b + \overline{b} = a\overline{a}$, and $c + \overline{c} = b\overline{b}$. We can choose such a, b, and c since the norm map $a \to a\overline{a}$ is onto $\mathbb{F}_q$. To see that the norm map is onto, recall that the units of a finite field form a cyclic multiplicative group and consider the image of a generator of $\mathbb{F}_{q^2}$ noting that the norm map is multiplicative. Note that

$$g^{-1} = \begin{pmatrix} 1 & a & \overline{b} & 0 & \overline{c} & 0 \\ 0 & 1 & \overline{a} & b & 0 & 0 \\ 0 & 0 & 1 & a & \overline{b} & 0 \\ 0 & 0 & 0 & 1 & \overline{a} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r \end{pmatrix}$$

and g is unipotent, since $g - 1$ is strictly upper triangular. It is also of type $(5, 1^r)$ (direct sum of a matrix with $(t-1)^5$ as an elementary divisor once and a matrix with $(t-1)$ an elementary divisor r times. $g^*J = Jg^{-1}$, so $g^*Jg = Jg^{-1}g = J$. We show that g is not strongly real. For the purpose of reaching a contradiction, suppose that g is strongly real. Then there is an s in the group for which $s^2 = 1$ and $sgs^{-1} = g^{-1}$. Since $s^2 = 1$, left multiplying both sides of the last equation by s gives $gs^{-1} = g^{-1}s$. Letting $s = (x_{ij})$ and comparing entries of gs^{-1} and $g^{-1}s$ (first compare the bottom row and first column, then every row moving up) and then comparing the diagonal entries of s^2 with 1, using the fact that if $a^2 = 1$ in an even characteristic field, then $a = 1$ $(0 = (a^2 - 1) = (a-1)(a+1) = (a-1)^2)$, we see that

$$s = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & X_{16} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & 0 \\ 0 & 0 & 1 & x_{34} & x_{35} & 0 \\ 0 & 0 & 0 & 1 & x_{45} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & X_{65} & I_r \end{pmatrix}$$

where $\overline{b} + ax_{23} = \overline{a}x_{34} + b$ and we are not using all that we know.

Since s is in the group, $s^*Js = J$. Using $s^2 = 1$, $s^*J = Js$. Comparing entries in these two yields $x_{45} = \overline{x_{12}}$, $x_{34} = \overline{x_{23}}$, and $X_{65} = X_{16}^*$ among other equalities so

$$s = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & X_{16} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & 0 \\ 0 & 0 & 1 & \overline{x_{23}} & x_{35} & 0 \\ 0 & 0 & 0 & 1 & \overline{x_{12}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & X_{16}^* & I_r \end{pmatrix}.$$

The (2,4) entry of s^2, $x_{23}\overline{x_{23}}$, is 0 since $s^2 = 1$, so $x_{23} = 0$. Thus, $x_{34} = \overline{x_{23}} = 0$. Now our earlier equation becomes $\overline{b} = b$. Finally, $a\overline{a} = b + \overline{b} = b + b = 0$ so $a = 0$, a contradiction. □

From Lemma 3.1 we now have

Corollary 3.5. *Unipotent elements of type $(l, (l-2k)^r)$ in $U(l + r(l-2k), q^2)$ with $l \geq 5$ odd and $k \geq 2$ are not sr.*

Proof. For the purpose of reaching a contradiction suppose that $(l, (l-2k)^r)$ is sr where $l \geq 5$ is odd and $k \geq 2$. By using the second bullet point of Lemma 3.1 $\frac{l-2k-1}{2}$ times, we see that $(2k+1, 1^r)$ is sr. Now using the third bullet point of Lemma 3.1 $k-2$ times gives that $(5, 1^r)$ is sr, a contradiction. □

What about when $k = 1$? Equivalently, is $(l, (l-2)^r)$ sr? This will be answered for $r = 1$ by the next proposition. First, a definition and a lemma. Let K/F be a finite field extension. K is a vector space over F for the same reasons that a field is a vector space over itself. That being the case, $m_\alpha : K \to K$ given by $m_\alpha(x) = \alpha x$ is a F-linear transformation for each $\alpha \in K$.

Definition 3.6. The field trace (relative to K/F), denoted $\mathrm{Tr}_{K/F}$, or Tr for short, maps each $\alpha \in K$ to the trace of m_α.

We consider $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Note that Tr is onto $\mathbb{F}_q$ since it straightforward to see that it is an $\mathbb{F}_q$ linear transformation and as such must be onto or 0. To be explicit, if it is not 0, then let $a \in$Im(Tr) so that Tr(b) $= a$. Then for any c in $\mathbb{F}_q$, $c = ca^{-1}\mathrm{Tr}(b)$ $= \mathrm{Tr}(ca^{-1}b)$ so that Tr is onto. Exercises in [1, Section 14.2] outline a proof that the the sum of Galois conjugates is the same as our trace. Using that fact, $\mathrm{Tr}(a) = a + \overline{a}$ is not 0 for any $a \in \mathbb{F}_{q^2} - \mathbb{F}_q$ and so the Tr map is not 0 and must be onto.

Lemma 3.7. *There is a $z \in \mathbb{F}_q$ such that $t^2 + zt + z$ is irreducible in $\mathbb{F}_q[t]$.*

Proof. Let z be such that $\mathrm{Tr}(z^{-1}) = 1$. Suppose $x^2 + x + z^{-1}$ is reducible in $\mathbb{F}_q[x]$ and its roots are a and b. Then $a + b = 1$ so $b = 1 + a$ and $ab = z^{-1}$ (characteristic is 2) giving $a + a^2 = 0$ (or $a = a^2$). This implies a is 1 or 0. If $a = 1$, $b = 1 + 1 = 0$. If $a = 0$, $b = 1$. Either way $ab = z^{-1} = 0$. But $\mathrm{Tr}(0) = tr(m_0) = tr(0) = 0 \neq 1$ where $tr(0)$ is the linear algebra trace of the zero map from $\mathbb{F}_{q^2}$ to itself. We have a contradiction and so $x^2 + x + z^{-1}$ is irreducible. Substituting $t = xz$ gives that $t^2 + zt + z$ is irreducible. □

Proposition 3.8. *If $l \geq 3$ is odd, then unipotent elements of type $(l, l-2)$ are sr.*

Proof. From Lemma 3.7 there is z such that $t^2 + zt + z$ is irreducible in $\mathbb{F}_q[t]$. We have then that $\mathbb{F}_q[t]/(t^2 + zt + z)$ is isomorphic to $\mathbb{F}_{q^2}$, and as we've seen before a and $a^q = \overline{a}$ are the roots $t^2 + zt + z$. Now, $t^2 + zt + z = (t+a)(t+\overline{a})$ giving $z = a + \overline{a} = a\overline{a}$. Let m be odd and

$$h_m = \begin{pmatrix} 1 & a & a & \cdots & \cdots & \cdots & a \\ & 1 & \overline{a} & \overline{a} & \cdots & \cdots & \overline{a} \\ & & 1 & a & a & \cdots & a \\ & & & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 1 & \overline{a} \\ & & & & & & 1 \end{pmatrix}.$$

Using $a + \overline{a} = a\overline{a}$,

$$h_m^{-1} = \begin{pmatrix} 1 & a & \overline{a} & \cdots & \cdots & \cdots & \overline{a} \\ & 1 & \overline{a} & a & \cdots & \cdots & \overline{a} \\ & & 1 & a & \overline{a} & \cdots & \overline{a} \\ & & & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 1 & \overline{a} \\ & & & & & & 1 \end{pmatrix}.$$

The upper triangular part of h_m has rows that alternate between all a and all $\overline{a}$. The upper triangular part of h_m^{-1} has columns that alternate between all a and all $\overline{a}$. Note that h_m is unipotent and unitary relative to the Hermitian form defined by N_m. Hence $g = h_l \oplus h_{l-2}$ is unipotent and unitary of type $(l, l-2)$ with form given by $J = N_l \oplus N_{l-2}$. Let

$$s = \begin{pmatrix} 1 & 1 & \cdots & & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ & 1 & \cdots & & \cdots & 1 & & 1 & \cdots & 1 \\ & & \ddots & & & \vdots & & & \ddots & \vdots \\ & & & 1 & 1 & 1 & & & & 1 \\ & & & & 1 & 1 & & & & \\ & & & & & 1 & & & & \\ & & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ & & & 1 & \cdots & 1 & & 1 & \cdots & 1 \\ & & & & \ddots & \vdots & & & \ddots & \vdots \\ & & & & & 1 & & & & 1 \end{pmatrix}$$

which is divided into 4 blocks, 3 of which are elementary matrices. The upper left is all ones upper triangular, the bottom left is $(l-2)$-by-2 all zeros connected on the right with an $(l-2)$-by-$(l-2)$ all ones upper triangular matrix, and the other two are $(l-2)$-by-$(l-2)$ upper triangular all ones matrices. Transposing s then reversing the first l and last $l-2$ columns gives the same as reversing the first l rows and then the last $l-2$ rows so $s^*J = Js$. Also, $s^2 = 1$. To see this, observe that most rows and columns have an even number of 1s. Therefore $s^*Js = J$ is equivalent to $s^*J = s^tJ = Js$. We just said $s^*J = Js$ so s is in $U(2l-2, q^2)$ along with g. $sg = \begin{pmatrix} R_1 & R_2 \\ R_1' & R_2 \end{pmatrix}$ where R_j has its ith row the sum of all but the first $i-1$ rows of $h_{l-2(j-1)}$ (j is 1 or 2) and R_1' has its ith row the sum of last $l-3-i$ rows of h_l. The sum of all but the first i rows of h_l is $(0, 0, \ldots, 0, 1, 1, \ldots, 1)(1, a+1, a\overline{a}+1, \overline{a}+1, 1, \overline{a}, 1, \overline{a}, \ldots, 1)$ where the product is entrywise and the first part of the product has i zeros and $l-i$ ones. The same is of course true of the ith row of R_2, the sum of all but the first i rows of h_{l-2}. Since the ith row of h_l (h_{l-2}) is the $(l-i)$th ($(l-2-i)$th) column of h_l (h_{l-2}), that

$sg = g^{-1}s$ follows after some computation. Therefore, g is sr. $\square$

The following is another generalization of Proposition 3.4.

Proposition 3.9. *A unipotent element of type* $(5^r, 2^k, 1^m)$ *in* $U(5r+2k+m, q^2)$ *with* r *odd and* k *and* m *nonnegative is not s.r..*

Proof. Consider $U(5r+2k+m, q^2)$ defined by the Hermitian matrix $J = N_{5r} \oplus N_{2k} \oplus I_m$. One unipotent element in this group is

$$g = \begin{pmatrix} I_r & aI_r & aI_r & aI_r & aI_r & & & \\ & I_r & \overline{a}I_r & \overline{a}I_r & \overline{a}I_r & & & \\ & & I_r & aI_r & aI_r & & & \\ & & & I_r & \overline{a}I_r & & & \\ & & & & I_r & & & \\ & & & & & I_k & I_k & \\ & & & & & & I_k & \\ & & & & & & & I_m \end{pmatrix},$$

where $a + \overline{a} = a\overline{a}$ and $a \neq 0$. Lemma 3.7 ensures such an a exists. Suppose it is strongly real and that s is an involution conjugating it to its inverse. Since $sg = g^{-1}s$, comparing the entries of these (bottom row, then first column, then each row moving up from the bottom as usual) gives that s is of the form

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & s_{18} \\ & s_{11} & s_{23} & s_{13} & s_{25} & & s_{27} & \\ & & s_{11} & s_{12} & s_{13} & & & \\ & & & s_{11} & s_{23} & & & \\ & & & & s_{11} & & & \\ & & & s_{64} & s_{65} & s_{66} & s_{67} & s_{68} \\ & & & & s_{75} & & s_{66} & \\ & & & & s_{85} & & s_{87} & s_{88} \end{pmatrix},$$

where $(a+\overline{a})s_{11} = a\overline{a}s_{11} = as_{23} + \overline{a}s_{12}$ and s is block conformal to g. Consider s^2. Since $s^2 = 1$, the top left block of s^2, s_{11}^2, is I_r. The second block of the first row of blocks is $s_{11}s_{12} + s_{12}s_{11}$, so $s^2 = 1$ gives $s_{11}s_{12} = s_{12}s_{11}$. We can also see that $s_{11}s_{23} = s_{23}s_{11}$ (third block of the second row), and $s_{11}s_{13} + s_{12}s_{23} + s_{13}s_{11} = 0$ (third block of first row). Since $s_{11}^2 = I_r$, $(s_{11} - I_r)^2 = s_{11}^2 - s_{11} - s_{11} + I_r = s_{11}^2 + I_r = s_{11}^2 - I_r = 0$. By definition s_{11} is idempotent, and so all of its eigenvalues are 1 (consider the eigenvalues of $s_{11} - I_r$). Then since s_{11} is r by r, its trace, the sum of its eigenvalues, is $1 + 1 + \cdots + 1 = 1$ where there are r 1s being summed (r is odd). Also recalling that the trace of AB is the trace of BA for matrices A and B whose products are defined, we see that $\mathrm{tr}(s_{11}s_{13}) = \mathrm{tr}(s_{13}s_{11})$. Since $s_{11}s_{13} + s_{12}s_{23} + s_{13}s_{11} = 0$, $0 = \mathrm{tr}(0) = \mathrm{tr}(s_{11}s_{13}) + \mathrm{tr}(s_{12}s_{23}) + \mathrm{tr}(s_{13}s_{11}) = \mathrm{tr}(s_{11}s_{13}) + \mathrm{tr}(s_{11}s_{13}) + \mathrm{tr}(s_{12}s_{23}) = \mathrm{tr}(s_{12}s_{23})$. Now since s_{11} commutes with s_{12}, the eigenvalues of $s_{11}s_{12}$ are the products of eigenvalues of s_{11} with eigenvalues of s_{12}. Since all the eigenvalues of s_{11} are 1, the eigenvalues of $s_{11}s_{12}$ are just the eigenvalues of s_{12}. Thus, $\mathrm{tr}(s_{11}s_{12}) = \mathrm{tr}(s_{12})$. Of course, s_{12} commutes with itself so the eigenvalues of s_{12}^2 are the squares of eigenvalues of s_{12}. Suppose $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ are the eigenvalues of s_{12}. Then $\mathrm{tr}(s_{12}^2) = \Lambda_1^2 + \Lambda_2^2 + \cdots + \Lambda_r^2 = (\Lambda_1 + \Lambda_2 + \cdots + \Lambda_r)^2 = \mathrm{tr}(s_{12})^2$ since there are are an even number of each cross term and q is even. Lastly, using $s^*J = Js$,

$s_{23} = N_r s_{12}^* N_r$, so $\mathrm{tr}(s_{23}) = \mathrm{tr}(N_r s_{12}^* N_r) = \mathrm{tr}(N_r N_r s_{12}^*) = \mathrm{tr}(s_{12}^*) = \mathrm{tr}(\overline{s_{12}}^t) = \mathrm{tr}(\overline{s_{12}})$ $= \overline{\mathrm{tr}(s_{12})}$. As the sum of diagonal entries, trace is preserved under taking a transpose and conjugated when the matrix is conjugated.

Recall that $a\overline{a}s_{11} = as_{23} + \overline{a}s_{12}$. Right multiplying by s_{12} and then taking the trace gives $a\overline{a}\mathrm{tr}(s_{11}s_{12}) = a\mathrm{tr}(s_{23}s_{12}) + \overline{a}\mathrm{tr}(s_{12}^2)$. Using $\mathrm{tr}(s_{11}s_{12}) = \mathrm{tr}(s_{12})$, $\mathrm{tr}(s_{23}s_{12})$ $= \mathrm{tr}(s_{12}s_{23}) = 0$, and $\mathrm{tr}(s_{12}^2) = \mathrm{tr}(s_{12})^2$ gives $a\overline{a}\mathrm{tr}(s_{12}) = \overline{a}\mathrm{tr}(s_{12})^2$. Therefore, $\mathrm{tr}(s_{12})$ $= 0$ or $\mathrm{tr}(s_{12}) = a$. Now taking the trace of $a\overline{a}s_{11} = as_{23} + \overline{a}s_{12}$ and using $\mathrm{tr}(s_{23}) =$ $\overline{\mathrm{tr}(s_{12})}$, we have $a\overline{a} = a\overline{\mathrm{tr}(s_{12})} + \overline{a}\mathrm{tr}(s_{12})$. Regardless of whether $\mathrm{tr}(s_{12}) = 0$ or $\mathrm{tr}(s_{12})$ $= a$, $a\overline{a} = 0$ so $a = 0$, a contradiction. □

The next result will be used many times in what follows. It is from [4, Proposition 6.2].

Proposition 3.10. *If an element of $U(n, q^2)$ has the property that every elementary divisor of the form $(t-1)^{2m+1}$, $m \geq 1$, has even multiplicity, then that element is sr. If an element of $U(n, q^2)$ has the property that every elementary divisor of the form $(t-1)^{2m+1}$, $m \geq 2$, has even multiplicity and also has $(t-1)$ as an elementary divisor, then that element is sr.*

Note here that the first part of this proposition *does not* imply the second part. The first part is not able to classify $(3^r, 1^m)$ with r odd as strongly real, but the second part is.

This immediately implies that elements corresponding to partitions with all odd parts greater than or equal to 3 having even multiplicity are sr, and that elements corresponding to partitions with 1 having positive multiplicity and with all odd parts greater than or equal to 5 having even multiplicity are sr.

3.3 Even Characteristic, Recent Results

We now present the results of this paper.

Proposition 3.11. *A unipotent element of type* $(4,3,2)$ *in* $U(9,q^2)$ *is not sr.*

Proof. Let g be defined as follows:

$$g = \begin{pmatrix} 1 & a & b & c & & & & & \\ & 1 & \overline{a} & \overline{b} & & & & & \\ & & 1 & a & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & a & b & & \\ & & & & & 1 & \overline{a} & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & 1 \\ & & & & & & & & 1 \end{pmatrix},$$

where $a \neq 0$, $b + \overline{b} = a\overline{a}$, and $c + \overline{c} = b\overline{b}$.

Then g is unipotent in $\mathrm{U}(9, q^2)$ defined by the Hermitian matrix $J = N_4 \oplus N_3 \oplus N_2$. Suppose g is strongly real and let s be an involution in $\mathrm{U}(9, q^2)$ that conjugates g to its inverse. Comparing sg and $g^{-1}s$ and then the diagonal entries of s^2 and 1, one sees that s is of the form

$$s = \begin{pmatrix} 1 & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & s_{18} & s_{19} \\ & 1 & s_{23} & s_{13} & & s_{15} & s_{16} & & a^{-1}s_{18} \\ & & 1 & s_{12} & & & s_{15} & & \\ & & & 1 & & & & & \\ & s_{52} & s_{53} & s_{54} & 1 & s_{56} & s_{57} & s_{58} & s_{59} \\ & & \overline{a}a^{-1}s_{52} & s_{53} & & 1 & s_{67} & & a^{-1}s_{58} \\ & & & s_{52} & & & 1 & & \\ & & s_{83} & s_{84} & & s_{86} & s_{87} & 1 & s_{89} \\ & & & s_{83} & & & \overline{a}s_{86} & & 1 \end{pmatrix},$$

where $as_{16} + a\overline{a}s_{15} = \overline{a}s_{16}$ and $s_{67} = \overline{a} + \overline{a}a^{-1}s_{56}$. Comparing s^*J with Js gives that $s_{15} = \overline{a^{-1}}as_{15}$ and $\overline{s_{56}} = \overline{a} + \overline{a}a^{-1}s_{56}$, where s is of the form

$$s = \begin{pmatrix} 1 & d_{12} & d_{13} & d_{14} & s_{15} & s_{16} & s_{17} & s_{18} & s_{19} \\ & 1 & s_{23} & d_{13} & & s_{15} & s_{16} & & a^{-1}s_{18} \\ & & 1 & d_{12} & & & s_{15} & & \\ & & & 1 & & & & & \\ & \overline{s_{15}} & \overline{s_{16}} & \overline{s_{17}} & 1 & s_{56} & d_{57} & s_{58} & s_{59} \\ & & \overline{s_{15}} & \overline{s_{16}} & & 1 & \overline{s_{56}} & & a^{-1}s_{58} \\ & & & \overline{s_{15}} & & & 1 & & \\ & & a^{-1}\overline{s_{18}} & \overline{s_{19}} & & \overline{a^{-1}}\overline{s_{58}} & \overline{s_{59}} & 1 & d_{89} \\ & & & \overline{s_{18}} & & & \overline{s_{58}} & & 1 \end{pmatrix},$$

where replacing s_{ij} with d_{ij} indicates that $\overline{s_{ij}} = s_{ij}$. Suppose $a \neq \overline{a}$. Then since $s_{15} = \overline{a^{-1}}as_{15}$, $s_{15} = 0$, as otherwise $1 = \overline{a^{-1}}a$ giving $a = \overline{a}$. If instead $a = \overline{a}$, then $as_{16} + a\overline{a}s_{15} = \overline{a}s_{16}$ becomes $a\overline{a}s_{15} = 0$, so $s_{15} = 0$ $(a \neq 0)$. In either case $s_{15} = 0$. Now using $s_{15} = 0$ and comparing the (6,7) entry of s^2 and 1, we see that $a^{-1}s_{58}\overline{s_{58}} = 0$ so $s_{58} = 0$. Now using $s_{58} = 0$ and looking at the (5,7) entry of s^2 and 1, we see that $s_{56} = 0$. But $\overline{s_{56}} = \overline{a} + \overline{a}a^{-1}s_{56}$, so $\overline{a} = 0$ giving $a = 0$, a contradiction. $\square$

We now present the main results of this paper. First we need to define a certain condition on partitions.

For an arbitrary partition, $(k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 2^{m_2}, 1^{m_1})$, the odd parts greater than or equal to 3 with odd multiplicities, are called the important parts. A partition is said to satisfy condition $\star$ if w is even and every odd number greater than an even index important part and less than an adjacent odd index important part has positive multiplicity; or if w is odd, the above is true, and every odd number less than the smallest important part has positive multiplicity where w is the number of important parts.

Theorem 3.12. *Let g be a unipotent element in $U(n,q^2)$ of type $(k^{m_k},(k-1)^{m_{k-1}},\ldots,2^{m_2},1^{m_1})$. Let $k_1 > k_2 > \ldots > k_w$ be the important parts. If g satisfies condition $\star$, then it is sr.*

Proof. Let $(k^{m_k},(k-1)^{m_{k-1}},\ldots,2^{m_2},1^{m_1})$ satisfy condition $\star$. The partition consisting of its even parts is already known to be strongly real since all of the its odd parts greater than or equal to 3 have multiplicity 0 which is even. Also, combining strongly real partitions gives new strongly real partitions, so we see that it suffices to show that the partition made up of the odd parts of $(k^{m_k},(k-1)^{m_{k-1}},\ldots,2^{m_2},1^{m_1})$ is strongly real. Since $(k^{m_k},(k-1)^{m_{k-1}},\ldots,2^{m_2},1^{m_1})$ satisfies the conditions given above, its odd parts do as well. In fact, its odd parts have the exact same important parts with the exact same multiplicities. Note that if $w=0$, then all odd parts greater than or equal to 3 have even multiplicities, and we know that the partition is sr. So suppose $w \geq 1$. Also note that if the largest odd part has even multiplicity, then that part as a partition by itself is strongly real. Therefore, we may assume that the largest odd part has odd multiplicity. In what follows, the fact that partitions of the form $(l,l-2)$ with l odd are sr (implying more generally that partitions of the form $(l^n,(l-2)^n)$ for any odd l and $n \geq 1$ are sr) will be used frequently.

Consider a partition consisting only of odd parts which satisfies condition $\star$ and whose largest part has odd multiplicity. We can do induction on the largest odd part. If the largest odd part is 3 and $w=1$, then our partition is of the form $(3^r,1^m)$ where r is odd and $m>0$. Here 1 has positive multiplicity and the multiplicity of all odd parts greater than or equal to 5 is even (they are all 0), so it is sr. Note that when the largest odd part is 3, w cannot be larger than 1. If the largest odd part is 5 and $w=1$, then our partition is of the form $(5^r,3^e,1^m)$ where r is odd, $e>0$ is even, and $m>0$. Now there are two cases. If $r>e$, then since $e>0$, $(5^{e-1},3^{e-1})$ and $(5^{r-e+1},3^1,1^m)$ are strongly real, and so our partition, which comes from combining them, is strongly real (the second of the two being combined is strongly real because 1 has positive multiplicity and all odd parts greater than or equal to 5 have even multiplicity). If $e > r$, then $(5^r, 3^r)$ and $(3^{e-r}, 1^m)$ are strongly real so our partition is as well (the second of these was strongly real because it is of a form already considered). Finally, if $w = 2$, then our partition is of the form $(5^r, 3^s, 1^m)$ where r and s are odd and $m \geq 0$. If $r = s$, then since $(5^r, 3^s)$ and (1^m) are strongly real, our partition is as well (if $m = 0$, then we already know that our partition is s.r.). If $r < s$, then since $(5^r, 3^r)$ and $(3^{s-r}, 1^m)$ are strongly real, our partition is as well. If $r > s$, then since $(5^s, 3^s)$ and $(5^{r-s}, 1^m)$ are strongly real, our partition is as well (the second of these has all odd parts at least 3 with even multiplicity).

Now suppose that the statement is true whenever the largest odd part is less than or equal to $2j - 1$ and suppose that the largest part of our partition is $2j + 1$.

If $w = 1$, then our partition is of the form $((2j+1)^r, (2j-1)^{e_1}, (2j-3)^{e_2} \ldots 3^{e_k}, 1^{m_1})$ with each e_i positive and even, and r odd. If $r < e_1$ then $((2j - 1)^{e_1-r}, (2j - 3)^{e_2} \ldots 3^{e_k}, 1^{m_1})$ is sr by the inductive assumption (it has largest odd part with odd multiplicity being $2j - 1$ and its parts satisfy the condition of the theorem) and we know $((2j+1)^r, (2j-1)^r)$ is sr, so our partition is sr. If $r > e_1$, then since $e_1 > 0$, e_1-1 is odd and greater than 0. Thus, since $((2j+1)^{r-e_1+1}, (2j-1)^1, (2j-3)^{e_2} \ldots 3^{e_k}, 1^{m_1})$ is strongly real by inductive assumption (the first part is by itself strongly real), and $((2j + 1)^{e_1-1}, (2j - 1)^{e_1-1})$ is strongly real, our partition is strongly real.

If $w = 2$, then our partition is of the form $((2j+1)^r, (2j-1)^{e_1}, (2j-3)^{e_2}, \ldots, (k_2+2)^{e_a}, k_2^s, (k_2 - 2)^{e_{a+1}}, (k_2 - 4)^{e_{a+2}}, \ldots, 3^{e_b}, 1^m)$ where r and s are odd, m is arbitrary, and every e_i is positive and even for $1 \leq i \leq a$ and even but possibly 0 for $a + 1 \leq i \leq b$. First, suppose $2j + 1$ and k_2 are adjacent. Since $((2j + 1), k_2)$ is sr and $((2j + 1)^{r-1}, k_2^{s-1}, (k_2 - 2)^{e_{a+1}}, (k_2 - 4)^{e_{a+2}}, \ldots, 3^{e_b}, 1^m)$ is sr (all odd parts have even multiplicity), our partition is sr. Now if $2j + 1$ and k_2 are not adjacent, then since e_1 is positive and even, $((2j - 1)^{e_1-1}, (2j - 3)^{e_2}, \ldots, (k_2 + 2)^{e_a}, k_2^s, (k_2 - 2)^{e_{a+1}}, (k_2 - 4)^{e_{a+2}}, \ldots, 3^{e_b}, 1^m)$ is strongly real by the inductive assumption on the largest odd part as this has largest odd part $2j-1$ and satisfies the condition of the theorem with $w=2$. Also, $((2j+1)^{r-1})$ is sr (has even multiplicity), and $((2j+1),(2j-1))$ is sr, so our partition is strongly real.

Suppose that our partition is strongly real whenever $w \leq k-1$, and suppose $w=k$. Then our partition is of the form $((2j+1)^r,(2j-1)^{m_{2j-1}},\ldots,1^{m_1})$. If m_{2j-1} is odd, then there are three cases. If $m_{2j-1}=r$, then $((2j+1)^r,(2j-1)^{m_{2j-1}})$ is strongly real and $((2j-3)^{m_{2j-3}},\ldots,1^{m_1})$ is strongly real by the inductive assumption on the largest odd part (its largest odd part is $2j-3$ and it satisfies the condition of the theorem with $w=k-2$) so our partition is strongly real. If $r<m_{2j-1}$, then since $((2j+1)^r,(2j-1)^r)$ is strongly real, $((2j-1)^{m_{2j-1}-r},(2j-3)^{m_{2j-3}},\ldots,(k_3+2)^{m_{k_3+2}})$, containing only odd parts with even multiplicities, is strongly real, and $(k_3^{m_{k_3}},\ldots,1^{m_1})$ is strongly real by the inductive assumption on w (with $w=k-2$), our partition is strongly real. Finally, if $r>m_{2j-1}$, then $((2j+1)^{m_{2j-1}},(2j-1)^{m_{2j-1}})$ is strongly real and $((2j+1)^{r-m_{2j-1}},(2j-1)^0,\ldots,(k_3+2)^{m_{k_3}+2})$ is strongly real since all of its odd parts have even multiplicity, and $(k_3^{m_{k_3}},\ldots,1^{m_1})$ is strongly real by the inductive assumption on w, so our partition is strongly real. If m_{2j-1} is even, then since $((2j+1)^{m_{2j+1}},\ldots,(k_3+2)^{m_{k_3+2}})$ is strongly real (since this has $w=2$ and satisfies the above conditions) and $(k_3^{m_{k_3}},\ldots,1^{m_1})$ is strongly real by the inductive assumption on w, our partition is strongly real.

By inducting on w, we conclude that all partitions with only odd parts, and whose largest part is $2j+1$ with odd multiplicity, are sr. By inducting on the largest odd part, we conclude that all partitions with only odd parts and whose largest part has odd multiplicity are sr. Recalling that since a partition consisting of even parts and odd parts with even multiplicity is sr, and that combing sr partitions gives sr partitions, the result follows. □

Corollary 3.13. *If g's parts are all odd, then g is strongly real if and only if its parts satisfy condition* $\star$.

Proof. Suppose that all of g's parts are odd and it satisfies the condition of the theorem. Then the statement just proved gives that g is sr. To prove the converse, suppose for the purpose of reaching a contradiction that the partition corresponding to g, $(k^{m_k}, (k-2)^{m_{k-2}}, \ldots, 2^{m_2}, 1^{m_1})$, does not satisfy the condition of the theorem, and yet g is strongly real. Again, if $w = 0$ then g automatically satisfies condition $\star$. Hence, $w \geq 1$. Assume for now that there is a smallest odd number with 0 multiplicity greater than an even index important part and less than an adjacent odd index important part (which must be the case when w is even and the partition doesn't satisfies condition $\star$). Call it a. By using Lemma 3.1 repeatedly with $l = k$, renaming the largest part k after each use, we see that an element of type $(k_1^{m_{k_1}}, \ldots, a^0, \ldots k_2^{m_{k_2}}, \ldots 1^{m_1})$ is s.r. (m_{k_1} is odd since it is the sum of an odd number of odd numbers with some even numbers). Now, using the lemma repeatedly and letting l be the largest part less than k_2, we see that an element with type of the form $(k_1^{m_{k_1}}, \ldots a^0, \ldots, k_2^{m_{k_2}}, (k_2-2)^0, (k_2-4)^0, \ldots, 3^0, 1^{m_1})$ is strongly real. Next, using the lemma with $l = k$ repeatedly we can get that an element with type of the form $(k_1^{m_{k_1}}, a^0, \ldots k_2^{m_{k_2}}, (k_2-2)^0, \ldots, 1^{m_1})$ is sr. Lastly, using the lemma repeatedly with $l = k_2$ and then with $l = k$ repeatedly, we get that an element with type of the form $(5^{m_5}, 1^{m_1})$ is sr with m_5 odd. This contradicts Proposition 3.9. If w is odd and there is no a as above, then there is some odd a with multiplicity 0 that is less than k_w. Therefore, using the lemma repeatedly with $l = k$, we get that an element with type of the form $(k_w^{m_{k_w}}, \ldots a^0, \ldots 1^{m_1})$ is sr where m_{k_w} is odd (as a sum of an odd number of odd numbers with some even numbers). We know that this leads to a contradiction so we are done. $\square$

We believe that $(5^r, 4^l, 2^k, 1^m)$ is strongly real when r is odd since this is a natural generalization of Proposition 3.9. We also believe that $(3^r, 2^k)$ is strongly real when r is odd for multiple reasons. First, $(3, 2)$ is not sr [4, Proposition 6.3]. Next, (3^r) is not sr [4, Proposition 6.4]. Also, Proposition 3.11 states that $(4, 3, 2)$ is not sr which would follow from $(3^r, 2^k)$ not being sr and Lemma 3.1. Finally, the computational results are special cases that would follow from $(3^r, 2^k)$ not being sr. If $(3^r, 2^k)$ and $(5^r, 4^l, 2^k, 1^m)$ are indeed not sr for r odd, then we can give a full classification.

Proposition 3.14. *If one assumes that $(3^r, 2^k)$ and $(5^r, 4^l, 2^k, 1^m)$ are not sr for all odd r and any l, k, and m, then g is strongly real if and only if its corresponding parts satisfy condition $\star$ (regardless of whether g's parts are all odd or not).*

Proof. We already know that g is strongly real if its parts satisfy condition $\star$. So we suppose that g's parts do not satisfy condition $\star$ and we show that g is not sr. For the purpose of reaching a contradiction, suppose g is strongly real. Letting $k_1 > k_2 > \ldots > k_w$ be g's important parts and for now assuming that w is even, we know that $\{i \in \mathbb{N} : \exists 2j+1 \text{ s.t } k_{2i} < 2j+1 < k_{2i-1} \text{ and } m_{2j+1} = 0\}$ is nonempty. By direct summing we may assume that $k > k_1$, and that g has every even number at most k and at least k_1 with positive multiplicity (since partitions consisting of only even parts are strongly real). Let i be the largest member of the set above (it is bounded since otherwise the partition has infinitely many odd parts and cannot sum to n). The maximality of i guarantees that the multiplicity of all odds that are greater than k_2 and less than k_1, or greater than k_4 and less than k_3, or $\ldots$, or greater than k_{2i-2} and less than k_{2i-3}, is positive. We have that g is of type $(k^{m_k}, (k-1)^{m_{k-1}}, \ldots, k_1^{m_{k_1}}, (k_1-1)^{m_{k_1-1}}, \ldots, k_2^{m_{k_2}}, \ldots, k_{2i-1}^{m_{k_{2i-1}}}, \ldots, (2j+1)^{m_{2j+1}}, \ldots k_{2i}^{m_{k_{2i}}}, \ldots, 1^{m_1})$ with $m_{2j+1} = 0$, $m_k > 0$, and the multiplicity of all the non important odd numbers that are at least 3, even. If k is even then use Lemma 3.1 repeatedly with l the largest part of whatever partition is concluded to be sr until you get that an element of the same form but with largest part odd is sr. Call the new largest part k in the previous application of Lemma 3.1. By the nature of this procedure, m_{k-1} is now greater than 0. First, if $k-2 > k_1$, then applying Lemma 3.1 with $l = k-1$ gives that an element whose type is of the form $((k-2)^{m_{k-2}+m_k}, (k-3)^{m_{k-3}+m_{k-1}}, (k-4)^{m_{k-4}}, (k-5)^{m_{k-5}}, \ldots, k_1^{m_{k_1}}, (k_1-1)^{m_{k_1-1}}, \ldots, k_2^{m_{k_2}}, \ldots, k_{2i-1}^{m_{k_{2i-1}}}, \ldots, (2j+1)^{m_{2j+1}}, \ldots, k_{2i}^{m_{k_{2i}}}, \ldots, 1^{m_1})$ is sr where the parity of $m_{k-2}+m_k$ is the same as the parity of m_{k-2} and $m_{k-3}+m_{k-1} \geq m_{k-1} > 0$. Redefine k as $k-2$. While $k > 2j+1$, apply Lemma 3.1 with $l = k-1$. Eventually k will be $2j+1$ and we have that an element with largest part $2j+1$ with odd multiplicity is sr. $2j+1$ has odd multiplicity because each odd number greater than 3 that is not an important part has even multiplicity and there are an odd number $(2i-1)$ of important parts at least k_{2i-1}. Hence the new multiplicity of k_{2i-1} is the sum of an odd number of odd numbers with some even numbers. This element also has $2j$ with positive multiplicity and $m_{2j-1} = 0$. All together we have that an element of type $(k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, \dots k_{2i}^{m_{k_{2i}}}, \dots 1^{m_1})$ is sr. Now, if $k = k_1$ we can also apply Lemma 3.1 while $k > 2j+1$ with $l = k-1$. Either way we have an element of type $(k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, \dots k_{2i}^{m_{k_{2i}}}, \dots 1^{m_1}) = (k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, (2j-2)^{m_{2j-2}}, (2j-3)^{e_1}, (2j-4)^{m_{2j-4}}, (2j-5)^{e_2}, \dots, (k_{2i}+4)^{e.5(2j-k_{2i}-3)}, (k_{2i}+1)^{m_{k_{2i}+1}}, k_{2i}^{m_{k_{2i}}}, \dots 1^{m_1})$ where each e_i is even, is sr.

If every e_i is 0, then maybe all the multiplicities of even numbers between $2j-1$ and k_{2i} are also 0. In that case, applying Lemma 3.1 with $l = k-1$ repeatedly, redefining $k-2$ as k each time gives that an element of type of the form $(k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, (2j-2)^0, k_{2i}^{m_{k_{2i}}}, (k_{2i}-1)^{m_{k_{2i}-1}}, (k_{2i}-2)^e, \dots, 1^{m_1})$ with e even (if $k_{2i}-2 \neq k_{2i+1}$) is sr. Now, using Lemma 3.1 with $l = k_{2i}$ gives that an element of type $((2j-1)^{k_{m_k}}, (2j-2)^{m_{2j}}, k_{2i}^0, (k_{2i}-1)^{m_{k_{2i}-1}}, \dots, k_{2i+1}^{m_{k_{2i+1}}}, 1^{m_1})$ is strongly real. If $k_{2i}-2 = k_{2i+1}$, then using the lemma with $l = k_{2i}$ gives that an element of type $((2j-1)^{m_k}, (2j-2)^{m_{2j}}, k_{2i}^0, (k_{2i}-1)^{m_{k_{2i}-1}}, k_{2i+1}^e, \dots 1^{m_1})$ is sr.

Now more generally: We have that an element of type $(k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, (2j-2)^{m_{2j-2}}, (2j-3)^{e_1}, (2j-4)^{m_{2j-4}}, (2j-5)^{e_2}, \dots, (k_{2i}+4)^{e.5(2j-k_{2i}-5)}, (k_{2i}+2)^{m_{k_{2i}+2}}, k_{2i}^{m_{k_{2i}}}, \dots 1^{m_1})$ is sr. Using Lemma 3.1 repeatedly with l the smallest odd part between $2j-1$ and k_{2i}, we eventually get that an element with type of the above form with each $e_i = 0$ is sr. Thus from the above we may suppose that there is some even part between $2j-1$ and k_{2i}. Using Lemma 3.1 repeatedly and letting l be the smallest even part each time, we get that an element of type $(k^{m_k}, (2j)^{m_{2j}}, (2j-1)^0, (2j-2)^0, \ldots, (k_{2i}+1)^0, k_{2i})$ is sr. Now from the above, we know that an element with type of the form $((2j-1)^{k_{m_k}}, (2j-2)^{m_{2j}}, (2j-3)^0, (2j-4)^{m_{2j-4}}, \ldots, k_{2i+1}^{m_{k_{2i+1}}}, 1^{m_1})$ is strongly real.

Repeating the above procedure, we eventually conclude that an element of type $(k_w^{m_{k_w}}, (k_w-1)^{m_{k_w-1}}, (k_w-2)^0, (k_w-3)^{m_{k_w-3}} \ldots, 1^{m_1})$ where m_{k_w} is odd and the multiplicities of even numbers between 3 and k_w are even is sr. If all numbers less than $k_w - 1$ have 0 multiplicity, then using Lemma 3.1 with $l = k-1$ repeatedly gives that $(3^r, 2^k)$ is strongly real where r is odd, a contradiction. If there is at least one odd part inclusively between 3 and $k_w - 4$, then repeatedly applying Lemma 3.1 with l the smallest of these odd parts gives that an element of type with all odd numbers less than k_w except 1 having 0 multiplicity is sr. Then applying Lemma 3.1 repeatedly with the smallest even part greater than 2 and less than $k_w - 1$ gives that an element with type of the form $(k_w^{m_{k_w}}, (k_w-1)^{m_{k_w-1}}, (k_w-2)^0, (k_w-3)^0, \ldots, 2^{m_2}, 1^{m_1})$ is sr. If one of m_2 or m_1 is nonzero, then we can conclude that an element of the form $(5^r, 4^l, 2^k, 1^m)$ is strongly real. If both are 0, we can conclude that an element of the form $(3^r, 2^k)$ is sr. Either way we have a contradiction.

In the case that w is odd, there must be some odd number greater than an even index important part and less than an adjacent odd index important part that has 0 multiplicity *or* an odd number less than k_w with 0 multiplicity. The first case is covered by the above, so suppose we are not in the first case. First, we may suppose that the largest part is odd from the procedure given near the beginning of this proof. Next applying the lemma with $l = k$ repeatedly we get that an element of type $(k_w^{m_{k_w}}, (k_w-1)^{m_{k_w-1}}, \ldots, 1^{m_1})$ is strongly real where m_{k_w} is odd and all odd numbers inclusively between 3 and $k_w - 2$ have even multiplicity and at least one odd number less than k_w has 0 multiplicity. Note that m_{k_w} is odd since it is the sum of what were the multiplicities of every odd number at least k_w. There were w odd multiplicities and w is odd, so this sum is odd. If all odd parts less than k_w have 0 multiplicity then we can Lemma 3.1 with $l = k - 1$ to get that $(3^r, 2^k)$ is sr, a contradiction. If 1 has nonzero multiplicity then we can conclude that an element of the form $(5^r, 4^l, 2^k, 1^m)$ is sr, a contradiction. The result follows. $\square$

Supposing that $(5^r, 4^l, 2^k, 1^m)$ and $(3^r, 2^k)$ are not strongly real for any odd r and nonnegative l, k, and m, we develop a generating function for the number of strongly real unipotent classes in $\mathrm{U}(n, q^2)$. First we find the number of partitions with a particular set of important parts, $k_1 > k_2 > \cdots > k_w$. It is useful to review the derivation of the generating function for the number of partitions of n. The number of partitions of n is the coefficient of x^n in $(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots = \prod_k \frac{1}{1-x^k}$. This is because this coefficient is the number of ways to choose a term from the first series corresponding to the amount 1s contribute to a partition, a term from the second series corresponding to the amount 2s contribute and so on. Now using similar reasoning we see that when w is even, the number of strongly real partitions has generating function

$$
\begin{gathered}
\prod_{k=2j+1>k_1} \frac{1}{1-x^{2k}} \prod_{k=2j>k_1} \frac{1}{1-x^k} \\
\cdot \prod_{2j+1 \le w-1} \left(\frac{x^{k_{2j+1}+k_{2j+2}}}{(1-x^{2k_{2j+1}})(1-x^{2k_{2j+2}})} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{x^{2(2j'+1)}}{1-x^{2(2j'+1)}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}} \right) \\
\cdot \prod_{2j<w} \left(\prod_{k_{2j}>2j'+1>k_{2j+1}} \frac{1}{1-x^{2(2j'+1)}} \prod_{k_{2j}>2j'>k_{2j+1}} \frac{1}{1-x^{2j'}} \right) \\
\cdot \prod_{1<2j+1<k_w} \frac{1}{1-x^{2(2j+1)}} \prod_{2j<k_w} \left(\frac{1}{1-x^{2j}} \right) \frac{1}{1-x}.
\end{gathered}
$$

If w is odd, then the generating function is the above times $\prod_{1<2j+1<k_w} x^{2(2j+1)}$.

Using difference of squares in the denominator of $\frac{1}{1-x^{2k}}$, we get that

$$\prod_{k=2j+1>k_1} \frac{1}{1-x^{2k}} \prod_{k=2j>k_1} \frac{1}{1-x^k} = \prod_{k>k_1} \frac{1}{1-x^k} \prod_{k=2j+1>k_1} \frac{1}{1+x^k}.$$

It is well known that the sum of the first i odd numbers is i^2, so since $k_{2j+1} - 2 = 2(\frac{k_{2j+1}-1}{2}) - 1$ is the $(\frac{k_{2j+1}-1}{2})$th positive odd number and k_{2j+2} is the $(\frac{k_{2j+2}}{2}+1)$th positive odd number, the sum of the odd numbers between k_{2j+2} and k_{2j+1} is $(\frac{k_{2j+1}-1}{2})^2 - (\frac{k_{2j+2}+1}{2})^2 = (\frac{k_{2j+1}-1}{2} + \frac{k_{2j+2}+1}{2})(\frac{k_{2j+1}-1}{2} - \frac{k_{2j+2}+1}{2}) = \frac{1}{4}(k_{2j+1}+k_{2j+2})(k_{2j+1} - k_{2j+2} - 2) = \frac{1}{4}(k_{2j+1}^2 - k_{2j+2}^2 - 2k_{2j+1} - 2k_{2j+2})$. Hence,

$$\prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{x^{2(2j'+1)}}{1-x^{2(2j'+1)}} = x^{\frac{1}{2}(k_{2j+1}^2 - k_{2j+2}^2 - 2k_{2j+1} - 2k_{2j+2})} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1-x^{2(2j'+1)}}.$$

Also,

$$\prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1-x^{2(2j'+1)}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}}$$
$$= \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1-x^{2j'+1}} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1+x^{2j'+1}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}}.$$

Finally, $\frac{1}{(1-x^{2k_{2j+1}})(1-x^{2k_{2j+2}})} = \frac{1}{1-x^{k_{2j+1}}} \frac{1}{1+x^{k_{2j+1}}} \frac{1}{1-x^{k_{2j+2}}} \frac{1}{1+x^{k_{2j+2}}}$ so

$$\prod_{k=2j+1>k_1} \frac{1}{1-x^{2k}} \prod_{k=2j>k_1} \frac{1}{1-x^k}$$

$$\cdot \prod_{2j+1\le w-1} \left(\frac{x^{k_{2j+1}+k_{2j+2}}}{(1-x^{2k_{2j+1}})(1-x^{2k_{2j+2}})} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{x^{2(2j'+1)}}{1-x^{2(2j'+1)}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}} \right)$$

$$= \prod_{k>k_1} \frac{1}{1-x^k} \prod_{k=2j+1>k_1} \frac{1}{1+x^k}$$

$$\cdot \prod_{2j+1\le w-1} \left(\frac{1}{1-x^{k_{2j+1}}} \frac{1}{1+x^{k_{2j+1}}} \frac{1}{1-x^{k_{2j+2}}} \frac{1}{1+x^{k_{2j+2}}} x^{\frac{1}{2}(k_{2j+1}^2-k_{2j+2}^2)} \right.$$

$$\left. \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1-x^{2j'+1}} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{1}{1+x^{2j'+1}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}} \right).$$

Since

$$\prod_{2j<w} \left(\prod_{k_{2j}>2j'+1>k_{2j+1}} \frac{1}{1-x^{2(2j'+1)}} \prod_{k_{2j}>2j'>k_{2j+1}} \frac{1}{1-x^{2j'}} \right)$$

$$= \prod_{2j<w} \left(\prod_{k_{2j}>2j'+1>k_{2j+1}} \frac{1}{1-x^{2j'+1}} \frac{1}{1+x^{2j'+1}} \prod_{k_{2j}>2j'>k_{2j+1}} \frac{1}{1-x^{2j'}} \right),$$

$$\prod_{k=2j+1>k_1} \frac{1}{1-x^{2k}} \prod_{k=2j>k_1} \frac{1}{1-x^k}$$

$$\prod_{2j+1\leq w-1} \left(\frac{x^{k_{2j+1}+k_{2j+2}}}{(1-x^{2k_{2j+1}})(1-x^{2k_{2j+2}})} \prod_{k_{2j+1}>2j'+1>k_{2j+2}} \frac{x^{2(2j'+1)}}{1-x^{2(2j'+1)}} \prod_{k_{2j+1}>2j'>k_{2j+2}} \frac{1}{1-x^{2j'}} \right)$$

$$\prod_{2j<w} \left(\prod_{k_{2j}>2j'+1>k_{2j+1}} \frac{1}{1-x^{2(2j'+1)}} \prod_{k_{2j}>2j'>k_{2j+1}} \frac{1}{1-x^{2j'}} \right)$$

$$= x^{\sum_{2j+1\leq w-1} \frac{1}{2}(k_{2j+1}^2 - k_{2j+2}^2)} \prod_{k\geq k_w} \frac{1}{1-x^k} \prod_{k=2j+1>k_w} \frac{1}{1+x^k}$$

Lastly,

$$\prod_{1<2j+1<k_w} \frac{1}{1-x^{2(2j+1)}} \prod_{2j<k_w} \left(\frac{1}{1-x^{2j}}\right) \frac{1}{1-x}$$

$$= \prod_{1<2j+1<k_w} \frac{1}{1-x^{2j+1}} \frac{1}{1+x^{2j+1}} \prod_{2j<k_w} \left(\frac{1}{1-x^{2j}}\right) \frac{1}{1-x}.$$

so the generating function for the number of unipotent strongly real classes with important parts $k_1 > k_2 > \cdots > k_w$ is $x^{\sum_{2j+1\leq w-1} \frac{1}{2}(k_{2j+1}^2 - k_{2j+2}^2)} \prod_k \frac{1}{1-x^k} \prod_{k=2j+1} \frac{1}{1+x^k}$ when w is even and has an extra factor of $\prod_{1<2j+1<k_w} x^{2(2j+1)} = x^{2\sum_{1<2j+1<k_w}(2j+1)} = x^{2((\frac{k_w-1}{2})^2-1)} = x^{\frac{1}{2}(k_w-1)^2-2}$ when w is odd. Defining

$$A(w, k_1, k_2, \ldots, k_w) = x^{\sum_{2j+1\leq w-1} \frac{1}{2}(k_{2j+1}^2 - k_{2j+2}^2)} \prod_k \frac{1}{1-x^k} \prod_{k=2j+1} \frac{1}{1+x^k},$$

we see that the number of unipotent strongly real classes has generating function

$$\sum_w \sum_{k_1>k_2>\cdots>k_w>1} A(w, k_1, k_2, \ldots, k_w) x^{(w \bmod 2)[\frac{1}{2}(k_w-1)^2-2]}$$

where each k_i is odd.

3.4 Computational Results

These results were achieved with GAP [9].

Proposition 3.15. *A unipotent element of type* $(3,2,2)$ *is not sr in* $U(7,4)$ *and* $U(7,16)$*. A unipotent element of type* $(3,2,2,2)$ *in* $U(9,4)$ *is not sr.*

Proof. A unipotent element of type $(3,2,2)$ in $\mathrm{U}(7,16)$ is

$$g = \begin{pmatrix} 1 & a & \overline{b} & & & & \\ & 1 & \overline{a} & & & & \\ & & 1 & & & & \\ & & & 1 & 1 & & \\ & & & & 1 & & \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{pmatrix},$$

where $a \neq 0$ and $b + \overline{b} = a\overline{a}$. Suppose s is an involution conjugating g to its inverse. We compare sg and $g^{-1}s$ and $s^*J = Js$ to determine some entries of s and get some equations that the entries of s must satisfy. We append to these those equations that come from $s^2 = 1$. By checking every possible value of a, b, and the entries of s, we see that these equations can never simultaneously be satisfied. The same procedure was used to get the other two results. $\square$

GAP Code for Computational Results

Here is the code for $(3,2,2)$ for $q^2 = 16$. Note that this implies the result for $q = 4$ since $\mathrm{U}(7,4)$ is a subgroup (actually identified with a subgroup) of $\mathrm{U}(7,16)$. Fx is $\mathbb{F}_{16} - \{0\}$, $F0$ is the fixed field under the Frobenius Automorphism. F is GF(16) but is more convenient to type. Note that the Frobenius automorphism in $\mathbb{F}_{16}$ is given by raising to the fourth and in $\mathbb{F}_4$ it is given by raising to the second.

```
Fx := [];

for x in GF(16) do
if not x = 0*Z(2) then
   Add(Fx,x);
    fi;
od;



F  := [0*Z(2)];

for x in Fx do
```

```
   Add(F,x);
od;



F0 := [0*Z(2), Z(2)^0];


for x in F do
  if x^4 = x then
     Add(F0,x)



for x12 in Fx do
  for x14 in F do
    for x15 in F do
      for x16 in F do
        for x17 in F do
          for d44 in F0 do
            for d45 in F0 do
              for x46 in F do
                for d47 in F0 do
                  for d54 in F0 do
                    for d56 in F0 do
                      for a in Fx do
                        for b in Fx do
                          if x15*x14^4 = x14*x15^4 and
x14*x17^4 + x15*x16^4+ x15^4*x16 + x14^4*x17 + Z(2)^0 = 0*Z(2) and
 x14 + x14*d44 + x15*d54 = 0*Z(2) and
```

```
x15 + x14*d45 + x15*d44 = 0*Z(2) and
x16 + Inverse(a)*x14*x12 + x14*x46 + x15*d56 + x16*d44 + x17*d54 = 0* Z(2) and
x17 + Inverse(a)*x15*x12 + x14*d47 + x15*x46^4 +x16*d45 + x17*d44 = 0*Z(2) and
d44 + d45*d54 = Z(2)^0 and
Inverse(a)^5*x14*x15^4 + d45*d56 + d47*d54 = 0*Z(2) and
Inverse(a)^4*x15^4*Inverse(a)*x15 + d45*x46^4 + x46*d45 = 0*Z(2) and
Inverse(a)^4*x14^4*Inverse(a)*x14 +
d54*x46 + x46^4*d54 = 0*Z(2) and b + a^4*x12 = b^4 + a*x12^4 and
b + b^4 = a^5 then
                          I := [x12,x14,x15,x16,x17,d44, d45,x46,d47,d54,d56,a,b];
                        fi;
                      od;
                    od;
                  od;
                od;
              od;
            od;
          od;
        od;
      od;
    od;
  od;
od;
od;

I;
```

Here is the code for (3,2,2,2):

```
Fx := [];

for x in GF(4) do
if not x = 0*Z(2) then
   Add(Fx,x);
    fi;
od;



F  := [0*Z(2)];

for x in Fx do
   Add(F,x);
od;



F0 := [0*Z(2), Z(2)^0];

for x12 in Fx do
  for x14 in F do
    for x15 in F do
      for x16 in F do
        for x17 in F do
          for d44 in F0 do
            for x45 in F do
              for x46 in F do
                for d55 in F0 do
```

```
                  for x56 in F do
                    for a in Fx do
                      for d47 in F0 do
                        for x18 in F do
                          for x19 in F do
                            for d66 in F0 do
                              for x48 in F do
                                for x49 in F do
                                  for d58 in F0 do
                                    for x59 in F do
                                      for d69 in F0 do
       if x14^3 + x15^3 + x16^3 = 0*Z(2) and
x12^3 + x17^2*x14 + x18^2*x15 + x19^2*x16 + x14^2*x17 + x15^2*x18 + x16^2*x19
 = 0*Z(2) and x14 + x14*d44 + x15*x45^2 + x16*x46^2 = 0*Z(2) and
x15 + x14*x45 + x15*d55 + x16*x56^2 = 0*Z(2) and
x16 + x14*x46 + x15*x56 + x16*d66 = 0*Z(2) and
x17 + Inverse(a)*x12*x14 + d47*x14 + x15*x48^2 +x16*x49^2 + d44*x17
+ x18*x45^2 + x19*x46^2= 0*Z(2) and x12 + a*Inverse(a^2)*x12^2 = a and
d44 + x45^3 + x46^3 = Z(2)^0 and
d44*x45 + x45*d55 +x46*x56^2 = 0*Z(2) and
d44*x46 + x45*x56 + x46*d66 = 0*Z(2) and
Inverse(a)*Inverse(a^2)*x14^3 + x45*x48^2 + x46*x49^2 + x48*x45^2
 + x49*x46^2 = 0*Z(2) and x45^3 + d55 + x56^3 = Z(2)^0 and
x45^2*x46 + d55*x56 + x56*d66 = 0*Z(2) and
x46^3 + x56^3 + d66 = Z(2)^0 and
x17 + Inverse(a)*x14 + x14*d47 + x15*x48^2 + x16*x49^2 + x17*d44 +
 x18*x45^2 + x19*x46^2 = 0*Z(2) and
```

```
Inverse(a^3)*x15*x14^2 + d44*x48 + x45*d58 + x46*x59^2 + d47*x45
 + x48*d55 + x49*x56^2 = 0*Z(2) and
Inverse(a^3)*x15^3 + x45^2*x48 + d55*d58 + x56*x59^2 + x48^2*x45
 + d58*d55 + x59*x56^2 = 0*Z(2) and
x19 + Inverse(a)*x12*x16 + x14*x49 + x15*x59 + x16*d69
 + x17*x46 + x18*x56 + x19*d66 = 0*Z(2) and
Inverse(a^3)*x14^2*x16 + x49*d44 + x45*x59 + x46*d69 + d47*x46
+x48*x56 + x49*d66 = 0*Z(2) and
Inverse(a^3)*x15^2*x16 + x49*x45^2 + d55*x59 + x56*d69 + x48^2*x46
 +d58*x56 + x59*d66 = 0*Z(2) and
Inverse(a^3)*x16^3 + x49*x46^2 + x56^2*x59 + x49^2*x46 + x59^2*x56
 + x59*d66 = 0*Z(2)
then
I := [x12, x14, x15, x16, x17, x18, x19, d44, x45, x46, d47, x48,
 x49, d55, x56, d58, x59, d66, d69];
                        fi;
                      od;
                    od;
                  od;
                od;
              od;
            od;
          od;
        od;
      od;
    od;
  od;
```

```
od;

od;

od;

od;

od;

od;

od;

od;

od;


I;
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