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Order invariant spectral properties for several matrices

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\textbf{ABSTRACT}

The collections of \(mn\)-by-\(n\) matrices over a field such that the products in any of the \(m!\) orders share a common similarity class (resp. spectrum, trace) are studied. The spectral and trace order invariant properties are characterized and the similarity invariant one is related to them in several cases. A complete explicit description is given in case \(m = 3\) and \(n = 2\).

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1. Introduction

We consider \(m\) \(n\)-by-\(n\) matrices \(A_1, \ldots, A_m\) over a field \(F\), and the \(m!\) products that may be formed by using each of them once. If the matrices are nonsingular, any two products that are related by cyclic permutation, e.g., \(A_1A_2A_3A_4\) and \(A_3A_4A_1A_2\) are necessarily similar, but in general, among the \(m!\) products, \((m - 1)\) different similarity classes may occur and even \((m - 1)!\) different traces.

The case \(m = 2\) has been thoroughly studied \([6,7]\) going back to the work of Flanders \([1]\). In this case, if one of the matrices is nonsingular, \(A_1A_2\) and \(A_2A_1\) are always similar, and if both are singular, the nonzero eigenvalues (and the corresponding Jordan structure) must be the same (counting multiplicity), and the precise possible relations between the Jordan structures associated with 0 are known. For larger \(m\), the determinants of all the products are the same, and the many different spectra

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\end{footnotesize}
Motivated in part by a curious recently appearing instance [2], we begin study here of a dual question suggested by Flanders’ observation. We call the collection \( A_1, \ldots, A_m \) similarity order invariant (SOI) if each of the \( m! \) possible products lies in the same similarity class. As noted, this always happens when \( m = 2 \) and the matrices are nonsingular; we will be primarily interested in the nonsingular case. Closely related are two weaker properties of interest.

We call \( A_1, \ldots, A_m \) eigenvalue order invariant (EOI) if each of the \( m! \) possible products has the same characteristic polynomial (i.e., each has the same eigenvalues, counting multiplicities, in an algebraically closed extension field). Of course, SOI implies EOI but not generally conversely (as we shall see).

Further, we call \( A_1, \ldots, A_m \) trace order invariant (TOI) if all the \( m! \) products have the same trace. Then, EOI implies TOI. When \( n = 2 \), they are equivalent, because of the common determinant but, for \( n > 2 \), they are not (in general).

Example 1. The real matrices

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & -2 & 1 \\
1 & 1 & -2 \\
0 & 1 & 1
\end{bmatrix}
\]

are TOI but not EOI.

We make several observations about SOI/EOI/TOI, but we suspect that a complete, effective characterization of SOI is very difficult. We say a great deal about TOI, which has some very nice structure, and relate it to the other two properties.

In the next section, we introduce some notation and make some general observations that will be used throughout. Then, in Section 3, we discuss TOI, especially for 3 matrices. EOI is discussed in Section 4, using TOI and compounds; then a complete picture of all three properties is given in Section 5 for \( m = 3 \) and \( n = 2 \), when the field \( F \) has characteristic different from 2.

2. General facts about TOI, EOI and SOI

It is clear that each of the properties: TOI, EOI and SOI is simultaneously similarity invariant. If \( A_1, \ldots, A_m \) are SOI (resp. TOI, EOI), then so are \( A'_1, \ldots, A'_m \) for \( A'_i = S^{-1} A_i S \) and \( S \) any fixed element of \( GL_n(F) \). Each of the properties is also invariant with respect to the order in which the matrices are presented; they are properties of the set. For this reason, we may assume, without loss of generality, that one of the matrices, for example the last one, is in any convenient form achievable by one of the matrices under similarity. Diagonal form, if achievable, is often convenient. Note also that SOI (resp. TOI, EOI) is invariant under multiplication of any matrix by a nonzero scalar. Thus, if a matrix has a nonzero eigenvalue, we can assume that it has an eigenvalue equal to 1.

Here, the field \( F \) becomes a minor issue. For some fields, a matrix may be diagonalizable over an extension field but not over the ground field (e.g., if the eigenvalues are distinct but are not elements of the ground field). Generally, whether or not an extension field is involved is not important, and we freely work over an extension without comment. Note that TOI and EOI (by virtue of being a statement about characteristic polynomials that are necessarily over the ground field) are more formal properties that should not depend upon the field. Even for the more subtle property, SOI, we remind the reader of the fact that if two matrices in \( M_n(F) \) are similar over an extension field, they are similar over the ground field \( F \).

We say that the \( n \) by \( n \) matrices \( A_1, \ldots, A_m \) over a field \( F \) are simultaneously symmetrizable (upper triangularizable) if there exists a nonsingular \( n \times n \) matrix \( P \) over an extension field of \( F \) such that \( PA_i P^{-1} \) is symmetric (upper triangular), for all \( i = 1, \ldots, m \).

If \( A_1, \ldots, A_m \) are nonsingular, then any cyclic permutation of \( A_1 \cdots A_m \) is similar to it. In fact, the nonsingularity of only \( m - 1 \) of \( A_1, \ldots, A_m \) is needed, but, generally, if the matrices are singular,
cyclic permutations need not be similar; this is already apparent in the case \( m = 2 \). If the matrices are nonsingular (or if \( m - 1 \) of them are), then each of the \((m - 1)!)\ cyclic permutation classes may belong to a different similarity class [4]. To check if \( A_1, \ldots, A_m \) are SOI it thus suffices to see if a representative of one cyclic class is similar to a representative of any other. For example, it suffices to check if the \((m - 1)!)\ matrices \( A_1A_2 \cdots A_m, \ldots, A_1A_mA_{m-1} \cdots A_2 \) (in which \( A_1 \) is followed by each of the permutations of \( A_2 \cdots A_m \)) are mutually similar. In case \( m = 3 \) and at least two of the matrices \( A_1, A_2, A_3 \) are nonsingular, they are SOI if and only if \( A_1A_2A_3 \) is similar to \( A_1A_3A_2 \). So, nonsingularity is an important assumption for SOI, but it is not so important for the more formal properties of TOI and EOI. In particular, note that any cyclic permutation of \( A_1, \ldots, A_m \) has the same trace, even when the matrices are singular.

Now, we note a curious fact about SOI that we shall use. Corresponding statements hold for EOI and TOI, as well. In case of TOI, the statement holds even if some of the matrices are singular.

**Theorem 2.** If \( A_1, A_2, A_3 \in GL_n(F) \), then \( A_1, A_2, A_3 \) are SOI (resp. TOI, EOI) if and only if \( A_1A_2A_3 \) is similar to \( A_1^T A_2^T A_3^T \) (resp. \( \text{Tr}(A_1A_2A_3) = \text{Tr}(A_1^T A_2^T A_3^T) \), \( A_1A_2A_3 \) and \( A_1^T A_2^T A_3^T \) are cospectral).

**Proof.** We verify only the SOI statement. The others are similar. Since \( m = 3 \) and the matrices are nonsingular, SOI is equivalent to the similarity of \( A_1A_2A_3 \) and \( A_1A_3A_2 \). But, since any matrix is similar to its transpose, \( A_1A_2A_3 \) is similar to \( A_1^T A_2^T A_3^T \), which, in turn, is similar to \( A_1^T A_2^T A_3^T \) via cyclic permutation and nonsingularity, which completes the proof. □

It follows that 3 nonsingular symmetric matrices are necessarily SOI, as are 3 nonsingular matrices that are simultaneously symmetrizable. But the condition given in Theorem 2 is even weaker, as the next example shows. There appears not to be an analogous result for larger \( m \). If \( m \) nonsingular matrices are simultaneously symmetrizable, it simply reduces the number of potentially different similarity classes to \((m - 1)!)^2/2\).

**Example 3.** Consider the real matrices

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.
\]

We will see that \( A_1 \) and \( A_2 \) are not simultaneously symmetrizable, but the triple \( A_1, A_2, A_3 \) is SOI. Suppose that there is a nonsingular \( P \) such that both \( PA_1P^{-1} \) and \( PA_2P^{-1} \) are symmetric. Then,

\[
PA_1P^{-1} = P^{-T}A_1^TP^T, \\
PA_2P^{-1} = P^{-T}A_2^TP^T.
\]

This implies that \( PA_1A_2P^{-1} = P^{-T}A_1^TA_2^TP^T \), or, equivalently, \((P^TP)A_1A_2 = (A_2A_1)^T(P^TP)\). It is easy to see that there is no nonsingular symmetric matrix \( Q \) such that \( QA_1A_2 = (A_2A_1)^TQ \). Now, note that the triple \( A_1, A_2, A_3 \) is SOI as

\[
A_1A_2A_3 = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad A_1A_3A_2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}.
\]

If \( A_1, \ldots, A_m \) are a commuting family (or if some \( m - 1 \) of them form a commuting family in case there are at least \( m - 1 \) nonsingular matrices), then they are SOI. Interestingly, they can also be SOI without any commutativity. In Example 3, no two of the three matrices commute and, still, SOI occurs.

Generally, SOI is equivalent to EOI if the eigenvalues of one (and, thus, all) of the matrix products are distinct. Though this is generic, there are important differences when eigenvalues coincide, as we shall see in several examples.

If \( A_1, \ldots, A_m \) are all upper (resp. lower) triangular, they are EOI. (So, if \( A_1A_2 \cdots A_m \) has distinct eigenvalues, they are SOI.) Thus, if \( A_1, \ldots, A_m \) are simultaneously upper triangularizable matrices, they are EOI. But, it can happen, however, that \( A_1, \ldots, A_m \) are EOI without being simultaneously triangularizable.
Matrix compounds are a useful tool, especially for EOI. Recall that the $k$th compound of $A \in M_n(F)$ is the $\binom{n}{k}$-by-$\binom{n}{k}$ matrix of $k$-by-$k$ minors of $A$, with the index sets of the minors ordered lexicographically. It is usually denoted by $C_k(A)$ and is defined for all $k \leq n$. As it respects most matrix operations, it has very nice structure. For example, $C_k(AB) = C_k(A) C_k(B)$ and $C_k(A^T) = C_k(A)^T$. Also, $\text{Tr}(C_k(A))$ is the sum of the $k$-by-$k$ principal minors of $A$, the $k$th elementary symmetric function of the eigenvalues or $+ \ldots +$ the coefficient of $x^{n-k}$ in the characteristic polynomial of $A$. If $A$ is diagonal, $C_k(A)$ is also diagonal. These properties will be used to link TOI and EOI in Section 4.

As usual, we denote by $e$ the vector each of whose entries is 1 and whose size is determined by the context. We denote by $\circ$ the Hadamard product of matrices. By $J_n(\lambda)$ we denote the $n$-by-$n$ Jordan block associated with the eigenvalue $\lambda$:

$$J_n(\lambda) = \lambda I_n + \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}.$$

3. Trace order invariance for 3 matrices

Let $A_1, A_2, A_3 \in M_n(F)$. Because any cyclic permutation of $A_1 A_2 A_3$ (resp. $A_1 A_3 A_2$) has the same trace, then $A_1, A_2, A_3$ are TOI if and only if $\text{Tr}(A_1 A_2 A_3) = \text{Tr}(A_1 A_3 A_2)$.

For $m$ matrices, if we regard one as variable and the other $m-1$ as fixed, TOI requires that $(m-1)!$, possibly different, traces be equal, which means that the one matrix must satisfy $(m-1)!-1$ linear homogeneous equations in its entries. If $m = 3$, this is a single linear equation, that we analyze in greater detail.

Lemma 4. For $X, Y \in M_n(F)$, $\text{Tr}(XY) = e^T(X^T \circ Y)e = e^T(X \circ Y^T)e$.

Proof. Suppose that $X = [x_{ij}]$ and $Y = [y_{ij}]$. Then, observe that

$$\text{Tr}(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ji}.$$  

On the other hand, $(X^T \circ Y)_{ij} = x_{ji} y_{ij}$ and $(X \circ Y^T)_{ij} = x_{ij} y_{ji}, i, j = 1, \ldots, n$. Since the sum of either coincides with $\text{Tr}(XY)$, the claim is verified. \qed

Theorem 5. Let $A_1, A_2, A_3 \in M_n(F)$. Then the following are equivalent:

(i) $A_1, A_2, A_3$ are TOI;
(ii) $e^T(A_1^T \circ A_2 A_3 - A_1 \circ A_2^T A_3^T)e = 0$;
(iii) $e^T(A_1^T \circ (A_2 A_3 - A_3 A_2))e = 0$.

Proof. By Lemma 4, we have that

$$\text{Tr}(A_1(A_2 A_3)) = e^T(A_1 \circ A_2 A_3)e$$

and

$$\text{Tr}(A_1(A_3 A_2)) = e^T(A_1 \circ (A_3 A_2)^T)e = e^T(A_1^T \circ A_3 A_2)e.$$  

Thus, $\text{Tr}(A_1 A_2 A_3) = \text{Tr}(A_1 A_3 A_2)$ if and only if

$$e^T(A_1^T \circ A_2 A_3)e = e^T(A_1 \circ (A_3 A_2)^T)e,$$

or, equivalently,

$$e^T(A_1^T \circ A_2 A_3)e = e^T(A_1^T \circ A_3 A_2)e,$$

and the result follows. \qed
We note that, if $A_2$ and $A_3$ commute, then $A_1^T \circ (A_2A_3 - A_3A_2) = 0$ and, so, condition (iii) in Theorem 5 is trivially satisfied, which implies that $A_1, A_2, A_3$ are TOI.

Also, if $A_1, A_2, A_3$ are symmetric or (upper) triangular then $A_1^T \circ A_2A_3 - A_1 \circ A_3^T A_2^T = 0$ and, therefore, condition (iii) in Theorem 5 is satisfied, which again implies that $A_1, A_2, A_3$ are TOI. The same happens if $A_3$ is symmetric and $A_2A_3 = A_3^T A_2^T$.

Observe also that, because TOI does not depend on the order of the matrices, conditions (ii) and (iii) in Theorem 5 can be phrased in other ways by permuting the matrices $A_1, A_2, A_3$. For example, condition (iii) is equivalent to

$$e^T (A_2^T \circ (A_1A_3 - A_3A_1))e = 0,$$

Also, because TOI is invariant under simultaneous similarity of $A_1, A_2, A_3$, then conditions (ii) and (iii) also are.

We now consider the case in which one of the matrices is in Jordan canonical form, which will allow us to get some nice corollaries.

**Theorem 6.** Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that $A_3 = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. Then, $A_1, A_2, A_3$ are TOI if and only if

$$e^T (A_1^T \circ A_2 - A_1 \circ A_3^T) = e^T (A_1^T \circ (JA_2 - A_2)) e = 0,$$

in which $v$ is the column vector corresponding to the diagonal of $A_3$ (the eigenvalues of $A_3$, in the indicated order) and $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$.

**Proof.** Let $D = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$. Then, $A_3 = D + J$. By Theorem 5, we have $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$ if and only if

$$e^T (A_1^T \circ A_2(D + J) - A_1 \circ A_3^T(D + J)^T) e = 0,$$

or, equivalently,

$$e^T (A_1^T \circ A_2D - A_1 \circ A_3^T D^T) e = e^T (A_1 \circ A_3^T - A_1 \circ A_2) e.$$

Since $D$ is diagonal and $v = De$, and because of Lemma 4, the last condition is equivalent to (1). \[\square\]

Note that if $A_2$ is diagonal, then $D = 0$ and (1) is a homogeneous equation in the eigenvalues of $A_3$. If $A_3$ is not diagonalizable, we again have a single linear equation (in the eigenvalues), which need no longer be homogeneous, that is equivalent to TOI. The coefficient vector may be reduced in dimension according to the multiple eigenvalues. In particular, if the coefficient vector is 0, there need be no solution.

**Example 7.** Consider the real matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$ 

Then, we have $Tr(A_1A_2A_3) = 4\lambda + 3$ and $Tr(A_1A_3A_2) = 4\lambda + 2$. Thus, for no $\lambda$ do we have $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$.

Note that in Theorem 6, if $A_3$ has only one distinct eigenvalue $\lambda$, then $v = \lambda e$. Also, since $A_1^T \circ A_2 = (A_1 \circ A_2^T)^T$, the matrix $A_1 \circ A_2^T - A_1 \circ A_2$ is skew-symmetric. Thus, $e^T (A_1^T \circ A_2 - A_1 \circ A_2) e = 0$. From Theorem 6, we have

**Corollary 8.** Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that $A_3 = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$. Then, $A_1, A_2, A_3$ are TOI if and only if

$$e^T (A_1^T \circ (JA_2 - A_2)) e = 0,$$

in which $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$. Moreover, if (2) holds, $A_1, A_2, A_3$ are TOI for any $\lambda$. 

1954.

If $A_3$ is diagonal, then $J = 0$ and from Theorem 6 we have

**Corollary 9.** Let $A_1, A_2 \in M_n(F), A_3 = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and let $\nu = A_3 e$, the vector of eigenvalues of $A_3$. Then $A_1, A_2, A_3$ are TOI if and only if

\[
e^T(A_1 \circ A_2 - A_1 \circ A_2^T)\nu = 0,
\]

that is, $e^T(A_1^T \circ A_2 - A_1 \circ A_2^T)$ is orthogonal to $\nu$.

It can happen that $A_1$ and $A_2$ are such that the triple $A_1, A_2, A_3$ is TOI for any diagonal $A_3$. Since $M = A_1 \circ A_2^2 - A_1^2 \circ A_2$ is skew-symmetric, $e^T M = 0$ if and only if $A_1 \circ A_2^2$ is line sum symmetric (LSS), i.e., the $i$th row sum is the same as the $i$th column sum. Since only the $0$ vector is orthogonal to all vectors, we have also in the above notation

**Corollary 10.** Let $A_1, A_2 \in M_n(F)$. Then the following are equivalent:

\begin{itemize}
  \item[(i)] $A_1, A_2, A_3$ are TOI for any diagonal matrix $A_3 \in M_n(F)$;
  \item[(ii)] $e^T(A_1 \circ A_2^2 - A_1^2 \circ A_2) = 0$;
  \item[(iii)] $A_1 A_2$ and $A_2 A_1$ have the same diagonal;
  \item[(iv)] $A_1 \circ A_2^2$ is LSS.
\end{itemize}

Since TOI is simultaneously similarity invariant and does not depend upon the order of the three matrices, we have from Theorem 6.

**Theorem 11.** Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that there is a nonsingular matrix $S$ such that $S^{-1} A_3 S = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. Then $A_1, A_2, A_3$ are TOI if and only if

\[
e^T((S^{-1} A_1 S)^T \circ (S^{-1} A_2 S) - (S^{-1} A_1 S) \circ (S^{-1} A_2 S)^T)\nu,
\]

in which $\nu$ is the column vector corresponding to the diagonal of $S^{-1} A_3 S$ (the eigenvalues of $A_3$) and $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$.

We also note that Corollary 10 may similarly be translated. The matrix $S^{-1} A_1 S \circ (S^{-1} A_2 S)^T$ is LSS if and only if any matrix $A_3$, diagonalizable via $S$, is, together with $A_1$ and $A_2$, a TOI triple.

We may now make some observations. Suppose that $A_3$ is diagonal. If $A_1$ and $A_2$ are symmetric, we have a symmetric triple, which is necessarily SOI if the matrices are nonsingular, and, thus, TOI. Of course, in general, the TOI property follows from Corollary 10, as $A_1 \circ A_2^2$ is symmetric and, thus, LSS. But $A_1 \circ A_2^2$ LSS is a much weaker condition than $A_1$ and $A_2$ being symmetric, which, though it implies TOI, does not imply SOI, even in the 2-by-2 case.

**Example 12.** Consider the real matrices

\[
A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then, $A_1 A_2 A_3 = 4I_2$ and

\[
A_1 A_2 A_3 = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix},
\]

which are not similar. Of course, because of triangularity, $A_1, A_2, A_3$ are EOI and, thus, TOI. Note that $A_1 \circ A_2^T$ is LSS.

This example also shows that, even for 2-by-2 matrices, TOI does not imply SOI, though very often a TOI triple is SOI.
In case the matrices are nonsingular, if \( A_1 \) and \( A_2 \) commute, then we know that \( A_1, A_2, A_3 \) are SOI and, thus, TOI. But, again, this condition can be significantly weakened for TOI. If \( A_1, A_2 \) commute, then \( A_1 A_2 \) and \( A_2 A_1 \), in particular, have the same diagonal. Again, condition (iii) of Corollary 10, which implies TOI if \( A_3 \) is diagonal, is much weaker, but is not sufficient for SOI, as shown by the same example above.

4. From TOI to EOI via compounds

For \( A_1, \ldots, A_m \in M_n(F) \) to be EOI they must be TOI. But this necessity may be turned into sufficiency via compounds. Since two matrices \( X, Y \in M_n(F) \) have the same characteristic polynomial if and only if

\[
\text{Tr}(C_k(X)) = \text{Tr}(C_k(Y)),
\]

\( k = 1, \ldots, n \), and since each product \( A_i \cdots A_m \) has the same determinant, we have (because the compounds are multiplicative)

**Theorem 13.** The matrices \( A_1, \ldots, A_m \in M_n(F) \) are EOI if and only if the compounds \( C_k(A_1), \ldots, C_k(A_m) \) are TOI, \( k = 1, \ldots, n - 1 \).

Based on Corollary 9, when \( m = 3 \) and one of the matrices is diagonalizable (say \( A_3 \) is diagonal), this takes a nice form

**Corollary 14.** Let \( A_2 \in M_n(F) \) be diagonal and let \( A_1, A_2 \in M_n(F) \). Then, \( A_1, A_2, A_3 \) are EOI if and only if

\[
e^T(C_k(A_1) \circ C_k(A_1^T) - C_k(A_2^T) \circ C_k(A_2))v_k = 0,
\]

in which \( v_k \) is the vector of diagonal entries of \( C_k(A_3) \), \( k = 1, \ldots, n - 1 \).

Of course, corresponding statements may be made for all diagonal \( A_3 \), for example by requiring that \( C_k(A_1) \circ C_k(A_2^T) \) be LSS for all \( k \).

5. TOI, EOI, SOI for 2-by-2 matrices

We now discuss the order invariance properties for three 2-by-2 matrices when \( F \) has characteristic different from 2 (\( \text{char}(F) \neq 2 \)). In the nonsingular case, a complete understanding of all three properties, and the relationships among them, is possible.

Of course, TOI is equivalent to EOI as EOI always implies TOI, and TOI, together with the fact that the determinants of all products are the same, implies that all products have the same characteristic polynomial. The equivalence of TOI and EOI remains true, even if there are more matrices (larger \( m \)) or they are singular. It only depends upon the requirement that \( n = 2 \). The property TOI is easily understood for \( m = 3 \) and \( n = 2 \) via Section 3, and the results there are especially simple in that case.

We next describe, in another explicit way, how TOI (EOI) occurs in this case, and this will allow us to describe explicitly how TOI occurs in the nonsingular case, as well as the precise relationship between TOI (EOI) and SOI. Recall that, when \( m = 3 \), the simultaneous symmetrizability of \( A_1, A_2 \) and \( A_3 \) is sufficient for TOI (in particular, if the matrices are nonsingular, it is sufficient for SOI). Also, in general, simultaneous (upper) triangularizability implies EOI. Interestingly, when \( \text{char}(F) \neq 2 \), these are the only ways EOI (and, thus, TOI) can occur in our case, even if some matrices are singular.

For \( a \in F \), we denote by \( \sqrt{a} \) a solution of the equation \( x^2 - a = 0 \) in an extension field of \( F \).

**Lemma 15.** Let \( A_1, A_2, A_3 \in M_2(F) \), with \( \text{char}(F) \neq 2 \). Suppose that among \( A_1, A_2, A_3 \) there are two matrices that commute. Then \( A_1, A_2, A_3 \) are simultaneously symmetrizable or simultaneously (upper) triangularizable.
Proof. Without loss of generality, suppose that $A_1$ and $A_2$ commute. If both $A_1$ and $A_2$ are scalar, the result is trivial. So, suppose that $A_1$ is nonscalar. Thus, by a possible simultaneous similarity, we can assume that $A_1$ has one of the following forms:

$$
A_1 = \begin{bmatrix}
1 & 0 \\
0 & a_{22}
\end{bmatrix} \quad \text{or} \quad A_1 = \begin{bmatrix}
a_{11} & 1 \\
0 & a_{11}
\end{bmatrix},
$$

(3)

with $a_{11}, a_{22} \in F, a_{22} \neq 1$. Let $A_3 = [c_{ij}]$.

Case 1: Suppose that $A_3$ has the first form in (3). A calculation shows that $A_2$ is diagonal. Then, if $A_3$ is triangular, by an additionally simultaneous permutation similarity, we can assume that $A_3$ is upper triangular, and then $A_1, A_2, A_3$ are all upper triangular. If $A_3$ is not triangular then $c_{12} \neq 0$ and $c_{21} \neq 0$. Let $D = \text{diag} (\sqrt{c_{21}} / \sqrt{c_{12}}, 1)$. Then $DA_3D^{-1}$ is symmetric, for $i = 1, 2, 3$.

Case 2: Suppose that $A_1$ has the second form in (3). A calculation shows that

$$
A_2 = \begin{bmatrix}
b_{11} & b_{12} \\
0 & b_{11}
\end{bmatrix},
$$

with $b_{11}, b_{12} \in F$. If $c_{21} = 0$ then the three matrices are upper triangular. Now suppose that $c_{21} \neq 0$. If $c_{11} \neq c_{22}$ let

$$
P = \begin{bmatrix}
1 & \sqrt{-1} & 0 \\
\sqrt{-1} & c_{22} - c_{11} & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

otherwise let

$$
P = \begin{bmatrix}
1 & -\sqrt{-1} & 0 \\
\sqrt{-1} & c_{22} - c_{11} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

Then $PA_ip^{-1}$ is symmetric, $i = 1, 2, 3$. □

**Corollary 16.** Let $A_1, A_2 \in M_2(F)$, with $\text{char}(F) \neq 2$. Then $A_1$ and $A_2$ are simultaneously symmetrizable or simultaneously (upper) triangularizable.

**Proof.** Let $A_3 = I_2$. Note that among $A_1, A_2$ and $A_3$ there are two matrices that commute. Now the result follows from Lemma 15. □

**Theorem 17.** Let $A_1, A_2, A_3 \in M_2(F)$, with $\text{char}(F) \neq 2$. The matrices $A_1, A_2, A_3$ are TOI if and only if at least one of the following occurs:

(i) $A_1, A_2$ and $A_3$ are simultaneously symmetrizable;
(ii) $A_1, A_2$ and $A_3$ are simultaneously (upper) triangularizable.

**Proof.** The sufficiency of any one of the conditions has already been established. Now suppose that $A_1, A_2, A_3$ are TOI (EOI).

Case 1. Suppose that one of the matrices is diagonalizable, say $A_3 = \text{diag}(\lambda_1, \lambda_2)$. Let $A_1 = [a_{ij}]$ and $A_2 = [b_{ij}]$. Then we have

$$
\text{Tr}(A_1A_2A_3) - \text{Tr}(A_1A_3A_2) = (\lambda_1 - \lambda_2)(a_{12}b_{21} - a_{21}b_{12}) = 0.
$$

Then, either $\lambda_1 = \lambda_2$ or $a_{12}b_{21} = a_{21}b_{12}$. If $\lambda_1 = \lambda_2$ or $a_{12} = a_{21} = 0$ or $b_{12} = b_{21} = 0$, then there are two matrices that commute and, by Lemma 15, one of the conditions (i) or (ii) holds. If $a_{12} = b_{12} = 0$ or $a_{21} = b_{21} = 0$ then condition (ii) holds. Now suppose that $a_{12}b_{21}a_{21}b_{12} \neq 0$. Then $A_1$ and $A_2$ are simultaneously symmetrizable via a diagonal matrix and condition (i) holds. In fact, in this case

$$
A_1 = \begin{bmatrix}
a_{11} & ka_{21} \\
a_{21} & a_{22}
\end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix}
b_{11} & kb_{21} \\
b_{21} & b_{22}
\end{bmatrix},
$$

with $k = a_{12}/a_{21} = b_{12}/b_{21}$. For $D = \text{diag} (\sqrt{k}, 1), D^{-1}A_1D, D^{-1}A_2D$ and $D^{-1}A_3D$ are symmetric.
Case 2. Suppose that none of the matrices $A_1, A_2, A_3$ is diagonalizable (which implies that each matrix has just one eigenvalue).

Case 2.1. Suppose that there are two matrices simultaneously upper triangularizable, say $A_2$ and $A_3$. Then $A_2$ and $A_3$ commute and, by Lemma 15, one of the conditions (i) or (ii) holds.

Case 2.2. Suppose that no two matrices are simultaneously upper triangularizable. First consider the case in which there are at least two nonsingular matrices. Without loss of generality, suppose that

$$A_1 = \begin{bmatrix} a & \frac{2a-b^2-1}{2} \\ b & -a \end{bmatrix}, \quad A_2 = \begin{bmatrix} c & \frac{2c-b^2-1}{2} \\ d & -c \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

with $b, d \neq 0$ and either $a \neq 1$ or $c \neq 1$. A calculation shows that

$$\text{Tr}(A_1A_2A_3 - A_1A_3A_2) = 2d - 2b - 2ad + 2bc.$$

Therefore, $\text{Tr}(A_1A_2A_3 - A_1A_3A_2) = 0$ if and only if $(a-1)d = (c-1)b$. Thus, $a, c \neq 1$ and $d = \frac{b(c-1)}{a-1}$, which implies that $A_1$ and $A_2$ commute and, by Lemma 15, one of the conditions (i) or (ii) holds.

Now suppose that there are at least two nilpotent matrices among $A_1, A_2, A_3$. Without loss of generality, suppose that

$$A_1 = \begin{bmatrix} a & -\frac{b^2}{b} \\ b & -a \end{bmatrix}, \quad A_2 = \begin{bmatrix} c & -\frac{c^2}{d} \\ d & -c \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

with $b, d \neq 0$. A calculation shows that

$$\text{Tr}(A_1A_2A_3 - A_1A_3A_2) = 2bc - 2ad.$$

Therefore, $\text{Tr}(A_1A_2A_3 - A_1A_3A_2) = 0$ if and only if $c = \frac{ad}{b}$. Thus, $A_1$ and $A_2$ commute and, by Lemma 15, one of the conditions (i) or (ii) holds. □

We note that the two possibilities (i) and (ii) of Theorem 17 are independent in general, in that any one may occur without the other. In fact, if $A_1, A_2$ and $A_3$ are the real matrices in Example 3, condition (ii) holds while condition (i) does not occur. Now consider the real matrices

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

The matrix $A_1$ has eigenvalues $-1, 3$, the matrix $A_2$ has eigenvalues $\pm \sqrt{10}$ and the matrix $A_1A_2$ has eigenvalues $6, \pm \sqrt{6}$. Thus, by McCoy’s Theorem [3], $A_1, A_2, A_3$, are not simultaneously triangularizable. So it may happen that condition (i) holds while (ii) does not hold.

The next example shows that the occurrence of at least one of the conditions in Theorem 17 is not necessary for TOI when $n > 2$.

**Example 18.** Consider the real matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A calculation shows that $A_1, A_2, A_3$ are TOI. However, since the eigenvalues of $A_1A_2$ are distinct from $-1$ and 1, by McCoy’s Theorem, $A_1$ and $A_2$ are not simultaneously triangularizable, which implies that $A_1, A_2, A_3$ also are not. Moreover, if $A_1, A_2, A_3$ were simultaneously symmetrizable there would exist a nonsingular symmetric matrix $Q$ such that $QA_2 = A_2^TQ$ and $QA_3 = A_3^TQ$, which is easily seen not to happen.

The next example shows that the first condition in Theorem 17 is not sufficient for TOI of $m$ 2-by-2 matrices, when $m > 3$. 


Example 19. Consider the real matrices

\[ A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = A_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \]

The matrices \( A_1, A_2, A_3, A_4 \) are symmetric. However, \( A_1, A_2, A_3, A_4 \) are not TOI as \( \text{Tr}(A_1A_2A_3A_4) = 4 \) and \( \text{Tr}(A_1A_3A_2A_4) = 0. \)

Since (i) is sufficient for SOI when the matrices are nonsingular, we see that

Corollary 20. Suppose that \( \text{char}(F) \neq 2. \) If \( A_1, A_2, A_3 \in GL_2(F) \) are EOI, then they are SOI, unless the three matrices are simultaneously upper triangularizable.

Example 22. Consider the following complex matrices:

\[ A_1 = A_2 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

These matrices are symmetric but are not SOI. Observe that \( A_1A_2A_3 = 0 \) and \( A_1A_3A_2 = 2iA_1. \)

Example 23. Consider the following matrices over a field \( F \) with \( \text{char}(F) = 2: \)

\[ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

The matrices \( A_1, A_2, A_3 \) are TOI. However, no two matrices commute. Also, \( A_1 \) and \( A_3 \) are not simultaneously triangularizable, otherwise the product

\[ A_1A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \]

would have the eigenvalue 1 with multiplicity 2 (as \( A_1 \) and \( A_3 \)), which does not happen. Also, any nonsingular matrix \( P \) such that \( PA_1P^{-1} \) is symmetric has the form
\[ P = \begin{bmatrix} b_{21} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} . \]

Now if \( PA_3 P^{-1} \) was symmetric then \( b_{12} = b_{22} \), which is not possible because \( P \) is nonsingular.

**Example 24.** Consider the following matrices over a field \( F \) with \( \text{char}(F) = 2 \):

\[ A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

Clearly, \( A_1 \) and \( A_2 \) commute. Also, it follows from Example 23 that \( A_1 \) and \( A_3 \) are neither simultaneously triangularizable nor simultaneously symmetrizable.

**References**