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Order invariant spectral properties for several matrices

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ABSTRACT

The collections of m n-by-n matrices over a field such that the products in any of the $m!$ orders share a common similarity class (resp. spectrum, trace) are studied. The spectral and trace order invariant properties are characterized and the similarity invariant one is related to them in several cases. A complete explicit description is given in case $m = 3$ and $n = 2$.

1. Introduction

We consider m n-by-n matrices $A_1, \ldots, A_m$ over a field F, and the $m!$ products that may be formed by using each of them once. If the matrices are nonsingular, any two products that are related by cyclic permutation, e.g., $A_1A_2A_3A_4$ and $A_3A_4A_1A_2$ are necessarily similar, but in general, among the $m!$ products, $(m-1)!$ different similarity classes may occur and even $(m-1)!$ different traces.

The case $m = 2$ has been thoroughly studied [6,7] going back to the work of Flanders [1]. In this case, if one of the matrices is nonsingular, A_1A_2 and A_2A_1 are always similar, and if both are singular, the nonzero eigenvalues (and the corresponding Jordan structure) must be the same (counting multiplicity), and the precise possible relations between the Jordan structures associated with 0 are known. For larger m, the determinants of all the products are the same, and the many different spectra

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(and similarity classes) are not independent, even subject to the common determinant condition; consistency conditions among them have been studied [4,5]. In complete generality, though, this question is still quite open.

Motivated in part by a curious recently appearing instance [2], we begin study here of a dual question suggested by Flanders' observation. We call the collection $A_1, \ldots, A_m$ *similarity order invariant* (SOI) if each of the $m!$ possible products lies in the same similarity class. As noted, this always happens when $m = 2$ and the matrices are nonsingular; we will be primarily interested in the nonsingular case. Closely related are two weaker properties of interest.

We call $A_1, \ldots, A_m$ *eigenvalue order invariant* (EOI) if each of the $m!$ possible products has the same characteristic polynomial (i.e., each has the same eigenvalues, counting multiplicities, in an algebraically closed extension field). Of course, SOI implies EOI but not generally conversely (as we shall see).

Further, we call $A_1, \ldots, A_m$ *trace order invariant* (TOI) if all the $m!$ products have the same trace. Then, EOI implies TOI. When $n = 2$, they are equivalent, because of the common determinant but, for $n > 2$, they are not (in general).

Example 1. *The real matrices*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

are TOI but are not EOI.

We make several observations about SOI/EOI/TOI, but we suspect that a complete, effective characterization of SOI is very difficult. We say a great deal about TOI, which has some very nice structure, and relate it to the other two properties.

In the next section, we introduce some notation and make some general observations that will be used throughout. Then, in Section 3, we discuss TOI, especially for 3 matrices. EOI is discussed in Section 4, using TOI and compounds; then a complete picture of all three properties is given in Section 5 for $m = 3$ and $n = 2$, when the field F has characteristic different from 2.

2. General facts about TOI, EOI and SOI

It is clear that each of the properties: TOI, EOI and SOI is simultaneously similarity invariant. If $A_1, \ldots, A_m$ are SOI (resp. TOI, EOI), then so are $A'_1, \ldots, A'_m$ for $A'_i = S^{-1}A_iS$ and S any fixed element of $GL_n(F)$. Each of the properties is also invariant with respect to the order in which the matrices are presented; they are properties of the set. For this reason, we may assume, without loss of generality, that one of the matrices, for example the last one, is in any convenient form achievable by one of the matrices under similarity. Diagonal form, if achievable, is often convenient. Note also that SOI (resp. TOI, EOI) is invariant under multiplication of any matrix by a nonzero scalar. Thus, if a matrix has a nonzero eigenvalue, we can assume that it has an eigenvalue equal to 1.

Here, the field F becomes a minor issue. For some fields, a matrix may be diagonalizable over an extension field but not over the ground field (e.g., if the eigenvalues are distinct but are not elements of the ground field). Generally, whether or not an extension field is involved is not important, and we freely work over an extension without comment. Note that TOI and EOI (by virtue of being a statement about characteristic polynomials that are necessarily over the ground field) are more formal properties that should not depend upon the field. Even for the more subtle property, SOI, we remind the reader of the fact that if two matrices in $M_n(F)$ are similar over an extension field, they are similar over the ground field F.

We say that the n-by-n matrices $A_1, \ldots, A_m$ over a field F are simultaneously symmetrizable (upper triangularizable) if there exists a nonsingular n-by-n matrix P over an extension field of F such that PA_iP^{-1} is symmetric (upper triangular), for all $i = 1, \ldots, m$.

If $A_1, \ldots, A_m$ are nonsingular, then any cyclic permutation of $A_{i_1} \cdots A_{i_m}$ is similar to it. In fact, the nonsingularity of only $m - 1$ of $A_1, \ldots, A_m$ is needed, but, generally, if the matrices are singular, cyclic permutations need not be similar; this is already apparent in the case $m = 2$. If the matrices are nonsingular (or if $m - 1$ of them are), then each of the $(m - 1)!$ cyclic permutation classes may belong to a different similarity class [4]. To check if $A_1, \ldots, A_m$ are SOI it thus suffices to see if a representative of one cyclic class is similar to a representative of any other. For example, it suffices to check if the $(m - 1)!$ matrices $A_1A_2 \cdots A_m, \ldots, A_1A_mA_{m-1} \cdots A_2$ (in which A_1 is followed by each of the permutations of $A_2 \cdots A_m$) are mutually similar. In case $m = 3$ and at least two of the matrices A_1, A_2, A_3 are nonsingular, they are SOI if and only if $A_1A_2A_3$ is similar to $A_1A_3A_2$. So, nonsingularity is an important assumption for SOI, but it is not so important for the more formal properties of TOI and EOI. In particular, note that any cyclic permutation of $A_{i_1} \cdots A_{i_m}$ has the same trace, even when the matrices are singular.

Now, we note a curious fact about SOI that we shall use. Corresponding statements hold for EOI and TOI, as well. In case of TOI, the statement holds even if some of the matrices are singular.

Theorem 2. *If $A_1, A_2, A_3 \in GL_n(F)$, then A_1, A_2, A_3 are SOI (resp. TOI, EOI) if and only if $A_1A_2A_3$ is similar to $A_1^TA_2^TA_3^T$ (resp. $Tr(A_1A_2A_3) = Tr(A_1^TA_2^TA_3^T)$, $A_1A_2A_3$ and $A_1^TA_2^TA_3^T$ are cospectral).*

Proof. We verify only the SOI statement. The others are similar. Since $m = 3$ and the matrices are nonsingular, SOI is equivalent to the similarity of $A_1A_2A_3$ and $A_1A_3A_2$. But, since any matrix is similar to its transpose, $A_1A_3A_2$ is similar to $A_2^TA_3^TA_1^T$, which, in turn, is similar to $A_1^TA_2^TA_3^T$ via cyclic permutation and nonsingularity, which completes the proof. □

It follows that 3 nonsingular symmetric matrices are necessarily SOI, as are 3 nonsingular matrices that are simultaneously symmetrizable. But the condition given in Theorem 2 is even weaker, as the next example shows. There appears not to be an analogous result for larger m. If m nonsingular matrices are simultaneously symmetrizable, it simply reduces the number of potentially different similarity classes to $(m - 1)!/2$.

Example 3. *Consider the real matrices*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

We will see that A_1 and A_2 are not simultaneously symmetrizable, but the triple A_1, A_2, A_3 is SOI. Suppose that there is a nonsingular P such that both PA_1P^{-1} and PA_2P^{-1} are symmetric. Then,

$$PA_1P^{-1} = P^{-T}A_1^TP^T,$$
$$PA_2P^{-1} = P^{-T}A_2^TP^T.$$

This implies that $PA_1A_2P^{-1} = P^{-T}A_1^TA_2^TP^T$, or, equivalently, $(P^TP)A_1A_2 = (A_2A_1)^T(P^TP)$. It is easy to see that there is no nonsingular symmetric matrix Q such that $QA_1A_2 = (A_2A_1)^TQ$. Now, note that the triple A_1, A_2, A_3 is SOI as

$$A_1A_2A_3 = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad A_1A_3A_2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}.$$

If $A_1, \ldots, A_m$ are a commuting family (or if some $m - 1$ of them form a commuting family in case there are at least $m - 1$ nonsingular matrices), then they are SOI. Interestingly, they can also be SOI without any commutativity. In Example 3, no two of the three matrices commute and, still, SOI occurs.

Generally, SOI is equivalent to EOI if the eigenvalues of one (and, thus, all) of the matrix products are distinct. Though this is generic, there are important differences when eigenvalues coincide, as we shall see in several examples.

If $A_1, \ldots, A_m$ are all upper (resp. lower) triangular, they are EOI. (So, if $A_1A_2 \cdots A_m$ has distinct eigenvalues, they are SOI.) Thus, if $A_1, \ldots, A_m$ are simultaneously upper triangularizable matrices, they are EOI. But, it can happen, however, that $A_1, \ldots, A_m$ are EOI without being simultaneously triangularizable.

Matrix compounds are a useful tool, especially for EOI. Recall that the kth compound of $A \in M_n(F)$ is the $\binom{n}{k}$-by-$\binom{n}{k}$ matrix of k-by-k minors of A, with the index sets of the minors ordered lexicographically. It is usually denoted by $C_k(A)$ and is defined for all $k \leqslant n$. As it respects most matrix operations, it has very nice structure. For example, $C_k(AB) = C_k(A)C_k(B)$ and $C_k(A^T) = C_k(A)^T$. Also, $Tr(C_k(A))$ is the sum of the k-by-k principal minors of A, the kth elementary symmetric function of the eigenvalues or $+/-$ the coefficient of x^{n-k} in the characteristic polynomial of A. If A is diagonal, $C_k(A)$ is also diagonal. These properties will be used to link TOI and EOI in Section 4.

As usual, we denote by e the vector each of whose entries is 1 and whose size is determined by the context. We denote by $\circ$ the Hadamard product of matrices. By $J_n(\lambda)$ we denote the n-by-n Jordan block associated with the eigenvalue λ:

$$J_n(\lambda) = \lambda I_n + \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}.$$

3. Trace order invariance for 3 matrices

Let $A_1, A_2, A_3 \in M_n(F)$. Because any cyclic permutation of $A_1A_2A_3$ (resp. $A_1A_3A_2$) has the same trace, then A_1, A_2, A_3 are TOI if and only if $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$.

For m matrices, if we regard one as variable and the other $m-1$ as fixed, TOI requires that $(m-1)!$, possibly different, traces be equal, which means that the one matrix must satisfy $(m-1)! - 1$ linear homogeneous equations in its entries. If $m = 3$, this is a single linear equation, that we analyze in greater detail.

Lemma 4. *For $X, Y \in M_n(F)$, $Tr(XY) = e^T(X^T \circ Y)e = e^T(X \circ Y^T)e$.*

Proof. Suppose that $X = [x_{ij}]$ and $Y = [y_{ij}]$. Then, observe that

$$Tr(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}y_{ji}.$$

On the other hand, $(X^T \circ Y)_{ij} = x_{ji}y_{ij}$ and $(X \circ Y^T)_{ij} = x_{ij}y_{ji}$, $i,j = 1, \ldots, n$. Since the sum of either coincides with $Tr(XY)$, the claim is verified. $\square$

Theorem 5. *Let $A_1, A_2, A_3 \in M_n(F)$. Then the following are equivalent:*

(i) A_1, A_2, A_3 *are TOI*;
(ii) $e^T(A_1^T \circ A_2A_3 - A_1 \circ A_2^TA_3^T)e = 0$;
(iii) $e^T(A_1^T \circ (A_2A_3 - A_3A_2))e = 0$.

Proof. By Lemma 4, we have that

$$Tr(A_1(A_2A_3)) = e^T(A_1^T \circ A_2A_3)e$$

and

$$Tr(A_1(A_3A_2)) = e^T(A_1 \circ (A_3A_2)^T)e = e^T(A_1^T \circ A_3A_2)e,$$

Thus, $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$ if and only if

$$e^T(A_1^T \circ A_2A_3)e = e^T(A_1 \circ (A_3A_2)^T)e,$$

or, equivalently,

$$e^T(A_1^T \circ A_2A_3)e = e^T(A_1^T \circ A_3A_2)e,$$

and the result follows. $\square$

We note that, if A_2 and A_3 commute, then $A_1^T \circ (A_2A_3 - A_3A_2) = 0$ and, so, condition (iii) in Theorem 5 is trivially satisfied, which implies that A_1, A_2, A_3 are TOI.

Also, if A_1, A_2, A_3 are symmetric or (upper) triangular then $A_1^T \circ A_2A_3 - A_1 \circ A_2^TA_3^T = 0$ and, therefore, condition (iii) in Theorem 5 is satisfied, which again implies that A_1, A_2, A_3 are TOI. The same happens if A_1 is symmetric and $A_2A_3 = A_2^TA_3^T$.

Observe also that, because TOI does not depend on the order of the matrices, conditions (ii) and (iii) in Theorem 5 can be phrased in other ways by permuting the matrices A_1, A_2, A_3. For example, condition (iii) is equivalent to

$$e^T(A_2^T \circ (A_1A_3 - A_3A_1))e = 0,$$

Also, because TOI is invariant under simultaneous similarity of A_1, A_2, A_3, then conditions (ii) and (iii) also are.

We now consider the case in which one of the matrices is in Jordan canonical form, which will allow us to get some nice corollaries.

Theorem 6. *Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that $A_3 = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. Then A_1, A_2, A_3 are TOI if and only if*

$$e^T(A_1^T \circ A_2 - A_1 \circ A_2^T)v = e^T(A_1^T \circ (JA_2 - A_2J))e, \tag{1}$$

in which v is the column vector corresponding to the diagonal of A_3 (the eigenvalues of A_3, in the indicated order) and $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$.

Proof. Let $D = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$. Then, $A_3 = D + J$. By Theorem 5, we have $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$ if and only if

$$e^T(A_1^T \circ A_2(D+J) - A_1 \circ A_2^T(D+J)^T)e = 0,$$

or, equivalently,

$$e^T(A_1^T \circ A_2D - A_1 \circ A_2^TD^T)e = e^T(A_1 \circ A_2^TJ^T - A_1^T \circ A_2J)e.$$

Since D is diagonal and $v = De$, and because of Lemma 4, the last condition is equivalent to (1). $\square$

Note that if A_3 is diagonal, then $J = 0$ and (1) is a homogeneous equation in the eigenvalues of A_3. If A_3 is not diagonalizable, we again have a single linear equation (in the eigenvalues), which need no longer be homogeneous, that is equivalent to TOI. The coefficient vector may be reduced in dimension according to the multiple eigenvalues. In particular, if the coefficient vector is 0, there need be no solution.

Example 7. *Consider the real matrices*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then, we have $Tr(A_1A_2A_3) = 4\lambda + 3$ and $Tr(A_1A_3A_2) = 4\lambda + 2$. Thus, for no λ do we have $Tr(A_1A_2A_3) = Tr(A_1A_3A_2)$.

Note that in Theorem 6, if A_3 has only one distinct eigenvalue λ, then $v = \lambda e$. Also, since $A_1^T \circ A_2 = (A_1 \circ A_2^T)^T$, the matrix $A_1 \circ A_2^T - A_1^T \circ A_2$ is skew-symmetric. Thus, $e^T(A_1^T \circ A_2 - A_1 \circ A_2^T)v = 0$. From Theorem 6, we have

Corollary 8. *Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that $A_3 = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$. Then A_1, A_2, A_3 are TOI if and only if*

$$e^T(A_1^T \circ (JA_2 - A_2J))e = 0, \tag{2}$$

in which $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$. Moreover, if (2) holds, A_1, A_2, A_3 are TOI for any λ.

If A_3 is diagonal, then $J = 0$ and from Theorem 6 we have

Corollary 9. *Let $A_1, A_2 \in M_n(F), A_3 = diag(\lambda_1, \ldots, \lambda_n)$ and let $v = A_3 e$, the vector of eigenvalues of A_3. Then A_1, A_2, A_3 are TOI if and only if*

$$e^T(A_1^T \circ A_2 - A_1 \circ A_2^T)v = 0,$$

that is, $e^T(A_1^T \circ A_2 - A_1 \circ A_2^T)$ is orthogonal to v.

It can happen that A_1 and A_2 are such that the triple A_1, A_2, A_3 is TOI for any diagonal A_3. Since $M = A_1 \circ A_2^T - A_1^T \circ A_2$ is skew-symmetric, $e^T M = 0$ if and only if $A_1 \circ A_2^T$ is line sum symmetric (LSS), i.e., the ith row sum is the same as the ith column sum. Since only the 0 vector is orthogonal to all vectors, we have also in the above notation

Corollary 10. *Let $A_1, A_2 \in M_n(F)$. Then the following are equivalent:*

(i) *A_1, A_2, A_3 are TOI for any diagonal matrix $A_3 \in M_n(F)$;*
(ii) *$e^T(A_1 \circ A_2^T - A_1^T \circ A_2) = 0$;*
(iii) *A_1A_2 and A_2A_1 have the same diagonal;*
(iv) *$A_1 \circ A_2^T$ is LSS.*

Since TOI is simultaneously similarity invariant and does not depend upon the order of the three matrices, we have from Theorem 6.

Theorem 11. *Let $A_1, A_2, A_3 \in M_n(F)$. Suppose that there is a nonsingular matrix S such that $S^{-1}A_3S = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. Then A_1, A_2, A_3 are TOI if and only if*

$$\begin{aligned} &e^T((S^{-1}A_1S)^T \circ (S^{-1}A_2S) - (S^{-1}A_1S) \circ (S^{-1}A_2S)^T)v, \\ &= e^T((S^{-1}A_1S)^T \circ (J(S^{-1}A_2S) - (S^{-1}A_2S)J))e, \end{aligned}$$

in which v is the column vector corresponding to the diagonal of $S^{-1}A_3S$ (the eigenvalues of A_3) and $J = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$.

We also note that Corollary 10 may similarly be translated. The matrix $S^{-1}A_1S \circ (S^{-1}A_2S)^T$ is LSS if and only if any matrix A_3, diagonalizable via S, is, together with A_1 and A_2, a TOI triple.

We may now make some observations. Suppose that A_3 is diagonal. If A_1 and A_2 are symmetric, we have a symmetric triple, which is necessarily SOI if the matrices are nonsingular, and, thus, TOI. Of course, in general, the TOI property follows from Corollary 10, as $A_1 \circ A_2^T$ is symmetric and, thus, LSS. But $A_1 \circ A_2^T$ LSS is a much weaker condition than A_1 and A_2 being symmetric, which, though it implies TOI, does not imply SOI, even in the 2-by-2 case.

Example 12. *Consider the real matrices*

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, $A_1A_2A_3 = 4I_2$ and

$$A_1A_3A_2 = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix},$$

which are not similar. Of course, because of triangularity, A_1, A_2, A_3 are EOI and, thus, TOI. Note that $A_1 \circ A_2^T$ is LSS.

This example also shows that, even for 2-by-2 matrices, TOI does not imply SOI, though very often a TOI triple is SOI.

In case the matrices are nonsingular, if A_1 and A_2 commute, then we know that A_1, A_2, A_3 are SOI and, thus, TOI. But, again, this condition can be significantly weakened for TOI. If A_1, A_2 commute, then A_1A_2 and A_2A_1, in particular, have the same diagonal. Again, condition (iii) of Corollary 10, which implies TOI if A_3 is diagonal, is much weaker, but is not sufficient for SOI, as shown by the same example above.

4. From TOI to EOI via compounds

For $A_1, \ldots, A_m \in M_n(F)$ to be EOI they must be TOI. But this necessity may be turned into sufficiency via compounds. Since two matrices $X, Y \in M_n(F)$ have the same characteristic polynomial if and only if

$$Tr(C_k(X)) = Tr(C_k(Y)),$$

$k = 1, \ldots, n$, and since each product $A_{i_1} \cdots A_{i_m}$ has the same determinant, we have (because the compounds are multiplicative)

Theorem 13. *The matrices $A_1, \ldots, A_m \in M_n(F)$ are EOI if and only if the compounds $C_k(A_1), \ldots, C_k(A_m)$ are TOI, $k = 1, \ldots, n-1$.*

Based on Corollary 9, when $m = 3$ and one of the matrices is diagonalizable (say A_3 is diagonal), this takes a nice form

Corollary 14. *Let $A_3 \in M_n(F)$ be diagonal and let $A_1, A_2 \in M_n(F)$. Then, A_1, A_2, A_3 are EOI if and only if*

$$e^T(C_k(A_1) \circ C_k(A_2^T) - C_k(A_1^T) \circ C_k(A_2))v_k = 0,$$

in which v_k is the vector of diagonal entries of $C_k(A_3)$, $k = 1, \ldots, n-1$.

Of course, corresponding statements may be made for all diagonal A_3, for example by requiring that $C_k(A_1) \circ C_k(A_2^T)$ be LSS for all k.

5. TOI, EOI, SOI for 2-by-2 matrices

We now discuss the order invariance properties for three 2-by-2 matrices when F has characteristic different from 2 ($\mathrm{char}(F) \neq 2$). In the nonsingular case, a complete understanding of all three properties, and the relationships among them, is possible.

Of course, TOI is equivalent to EOI as EOI always implies TOI, and TOI, together with the fact that the determinants of all products are the same, implies that all products have the same characteristic polynomial. The equivalence of TOI and EOI remains true, even if there are more matrices (larger m) or they are singular. It only depends upon the requirement that $n = 2$. The property TOI is easily understood for $m = 3$ and $n = 2$ via Section 3, and the results there are especially simple in that case.

We next describe, in another explicit way, how TOI (EOI) occurs in this case, and this will allow us to describe explicitly how SOI occurs in the nonsingular case, as well as the precise relationship between TOI (EOI) and SOI. Recall that, when $m = 3$, the simultaneous symmetrizability of A_1, A_2 and A_3 is sufficient for TOI (in particular, if the matrices are nonsingular, it is sufficient for SOI). Also, in general, simultaneous (upper) triangularizability implies EOI. Interestingly, when $\mathrm{char}(F) \neq 2$, these are the only ways EOI (and, thus, TOI) can occur in our case, even if some matrices are singular.

For $a \in F$, we denote by $\sqrt{a}$ a solution of the equation $x^2 - a = 0$ in an extension field of F.

Lemma 15. *Let $A_1, A_2, A_3 \in M_2(F)$, with $\mathrm{char}(F) \neq 2$. Suppose that among A_1, A_2, A_3 there are two matrices that commute. Then A_1, A_2, A_3 are simultaneously symmetrizable or simultaneously (upper) triangularizable.*

Proof. Without loss of generality, suppose that A_1 and A_2 commute. If both A_1 and A_2 are scalar, the result is trivial. So, suppose that A_1 is nonscalar. Thus, by a possible simultaneous similarity, we can assume that A_1 has one of the following forms:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \quad \text{or} \quad A_1 = \begin{bmatrix} a_{11} & 1 \\ 0 & a_{11} \end{bmatrix}, \tag{3}$$

with $a_{11}, a_{22} \in F, a_{22} \neq 1$. Let $A_3 = [c_{ij}]$.

Case 1: Suppose that A_1 has the first form in (3). A calculation shows that A_2 is diagonal. Then, if A_3 is triangular, by an additionally simultaneous permutation similarity, we can assume that A_3 is upper triangular, and then A_1, A_2, A_3 are all upper triangular. If A_3 is not triangular then $c_{12} \neq 0$ and $c_{21} \neq 0$. Let $D = diag\left(\sqrt{c_{21}}/\sqrt{c_{12}}, 1\right)$. Then DA_iD^{-1} is symmetric, for $i = 1, 2, 3$.

Case 2: Suppose that A_1 has the second form in (3). A calculation shows that

$$A_2 = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{bmatrix},$$

with $b_{11}, b_{12} \in F$. If $c_{21} = 0$ then the three matrices are upper triangular.
Now suppose that $c_{21} \neq 0$. If $c_{11} \neq c_{22}$ let

$$P = \begin{bmatrix} 1 & 0 \\ \sqrt{-1} & \frac{\sqrt{-1}}{c_{21}}(c_{22} - c_{11}) \end{bmatrix},$$

otherwise let

$$P = \begin{bmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{bmatrix}.$$

Then PA_iP^{-1} is symmetric, $i = 1, 2, 3$. $\square$

Corollary 16. *Let $A_1, A_2 \in M_2(F)$, with $\mathrm{char}(F) \neq 2$. Then A_1 and A_2 are simultaneously symmetrizable or simultaneously (upper) triangularizable.*

Proof. Let $A_3 = I_2$. Note that among A_1, A_2 and A_3 there are two matrices that commute. Now the result follows from Lemma 15. $\square$

Theorem 17. *Let $A_1, A_2, A_3 \in M_2(F)$, with $\mathrm{char}(F) \neq 2$. The matrices A_1, A_2, A_3 are TOI if and only if at least one of the following occurs:*

(i) *A_1, A_2 and A_3 are simultaneously symmetrizable;*
(ii) *A_1, A_2 and A_3 are simultaneously (upper) triangularizable.*

Proof. The sufficiency of any one of the conditions has already been established. Now suppose that A_1, A_2, A_3 are TOI (EOI).

Case 1. Suppose that one of the matrices is diagonalizable, say $A_3 = diag(\lambda_1, \lambda_2)$. Let $A_1 = [a_{ij}]$ and $A_2 = [b_{ij}]$. Then we have

$$Tr(A_1A_2A_3) - Tr(A_1A_3A_2) = (\lambda_1 - \lambda_2)(a_{12}b_{21} - a_{21}b_{12}) = 0.$$

Then, either $\lambda_1 = \lambda_2$ or $a_{12}b_{21} = a_{21}b_{12}$. If $\lambda_1 = \lambda_2$ or $a_{12} = a_{21} = 0$ or $b_{12} = b_{21} = 0$, then there are two matrices that commute and, by Lemma 15, one of the conditions (i) or (ii) holds. If $a_{12} = b_{12} = 0$ or $a_{21} = b_{21} = 0$ then condition (ii) holds. Now suppose that $a_{12}b_{21}a_{21}b_{12} \neq 0$. Then A_1 and A_2 are simultaneously symmetrizable via a diagonal matrix and condition (i) holds. In fact, in this case

$$A_1 = \begin{bmatrix} a_{11} & ka_{21} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} b_{11} & kb_{21} \\ b_{21} & b_{22} \end{bmatrix},$$

with $k = a_{12}/a_{21} = b_{12}/b_{21}$. For $D = diag\left(\sqrt{k}, 1\right)$, $D^{-1}A_1D, D^{-1}A_2D$ and $D^{-1}A_3D$ are symmetric.

Case 2. Suppose that none of the matrices A_1, A_2, A_3 is diagonalizable (which implies that each matrix has just one eigenvalue).

Case 2.1. Suppose that there are two matrices simultaneously upper triangularizable, say A_2 and A_3. Then A_2 and A_3 commute and, by Lemma 15, one of the conditions (i) or (ii) holds.

Case 2.2. Suppose that no two matrices are simultaneously upper triangularizable. First consider the case in which there are at least two nonsingular matrices. Without loss of generality, suppose that

$$A_1 = \begin{bmatrix} a & \frac{2a-a^2-1}{b} \\ b & 2-a \end{bmatrix}, \quad A_2 = \begin{bmatrix} c & \frac{2c-c^2-1}{d} \\ d & 2-c \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

with $b, d \neq 0$ and either $a \neq 1$ or $c \neq 1$. A calculation shows that

$$Tr(A_1A_2A_3 - A_1A_3A_2) = 2d - 2b - 2ad + 2bc.$$

Therefore, $Tr(A_1A_2A_3 - A_1A_3A_2) = 0$ if and only if $(a-1)d = (c-1)b$. Thus, $a, c \neq 1$ and $d = \frac{b(c-1)}{a-1}$, which implies that A_1 and A_2 commute and, by Lemma 15, one of the conditions (i) or (ii) holds.

Now suppose that there are at least two nilpotent matrices among A_1, A_2, A_3. Without loss of generality, suppose that

$$A_1 = \begin{bmatrix} a & -\frac{a^2}{b} \\ b & -a \end{bmatrix}, \quad A_2 = \begin{bmatrix} c & -\frac{c^2}{d} \\ d & -c \end{bmatrix}, \quad A_3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

with $b, d \neq 0$. A calculation shows that

$$Tr(A_1A_2A_3 - A_1A_3A_2) = 2bc - 2ad.$$

Therefore, $Tr(A_1A_2A_3 - A_1A_3A_2) = 0$ if and only if $c = \frac{ad}{b}$. Thus, A_1 and A_2 commute and, by Lemma 15, one of the conditions (i) or (ii) holds. $\square$

We note that the two possibilities (i) and (ii) of Theorem 17 are independent in general, in that any one may occur without the other. In fact, if A_1, A_2 and A_3 are the real matrices in Example 3, condition (ii) holds while condition (i) does not occur. Now consider the real matrices

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

The matrix A_1 has eigenvalues $-1, 3$, the matrix A_2 has eigenvalues $\pm\sqrt{10}$ and the matrix A_1A_2 has eigenvalues $6 \pm \sqrt{6}$. Thus, by McCoy's Theorem [3], A_1, A_2, A_3, are not simultaneously triangularizable. So it may happen that condition (i) holds while (ii) does not hold.

The next example shows that the occurrence of at least one of the conditions in Theorem 17 is not necessary for TOI when $n > 2$.

Example 18. *Consider the real matrices*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A calculation shows that A_1, A_2, A_3 are TOI. However, since the eigenvalues of A_1A_2 are distinct from -1 and 1, by McCoy's Theorem, A_1 and A_2 are not simultaneously triangularizable, which implies that A_1, A_2, A_3 also are not. Moreover, if A_1, A_2, A_3 were simultaneously symmetrizable there would exist a nonsingular symmetric matrix Q such that $QA_2 = A_2^TQ$ and $QA_3 = A_3^TQ$, which is easily seen not to happen.

The next example shows that the first condition in Theorem 17 is not sufficient for TOI of m 2-by-2 matrices, when $m > 3$.

Example 19. *Consider the real matrices*

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } A_3 = A_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The matrices A_1, A_2, A_3, A_4 are symmetric. However, A_1, A_2, A_3, A_4 are not TOI as $Tr(A_1A_2A_3A_4) = 4$ and $Tr(A_1A_3A_2A_4) = 0$.

Since (i) is sufficient for SOI when the matrices are nonsingular, we see that

Corollary 20. *Suppose that* $\mathrm{char}(F) \neq 2$. *If $A_1, A_2, A_3 \in GL_2(F)$ are EOI, then they are SOI, unless the three matrices are simultaneously upper triangularizable.*

Of course, three triangular matrices in $GL_2(F)$ that are EOI need not be SOI, as Example 12 shows.

According to Corollary 20, it suffices to characterize the nonsingular triangular SOI triples in order to explicitly describe all nonsingular SOI triples. The only difficulty arises when the product has two equal eigenvalues and one of $A_1A_2A_3$ and $A_1A_3A_2$ is scalar while the other is not.

Now we know that EOI (TOI) implies (is equivalent to) SOI for $A_1, A_2, A_3 \in GL_2(F)$, with a certain few identified, simultaneously (upper) triangular exceptions.

From these observations and Theorem 17, we have

Theorem 21. *Let $A_1, A_2, A_3 \in GL_2(F)$, with* $\mathrm{char}(F) \neq 2$. *The matrices A_1, A_2, A_3 are SOI if and only if at least one of the following occurs*:

(i) *A_1, A_2 and A_3 are simultaneously symmetrizable*;
(ii) *A_1, A_2 and A_3 are simultaneously upper triangularizable*, *except if there is exactly one scalar matrix among $A_1A_2A_3$ and $A_1A_3A_2$*.

We note here that all our results, including Theorem 2, concerning SOI of three nonsingular matrices A_1, A_2, A_3 also hold if there is one singular matrix among A_1, A_2, A_3. For the sake of simplicity, we stated them in case all the matrices are nonsingular.

Though the occurrence of at least one of the conditions (i) or (ii) of Theorem 21 is still necessary for SOI when there are at least two singular matrices among A_1, A_2, A_3, in general it is not sufficient, as the following example shows.

Example 22. *Consider the following complex matrices:*

$$A_1 = A_2 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These matrices are symmetric but are not SOI. Observe that $A_1A_2A_3 = 0$ and $A_1A_3A_2 = 2iA_1$.

We finish by noting that, if $char(F) = 2$, it may happen that $A_1, A_2, A_3 \in M_2(F)$ are TOI and among A_1, A_2, A_3 no two matrices commute and none of the conditions (i) or (ii) in Theorem 17 is satisfied (Example 23). Also, Lemma 15 is not true if $char(F) = 2$ (Example 24).

Example 23. *Consider the following matrices over a field F with* $\mathrm{char}(F) = 2$*:*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrices A_1, A_2, A_3 are TOI. However, no two matrices commute. Also, A_1 and A_3 are not simultaneously triangularizable, otherwise the product

$$A_1A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

would have the eigenvalue 1 with multiplicity 2 (as A_1 and A_3), which does not happen. Also, any nonsingular matrix P such that PA_1P^{-1} is symmetric has the form

$$P = \begin{bmatrix} b_{21} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Now if PA_3P^{-1} was symmetric then $b_{12} = b_{22}$, which is not possible because P is nonsingular.

Example 24. *Consider the following matrices over a field F with $\mathrm{char}(F) = 2$:*

$$A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, A_1 and A_2 commute. Also, it follows from Example 23 that A_1 and A_3 are neither simultaneously triangularizable nor simultaneously symmetrizable.

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