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Junping Shi
William & Mary, jxshix@wm.edu

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Zhifu Xie

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Classification of four-body central configurations with three equal masses

Junping Shi [a,b,*,1], Zhifu Xie [c,2]

[a] Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
[b] School of Mathematics, Harbin Normal University, Harbin, Heilongjiang 150080, PR China
[c] Department of Mathematics and Computer Science, Virginia State University, Petersburg, VA 23806, USA

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ABSTRACT

It is known that a central configuration of the planar four body problem consisting of three particles of equal mass possesses a symmetry if the configuration is convex or is concave with the unequal mass in the interior. We use analytic methods to show that besides the family of equilateral triangle configurations, there are exactly one family of concave and one family of convex central configurations, which completely classifies such central configurations.

1. Introduction

A classical problem in celestial mechanics is to find all the central configurations in the planar n-body problem. Given a set of masses $\{m_i\colon\ 1 \leqslant i \leqslant n\}$, a central configuration is a set of positions in $\mathbf{R}^3$: $\{q_i = (x_i, y_i, z_i)\colon\ 1 \leqslant i \leqslant n\}$ such that

$$\sum_{j \neq i} \frac{m_i m_j}{|q_j - q_i|^3}(q_j - q_i) = -2\lambda m_i (q_i - C), \quad 1 \leqslant i \leqslant n, \tag{1}$$

where $C = (\sum_{i=1}^n m_i q_i)/(\sum_{i=1}^n m_i)$ is the center of mass for $\{m_i\}$, and λ is a constant. In the following, $|q|$ always denote the Euclidean distance. A translation can always make $C = 0$, thus

$$\sum_{j \neq i} \frac{m_i m_j}{|q_j - q_i|^3}(q_j - q_i) = -2\lambda m_i q_i, \quad 1 \leqslant i \leqslant n. \tag{2}$$

There are several variational formulations for the central configurations expressed by (2). Define

$$U = \sum_{1 \leqslant j < i \leqslant n} \frac{m_i m_j}{|q_i - q_j|} \quad \text{and} \quad I = \sum_{i=1}^n m_i |q_i|^2, \tag{3}$$

* Corresponding author at: Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA.
E-mail addresses: shij@math.wm.edu (J. Shi), zxie@vsu.edu (Z. Xie).

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which are the Newton potential and the moment of inertial of the system, respectively. By the homogeneity of U of degree -1, one obtain

$$2\lambda I = U \quad \text{or} \quad \lambda = \frac{U}{2I}. \tag{4}$$

Here are several possible variational formulations:

1. A critical point $q \in \mathbf{R}^{3n}$ of $U(q)$ under the constraint $I = 1$ satisfies (2). This is apparent by considering the augmented energy function $G(q) = U(q) - \lambda(I(q) - 1)$, and λ is the Lagrange multiplier.
2. A critical point $q \in \mathbf{R}^{3n}$ of $E = IU^2$. Suppose that q is a critical point of E. Let $t = x_i$, y_i or z_i, then
$$\frac{\partial E}{\partial t} = \frac{\partial I}{\partial t}U^2 + 2UI\frac{\partial U}{\partial t} = 0.$$
By using (4), we reach (2) again.
3. One can also use relative distances between q_i as variables, see [6,7,11].

All these observations still hold if $q_i \in \mathbf{R}^1$ or $\mathbf{R}^2$. It is generally believed that the set of central configurations for a given positive mass vector $m = (m_1, m_2, \ldots, m_n)$ is finite. However, the number of planar central configurations of n-body problem for an arbitrary given set of positive masses has been established only for $n = 3$: there are always five central configurations (two Lagrange's equilateral triangles and three Euler's collinear central configurations). The exact number and classification of central configurations is still not known for four-body problem and only some partial results are obtained although it has been extensively studied in the past. It is well known that for given n positive masses there are precisely $n!/2$ collinear central configurations (see Moulton [13]) and Smale [19] reconfirmed the result by a variational method in 1970. Recently, the central configurations in collinear n-body problem are reinvestigated in [14,15,20,22]. Albouy [1,2] established a complete classification for the case of four equal masses by using Dziobek's coordinates and a symbolic computation program. The finiteness for the general four-body problem was settled by Hampton and Moeckel [6]. Long and Sun [9] studied the convex central configurations with $m_1 = m_2$ and $m_3 = m_4$, and they proved symmetry and uniqueness under some restrictions which were later removed by Perez-Chavela and Santoprete [18]. Perez-Chavela and Santoprete proved that there is a unique convex non-collinear central configuration of planar four-body problem when two equal masses are located at opposite vertices of a quadrilateral and, at most, only one of the remaining masses is larger than the equal masses. Leandro [7,8] applied a combination of numerical and analytical methods to provide the solutions to the problem of central configuration for symmetrical classes or for one zero mass in planar 4-body problem. Based on numerical experiments, he used the method of rational parametrization and the method of resultants to give the exact numbers of central configuration for planar and spatial symmetrical classes. Bernat, Llibre and Perez-Chavela [4] numerically studied the central configurations of the planar 4-body problem with three equal masses. They observed that there is exactly one class of convex central configurations and there is one or two classes of concave central configurations. Celli [5] established the exact number of central configurations for masses x, $-x$, y, $-y$. Albouy, Fu and Sun [3] recently proved the symmetric properties of convex central configurations for two equal masses and they conjecture that there is exactly one convex central configuration for any choice of four positive masses.

Here we consider a special case when $n = 4$, $q_i \in \mathbf{R}^2$, $m_1 = m_3 = m_4 = 1$, and $m_2 \in \mathbf{R}$. A central configuration in this case is either concave or convex, depending on whether the unequal mass m_2 is inside or outside of the triangle formed by the other three masses. Notice that a degenerate concave or convex configuration, *i.e.* three masses are collinear but not the fourth one, cannot be a central configuration, see for example, Xia [21]. For concave or convex central configurations, recent results of Long and Sun [9], and Albouy, Fu and Sun [3] stated that such central configuration must possess a symmetry. More precisely, they proved

Lemma 1.1. *(See Long and Sun [9].) Let $\alpha, \beta > 0$ be any two given real numbers. Let $q = (q_1, q_2, q_3, q_4) \in (\mathbf{R}^2)^4$ be a concave non-collinear central configuration with masses $(\beta, \alpha, \beta, \beta)$ respectively, with q_2 located inside the triangle formed by q_1, q_3, and q_4. Then the configuration q must possess a symmetry, so either q_1, q_3, and q_4 form an equilateral triangle and q_2 is located at the center of the triangle, or q_1, q_3, and q_4 form an isosceles triangle, and q_2 is on the symmetrical axis of the triangle.*

Lemma 1.2. *(See Albouy, Fu and Sun [3].) Let four particles (q_1, q_2, q_3, q_4) form a planar central configuration, which is a convex quadrilateral having $[q_1, q_2]$ and $[q_3, q_4]$ as diagonals. This configuration is symmetric with respect to the axis $[q_1, q_2]$ if and only if $m_3 = m_4$.*

Palmore [17] considered the one-parameter family consisting of three bodies of mass 1 at the vertices of an equilateral triangle and a fourth body of arbitrary mass m_2 at the centroid. We shall call this the equilateral central configuration. He showed that $m_2 = m_2^* = (64\sqrt{3} + 81)/249$ is the unique value of the mass parameter m_2 for which this central configuration is degenerate. Meyer and Schmidt [11] reproduced this result and further proved that another family of central configurations bifurcates from the equilateral central configuration when $m_2 = m_2^*$. The other family, called the isosceles central configuration, has three bodies of mass 1 at the vertices of an isosceles triangle and a fourth body of mass m_2 near the centroid and on the line of symmetry of the triangle.

Our goal here is to classify the concave (and convex) central configurations with masses $(1, m_2, 1, 1)$ in $\mathbf{R}^2$ with q_2 located inside (or outside of) the triangle formed by q_1, q_3, and q_4 following the symmetry properties in Lemmas 1.1 and 1.2. For the concave case, an obvious solution is the equilateral triangle $q_1q_3q_4$, and q_2 is located at the center of the triangle. In this case, q is a central configuration for any mass $m_2 > 0$. This fact can be easily verified through Eq. (2). We use a different variational approach to reproduce Meyer and Schmidt's results and further prove that for $m_2 = m_2^*$ there is exactly one concave central configuration (the isosceles family coincides with the equilateral family). We also proved the uniqueness of the convex central configuration with masses $(1, m_2, 1, 1)$ in $\mathbf{R}^2$. Our bifurcation analysis involves a subtle rigorous analysis. The novelty is the use of symmetries ascertained in Lemmas 1.1, 1.2 and symbolic computation software `Maple` to handle the more tedious calculations.

For the stability of critical point q of E, we recall that a critical point q is *non-degenerate* if all eigenvalues of the Hessian D^2E at q are non-zero. Due to the invariance of E with respect to the scaling and rotation, E is *degenerate* at each critical point. Hence we call E *essentially degenerate* if the degeneracy is not only due to the invariance. For planar central configurations, q is essentially degenerate if the multiplicity of the zero eigenvalue of $D^2E(q)$ is greater than 2; and for central configurations in the space, q is essentially degenerate if the multiplicity of the zero eigenvalue of $D^2E(q)$ is greater than 3.

Our main results are summarized as follows.

Theorem 1.3. *Let $E = IU^2$ and let q be the equilateral triangle central configuration with central mass m_2. Then E becomes essentially degenerate for only one value $m_2 = m_2^* = (64\sqrt{3} + 81)/249$. For given $0 < m_2 < m_2^*$, the equilateral central configuration is a local minimum of the energy functional E; for $m_2 > m_2^*$, the equilateral central configuration is a saddle point of E.*

Theorem 1.4. *For any fixed isosceles triangle $q_2q_3q_4$ with $\angle q_2q_3q_4 = \angle q_2q_4q_3 < \pi/3$, except when the height b of the triangle is $b_2 \approx 1.14090$ times $|q_3 - q_4|/2$ (see Theorem 3.1 for the definition of b_2), there is a unique q_1 (on the plane formed by $q_2q_3q_4$) and a unique (possibly negative) m_2, such that (q_1, q_2, q_3, q_4) with masses $(1, m_2, 1, 1)$ form a planar central configuration, with q_2 located inside the isosceles triangle $\triangle q_1q_3q_4$. Moreover, any concave planar central configuration (q_1, q_2, q_3, q_4) with masses $(1, m_2, 1, 1)$ and q_2 located inside the isosceles triangle $\triangle q_1q_3q_4$ is given by such an isosceles triangle one provided $m_2 > 0$.*

Theorem 1.5. *For any fixed isosceles triangle $q_1q_3q_4$, except when the height a of the triangle is $a_1 \approx 0.43856$ times $|q_3 - q_4|/2$ (see Theorem 4.1 for the definition of a_1), there is a unique q_2 (on the plane formed by $q_1q_3q_4$) and a unique (possibly negative) m_2, such that (q_1, q_2, q_3, q_4) with masses $(1, m_2, 1, 1)$ form a planar central configuration, with q_2 located outside the isosceles triangle $\triangle q_1q_3q_4$. Moreover, any convex planar central configuration (q_1, q_2, q_3, q_4) with masses $(1, m_2, 1, 1)$ and q_2 located outside the isosceles triangle $\triangle q_1q_3q_4$ is given by such a kite-shape central configuration provided $m_2 > 0$.*

We remark that our uniqueness result is for a fixed isosceles triangle, then the position and the mass of the fourth particle is uniquely determined by the fixed isosceles triangle. Our method also provides efficient numerical algorithms for computing all these central configurations, so that the unique one-parameter concave and the unique one-parameter convex can be clearly calculated. The numerical results imply another type of uniqueness: for any m_2 in the admissible mass set, there is a unique isosceles triangle convex or concave with m_2 in the interior central configuration. However we cannot prove the latter result algebraically. See Sections 3 and 4 for more details.

We prove Theorem 1.3 in Section 2. Theorem 1.4 would be restated and proved in more details in Section 3. The dependence of q_1 and m_2 on q_2 is implicitly given by an equation. In Section 4 Theorem 1.5 will be proved and the relation between m_2 and position q_1 will be established.

2. Bifurcations of the equilateral triangle configuration

We prove Theorem 1.3 in this section. We use a bifurcation approach with bifurcation parameter m_2, and we call the equilateral triangle solution q^0 to be the trivial solution. Then (m_2, q^0) is the branch of trivial solutions. Notice that here q^0 is an equivalent class of configurations in $\mathbf{R}^8$, since the functional $E = IU^2$ possesses two invariances:

$$E(kq) = E(q), \quad k \in \mathbf{R}\backslash\{0\}, \quad \text{and} \quad E\big(R(q)\big) = E(q), \tag{5}$$

here kq is the scalar multiplication, and R is a rotation about the origin (where q_2 is located). E is a functional of eight variables which we order by introducing the 8-vector $q = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ and $q_i = (x_i, y_i)$, $i = 1, 2, 3, 4$.

To find bifurcation points along the trivial solution branch, we notice that for $m_2 > 0$, $\nabla E(q^0) = 0$, and we consider the Hessian matrix $H_0(m_2)$ of E at q^0. The 8×8 matrix $H_0(m_2)$ has at least two zero eigenvalues from (5), but we look for values of m_2 such that $H_0(m_2)$ has additional zero eigenvalues. We fix the equilateral triangle configuration at

$$q_1 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \qquad q_2 = (0, 0), \qquad q_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad q_4 = (-1, 0), \tag{6}$$

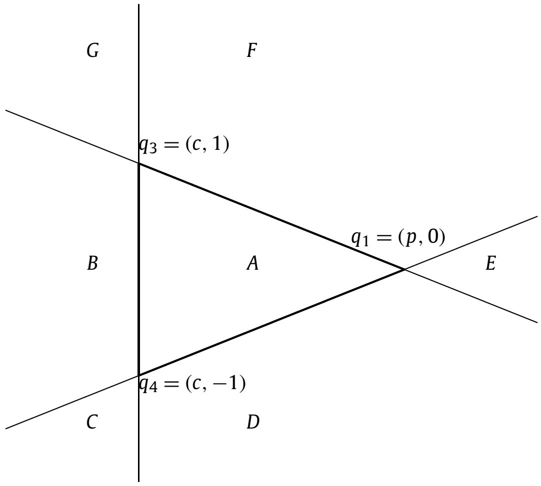

Fig. 1. Triangle configuration.

and use symbolic computation software `Maple` (version 9.5) to calculate the characteristic polynomial $P(m_2, x)$ of $H_0(m_2)$. The result shows that $P(m_2, x) = x^2[Q(m_2, x)]^2$, where $Q(m_2, x)$ is a cubic polynomial in x:

$$Q(m_2, x) = Q_0(m_2) + Q_1(m_2)x + Q_2(m_2)x^2 + x^3, \tag{7}$$

where

$$Q_0(m_2) = -\frac{6}{83}(5\sqrt{3} - 18)(249m_2 - 81 - 64\sqrt{3})m_2(3m_2 + \sqrt{3})^4,$$
$$Q_1(m_2) = 6\big(18m_2^3 + (11\sqrt{3} - 18)m_2^2 + 5(1 + 3\sqrt{3})m_2 + 3\big)(3m_2 + \sqrt{3})^2,$$
$$Q_2(m_2) = -\big(6m_2^2 + (27 + 2\sqrt{3})m_2 + 5\sqrt{3}\big)(3m_2 + \sqrt{3}).$$

This implies that all eigenvalues of $H_0(m_2)$ have even multiplicity. Since we look for m_2 such that $P(m_2, x)$ has more than two zero eigenvalues, we consider the roots m_2 of $Q(m_2, 0) = 0$. Indeed `Maple` shows that

$$Q(m_2, 0) = -\frac{6}{83}(-18 + 5\sqrt{3})m_2(249m_2 - 81 - 64\sqrt{3})(3m_2 + \sqrt{3})^4. \tag{8}$$

Hence the positive bifurcation point is $m_2 = m_2^* \equiv (81 + 64\sqrt{3})/249 \approx 0.77$. This value has been discovered by Palmore [16], Meyer and Schmidt [11] by using different formulations.

Note that $H_0(m_2)$ has eight real eigenvalues because the 8×8 Hessian matrix is symmetric. Note that $Q_2(m_2) < 0$ and $Q_1(m_2) > 0$ for all $m_2 > 0$, which implies that there are at least two positive roots by using Descartes' Rule of signs. $Q_0(m_2)$ will only change sign once at m_2^* for $m_2 > 0$. For given $0 < m_2 < m_2^*$, $Q(m_2, x) = 0$ has exactly three positive roots by using Descartes' Rule of signs, which implies the equilateral central configuration is a local minimum for the energy functional E. For $m_2 > m_2^*$, $Q(m_2, x) = 0$ has exactly two positive roots and one negative root, which implies the equilateral central configuration is a saddle point. This completes the proof of Theorem 1.3.

3. Concave central configurations

From the symmetry property shown in Lemmas 1.1 and 1.2, to better understand the bifurcation occurring near m_2^*, we only need to consider the configuration

$$q_1 = (p, 0), \qquad q_2 = (k, 0), \qquad q_3 = (c, 1), \qquad q_4 = (c, -1), \tag{9}$$

where $c < p$. Here q_1, q_3 and q_4 form an isosceles triangle in clockwise order, q_2 is in the symmetric axis of the triangle. (see Fig. 1). With this configuration, the degeneracy caused by (5) no longer exists.

Here we do not necessarily assume that q_2 is in the interior of the isosceles triangle $q_1q_3q_4$. Let $a = p - c$, $b = k - c$, then $a - b = p - k$. Then with given isosceles triangle $q_1q_3q_4$, the location of q_2 can be illustrated by the diagram in Fig. 1 (here we assume that $\triangle q_1q_3q_4$ is isosceles but not equilateral):

1. *(Concave case)* If q_2 is in the interior of $\triangle q_1q_3q_4$ (region A in Fig. 1), then Lemma 1.1 asserts that m_2 is on the interior symmetric axis of the triangle. This corresponds to be configuration (9) with $a > b > 0$, or equivalently $p > k > c$.

2. *(Convex case)* If q_2 is in region B, then Lemma 1.2 shows that all possible convex central configurations must have m_2 on the extended symmetric axis of $\triangle q_1q_3q_4$. This corresponds to be configuration (9) with $a > 0 > b$, or equivalently $p > c > k$.
3. *(Impossible cases)* If q_2 is in region D or F, then the configuration is convex but violates the result in Lemma 1.2, thus impossible. If q_2 is in region C or G, then the configuration is concave but violates the perpendicular bisector theorem of Conley and Moeckel [12], thus impossible.
4. *(Other cases)* If q_2 is in region E, there exist central configurations since Albouy's isosceles configuration of four equal masses with one on the symmetric axis is such an example (see [1,2], or [10]), where q_1 is on the interior symmetric axis of the triangle $\triangle q_2q_3q_4$. By the perpendicular bisector theorem, q_2 must be on the symmetric axis of the isosceles triangle $\triangle q_1q_3q_4$. But it is not known whether central configurations exist or not if q_2 is in region E and $\triangle q_1q_3q_4$ is not isosceles, and we do not consider these cases in this article.

Using the assumption of center of mass at origin, we have $p + m_2k + 2c = 0$. Therefore,

$$p = \frac{-m_2b + 2a + m_2a}{m_2+3}, \qquad k = \frac{-(a-3b)}{m_2+3}, \qquad c = \frac{-(m_2b+a)}{m_2+3},$$
$$U = \frac{m_2}{a-b} + \frac{4}{\sqrt{a^2+1}} + \frac{2m_2}{\sqrt{b^2+1}} + \frac{1}{8},$$
$$I = \frac{(-m_2b + 2a + m_2a)^2 + m_2(a-3b)^2 + 2(m_2b+a)^2 + 2(m_2+3)^2}{(m_2+3)^2}.$$

By using these identities and $\lambda = U/2I$ in (4), the eight equations of central configuration in (2) are reduced to two equations with m_2 as a parameter:

$$f_1(a,b,m_2) \equiv \frac{3b-a}{(a^2+1)^{3/2}} + \frac{m_2(b-a) - 6b}{(b^2+1)^{3/2}} + \frac{1}{4}(3b-a) + \frac{m_2+3}{(a-b)^2} = 0, \tag{10}$$
$$f_2(a,b,m_2) \equiv \frac{m_2(a-b) + 2a}{(a^2+1)^{3/2}} + \frac{m_2(3b-a)}{(b^2+1)^{3/2}} - \frac{1}{4}(m_2b + a) = 0. \tag{11}$$

With a linear elimination, we convert (10) and (11) to

$$-\frac{m_2(b-a)^2 + 4ab}{(b^2+1)^{3/2}} + \frac{a(3b-a)}{4} + \frac{m_2(a-b)+2a}{(a-b)^2} = 0, \tag{12}$$
$$\frac{m_2(b-a)^2 + 4ab}{(a^2+1)^{3/2}} - \frac{b(m_2(a-b)+2a)}{4} + \frac{m_2(3b-a)}{(a-b)^2} = 0. \tag{13}$$

Then we rewrite (12) and (13) into

$$s_1m_2 - t_1 = 0, \qquad s_2m_2 - t_2 = 0, \tag{14}$$

where

$$s_1 := -\frac{(a-b)^2}{(b^2+1)^{3/2}} + \frac{1}{a-b}, \tag{15}$$
$$t_1 := \frac{4ab}{(b^2+1)^{3/2}} + \frac{a(a-3b)}{4} - \frac{2a}{(a-b)^2}, \tag{16}$$
$$s_2 := \frac{(a-b)^2}{(a^2+1)^{3/2}} + \frac{3b-a}{(a-b)^2} - \frac{b(a-b)}{4}, \tag{17}$$
$$t_2 := -\frac{4ab}{(a^2+1)^{3/2}} + \frac{ab}{2}. \tag{18}$$

It is necessary that a solution of (12) and (13) satisfies $s_1t_2 - s_2t_1 = 0$. With some elementary but tedious calculations, we obtain

$$\Delta := s_1t_2 - s_2t_1 = -\frac{a(a-3b)(a-b-2)[(a-b)^2 + 2(a-b) + 4]}{16(a^2+1)^{3/2}(a-b)^4(b^2+1)^{3/2}} P(a,b), \tag{19}$$

where

$$P(a,b) := 4(a-b)^3\big(b^2+1\big)^{3/2} + 8b(a-b)^2\big(a^2+1\big)^{3/2} - \big[4 + b(a-b)^2\big]\big(b^2+1\big)^{3/2}\big(a^2+1\big)^{3/2}. \tag{20}$$

We shall consider the concave case in this section, and the convex case in the next section. For the concave case, we assume $0 < b < a$ in this section. Theorem 1.4 can be stated more precisely as follows (note that here we include the case of negative mass m_2)

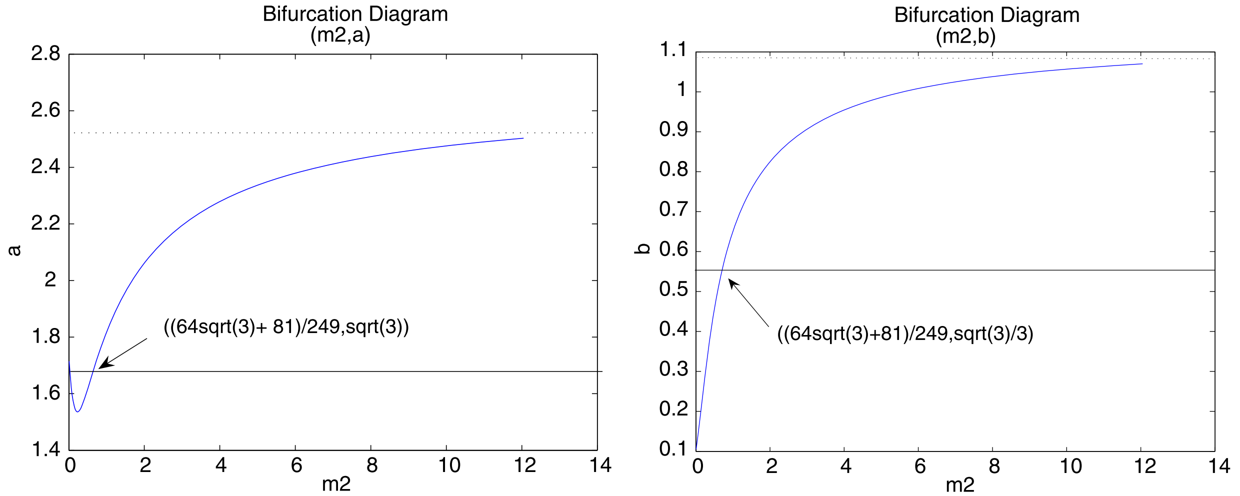

Fig. 2. Bifurcation diagram of concave central configurations. Left: bifurcation diagram of a with respect to m_2. Right: bifurcation diagram of b with respect to m_2. $m_2^* = (64\sqrt{3} + 81)/249$ is the unique bifurcation point.

Theorem 3.1. *Let $m = (1, m_2, 1, 1)$, $q = (q_1, q_2, q_3, q_4)$ (defined as in (9)) be a concave configuration with q_2 located inside the triangle $\triangle q_1q_3q_4$ (region A in Fig. 1). Then all such central configurations are determined by the parameters (a, b, m_2) as follows (see Fig. 3):*

(A) *The set of concave central configurations are contained in two curves:*

$$\Gamma_0 = \{(\sqrt{3}, \sqrt{3}/3, m_2) \colon m_2 \in \mathbf{R}\},$$
$$\Gamma_* = \{(a_0(b), b, m_2(a_0(b), b)) \colon 0 < b < \sqrt{3}\},$$

where $a_0(b)$ is a function implicitly determined by $P(a_0(b), b) = 0$, and $m_2(a_0(b), b)$ is computed as $m_2 = t_1/s_1 = t_2/s_2$ whenever $(a, b) \neq (\sqrt{3}, \sqrt{3}/3)$, $(\sqrt{3}, b_1)$ or $(a_0(b_2), b_2)$, where $b_1 \approx 0.09943364929$ is the only positive root of $t_1(a, b) = t_2(a, b) = 0$ other than $b = \sqrt{3}/3$, and $b_2 \approx 1.140903160$ is the only positive root of $s_1(a, b) = s_2(a, b) = 0$ other than $b = \sqrt{3}/3$; Γ_0 and Γ_* has exactly one intersection point at $(\sqrt{3}, \sqrt{3}/3, m_2^*)$ which is the bifurcation point found in Theorem 1.3 (see Fig. 2).

(B) *If (a, b, m_2) is on the portion $\Gamma_1 = \{(a_0(b), b, m_2(a_0(b), b)) \colon b_1 < b < b_2\}$, then q is an isosceles triangular central configuration with positive mass $\infty > m_2 > 0$; for any $m \in (0, \infty)$, there exists at least one $b \in (b_1, b_2)$ such that $m = m_2(a_0(b), b)$.*

(C) *If (a, b, m_2) is on the portion $\Gamma_2 = \{(a_0(b), b, m_2(a_0(b), b)) \colon 0 < b < b_1\}$, then q is an isosceles triangular central configuration with negative mass $0 > m_2 > -0.25$; for any $m \in (-0.25, 0)$, there exists at least one $b \in (0, b_1)$ such that $m = m_2(a_0(b), b)$.*

(D) *If (a, b, m_2) is on the portion $\Gamma_3 = \{(a_0(b), b, m_2(a_0(b), b)) \colon b_2 < b < \sqrt{3}\}$, then q is an isosceles triangular central configuration with negative mass $0 > m_2 > -2$; for any $m \in (-2, 0)$, there exists at least one $b \in (b_2, \sqrt{3})$ such that $m = m_2(a_0(b), b)$.*

Note that $t_1(a, b) = t_2(a, b) = 0$ is equivalent to $a = \sqrt{3}$ and $t_1(\sqrt{3}, b) = 0$, which is equivalent to the algebraic expression:

$$(3b - \sqrt{3})(b^2 + 1)^{3/2}(\sqrt{3} - b)^2 - 16b(\sqrt{3} - b)^2 + 8(b^2 + 1)^{3/2} = 0. \tag{21}$$

Similarly $s_1(a, b) = s_2(a, b) = 0$ is equivalent to

$$\frac{\sqrt{b^2 + 1}}{((b + \sqrt{b^2 + 1})^2 + 1)^{3/2}} + \frac{2b - \sqrt{b^2 + 1}}{(b^2 + 1)^{3/2}} - \frac{b}{4} = 0. \tag{22}$$

The limit -0.25 of m_2 on Γ_2 when $b \to 0^+$ can be obtained with (14), $b \to 0^+$ and $a \to \infty$; similarly the limit -2 of m_2 on Γ_3 when $b \to (\sqrt{3})^-$ can be obtained with (14), $b \to (\sqrt{3})^-$ and $a \to \infty$.

For the proof of Theorem 3.1, we prove the following key lemma.

Lemma 3.2. *Suppose that $P(a, b)$ is defined as in (20). Then for any $b \geqslant \sqrt{3}$ and $a \geqslant b$, $P(a, b) < 0$; for any $0 < b < \sqrt{3}$, there exists a unique $a = a_0(b)$ such that*

(i) *$P(a, b) < 0$ for $b \leqslant a < a_0(b)$,*
(ii) *$P(a_0(b), b) = 0$, and*
(iii) *$P(a, b) > 0$ for $a > a_0(b)$.*

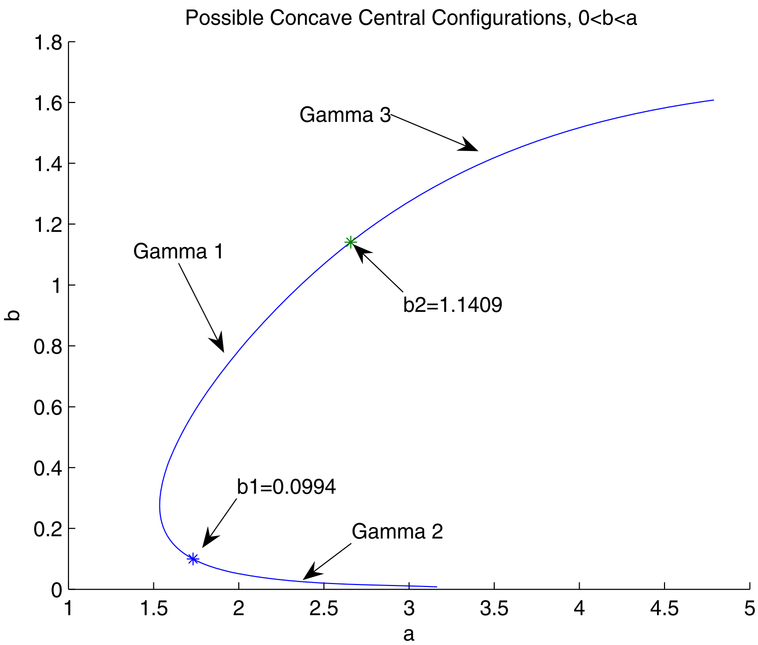

Fig. 3. Possible concave central configurations on curve Γ_*.

Moreover $\Gamma_* = \{(a_0(b), b)\colon\ 0 < b < \sqrt{3}\}$ *is a smooth curve in* $\mathbf{R}^2_+$, *and* $\lim_{b\to 0^+} a_0(b) = \infty$, $\lim_{b\to(\sqrt{3})^-} a_0(b) = \infty$.

Proof. We rewrite $P(a,b)$ in the form:

$$P(a,b) = b\bigl[8 - \bigl(b^2+1\bigr)^{3/2}\bigr](a-b)^2\bigl(a^2+1\bigr)^{3/2} - 4\bigl(b^2+1\bigr)^{3/2}\bigl[\bigl(a^2+1\bigr)^{3/2} - (a-b)^3\bigr]. \tag{23}$$

When $b \geqslant \sqrt{3}$, we have $8 - (b^2+1)^{3/2} \leqslant 0$, then $P(a,b) < 0$ for $a \geqslant b$ since $(a^2+1)^{3/2} - (a-b)^3 > 0$. For $b \in (0,\sqrt{3})$, $P(b,b) < 0$ and $P(a,b) \to \infty$ as $a \to \infty$, then there exists $a \in (b,\infty)$ such that $P(a,b) = 0$. We calculate $\partial P/\partial a$:

$$\begin{aligned}\frac{\partial P}{\partial a}(a,b) = {} & b\bigl[8 - \bigl(b^2+1\bigr)^{3/2}\bigr] \cdot \bigl[(a-b)\bigl(a^2+1\bigr)^{1/2}\bigl(5a^2 - 3ab + 2\bigr)\bigr] \\ & - 4\bigl(b^2+1\bigr)^{3/2} \cdot 3\bigl[a\bigl(a^2+1\bigr)^{1/2} - (a-b)^2\bigr].\end{aligned} \tag{24}$$

We notice that (23) and (24) can be written as

$$\begin{pmatrix} P \\ \partial P/\partial a \end{pmatrix} = M \cdot \begin{pmatrix} b[8 - (b^2+1)^{3/2}] \\ -4(b^2+1)^{3/2} \end{pmatrix}, \tag{25}$$

where

$$M := (M_{ij}) := \begin{pmatrix} (a-b)^2(a^2+1)^{3/2} & (a^2+1)^{3/2} - (a-b)^3 \\ (a-b)(a^2+1)^{1/2}(5a^2 - 3ab + 2) & 3[a(a^2+1)^{1/2} - (a-b)^2] \end{pmatrix}, \tag{26}$$

and M_{ij} is the entry of the matrix M. A simple calculation shows that

$$Det(M) = (a-b)\bigl(a^2+1\bigr)^{3/2}\bigl\{2\bigl(a^2+1\bigr)\bigl[-\bigl(a^2+1\bigr)^{3/2} + (a-b)^3\bigr] - 3ab - 3\bigr\} < 0, \tag{27}$$

for any $0 < b < \sqrt{3}$ and $a \geqslant b$. This implies that for any $0 < b < \sqrt{3}$ and $a \geqslant b$,

$$M_{21}P(a,b) - M_{11}\frac{\partial P}{\partial a}(a,b) = Det(M) \cdot 4\bigl(b^2+1\bigr)^{3/2} < 0. \tag{28}$$

Fix $b \in (0,\sqrt{3})$. Let $a = a_0(b)$ be the smallest $a > b$ such that $P(a,b) = 0$, then $\partial P(a_0(b),b)/\partial a > 0$ from (28). In fact whenever $P(a,b) \geqslant 0$ then $\partial P(a,b)/\partial a > 0$ from (28). This implies $P(a,b) > 0$ for all $a > a_0(b)$. Finally since $\partial P(a_0(b),b)/\partial a > 0$ for any $b \in (0,\sqrt{3})$, then the set $\Gamma_* = \{(a_0(b),b)\colon\ 0 < b < \sqrt{3}\}$ is a smooth curve from the implicit function theorem, and the asymptotes of $a_0(b)$ follows easily from calculus. $\square$

Proof of Theorem 3.1. Γ_* contains the trivial equilateral central configuration, which is well known. Hence we consider other possible central configurations by using (10) and (11), or equivalently (12) and (13). From discussion above, the set of

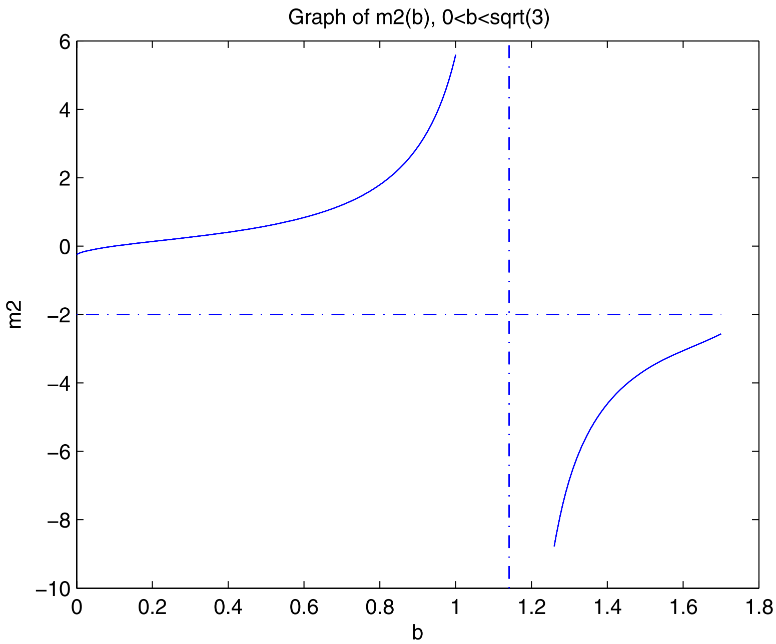

Fig. 4. Graph of $m_2(b)$, $0 < b < \sqrt{3}$ along Γ.

concave central configurations must satisfy $\Delta = s_1 t_2 - s_2 t_1 = 0$. From (19), for $b < a$, $\Delta = s_1 t_2 - s_2 t_1 = 0$ has the following possibilities:

1. $a = 0$. This does not give rise to a central configuration (see [21]).
2. $a - 3b = 0$. With elementary computation, we can prove that it gives a central configuration only when $a = \sqrt{3}$, $b = \sqrt{3}/3$ which is the equilateral central configuration for any mass $m_2 \in \mathbf{R}$, and it is included in Γ_0.
3. $a - b - 2 = 0$. We can prove that it gives a central configuration only when $b = -1$ and $a = 1$ which is a square configuration for mass $m_2 = 1$, and this is a convex central configuration which will be discussed in the next section.
4. $(a-b)^2 + 2(a-b) + 4 = (a-b+1)^2 + 3 = 0$ has no real valued solutions.
5. $P(a,b) = 0$, which we shall discuss more below.

From Lemma 3.2, the level set $P(a,b) = 0$ for $b > 0$ is indeed a curve $\Gamma_* = \{(a_0(b), b)\colon\ 0 < b < \sqrt{3}\}$. For any $(a,b) \in \Gamma_*$, if $s_1 = 0$, then $t_1 = 0$ or $s_2 = 0$. In case $t_1 = 0$, one can deduce that $a = \sqrt{3}$ and $b = \sqrt{3}/3$. In case $s_2 = 0$, one obtain positive values $(a,b) = (\sqrt{3}, \sqrt{3}/3)$ and $(a,b) = (2.658025443, 1.140903160 = b_2)$ from the equations. Similarly if $t_2 = 0$ and $ab \neq 0$, then $a = \sqrt{3}$, and $t_1 = 0$ or $s_2 = 0$. In case $s_2 = 0$, one obtain $(a,b) = (\sqrt{3}, \sqrt{3}/3)$ and $(a,b) = (\sqrt{3}, \sqrt{3} - 2)$. In case $t_1 = 0$, one obtain positive values $(a,b) = (\sqrt{3}, \sqrt{3}/3)$ and $(a,b) = (\sqrt{3}, 0.09943364929 = b_1)$. Hence except at the $(a,b) = (\sqrt{3}, \sqrt{3}/3)$, $(a,b) = (\sqrt{3}, b_1)$ and $(a,b) = (2.658014, b_2)$, for any other $(a,b) \in \Gamma_*$, the mass $m_2(a,b) = t_1/s_1 = t_2/s_2$ is uniquely determined and we denote by $m_2(b) := m_2(a_0(b), b)$ with the domain $b \in (0, \sqrt{3}) \backslash \{b_1, b_2, \sqrt{3}/3\}$. The asymptotes of $m_2(b)$ for b near $b = 0, b_1, \sqrt{3}/3, b_2, \sqrt{3}$ can be easily determined from calculus, which we omit the details. The function $m_2(b)$ is continuous in $(0, b_1) \cup (b_1, \sqrt{3}/3) \cup (\sqrt{3}/3, b_2) \cup (b_2, \sqrt{3})$, which implies all results stated in theorem. □

The numerical graph of $a_0(b)$ is shown in Fig. 3, and the numerical graph of $m_2(b)$ is shown in Fig. 4. We can observe the following facts from the numerical graphs of $a_0(b)$ and $m_2(b)$. When $b_1 \leqslant b < b_2$, $m_2(b)$ increase from 0 to infinity. $b = b_2$ is a vertical asymptote of the curve $m_2 = m_2(b)$. Each solution on the curve $\Gamma_1 = \{(a_0(b), b, m_2(b))\colon\ b_1 < b < b_2\}$ gives an isosceles triangle central configuration. If $b = \sqrt{3}/3$, then $a_0(b) = \sqrt{3}$ and $m_2(b) = m_2^*$, which coincides to an equilateral triangle central configuration. $(\sqrt{3}, \sqrt{3}/3, m_2^*)$ is the only intersection point between the curve Γ_0 and Γ_*. Let angles of isosceles triangle be ϕ, θ, θ, the angle ϕ can be computed as $\phi = 2\arctan(1/a)$. For $b_1 < b < \sqrt{3}/3$, $0 < m_2 < m_2^*$ and $a < \sqrt{3}$, so $\phi < \pi/3$. For $\sqrt{3}/3 < b < b_2$, $m_2 > m_2^*$ and $a > \sqrt{3}$, so $\phi > \pi/3$.

When $0 < b < b_1$, $a_0(b)$ decreases from infinity to $\sqrt{3}$ as shown in Fig. 3 and $m_2(b)$ increases from -0.25 to 0 as shown in Fig. 4. When $b_2 < b < \sqrt{3}$, $a_0(b)$ increases from 2.658014 to infinity. But $m_2(b)$ is negative and is increasing from negative infinity to -2.

Our result in this section classifies all concave central configurations with mass $(1, m_2, 1, 1)$. It is shown that besides the equilateral triangular ones, all other isosceles ones lie on a smooth curve which can be parameterized by b, the distance

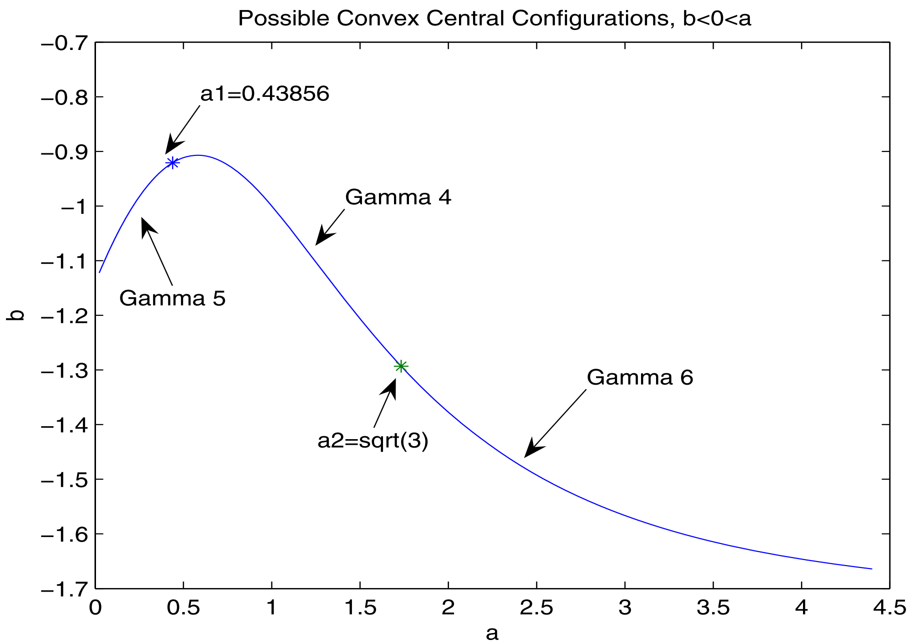

Fig. 5. Possible convex central configuration on curve Γ'.

from mass m_2 to the midpoint of the two symmetric vertices. The numerical graph (Fig. 4) suggest that $m_2(b)$ is always monotone.

4. Convex central configurations

In this section, we consider the convex configurations, *i.e.* $b < 0 < a$ in the discussion in Section 3. In fact the analysis in Section 3 remain valid for convex case until Eq. (20). Theorem 1.5 can be stated more precisely as follows (note that we include the case of negative mass m_2).

Theorem 4.1. *Let $m = (1, m_2, 1, 1)$, $q = (q_1, q_2, q_3, q_4)$ (defined as in (9)) be a convex configuration with q_2 located in the region B in Fig. 1. Then all such convex central configurations are determined by the parameters (a, b, m_2) as follows (see Fig. 5):*

(A) *The set of convex central configurations are contained in the curve*

$$\Gamma' = \big\{\big(a, b_0(a), m_2\big(a, b_0(a)\big)\big)\colon\ 0 < a < \infty\big\},$$

where $b_0(a)$ is a function implicitly determined by $P(a, b_0(a)) = 0$ such that $b_0(a) < 0 < a$, where P is defined in (20), and $m_2(a, b_0(a))$ is computed as $m_2 = t_1/s_1 = t_2/s_2$ whenever $(a, b) \neq (a_1, b_0(a_1))$, or $(a_2, b_0(a_2))$, where $a_1 \approx 0.4385619887$ is the only positive root of $s_1(a, b) = s_2(a, b) = 0$ such that $b_0(a_1) \approx -0.9208086952$ is negative, and $a_2 = \sqrt{3}$ is the only positive root of $t_1(a, b) = t_2(a, b) = 0$ such that $b_0(a_2) \approx -1.293021668$ is negative.

(B) *If (a, b, m_2) is on the portion $\Gamma_4 = \{(a, b_0(a), m_2(a, b_0(a)))\colon\ a_1 < a < a_2\}$, then q is a convex central configuration with positive mass $\infty > m_2 > 0$; for any $m \in (0, \infty)$, there exists at least one $a \in (a_1, a_2)$ such that $m = m_2(a, b_0(a))$.*

(C) *If (a, b, m_2) is on the portion $\Gamma_5 = \{(a, b_0(a), m_2(a, b_0(a)))\colon\ 0 < a < a_1\}$, then q is a convex central configuration with negative mass $-\infty < m_2 < 0$; for any $m \in (-\infty, 0)$, there exists at least one $a \in (0, a_1)$ such that $m = m_2(a, b_0(a))$.*

(D) *If (a, b, m_2) is on the portion $\Gamma_6 = \{(a, b_0(a), m_2(a, b_0(a)))\colon\ a_2 < a < \infty\}$, then q is a convex central configuration with negative mass $0 > m_2 > -2$; for any $m \in (-2, 0)$, there exists at least one $a \in (a_2, \infty)$ such that $m = m_2(a, b_0(a))$.*

Note that $s_1(a, b) = s_2(a, b) = 0$ is equivalent to

$$\frac{1}{2a\sqrt{a^2+1}} + \frac{4a^2(a^2-3)}{(a^2+1)^3} - \frac{a^2-1}{8a} = 0. \tag{29}$$

Similarly $t_1(a, b) = t_1(a, b) = 0$ is equivalent to $a = \sqrt{3}$ and $t_1(\sqrt{3}, b) = 0$ which is equivalent to the algebraic expression

$$(3b - \sqrt{3})\big(b^2+1\big)^{3/2}(\sqrt{3}-b)^2 - 16b(\sqrt{3}-b)^2 + 8\big(b^2+1\big)^{3/2} = 0. \tag{30}$$

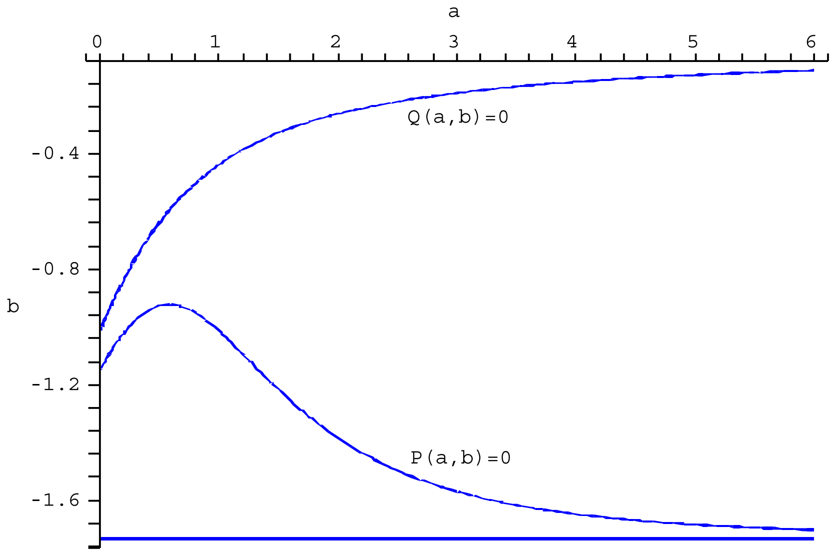

Fig. 6. Graph of $Q(a,b)=0$, $P(a,b)=0$ and $b=-\sqrt{3}$.

Here we choose the positive zeros a such that $b_0(a)$ is negative. The limit 0 of m_2 on Γ_5 when $a \to 0^+$ can be obtained with (14), $a \to 0^+$ and $b \to -1.139428225$; similarly the limit -2 of m_2 on Γ_6 when $a \to \infty$ can be obtained with (14), $b \to (-\sqrt{3})^+$ and $a \to \infty$.

Before we give the proof, we first show the following key lemma.

Lemma 4.2. *Recall that $P(a,b)$ is defined in* (20) *or equivalently* (23)*. Then there exists a unique $b = b_0(a)$ such that*

(i) $-\sqrt{3} < b = b_0(a) < 0$;
(ii) $P(a,b) < 0$ *for* $b_0(a) \leqslant b < 0$;
(iii) $P(a, b_0(a)) = 0$; *and*
(iv) $P(a,b) > 0$ *for* $b < b_0(a)$.

Moreover $\Gamma' = \{(a, b_0(a))\colon 0 < a < \infty\}$ *is a smooth curve in* $\mathbf{R}^2_+$ *as shown in Fig.* 5, *and* $\lim_{a\to\infty} b_0(a) = -\sqrt{3}$.

Proof. For $b \leqslant -\sqrt{3}$ and $0 \leqslant a$, $P(a,b) > 0$ because $8 - (b^2+1)^{3/2} < 0$ and $(a-b)^3 - (a^2+1)^{3/2} > 0$. For any $a > 0$, $P(a,0) = 4a^3 - 4(a^2+1)^{3/2} < 0$. Then for any $a > 0$, there exists at least one $b \in (-\sqrt{3}, 0)$ such that $P(a,b) = 0$. Let

$$Q(a,b) = (a-b)^3 - \left(a^2+1\right)^{3/2}.$$

In the fourth quadrant, $Q(a,b) = 0$ is equivalent to $b^2 - 2ab - 1 = 0$, which is implicitly defined a smooth function $b = b_Q(a)$ such that $Q(a, b_Q(a)) = 0$ (see Fig. 6). By taking implicit derivative, we have $\frac{d}{da} b_Q(a) > 0$. So $b_Q(a) > b_Q(0) = -1$ for any $a > 0$.

It is easy to check that $Q(a,b) < 0$ when $0 > b > b_Q(a)$, and $Q(a,b) > 0$ when $-\sqrt{3} < b < b_Q(a)$. Therefore, $P(a,b) < 0$ for $(a,b) \in \{a > 0,\ b_Q(a) < b < 0\}$. If $P(a,b) = 0$, b must be in the interval $(-\sqrt{3}, b_Q(a))$. We calculate $\partial P/\partial b$:

$$\begin{aligned}\frac{\partial P}{\partial b}(a,b) = {}& \left[-12(a-b)^2\left(b^2+1\right) + 12\left((a-b)^3 - \left(a^2+1\right)^{3/2}\right)b\right] \cdot \left(b^2+1\right)^{1/2} \\ & + \left[(a-b)(a-3b)\left(8 - \left(b^2+1\right)^{3/2}\right) - 3b^2(a-b)^2\left(b^2+1\right)^{1/2}\right] \cdot \left(a^2+1\right)^{3/2}. \end{aligned} \tag{31}$$

We notice that (23) and (31) can be written as

$$\begin{pmatrix} P \\ \partial P/\partial b \end{pmatrix} = M \cdot \begin{pmatrix} (b^2+1)^{1/2} \\ (a^2+1)^{3/2} \end{pmatrix}, \tag{32}$$

where

$$M := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{33}$$

and M_{ij} are

$$M_{11} = 4\big((a-b)^3 - \big(a^2+1\big)^{3/2}\big)\big(b^2+1\big),$$
$$M_{12} = \big(8 - \big(b^2+1\big)^{3/2}\big)b(a-b)^2,$$
$$M_{21} = -12(a-b)^2\big(b^2+1\big) + 12\big((a-b)^3 - \big(a^2+1\big)^{3/2}\big)b,$$
$$M_{22} = (a-b)(a-3b)\big(8 - \big(b^2+1\big)^{3/2}\big) - 3b^2(a-b)^2\big(b^2+1\big)^{1/2}.$$

We claim that

$$Det(M) = M_{11}M_{22} - M_{12}M_{21} < 0, \tag{34}$$

for any $a > 0$ and $-\sqrt{3} < b < b_Q(a)$, this implies that for any $a > 0$ and $-\sqrt{3} < b < b_Q(a)$,

$$-M_{21}P(a,b) + M_{11}\frac{\partial P}{\partial b}(a,b) = Det(M)\cdot\big(a^2+1\big)^{3/2} < 0. \tag{35}$$

Assuming (34) and consequently (35), we prove the statements in Lemma 4.2. Fix $a > 0$. Let $b = b_0(a)$ be the largest value such that $-\sqrt{3} < b < b_Q(a)$ and $P(a,b) = 0$. Then $\partial P(a, b_0(a))/\partial b < 0$ from (35) (note that $M_{11} > 0$ and $M_{21} < 0$ if $-\sqrt{3} < b < b_Q(a)$). In fact whenever $P(a,b) \geqslant 0$ then $\partial P(a,b)/\partial b < 0$ from (35). This implies $P(a,b) > 0$ for all $b < b_0(a)$. Finally since $\partial P(a, b_0(a))/\partial b < 0$ for any $b \in (-\sqrt{3}, b_Q(a))$, then the set $\Gamma' = \{(a, b_0(a))\colon\ -\sqrt{3} < b < b_Q(a),\ a > 0\}$ is a smooth curve from the implicit function theorem, and the asymptotes of $b_0(a)$ follows easily from calculus.

It remains to prove the claim that $Det(M) = M_{11}M_{22} - M_{12}M_{21} < 0$. Let

$$A = \big(a^2+1\big)^{1/2}, \qquad B = \big(b^2+1\big)^{1/2}, \qquad C = a - b.$$

A simple calculation shows that

$$Det(M) = 4C\cdot S,$$

where

$$S = B^2\big(8 - B^3\big)\big[\big(C^3 - A^3\big)a + 3bA^3\big] - 24b^2C\big(C^3 - A^3\big).$$

Then $Det(M) < 0$ is equivalent to $S < 0$ because $C > 0$. We compute $\partial S/\partial a$:

$$\frac{\partial S}{\partial a} = \big[3\big(C^2 - Aa\big)a + \big(C^3 - A^3\big) + 9Aba\big]\cdot B^2\big(8 - B^3\big) - 24b^2\cdot\big[\big(C^3 - A^3\big) + 3C\big(C^2 - Aa\big)\big]. \tag{36}$$

S and $\partial S/\partial a$ can be written as

$$\begin{pmatrix} S \\ \partial S/\partial a \end{pmatrix} = T\cdot\begin{pmatrix} B^2(8 - B^3) \\ -24b^2 \end{pmatrix}, \tag{37}$$

where

$$T := (T_{ij}) := \begin{pmatrix} (C^3 - A^3)a + 3bA^3 & C(C^3 - A^3) \\ 3(C^2 - Aa)a + (C^3 - A^3) + 9Aba & (C^3 - A^3) + 3C(C^2 - Aa) \end{pmatrix}.$$

Note that $Q(a,b) = C^3 - A^3 > 0$ and $C^2 - Aa > 0$ for any $a > 0$, $-\sqrt{3} < b < b_Q(a) < 0$. So $T_{22} > 0$, $T_{12} > 0$ and

$$Det(T) = 3b\big(A^3 + 3aCA\big)\big(C^3 - A^3\big) + 9bCA^3\big(C^2 - Aa\big) + b\big(C^3 - A^3\big)^2 < 0$$

for any $a > 0$, $-\sqrt{3} < b < b_Q(a) < 0$. Thus

$$T_{22}S - T_{12}\frac{\partial S}{\partial a} = Det(T)\cdot B^2\big(8 - B^3\big) < 0. \tag{38}$$

If $S \geqslant 0$ at (a_0, b_0) for some $a_0 > 0$ and $-\sqrt{3} < b_0 < b_Q(a)$, then $\partial S/\partial a > 0$ from (38). In fact, whenever $S \geqslant 0$ then $\partial S/\partial a > 0$ from (38). This implies that $S > 0$ for all $a > a_0$. On the other hand, we note that for any fixed $-\sqrt{3} < b < 0$, $S < 0$ for sufficient large $a > 0$. This is a contradiction. So $S < 0$ for any $a > 0$ and $-\sqrt{3} < b_0 < b_Q(a)$. We complete the proof. $\square$

Proof of Theorem 4.1. By a similar argument as in the proof of Theorem 3.1, the set of convex central configurations must satisfy $\Delta = s_1t_2 - s_2t_1 = 0$ and $b < 0 < a$. From (19), for $b < 0 < a$, $\Delta = s_1t_2 - s_2t_1 = 0$ has the following possibilities: (i) $a = 0$; (ii) $a - 3b = 0$; (iii) $a - b - 2 = 0$; (iv) $(a-b)^2 + 2(a-b) + 4 = (a-b+1)^2 + 3 = 0$; (v) $P(a,b) = 0$. From the

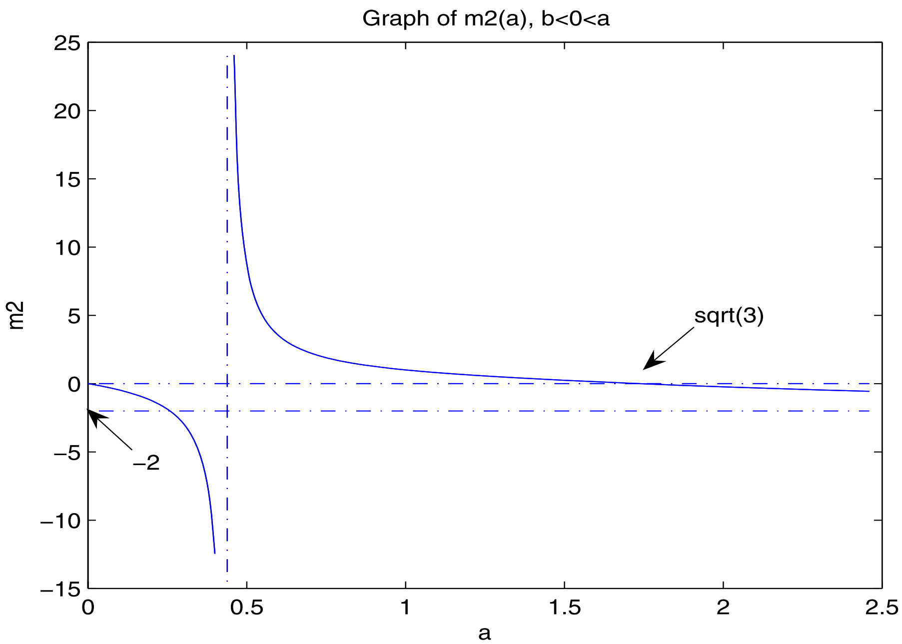

Fig. 7. Graph of $m_2(a)$, $-\sqrt{3} < b < 0$ along $p(a,b) = 0$.

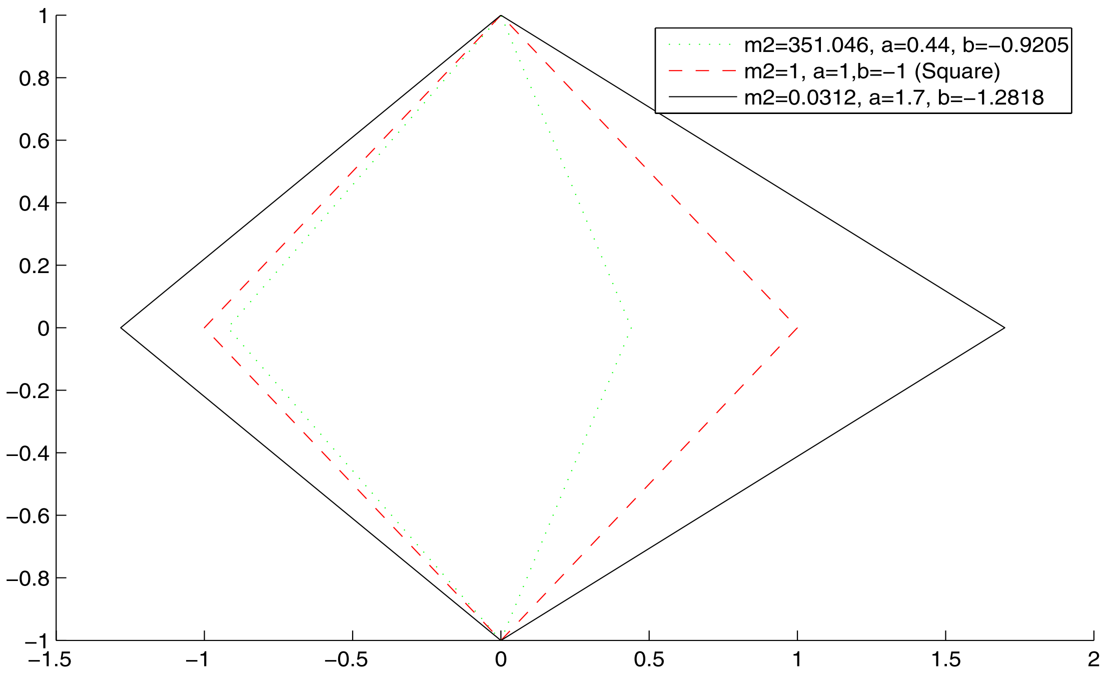

Fig. 8. Convex central configurations for $a_1 < a < a_2$.

arguments for (i) to (iv) in the proof of Theorem 3.1, only case (iii) gives a convex central configuration when $b = -1$ and $a = 1$ which is a square configuration for mass $m_2 = 1$, and it is included in Γ'.

From Lemma 4.2, the level set $P(a,b) = 0$ for $b < 0 < a$ is a parameterized curve $\Gamma' = \{(a, b_0(a)):\ 0 < a < \infty\}$. Then all the possible convex central configurations are along the curve Γ'. For any $(a,b) \in \Gamma'$ and $b < 0 < a$, if $s_1 = 0$, then $t_1 = 0$ or $s_2 = 0$. In case $t_1 = 0$, one can deduce that $a = 2 - \sqrt{3}$ and $b = -\sqrt{3}$ which is not on Γ'. In case $s_2 = 0$, one obtain $(a,b) = (0.4385619887 = a_1, -0.9208086915)$ from the equations. Similarly if $t_2 = 0$ and $ab \neq 0$, then $a = \sqrt{3}$, and $t_1 = 0$ or $s_2 = 0$. In case $t_1 = 0$, one obtain $(a,b) = (\sqrt{3} = a_2, -1.293021665)$ which is on Γ'. In case $s_2 = 0$, $(a,b) = (\sqrt{3}, \sqrt{3} - 2)$ which is not on Γ' but on $a - b - 2 = 0$. Hence only except at the point $(a,b) = (a_1, b_0(a_1))$, for any other $(a,b) \in \Gamma'$, the mass $m_2(a,b) = t_1/s_1 = t_2/s_2$ is uniquely determined and we denote by $m_2(a) := m_2(a, b_0(a))$ with the domain $a \in (0, a_1) \cup (a_1, \infty)$. The asymptotes of $m_2(a)$ for a near $a = 0, a_1, a_2, \infty$ can be easily determined from calculus, which we omit the details. The function $m_2(b)$ is continuous in $(0, b_1) \cup (b_1, \sqrt{3}/3) \cup (\sqrt{3}/3, b_2) \cup (b_2, \sqrt{3})$, which implies all results stated in theorem. $\square$

The numerical graph of $b_0(a)$ is shown in Fig. 5, and the numerical graph of $m_2(a)$ is shown in Fig. 7.

We can observe the following facts from the numerical graphs of $b_0(a)$ and $m_2(a)$. When $a_1 \leqslant a < a_2$, $m_2(a)$ decrease from infinity to 0. $a = a_1$ is a vertical asymptote of the curve $m_2 = m_2(a)$. Each solution on the curve $\Gamma_4 = \{(a, b_0(a), m_2(a)) : a_1 < a < a_2\}$ gives a convex central configuration. When $a_1 < a < 1$, $m_2(a) > 1$ and $|b_0(a)| > a$, which means that the heavier mass is further away the diagonal (see Fig. 8). When $a = 1$, $b = -1$ and $m_2(a) = 1$ which is a square central configuration. When $1 < a < a_2$, $m_2(a) < 1$ and $|b_0(a)| < a$, which means that the smaller mass is closer to the diagonal.

When $0 < a < a_1$, $b_0(a)$ increase from -1.139428225 to -0.9208086915 as shown in Fig. 5 and $m_2(a)$ decrease from 0 to $-\infty$ as shown in Fig. 7. When $a_2 \leqslant a < \infty$, $m_2(a)$ decrease from 0 to -2 and $b_0(a)$ decrease to $-\sqrt{3}$.

Our result in this section classifies all convex central configurations with mass $(1, m_2, 1, 1)$. It is shown that all convex central configurations lie on a smooth curve which can be parameterized by a, the distance from mass m_2 to the midpoint of the two symmetric vertices. The numerical graph (Fig. 7) shows that $m_2(a)$ is always monotone.

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