

2010

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Recommended Citation

Böttcher, A., & Spitkovsky, I. M. (2010). A gentle guide to the basics of two projections theory. *Linear Algebra and its Applications*, 432(6), 1412-1459.

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Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa



A gentle guide to the basics of two projections theory[☆]

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ARTICLE INFO

Article history:

Received 12 October 2009

Accepted 3 November 2009

Available online 6 December 2009

Submitted by R.A. Brualdi

To the memory of Israel Gohberg
(August 23, 1928–October 12, 2009)

AMS classification:

Primary: 47L15

Secondary: 15A21, 15A30, 16S50, 46H15,
46L89, 47C15

Keywords:

Two projections

Two subspaces

Idempotent

Skew and oblique projection

Angle and gap

Symbol calculus

ABSTRACT

This paper is a survey of the basics of the theory of two projections. It contains in particular the theorem by Halmos on two orthogonal projections and Roch, Silbermann, Gohberg, and Krupnik's theorem on two idempotents in Banach algebras. These two theorems, which deliver the desired results usually very quickly and comfortably, are missing or wrongly cited in many recent publications on the topic. The paper is intended as a gentle guide to the field. The basic theorems are precisely stated, some of them are accompanied by full proofs, others not, but precise references are given in each case, and many examples illustrate how to work with the theorems.

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[☆] On October 13, 2009, we received the sad news that Israel Gohberg has passed away on October 12, 2009, the day we submitted this article. We both have learned a great deal of mathematics from Israel Gohberg and had numerous opportunities to talk and to work with him. We lost a really outstanding mathematician, colleague, and friend. Israel Gohberg has left his imprint on many fields of mathematics, including the theory of two projections, which is the topic of this guide. We dedicate this article to him with admiration and gratitude.

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1. Halmos and Afriat

Throughout this paper, \mathcal{H} is a complex separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the C^* -algebra of all bounded linear operators on \mathcal{H} . Let P and Q be two bounded orthogonal projections on \mathcal{H} . Thus, $P = P^2 = P^*$ and $Q = Q^2 = Q^*$. The ranges of P and Q will be denoted by L and N , respectively. The sets L and N are closed subspaces of \mathcal{H} . Given a closed subspace K of \mathcal{H} , we denote by K^\perp its orthogonal complement and by P_K the orthogonal projection of \mathcal{H} onto K . In this terminology, $P = P_L$ and $Q = P_N$.

In general, $(L \cap N) \oplus (L \cap N^\perp)$ is a proper closed subspace of L . We therefore have

$$L = (L \cap N) \oplus (L \cap N^\perp) \oplus M_0,$$

with some closed subspace M_0 of L . Analogously,

$$L^\perp = (L^\perp \cap N) \oplus (L^\perp \cap N^\perp) \oplus M_1,$$

with some closed subspace M_1 of L^\perp . Letting

$$M_{00} = L \cap N, \quad M_{01} = L \cap N^\perp, \quad M_{10} = L^\perp \cap N, \quad M_{11} = L^\perp \cap N^\perp,$$

we obtain the orthogonal decomposition

$$\mathcal{H} = M_{00} \oplus M_{01} \oplus M_{10} \oplus M_{11} \oplus M_0 \oplus M_1. \tag{1}$$

Given four complex numbers α_{jk} , we use the abbreviation

$$(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) = \alpha_{00}I_{M_{00}} \oplus \alpha_{01}I_{M_{01}} \oplus \alpha_{10}I_{M_{10}} \oplus \alpha_{11}I_{M_{11}}, \tag{2}$$

where I_K denotes the identity operator on K . In the case where $M_{jk} = \{0\}$, we may take an arbitrary value for α_{jk} and we may alternatively assume that the corresponding term in (2) is absent. To be absolutely precise, on defining $\Lambda = \{(j, k) : M_{jk} \neq \{0\}\}$, we have

$$(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) = \bigoplus_{(j,k) \in \Lambda} \alpha_{jk}I_{M_{jk}}.$$

Clearly, if $M_0 = M_1 = \{0\}$, we get the orthogonal sum $\mathcal{H} = M_{00} \oplus M_{01} \oplus M_{10} \oplus M_{11}$ and accordingly P and Q may be written as

$$P = (1, 1, 0, 0), \quad Q = (1, 0, 1, 0). \tag{3}$$

The following is usually referred to as Halmos’ two projections theorem and sometimes also as the CS decomposition of two projections (see Remark 1.4 below). It provides us with a canonical representation for P and Q in the orthogonal sum (1) in the case where M_0 or M_1 are nontrivial. For real numbers α and β , we write $\alpha I \leq A \leq \beta I$ if A is selfadjoint and $\alpha(x, x) \leq (Ax, x) \leq \beta(x, x)$ for all x in the underlying Hilbert space. As usual, we denote the kernel (= null space) and range (= image) of an operator A by $\text{Ker } A$ and $\text{Ran } A$, respectively.

Theorem 1.1 (Halmos). *If one of the spaces M_0 and M_1 is nontrivial, then these two spaces have the same dimension and there exist a unitary operator $R : M_1 \rightarrow M_0$ and selfadjoint operators S and C of M_0 into itself such that $0 \leq S \leq I, 0 \leq C \leq I, S^2 + C^2 = I, \text{Ker } S = \text{Ker } C = \{0\}$, and*

$$P = (1, 1, 0, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}, \tag{4}$$

$$Q = (1, 0, 1, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}. \tag{5}$$

Here is Halmos’ proof from [57]. We use the abbreviation $M := M_0 \oplus M_1$. Each of the five spaces $M_{00}, M_{01}, M_{10}, M_{11}, M$ is invariant under both P and Q , and hence we may write

$$P = (1, 1, 0, 0) \oplus P|M, \quad Q = (1, 0, 1, 0) \oplus Q|M, \tag{6}$$

where $|$ denotes restriction to an invariant subspace. The restrictions $P|M$ and $Q|M$ may be represented by 2×2 operator matrices according to the decomposition $M = M_0 \oplus M_1$. It is clear that the matrix representation for $P|M$ is

$$P|M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

For $Q|M$ we can write

$$Q|M = \begin{pmatrix} B & E \\ E^* & D \end{pmatrix}, \tag{8}$$

with selfadjoint operators B and D . Letting L_0 and N_0 denote the ranges of the restrictions $P|M$ and $Q|M$, that is, $L_0 = PM$ and $N_0 = QM$, and taking orthogonal complements in M , we have

$$L_0 \cap N_0 = \{0\}, \quad L_0 \cap N_0^\perp = \{0\}, \quad L_0^\perp \cap N_0 = \{0\}, \quad L_0^\perp \cap N_0^\perp = \{0\}. \tag{9}$$

Indeed, suppose, for example, $y \in L_0 \cap N_0$. Then $y \in L_0 = M_0$ and hence $y \in L$ and $y \perp L \cap N$. On the other hand, $y \in N_0$ and therefore $y = Qz = P_N z$ for some $z \in M$, which implies that $y \in N$. Thus, $y \in L \cap N$ and $y \in L \cap N^\perp$, which is only possible for $y = 0$. This shows that $L_0 \cap N_0 = \{0\}$. The remaining three equalities can be proved similarly.

We now return to the spaces L and N and claim that if

$$L \cap N = \{0\}, \quad L \cap N^\perp = \{0\}, \quad L^\perp \cap N = \{0\}, \quad L^\perp \cap N^\perp = \{0\}, \tag{10}$$

then the spaces L, L^\perp, N, N^\perp are mutually isomorphic, $L \cong L^\perp \cong N \cong N^\perp$. This is clear if all four spaces are infinite-dimensional. So assume at least one of them, say L , has finite dimension. Then the direct sum $L \dot{+} N$ is closed, and since $(L \dot{+} N)^\perp = L^\perp \cap N^\perp = \{0\}$, it follows that $L \dot{+} N = \mathcal{H}$. As also $L \oplus L^\perp = \mathcal{H}$, we conclude that $L^\perp \cong N$. The direct sum $L \dot{+} N^\perp$ is also closed and $(L \dot{+} N^\perp)^\perp = L^\perp \cap N = \{0\}$, whence $L \dot{+} N^\perp = \mathcal{H}$. This in conjunction with the equalities $L \oplus L^\perp = \mathcal{H}$ and $N \oplus N^\perp = \mathcal{H}$ implies that $L^\perp \cong N^\perp$ and $L \cong N$, as desired. The argument is completely analogous if one of the spaces L^\perp, N, N^\perp is finite-dimensional.

By virtue of (6) and (9) we may assume from the very beginning that (10) is valid. As shown in the previous paragraph, then $M_0 = L$ and $M_1 = L^\perp$ are isomorphic. The operators B, E, D in (8) are

$$B = PQP|L = P_L P_N P_L|L = (I - P_L P_N P_L)|L, \tag{11}$$

$$E = PQ(I - P)|L^\perp = P_L P_N P_{L^\perp}|L^\perp,$$

$$E^* = (I - P)QP|L = P_{L^\perp} P_N P_L|L,$$

$$D = (I - P)(I - Q)(I - P)|L^\perp = P_{L^\perp} P_{N^\perp} P_{L^\perp}|L^\perp. \tag{12}$$

We claim that $\text{Ker } E = \text{Ker } E^* = \{0\}$. So let $Ey = 0$ for some $y \in L^\perp$. Then $P_N P_{L^\perp} y \in L^\perp$, and since at the same time $P_N P_{L^\perp} y \in N$, we see from (10) that $P_N P_{L^\perp} y = 0$. This implies that $P_{L^\perp} y \in N^\perp$, and as $P_{L^\perp} y$ is also in L^\perp , we infer again from (10) that $P_{L^\perp} y = 0$. Consequently, $y \in L \cap L^\perp$ and thus $y = 0$. It can be shown analogously that $\text{Ker } E^* = \{0\}$. Since E and E^* have trivial kernels, the partial isometry $W : L \rightarrow L^\perp$ in the polar decomposition $E^* = WA$ is in fact unitary. Then $R = W^* : L^\perp \rightarrow L$ is also unitary. We get

$$\begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} (Q|M) \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} = \begin{pmatrix} B & ER^* \\ RE^* & RDR^* \end{pmatrix}.$$

By (11) and (12), $0 \leq B \leq I$ and $0 \leq RDR^* \leq I$. Hence $B = C^2$ and $RDR^* = S^2$ with $0 \leq C \leq I$ and $0 \leq S \leq I$. Since $ER^* = A$ and $RE^* = A$, it follows that

$$\begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} (Q|M) \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} = \begin{pmatrix} C^2 & A \\ A & S^2 \end{pmatrix}$$

and the equality $(Q|M)^2 = Q|M$ yields that

$$C^4 + A^2 = C^2, \quad C^2A + AS^2 = A, \quad A^2 + S^4 = S^2. \tag{13}$$

The first of these equalities gives $A = C\sqrt{I - C^2}$ (note that $A \geq 0$). This implies that A commutes with C and hence the second equality in (13) shows that $A(C^2 + S^2 - I) = 0$. Since $\text{Ker } A = \text{Ker } (ER^*) = \{0\}$, it results that $C^2 + S^2 - I = 0$ and $A = C\sqrt{I - C^2} = CS$. Finally, from the first and third equations in (13) we conclude that if y is in $\text{Ker } C$ or $\text{Ker } S$, then $A^2y = 0$. As $\text{Ker } A = \{0\}$, this can only happen if $y = 0$. Thus, $\text{Ker } C = \text{Ker } S = \{0\}$. This completes the proof.

The operators S and C are called the operator sine and cosine of the pair (M_0, M_1) . This terminology was introduced in [72]. Denoting S^2 by H we immediately get

$$Q = (1, 0, 1, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ \sqrt{H(I - H)} & H \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}, \tag{14}$$

whereas the substitution $C^2 = H$ leads to

$$Q = (1, 0, 1, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} H & \sqrt{H(I - H)} \\ \sqrt{H(I - H)} & I - H \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}. \tag{15}$$

In part of the literature one sees (14) and in the other part authors work with (15). We agreed upon taking (14) throughout the rest of the paper. In terms of H , Theorem 1.1 reads as follows.

Theorem 1.2 (Halmos). *We have $M_0 \neq \{0\} \iff M_1 \neq \{0\}$, and if one of these spaces is nontrivial, then*

$$P = (1, 1, 0, 0) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U, \tag{16}$$

$$Q = (1, 0, 1, 0) \oplus U^* \begin{pmatrix} I - H & W \\ W & H \end{pmatrix} U, \tag{17}$$

where $U = \text{diag } (I, R)$, $W = \sqrt{H(I - H)}$, $R : M_1 \rightarrow M_0$ is a unitary operator and $H : M_0 \rightarrow M_0$ is a selfadjoint operator such that $0 \leq H \leq I$ and $\text{Ker } H = \text{Ker } (I - H) = \{0\}$.

Remark 1.3 (Historical). Theorem 1.1 in almost exactly the form cited here appeared first in Halmos' paper [57] and nowhere before. The name "Halmos' two projections theorem" is nowadays in common use. A special argument justifying this name is that, in our opinion, Halmos' paper [57] in unrivalled in its extremely lucid exposition of the matter. However, other authors had the theorem or were very close to it independently of Halmos and even before him.

In 1948, Krein, Krasnoselski, and Milman [72] showed that M_0 and M_1 have the same dimension and called the operators S and C defined by

$$S^2 = P(I - Q)|L, \quad C^2 = (I - P)Q|L^\perp,$$

the operator sine and operator cosine of the pair (L, N) . Clearly, $P(I - Q)|L$ and $(I - P)Q|L^\perp$ are up to unitary equivalence equal to

$$I_{L \cap N^\perp} \oplus S^2 = I_{L \cap N^\perp} \oplus H, \quad I_{L^\perp \cap N} \oplus C^2 = I_{L^\perp \cap N} \oplus (I - H),$$

respectively, with S and C as in Theorem 1.1 and H as in Theorem 1.2.

Dixmier [34] and Davis [26] also had results like Theorems 1.1 and 1.2. They considered the operators

$$t := (P - Q)^2, \quad s := PQP + (I - P)(I - Q)(I - P),$$

which will make their debut in this guide in Example 4.5. Dixmier used the notation $t =: B^2$, $s =: A^2$, while Davis wrote $t =: S$, $s =: C$ and referred to S and C as the separation and closedness operators, respectively. Clearly, Dixmier's B^2 and A^2 and Davis' S and C are just Krein, Krasnoselski, and Milman S^2 and C^2 . Theorem 6.2 of Davis' paper [26] from 1958 may be restated as follows.

The spaces M_0 and M_1 have the same dimension, and if these spaces are nontrivial, then

$$P = (1, 1, 0, 0) \oplus V^* \begin{pmatrix} F & -\sqrt{F(I-F)} \\ -\sqrt{F(I-F)} & I-F \end{pmatrix} V,$$

$$Q = (1, 0, 1, 0) \oplus V^* \begin{pmatrix} F & \sqrt{F(I-F)} \\ \sqrt{F(I-F)} & I-F \end{pmatrix} V,$$

with some unitary operator $V : M_0 \oplus M_1 \rightarrow M_0 \oplus M_0$ and some selfadjoint operator $F : M_0 \rightarrow M_0$ such that $0 \leq F \leq I/2$ and $\text{Ker } F = \text{Ker } (I/2 - F) = \{0\}$.

The theorem also holds with $0 \leq F \leq I/2$ and $\text{Ker } F = \text{Ker } (I/2 - F) = \{0\}$ replaced by $I/2 \leq F \leq I$ and $\text{Ker } (I/2 - F) = \text{Ker } (I - F) = \{0\}$. Davis' theorem is almost Theorem 1.2, but in different language. Indeed, let

$$H = 4F(I - F), \quad Z = \begin{pmatrix} \sqrt{F} & -\sqrt{I-F} \\ -\sqrt{I-F} & -\sqrt{F} \end{pmatrix}.$$

The operator H has the properties listed in Theorem 1.2, and $Z = Z^* = Z^{-1}$. A straightforward computation yields

$$\begin{pmatrix} F & -\sqrt{F(I-F)} \\ -\sqrt{F(I-F)} & I-F \end{pmatrix} = Z \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Z,$$

$$\begin{pmatrix} F & \sqrt{F(I-F)} \\ \sqrt{F(I-F)} & I-F \end{pmatrix} = Z \begin{pmatrix} I-H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & H \end{pmatrix} Z.$$

Consequently, Davis' theorem yields that

$$P = (1, 1, 0, 0) \oplus V^* Z \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Z V,$$

$$Q = (1, 0, 1, 0) \oplus V^* Z \begin{pmatrix} I-H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & H \end{pmatrix} Z V,$$

which coincides with Theorem 1.2, the only difference being that the unitary operator $U = ZV$ is not guaranteed to be of the form $\text{diag}(I, R)$.

We should mention that Davis [26] also proved that if H is a selfadjoint operator on M_0 and $0 \leq H \leq I$, then there exist orthogonal projections P and Q on $M_0 \oplus M_0$ such that

$$\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = PQP + (I - P)(I - Q)(I - P).$$

To do this, he put

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix},$$

attributing this construction to Michael and referring to [100,89]. In [100], one finds a reference to Halmos' 1950 paper [56] for the construction by Michael.

Davis and Kahan's paper [27] also contains several kinds of two projections theorems. Their paper was received by the editors on December 9, 1968 and hence they could not have known of Halmos' paper [57] then. However, Davis and Kahan refer to Jordan [62], Dixmier [34,35], Krein, Krasnoselski, and Milman [72], Sz-Nagy [100], Afriat [1], Kato [65] and also mention Seidel [93], Suschowk [98], Vitner [104], and Zassenhaus [108].

In his paper [81] (which was received by the editors on November 1, 1966), Pedersen stated Theorem 1.2 in the language of representation theory and described the C^* -algebra generated by P and Q . He already then obtained what will become Theorem 4.7 later in this survey. Giles and Kummer [48] also

had Theorem 1.2 in slightly disguised form and derived a description of the W^* -algebra generated by P and Q (see Theorem 7.1 later in this survey). Note that the Halmos paper [57] and the Giles/Kummer paper [48] were received by the editors on April 7, 1969 and April 14, 1969, respectively. We also want to mention the papers [14,15] by Behncke. He knew of Halmos' paper [57] and gave a very short proof of the theorem using group representation theory in [15].

Remark 1.4 (The connection with the CS decomposition). The following is a special case of what is usually called the CS decomposition; see, for example, [27,53,79,97].

If $F \in \mathbf{C}^{2r \times 2r}$ is a unitary matrix, then there exist unitary matrices $U_1, U_2, V_1, V_2 \in \mathbf{C}^{r \times r}$ and commuting Hermitian matrices $C_0, S_0 \in \mathbf{C}^{r \times r}$ such that $0 \leq C_0 \leq I, 0 \leq S_0 \leq I, C_0^2 + S_0^2 = I,$

$$F = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C_0 & S_0 \\ -S_0 & C_0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}. \tag{18}$$

The finite-dimensional version of Theorem 1.1 can be derived from the CS decomposition as follows. We start as in Halmos' proof quoted above. It suffices to consider $P|M$ and $Q|M$. We think of M as the column space \mathbf{C}^{2r} and freely identify operators on M with $2r \times 2r$ matrices. In particular, we may assume that $P|M$ and $Q|M$ are given by the matrices (7) and (8), the blocks of these matrices being $r \times r$. We know from Halmos' proof that $\text{rank}(Q|M) = r$ (because $\dim N_0 = \dim L_0 = r$) and that E is nonsingular (because $\text{Ker } E = \{0\}$). Let $F = (F_1 \ F_2) \in \mathbf{C}^{2r \times 2r}$ be a unitary matrix whose first r columns, constituting the $2r \times r$ matrix F_1 , span the range of $Q|M$. We then have the decomposition (18),

$$(F_1 \ F_2) = \begin{pmatrix} U_1 C_0 V_1 & U_1 S_0 V_2 \\ -U_2 S_0 V_1 & U_2 C_0 V_2 \end{pmatrix}.$$

Since $Q|M = F_1 F_1^*$, it follows that

$$\begin{aligned} Q|M &= \begin{pmatrix} U_1 C_0 V_1 \\ -U_2 S_0 V_1 \end{pmatrix} (V_1^* C_0 U_1^* \quad -V_1^* S_0 U_2^*) \\ &= \begin{pmatrix} I & 0 \\ 0 & -U_2 U_1^* \end{pmatrix} \begin{pmatrix} (U_1 C_0 U_1^*) (U_1 C_0 U_1^*) & (U_1 C_0 U_1^*) (U_1 S_0 U_1^*) \\ (U_1 C_0 U_1^*) (U_1 S_0 U_1^*) & (U_1 S_0 U_1^*) (U_1 S_0 U_1^*) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -U_1 U_2^* \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}, \end{aligned}$$

with $R = -U_2 U_1^*, C = U_1 C_0 U_1^*, S = U_1 S_0 U_1^*$. As $E = CSR$ is nonsingular, so also are C and S . This completes the proof.

Proposition 1.5. We have $M_0 = M_1 = \{0\}$ if and only if $PQ = QP$.

The “only if” part is trivial, since the two operators (3) obviously commute. To get the “if portion”, assume $PQ = QP$ but $M_0 \neq \{0\}$. Theorem 1.2 then gives

$$\begin{aligned} 0 &= PQ - QP \\ &= (0, 0, 0, 0) \oplus U^* \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I-H & W \\ W & H \end{pmatrix} - \begin{pmatrix} I-H & W \\ W & H \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right] U \\ &= (0, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & W \\ -W & 0 \end{pmatrix} U. \end{aligned}$$

It follows that $W = 0$ and thus $H(I - H) = W^2 = 0$, which is impossible because H and $I - H$ are injective. This proves Proposition 1.5.

We do not define H in case $M_0 = M_1 = \{0\}$. Equivalently, in the case where $M_0 = M_1 = \{0\}$ we interpret (16) and (17) as (3). Note that U is unitary and that $0 \leq W \leq I$ with $\text{Ker } W = \text{Ker } (I - W) = \{0\}$. In particular, the spectra $\sigma(H)$ and $\sigma(W)$ are both subsets of $[0, 1]$.

Now let $\Pi \in \mathcal{B}(\mathcal{H})$ be an arbitrary (not necessarily orthogonal) projection, $\Pi = \Pi^2$. Such projections are called skew or oblique. The closed subspaces $L := \text{Ran } \Pi$ and $N := \text{Ker } \Pi$ are complementary, that is, $L \cap N = \{0\}$ and $L + N = \mathcal{H}$. It follows also that $L^\perp \cap N^\perp = (L + N)^\perp = \{0\}$. Let $P = P_L$ and $Q = P_N$ be the orthogonal projections onto L and N , respectively.

Proposition 1.6 (Afriat). *If $L = \text{Ran } \Pi$ and $N = \text{Ker } \Pi$ for some skew projection Π , then $\|\Pi Q\| < 1$ and*

$$\Pi = (I - PQ)^{-1}P(I - PQ).$$

This result is from Afriat’s paper [1]. Proposition 1.6 is not of the depth of Theorems 1.1 and 1.2, but it is a key result in work with skew projections. Here is a proof of Proposition 1.6. The result is trivial if Π is the zero operator. So assume $\Pi \neq 0$. If $x \in L$ is a unit vector, then $x \in \text{Ran } \Pi$, $Qx \in \text{Ker } \Pi$, and hence

$$1 = \|x\| = \|\Pi x - \Pi Qx\| \leq \|\Pi\| \|x - Qx\|,$$

which gives

$$\begin{aligned} 1/\|\Pi\|^2 &\leq \inf_{x \in L, \|x\|=1} \|x - Qx\|^2 = \inf_{x \in L, \|x\|=1} (1 - \|Qx\|^2) = 1 - \sup_{x \in L, \|x\|=1} \|Qx\|^2 \\ &= 1 - \sup_{x \in L, \|x\| \leq 1} \|Qx\|^2 = 1 - \sup_{u \in \mathcal{H}, \|u\| \leq 1} \|\Pi Q u\|^2 = 1 - \|\Pi Q\|^2 \end{aligned}$$

and since $\|\Pi Q\| = \|(PQ)^* \Pi\| = \|\Pi Q\|$, it follows that $\|\Pi Q\| < 1$. The last inequality implies that $I - PQ$ is invertible, and we are left to prove that $(I - PQ)\Pi = P(I - PQ)$. Every $u \in \mathcal{H}$ can be written as $u = x + y$ with $x = \Pi u \in L$ and $y = (I - \Pi)u \in N$. This yields

$$(I - PQ)\Pi u = (I - PQ)x = x - PQx$$

and

$$P(I - PQ)u = P(x + y) - PQ(x + y) = Px + Py - PQx - Py = x - PQx,$$

that is, $(I - PQ)\Pi = P(I - PQ)$.

Representing P and Q by (16) and (17) we obtain the following representation for Π .

Corollary 1.7. *Let $L = \text{Ran } \Pi$ and $N = \text{Ker } \Pi$ for some skew projection Π . If $M_0 = \{0\}$, then $\Pi = I_{L \cap N^\perp}$ is simply the orthogonal projection onto $L \cap N^\perp$, while if $M_0 \neq \{0\}$, then H is invertible and*

$$\Pi = I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ 0 & 0 \end{pmatrix} U.$$

Indeed, we have

$$I = (1, 1, 1, 1) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} U, \quad PQ = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} I - H & W \\ 0 & 0 \end{pmatrix} U$$

and hence

$$I - PQ = (0, 1, 1, 1) \oplus U^* \begin{pmatrix} H & -W \\ 0 & I \end{pmatrix} U. \tag{19}$$

Since $L \cap N = L^\perp \cap N^\perp = \{0\}$, we may replace $(0, 1, 1, 1)$ by $I_{L \cap N^\perp} \oplus I_{L^\perp \cap N}$. The operator H is invertible together with $I - PQ$. The entries of the 2×2 matrix on the right of (19) commute and therefore this matrix can be inverted as in the scalar case. What results is that

$$(I - PQ)^{-1} = I_{L \cap N^\perp} \oplus I_{L^\perp \cap N} \oplus U^* \begin{pmatrix} H^{-1} & H^{-1}W \\ 0 & I \end{pmatrix} U.$$

Proposition 1.6 therefore yields

$$\Pi = I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} H^{-1} & H^{-1}W \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} H & -W \\ 0 & I \end{pmatrix} U,$$

which instantly gives the asserted formula.

The reader is referred to Galántai’s book [45] for numerous results on orthogonal and skew projections, ranging from elementary observations up to rather sophisticated properties and, in particular, for various representations of skew projections.

We remark that several representations that can be found in the literature are nothing but Halmos’ or Afriat’s formulas in disguise. For example, Groß [54] showed that if P and Q are orthogonal projections on \mathbf{C}^n , then

$$PQ = V^* \begin{pmatrix} T & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} V, \tag{20}$$

where V is unitary, T is diagonal with all diagonal entries in $(0, 1)$, and $XX^* = T(I - T)$. This is the same as the formula

$$PQ = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} I - H & W \\ 0 & 0 \end{pmatrix} U,$$

which is immediate from Theorem 1.2. The last formula even implies that the X in (20) may be chosen to be a Hermitian diagonal matrix.

2. Wedin, Doković, and Jordan

We begin by citing two versions of the theorems of the previous section in the case of finite-dimensional spaces. Thus, let $\mathcal{H} = \mathbf{C}^n$ with a natural number n . We freely identify operators on \mathbf{C}^n with their matrices in the standard basis. Let P and Q be two orthogonal projections on \mathbf{C}^n (= Hermitian and idempotent matrices). The trivial case where $M_0 = M_1 = \{0\}$ may be excluded. Thus, let $r := \dim M_0 = \dim M_1 \geq 1$. Since H is an Hermitian $r \times r$ matrix with all eigenvalues in $(0, 1)$, we have

$$H = S^* \text{diag} (\mu_1, \dots, \mu_r) S, \tag{21}$$

with a unitary matrix S and $0 < \mu_1 \leq \dots \leq \mu_r < 1$. Evidently, μ_1, \dots, μ_r are just the eigenvalues of H labeled in nondecreasing order and repeated according to their multiplicity. The angles $\theta_1, \dots, \theta_r \in (0, \pi/2)$ defined by

$$\sin^2 \theta_j = \mu_j \quad (j = 1, \dots, r) \tag{22}$$

are referred to as the principal angles of the pair (M_0, M_1) . The following was established in [106] by different methods and is called the Wedin canonical form of P and Q .

We denote by $\det A$ the usual determinant of a matrix $A \in \mathbf{C}^{n \times n}$. Given a 2×2 operator matrix with commuting entries, we define the operator determinant by

$$\text{Det} \begin{pmatrix} B & C \\ D & E \end{pmatrix} = BE - CD.$$

The operator matrix is invertible if and only if so is its operator determinant.

Example 2.1 (The sum of two orthogonal projections). By Theorem 1.2,

$$P + Q - \lambda I = (2 - \lambda, 1 - \lambda, 1 - \lambda, -\lambda) \oplus U^* \begin{pmatrix} 2I - H - \lambda I & W \\ W & H - \lambda I \end{pmatrix} U,$$

with

$$\text{Det} \begin{pmatrix} 2I - H - \lambda I & W \\ W & H - \lambda I \end{pmatrix} = (\lambda^2 - 2\lambda)I + H. \tag{23}$$

Consequently,

$$\sigma(P + Q) = \sigma((2, 1, 1, 0)) \cup \{1 \pm \sqrt{1 - x} : x \in \sigma(H)\}. \tag{24}$$

Note that $\sigma((2, 1, 1, 0)) \subset \{0, 1, 2\}$. If $\dim \mathcal{H} < \infty$, then (21) gives $\sigma(H) = \{\mu_1, \dots, \mu_r\}$, and taking into account (22) we get

$$\sigma(P + Q) \setminus \{0, 1, 2\} = \{1 \pm \cos \theta_1, \dots, 1 \pm \cos \theta_r\}.$$

Thus, at least theoretically the problem of finding the principle angles $\theta_1, \dots, \theta_r$ simply amounts to the determination of the eigenvalues of the Hermitian and positive matrix $P + Q$.

We remark that formula (24) also provides us with a quick solution of the problem considered by Holland in [59], namely, the construction of orthogonal projections P and Q such that $P + Q$ has prescribed eigenvalues.

Corollary 2.2 (Wedin). *There exists a unitary $n \times n$ matrix V such that*

$$\begin{aligned} VPV^* &= (1, 1, 0, 0) \oplus \text{diag} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]_{j=1}^r, \\ VQV^* &= (1, 0, 1, 0) \oplus \text{diag} \left[\begin{pmatrix} 1 - \mu_j & \sqrt{\mu_j(1 - \mu_j)} \\ \sqrt{\mu_j(1 - \mu_j)} & \mu_j \end{pmatrix} \right]_{j=1}^r \\ &= (1, 0, 1, 0) \oplus \text{diag} \left[\begin{pmatrix} \cos^2 \theta_j & \cos \theta_j \sin \theta_j \\ \cos \theta_j \sin \theta_j & \sin^2 \theta_j \end{pmatrix} \right]_{j=1}^r. \end{aligned}$$

To see this, put $D = \text{diag}(\mu_1, \dots, \mu_r)$. Then

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \\ \begin{pmatrix} I - H & W \\ W & H \end{pmatrix} &= \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} I - D & \sqrt{D(I - D)} \\ \sqrt{D(I - D)} & D \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \end{aligned}$$

and an obvious choice of a permutation matrix T yields

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} &= T^* \text{diag} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]_{j=1}^r T, \\ \begin{pmatrix} I - D & \sqrt{D(I - D)} \\ \sqrt{D(I - D)} & D \end{pmatrix} &= T^* \text{diag} \left[\begin{pmatrix} 1 - \mu_j & \sqrt{\mu_j(1 - \mu_j)} \\ \sqrt{\mu_j(1 - \mu_j)} & \mu_j \end{pmatrix} \right]_{j=1}^r T. \end{aligned}$$

Since $\mu_j = \sin^2 \theta_j$, $1 - \mu_j = \cos^2 \theta_j$, $\sqrt{\mu_j(1 - \mu_j)} = \cos \theta_j \sin \theta_j$, we obtain the desired representations with

$$V = (1, 1, 1, 1) \oplus T \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} U$$

from Theorem 1.2.

Now let Π be a skew projection on \mathbb{C}^n (= an idempotent matrix), $L = \text{Ran } \Pi$, $N = \text{Ker } \Pi$, $P = P_L$, $Q = P_N$, define μ_j by (21) and the angles θ_j by (22). The following representation is from [36], where it was proved in a completely elementary fashion. It is referred to as the Doković canonical form of Π .

Corollary 2.3 (Doković). *With the same unitary $n \times n$ matrix V as in Corollary 2.2,*

$$\begin{aligned} V\Pi V^* &= I_{L \cap N^\perp} \oplus \operatorname{diag} \left[\left(\begin{array}{cc} 1 & -\sqrt{(1-\mu_j)/\mu_j} \\ 0 & 0 \end{array} \right) \right]_{j=1}^r, \\ &= I_{L \cap N^\perp} \oplus \operatorname{diag} \left[\left(\begin{array}{cc} 1 & -\cot \theta_j \\ 0 & 0 \end{array} \right) \right]_{j=1}^r. \end{aligned}$$

This can be derived from Corollary 1.7 in the same way we derived Corollary 2.2 from Theorem 1.2.

Davis begins his paper [26] as follows. “A pair of non-trivial linear subspaces of Euclidean 3-space, whose dimensionalities are known, form a geometrical figure which is determined up to Euclidean congruence by the non-obtuse angle between them – single number between 0 and $\pi/2$.” In a sense, the whole two projections business since Jordan’s 1875 paper [62] has its origin in the endeavor to get an understanding of the corresponding picture in higher dimensions. We here confine ourselves to complex separable Hilbert spaces \mathcal{H} , and in this context we have the following definitions.

Let (L_1, N_1) and (L_2, N_2) be two pairs of closed subspaces of \mathcal{H} and denote by

$$P_1 = P_{L_1}, \quad Q_1 = P_{N_1}, \quad P_2 = P_{L_2}, \quad Q_2 = P_{N_2},$$

the associate orthogonal projections. If the pairs are formed by complementary subspaces, we let Π_1 and Π_2 stand for the projections of \mathcal{H} onto L_1 and L_2 parallel to N_1 and N_2 , respectively. The pairs (L_1, N_1) and (L_2, N_2) are said to be unitarily equivalent if there exists a unitary operator $V : \mathcal{H} \rightarrow \mathcal{H}$ such that $VL_1 = L_2$ and $VN_1 = N_2$. The pairs (P_1, Q_1) and (P_2, Q_2) are called unitarily equivalent if $P_2 = VP_1V^*$ and $Q_2 = VQ_1V^*$ for some unitary operator $V : \mathcal{H} \rightarrow \mathcal{H}$. Finally, Π_1 and Π_2 are unitarily equivalent by definition if there is a unitary operator $V : \mathcal{H} \rightarrow \mathcal{H}$ such that $\Pi_2 = V\Pi_1V^*$. Note that instead of unitary equivalence one frequently also speaks of unitary similarity or simply of congruence. The following proposition reveals that all these notions are one and the same thing in different disguise.

Proposition 2.4. *Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be a unitary operator. Then the following are equivalent:*

- (i) $VL_1 = L_2$ and $VN_1 = N_2$,
- (ii) $P_2 = VP_1V^*$ and $Q_2 = VQ_1V^*$.

If the pairs (L_1, N_1) and (L_2, N_2) are constituted by complementary subspaces, then (i) and (ii) are also equivalent to the equality

- (iii) $\Pi_2 = V\Pi_1V^*$.

This can be shown as follows. If K_1 and K_2 are closed subspaces of \mathcal{H} , then

$$K_2 = VK_1 \iff P_{K_2} = VP_{K_1}V^* \tag{25}$$

because $VP_{K_1}V^*$ is obviously an orthogonal projection and its range is VK_1 . The equivalence of (i) and (ii) is immediate from (25). To see that (i) and (iii) are equivalent, note that $V\Pi_1V^*$ is a projection with range VL_1 and kernel VN_1 . That’s all.

Now let $\mathcal{H} = \mathbf{C}^n$. For $i = 1, 2$, we put

$$\begin{aligned} \ell_i &= \dim(L_i \cap N_i), \quad k_i = \dim(L_i \cap N_i^\perp), \\ \ell_i^\perp &= \dim(L_i^\perp \cap N_i), \quad k_i^\perp = \dim(L_i^\perp \cap N_i^\perp), \\ M_0^{(i)} &= L_i \ominus ((L_i \cap N_i) \oplus (L_i \cap N_i^\perp)), \quad M_1^{(i)} = L_i^\perp \ominus ((L_i^\perp \cap N_i) \oplus (L_i^\perp \cap N_i^\perp)), \\ r_i &= \dim M_0^{(i)} = \dim M_1^{(i)} \end{aligned}$$

and we denote by $0 < \theta_1^{(i)} \leq \dots \leq \theta_r^{(i)} < \pi/2$ the principal angles of the pair $(M_0^{(i)}, M_1^{(i)})$. The following theorem is the n -dimensional version of Davis' introductory sentence cited above. This theorem is basically due to Jordan [62].

Theorem 2.5 (Jordan). *The pairs (L_1, N_1) and (L_2, N_2) are unitarily equivalent if and only if $\ell_1 = \ell_2$, $k_1 = k_2$, $\ell_1^\perp = \ell_2^\perp$, $k_1^\perp = k_2^\perp$, $r_1 = r_2$, and the principal angles $0 < \theta_1^{(1)} \leq \dots \leq \theta_{r_1}^{(1)} < \pi/2$ coincide with the principal angles $0 < \theta_1^{(2)} \leq \dots \leq \theta_{r_2}^{(2)} < \pi/2$.*

The “if” portion follows easily from Corollary 2.2 and Proposition 2.4: the corollary shows that $V_1 P_1 V_1^* = V_2 P_2 V_2^*$ and $V_1 Q_1 V_1^* = V_2 Q_2 V_2^*$ with unitary matrices V_1, V_2 , whence $P_2 = V P_1 V^*$ and $Q_2 = V Q_1 V^*$ with $V = V_2^* V_1$, and the proposition then gives $V L_1 = L_2$ and $V N_1 = N_2$. To prove the “only if” part, suppose (L_1, N_1) and (L_2, N_2) are unitarily equivalent. Then, by Proposition 2.4, $P_2 = V P_1 V^*$ and $Q_2 = V Q_1 V^*$ with some unitary matrix V . It follows that $P_2 + Q_2 = V(P_1 + Q_1)V^*$, and therefore $P_1 + Q_1$ and $P_2 + Q_2$ must have the same collection of eigenvalues. From Example 2.1 we deduce that $r_1 = r_2 =: r$, $\theta_j^{(1)} = \theta_j^{(2)}$ for $1 \leq j \leq r$, $\ell_1 = \ell_2$ (multiplicity of the eigenvalue 2), $k_1^\perp = k_2^\perp$ (multiplicity of the eigenvalue 0), and $k_1 + \ell_1^\perp + r = k_2 + \ell_2^\perp + r$ (multiplicity of the eigenvalue 0). Since P_1 and P_2 must also have the same eigenvalues, Corollary 2.2 implies that $\ell_1 + k_1 + r = \ell_2 + k_2 + r$ (multiplicity of the eigenvalue 1). Consequently, $k_1 = k_2$ and $\ell_1^\perp = \ell_2^\perp$, which completes the proof.

The (infinite-dimensional) Hilbert space analogue of Theorem 2.5 is in [26]. It characterizes unitary equivalence in terms of the dimensions of the four spaces $L \cap N, L \cap N^\perp, L^\perp \cap N, L^\perp \cap N^\perp$, and the spectral decomposition of the operator H .

Remark 2.6. In the literature, the principal angles are usually defined as follows, that is, in a fashion different from ours. Suppose $m := \dim L \leq \dim N$. The first principal angle φ_1 is defined by

$$\cos \varphi_1 = \max\{|(x, y)| : x \in L, \|x\| = 1, y \in N, \|y\| = 1\}.$$

Assume the maximum is attained at x_1 and y_1 , that is, $\cos \varphi_1 = |(x_1, y_1)|$. The second principal angle φ_2 is then given by

$$\cos \varphi_2 = \max\{|(x, y)| : x \in L, x \perp x_1, \|x\| = 1, y \in N, y \perp y_1, \|y\| = 1\}$$

and if $\cos \varphi_2 = |(x_2, y_2)|$, the next principal angle φ_3 is the angle whose cosine is the maximum of

$$\{|(x, y)| : x \in L, x \perp x_1, x \perp x_2, \|x\| = 1, y \in N, y \perp y_1, y \perp y_2, \|y\| = 1\},$$

and so on. At step $m + 1$ we meet the requirement

$$x \in L, x \perp x_1, \dots, x \perp x_m, \|x\| = 1,$$

which cannot be fulfilled. Thus, the procedure stops at the m th step and yields the m principal angles

$$0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_m \leq \pi/2.$$

Note that in the preceding recursive definition the equalities $\|x\| = 1$ and $\|y\| = 1$ can everywhere be replaced by the inequalities $\|x\| \leq 1$ and $\|y\| \leq 1$ without changing the result. The connection between the angles just defined and our principal angles

$$0 < \theta_1 \leq \dots \leq \theta_r < \pi/2$$

is that our θ 's are just the φ 's lying in $(0, \pi/2)$ or, equivalently, the φ 's whose cosines are neither 0 nor 1. To be more precise, let $\ell = \dim(L \cap N)$ and $k = \dim(L \cap N^\perp)$. Since

$$L = (L \cap N) \oplus (L \cap N^\perp) \oplus M_0,$$

we have $m = \ell + k + r$. One can show (see, for instance, [17]) that

$$\varphi_1 = \dots = \varphi_\ell = 0, \quad \varphi_{\ell+1} = \theta_1, \dots, \varphi_{\ell+r} = \theta_r, \quad \varphi_{\ell+r+1} = \dots = \varphi_{\ell+r+k} = \frac{\pi}{2}.$$

In terms of the principal angles φ_j , Theorem 2.5 reads as follows. Let $m_1 = \dim L_1 \leq \dim N_1$ and $m_2 = \dim L_2 \leq \dim N_2$. The pairs (L_1, N_1) and (L_2, N_2) are unitarily equivalent if and only if $m_1 = m_2 =: m$ and $\varphi_j^{(1)} = \varphi_j^{(2)}$ for $1 \leq j \leq m$.

In connection with the topic of this section, we recommend Galántai’s book [45] and his recent article [46], which contain all results of this section along with many references to original works on principal angles. In [46], Galántai actually starts with the definition of the principal angles as in Remark 2.6 and uses the resulting characterizations of the relative positions of subspaces to derive Wedin’s representation and subsequently Halmos’ two projection theorem (in finite dimensions) and Doković’s canonical form. We here proceed in the reverse direction: we deduce Wedin and Doković from Halmos and not vice versa. We also want to mention that Rakočević and Wimmer [85] proved a min–max characterization of the principal angles, namely,

$$\cos \varphi_j = \min_U \max_{x,y} |(x,y)| \quad (j = 1, \dots, m),$$

the minimum over all subspaces $U \subset L$ of dimension $j - 1$ and the maximum over $x \in L \cap U^\perp, y \in N, \|x\| = 1, \|y\| = 1$.

3. Some simple consequences

In a sense, Theorem 1.2 does for geometry involving two subspaces or operator theory connected with two orthogonal projections the same as analytical geometry does for Euclidean geometry: after expressing everything in terms of the operator H (the “coordinates”), we are left with more or less straightforward computations. It is the purpose of this section to demonstrate this strategy by several concrete problems.

Suppose P and Q are orthogonal projections on \mathcal{H} with the ranges L and N , respectively. Let $f(p, q)$ be a polynomial in two non-commuting variables p and q of the form

$$\begin{aligned} f(p, q) = & f_{00} + f_{11}p + f_{21}pq + f_{31}pqp + f_{41}pqppq + f_{51}pqppqp + \dots \\ & + f_{12}q + f_{22}qp + f_{32}qpq + f_{42}qpqp + f_{52}qpqpq + \dots \end{aligned} \tag{26}$$

Then

$$\begin{aligned} f(P, Q) = & f_{00}I + f_{11}P + f_{21}PQ + f_{31}PQP + f_{41}PQPQ + f_{51}PQPQP + \dots \\ & + f_{12}Q + f_{22}QP + f_{32}QPQ + f_{42}QPQP + f_{52}QPQPQ + \dots \end{aligned}$$

and Theorem 1.2 shows that $f(P, Q)$ may be written as

$$(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) \oplus U^* \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix} U. \tag{27}$$

The operator $(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$ is invertible on $M_{00} \oplus M_{01} \oplus M_{10} \oplus M_{11}$ if and only if $\alpha_{jk} \neq 0$ for $M_{jk} \neq \{0\}$, and the norm of this operator is $\max |\alpha_{jk}|$, the maximum over the (j, k) with $M_{jk} \neq \{0\}$.

Since the operators $\varphi_{jk}(H)$ commute, the matrix in (27) is invertible if and only if so is its operator determinant

$$\text{Det} \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix} := \varphi_{00}(H)\varphi_{11}(H) - \varphi_{01}(H)\varphi_{10}(H),$$

which is in turn equivalent to the condition

$$\det \begin{pmatrix} \varphi_{00}(x) & \varphi_{01}(x) \\ \varphi_{10}(x) & \varphi_{11}(x) \end{pmatrix} \neq 0 \quad \text{for all } x \in \sigma(H).$$

In this way one can compute the spectrum of $f(P, Q)$. One can also determine the spectrum of $f(P, Q)f(P, Q)^*$ and thus the singular values and in particular the norm of $f(P, Q)$.

To exclude the permanent distinction between the cases $M_0 = \{0\}$ and $M_0 \neq \{0\}$, we make the following convention. If $M_0 \neq \{0\}$, we denote by $\min \sigma(H)$ the minimum of the set $\sigma(H)$, while in

the case $M_0 = \{0\}$ (where H is not defined), we define $\min \sigma(H) := 1$. With this convention, $\min \sigma(H) > 0$ if and only if $M_0 = \{0\}$ or if $M_0 \neq \{0\}$ and H is invertible.

Example 3.1 (Duncan and Taylor). We have $\|P + Q\| = 1 + \|PQ\|$ unless $P = Q = 0$.

To see this, suppose first that $M_0 = \{0\}$. Then $P = (1, 1, 0, 0)$ and $Q = (1, 0, 1, 0)$ and thus $P + Q = (2, 1, 1, 0)$ and $PQ = (1, 0, 0, 0)$. This implies that $\|P + Q\| = 1 + \|PQ\| = 2$ if $L \cap N \neq \{0\}$. So let $L \cap N = \{0\}$. Then PQ is the zero operator and hence $1 + \|PQ\| = 1$. If $L \cap N^\perp$ and $L^\perp \cap N$ would be $\{0\}$, then P and Q would be the zero operators, which case is excluded. Therefore one of the spaces $L \cap N^\perp$ and $L^\perp \cap N$ is nontrivial, which gives $\|P + Q\| = 1$. This completes the proof in the case where $M_0 = \{0\}$.

Assume $M_0 \neq \{0\}$. Then P and Q may be written as in Theorem 1.2. It follows that

$$(PQ)(PQ)^* = PQP = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} I - H & 0 \\ 0 & 0 \end{pmatrix} U. \tag{28}$$

Since $\|PQ\|^2 = \|(PQ)(PQ)^*\|$ and $\|I - H\| \leq 1$, we get

$$\|PQ\| = \begin{cases} 1 & \text{if } L \cap N \neq \{0\}, \\ \sqrt{\|I - H\|} & \text{if } L \cap N = \{0\}. \end{cases}$$

On the other hand, from (24) we see that the norm of the selfadjoint operator $P + Q$ is

$$\|P + Q\| = \begin{cases} 2 & \text{if } L \cap N \neq \{0\}, \\ 1 + \sqrt{1 - \min \sigma(H)} & \text{if } L \cap N = \{0\}. \end{cases}$$

As, obviously, $\|I - H\| = 1 - \min \sigma(H)$, this completes the proof.

The equality of this example was first established by Duncan and Taylor [39]. An algebraic proof of it is in Vidav’s paper [103].

Example 3.2 (Closedness of the sum of two subspaces). The sum $L + N$ of two closed subspaces of \mathcal{H} is closed if and only if $\min \sigma(H) > 0$. In different but equivalent terms, this was stated without proof by Krein, Krasnoselski, and Milman in [72]. A full proof was first published by Ljance [76]. Here is a proof that is based on Theorem 1.2. The first part of this proof, until the equality $L + N = \text{Ran}(P + Q)^{1/2}$, is due to Anderson and Schreiber [4].

The assertion is trivial if $M_0 = \{0\}$, in which case

$$L + N = (L \cap N) \oplus (L \cap N^\perp) \oplus (L^\perp \cap N).$$

So suppose $M_0 \neq \{0\}$. On $\mathcal{H} \oplus \mathcal{H}$, let

$$A := \begin{pmatrix} P & -Q \\ 0 & 0 \end{pmatrix} \text{ and hence } AA^* = \begin{pmatrix} P + Q & 0 \\ 0 & 0 \end{pmatrix}.$$

It is well known that $\text{Ran } A = \text{Ran}(AA^*)^{1/2}$ for every Hilbert space operator A . Since

$$\text{Ran } A = (\text{Ran } P + \text{Ran } Q) \oplus \{0\}, \quad \text{Ran}(AA^*)^{1/2} = \text{Ran}(P + Q)^{1/2} \oplus \{0\},$$

we conclude that

$$L + N = \text{Ran } P + \text{Ran } Q = \text{Ran}(P + Q)^{1/2}.$$

Consequently, $L + N$ is closed if and only if $(P + Q)^{1/2}$ has closed range. But the range of an arbitrary Hilbert space operator B is closed if and only if

$$\sigma(BB^*) \subset \{0\} \cup [\varepsilon^2, \infty) \tag{29}$$

for some $\varepsilon > 0$ (see, e.g. [20, Theorem 4.21]). With $B = (P + Q)^{1/2}$, we deduce from Theorem 1.2 that

$$BB^* = B^2 = P + Q = (2, 1, 1, 0) \oplus U^* \begin{pmatrix} 2I - H & W \\ W & H \end{pmatrix} U$$

and thus

$$\sigma(BB^*) = \sigma((2, 1, 1, 0)) \cup \sigma \left(\begin{pmatrix} 2I - H & W \\ W & H \end{pmatrix} \right). \tag{30}$$

The spectrum of $(2, 1, 1, 0)$ is contained in $\{0, 1, 2\}$, and formula (23) implies that the spectrum of the matrix in (30) is $\{1 \pm \sqrt{1 - x} : x \in \sigma(H)\}$. If $\min \sigma(H) > 0$, this set is bounded away from zero and thus (29) holds. However, if $\min \sigma(H) = 0$, then 0 is a cluster point of H (since it isn't an eigenvalue) and therefore (29) cannot be true. This completes the proof.

Example 3.3 (The minimal angle between two subspaces). Suppose $L \neq \{0\}$ and $N \neq \{0\}$, or equivalently, $P \neq 0$ and $Q \neq 0$. The minimal angle $\theta_{\min}(L, N)$ between L and N is the angle in $[0, \pi/2]$ that is given by

$$\sin \theta_{\min}(L, N) := \inf_{x \in L, \|x\|=1} \text{dist}(x, N) = \inf_{x \in L, \|x\|=1} \sqrt{1 - \|Qx\|^2}.$$

This definition goes back to Dixmier [35]. An argument we employed to prove Proposition 1.6 shows that

$$\sin^2 \theta_{\min}(L, N) = 1 - \|QP\|^2 = 1 - \|PQ\|^2 = \sin^2 \theta_{\min}(N, L). \tag{31}$$

In Example 3.1 we observed that

$$\|PQ\|^2 = \begin{cases} 1 & \text{if } L \cap N \neq \{0\}, \\ 1 - \min \sigma(H) & \text{if } L \cap N = \{0\}. \end{cases}$$

Consequently,

$$1 - \|PQ\|^2 = \begin{cases} 0 & \text{if } L \cap N \neq \{0\}, \\ \min \sigma(H) & \text{if } L \cap N = \{0\}. \end{cases}$$

Now let $\dim \mathcal{H} = n < \infty$ and put $\dim(L \cap N) = \ell$, $\dim M_0 = r$. From (28) we get

$$\sigma((PQ)(PQ)^*) = \sigma((1, 0, 0, 0)) \cup \sigma(\text{diag}(I - H, 0)).$$

Thus, PQ has ℓ times the singular value 1, $n - \ell - r$ times the singular value 0, and the remaining r singular values are $\sqrt{1 - \mu_j} = \cos \theta_j$ ($j = 1, \dots, r$).

Example 3.4 (Ljance's formula). Let Π be a skew projection and suppose $\Pi \neq 0$ and $\Pi \neq I$. Put $L = \text{Ran } \Pi$, $N = \text{Ker } \Pi$ and $P = P_L$, $Q = P_N$. By Proposition 1.6, $\|PQ\| < 1$. Example 3.3 therefore reveals that $\sin \theta_{\min}(L, N) > 0$ and $\min \sigma(H) > 0$. Ljance [76] showed that

$$\|\Pi\| = \frac{1}{\sin \theta_{\min}(L, N)} = \frac{1}{\sqrt{1 - \|PQ\|^2}} = \frac{1}{\sqrt{\min \sigma(H)}}.$$

This follows easily from Corollary 1.7. Indeed, assume first that $M_0 \neq \{0\}$. Then the corollary gives

$$\begin{aligned} \Pi \Pi^* &= I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -H^{-1}W & 0 \end{pmatrix} U \\ &= I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} H^{-1} & 0 \\ 0 & 0 \end{pmatrix} U. \end{aligned}$$

The norm of $I_{L \cap N^\perp}$ is at most 1 and $\|H^{-1}\| = 1/\min \sigma(H) \geq 1$. Thus $\|\Pi\|^2 = 1/\min \sigma(H)$. On the other hand, if $M_0 = \{0\}$, then $L \cap N^\perp = L \neq \{0\}$ and hence $\Pi \Pi^* = I_{L \cap N^\perp}$ and $\|\Pi\| = 1$. As we made the convention to put $\min \sigma(H) = 1$ in the case $M_0 = \{0\}$, we get again $\|\Pi\|^2 = 1/\min \sigma(H)$.

Since $\text{Ran}(I - \Pi) = N$ and $\text{Ker}(I - P) = L$, Ljance's formula applied to $I - \Pi$ yields $\|I - \Pi\|^2 = 1/(1 - \|PQ\|^2)$. From (31) we therefore obtain that $\|\Pi\| = \|I - \Pi\|$. We will say more about this identity in Example 5.8.

In the case where $\| \cdot \|$ is an arbitrary unitarily invariant matrix norm in \mathbf{C}^n with a symmetric gauge function, the norm $\|T\|$ is computed in [45, Proposition 2.55].

Example 3.5 (The maximal angle between two subspaces). Again suppose $L \neq \{0\}$ and $N \neq \{0\}$. The maximal angle $\theta_{\max}(L, N)$ between L and N of \mathcal{H} was introduced in [72] and is defined as the angle in $[0, \pi/2]$ given by

$$\sin \theta_{\max}(L, N) = \sup_{x \in L, \|x\|=1} \text{dist}(x, N) = \sup_{x \in L, \|x\|=1} \sqrt{1 - \|Qx\|^2}.$$

We have

$$\begin{aligned} \sin \theta_{\max}(L, N) &= \sup_{x \in L, \|x\|=1} \|(I - Q)x\| = \sup_{x \in L, \|x\| \leq 1} \|(I - Q)x\| \\ &= \sup_{u \in \mathcal{H}, \|u\| \leq 1} \|(I - Q)Pu\| = \|(I - Q)P\| = \|P - PQ\| = \|P - PQ\|. \end{aligned}$$

Using Theorem 1.2 one can show as in the previous examples that

$$\|P - PQ\|^2 = \max(\|(0, 1, 0, 0)\|, \|H\|)$$

and that, analogously,

$$\sin \theta_{\max}(N, L) = \|Q - PQ\|, \quad \|Q - PQ\|^2 = \max(\|(0, 0, 1, 0)\|, \|H\|).$$

In the same vein,

$$\|P - Q\|^2 = \max(\|(0, 1, 1, 0)\|, \|H\|).$$

(In these formulas, $\|H\|$ is absent if $M_0 = \{0\}$.) Consequently,

$$\max(\sin \theta_{\max}(L, N), \sin \theta_{\max}(N, L)) = \|P - Q\|. \tag{32}$$

Example 3.6 (Complementary subspaces). Two closed subspaces L and N of \mathcal{H} are complementary if and only if $L \cap N = \{0\}$, $L^\perp \cap N^\perp = \{0\}$, and $\min \sigma(H) > 0$. The “only if” part follows from Corollary 1.7 and the “if” portion is an immediate consequence of Example 3.2.

The following was shown by Ipsen and Meyer [61] for $\dim \mathcal{H} < \infty$ using different methods and independently by Buckholtz [23] for general \mathcal{H} , also without employing Halmos’ two projections theorem.

Two closed subspaces L and N of \mathcal{H} are complementary if and only if $P - Q$ is invertible. In that case the norm of the projection Π of \mathcal{H} onto L parallel to N is given by

$$\|\Pi\| = \|(P - Q)^{-1}\|.$$

Using Theorem 1.2, this can be shown as follows. We have

$$P - Q = (0, 1, -1, 0) \oplus U^* \begin{pmatrix} H & -W \\ -W & -H \end{pmatrix} U \tag{33}$$

and hence $P - Q$ is invertible if and only if $L \cap N = \{0\}$, $L^\perp \cap N^\perp = \{0\}$, and

$$\text{Det} \begin{pmatrix} H & -W \\ -W & -H \end{pmatrix} = -H$$

is invertible. The last requirement is equivalent to saying that $\min \sigma(H) > 0$. This proves the first part of the assertion. The norm equality is trivial for $M_0 = \{0\}$. So assume $M_0 \neq \{0\}$. If $P - Q$ is invertible, we get from (33) that

$$(P - Q)^{-1} = (0, 1, -1, 0) \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ -H^{-1}W & -I \end{pmatrix} U \tag{34}$$

and since

$$\text{Det} \begin{pmatrix} I - \lambda I & -H^{-1}W \\ -H^{-1}W & -I - \lambda I \end{pmatrix} = \lambda^2 I - H^{-1},$$

the norm of the selfadjoint operator matrix on the right of (34) is

$$\max\{|\lambda| : \lambda^2 \in \sigma(H^{-1})\} = \frac{1}{\sqrt{\min \sigma(H)}}.$$

Because $\min \sigma(H) < 1$, we obtain that

$$\|(P - Q)^{-1}\| = \max \left(\|(0, 1, -1, 0)\|, \frac{1}{\sqrt{\min \sigma(H)}} \right) = \frac{1}{\sqrt{\min \sigma(H)}},$$

which in conjunction with Example 3.4 yields the equality $\|T\| = \|(P - Q)^{-1}\|$.

Example 3.7 (The gap between two subspaces). The number (32) is referred to as the gap between L and N and will be denoted by $\delta(L, N)$. This notion was introduced by Krein and Krasnoselski in [71]. Letting $\delta(L, N) := \|P_L - P_N\|$ we may extend the definition of $\delta(L, N)$ also to the case where one of the spaces L and N is trivial. Obviously, δ is a metric on the set of all closed subspaces of \mathcal{H} .

If $\delta(L, N) < 1$, then P and Q are unitarily equivalent and, in particular, $\dim L = \dim N$.

This was established independently by Sz-Nagy [99], Krein, Krasnoselski, and Milman [72], and Kato [63] (see also [65, I. § 4. Section 6 and I. § 6. Section 8]). Proofs are also in the books [45, 51]. A proof based on Theorem 1.2 is as follows. We have representation (33) for $P - Q$, and if $\delta(L, N) = \|P - Q\| < 1$, then 1 and -1 in $(0, 1, -1, 0)$ must be absent (which happens if and only if $L \cap N^\perp = L^\perp \cap N = \{0\}$). Put

$$V = (1, 1, 1, 1) \oplus U^* \begin{pmatrix} \sqrt{I - H} & -\sqrt{H} \\ \sqrt{H} & \sqrt{I - H} \end{pmatrix} U.$$

It can be checked straightforwardly that V is unitary and that $Q = VPV^*$ (note that $(1, 1, 0, 0) = (1, 0, 1, 0)$). From (25) we infer that $N = VL$. This completes the proof.

Here is a simple application of the concepts of the gap and the minimal angle.

Suppose $\dim \mathcal{H} < \infty$ and let $L \neq \{0\}$ and $N \neq \{0\}$ be complementary subspaces of \mathcal{H} . If N' is a subspace of \mathcal{H} such that

$$\delta(N, N') < \sin \theta_{\min}(L, N),$$

then L and N' are also complementary.

This is a special case of a result by Berkson [16]. See also [51, Theorem 13.1.3]. The following simple proof is from Schumacher's paper [92]. Let $\delta(N, N') < 1$. We have just shown that then N and N' have the same dimension. It therefore suffices to prove that $L \cap N' = \{0\}$. Assume the contrary, that is, assume there is a $z_0 \in L \cap N'$ with $\|z_0\| = 1$. We then obtain

$$\begin{aligned} \sin \theta_{\min}(L, N) &= \sin \theta_{\min}(N, L) = \inf_{x \in N, \|x\|=1} \inf_{z \in L} \|x - z\| \\ &\leq \inf_{x \in N, \|x\|=1} \|x - z_0\| \leq \sup_{z \in N', \|z\|=1} \inf_{x \in N, \|x\|=1} \|x - z\| \\ &\leq \sup_{z \in N', \|z\|=1} \inf_{x \in N} \|x - z\| = \sin \theta_{\max}(N', N) \leq \delta(N', N) = \delta(N, N'), \end{aligned}$$

which is a contradiction.

The previous result is of interest in connection with controller robustness, for example. Schumacher [92] associates a family $L(s)$ and $N(s)$ (s in the closed right half-plane) of subspaces with a system R and the controller Q . The feedback loop (R, Q) turns out to be stable and well-posed if and only if all pairs

$(L(s), N(s))$ are complementary and a certain matrix function has no zeros. Considering a perturbation R' of the system, the question whether the feed-back loop (R', Q) is still stable and well-posed includes the problem of whether all pairs $(L'(s), N(s))$ remain complementary. From what was shown above, we know that this can be guaranteed if $\delta(L'(s), L(s)) < \sin \theta_{\min}(L(s), N(s))$ for all s .

In the above proof, we basically showed that if $\delta(N, N') < \sin \theta_{\min}(L, N)$, then $\dim(L \cap N') < 1$. Let $m = \min(\dim L, \dim N)$ and define the principal angles $0 \leq \varphi_1 \leq \dots \leq \varphi_m \leq \pi/2$ as in Remark 2.6. Wimmer [107] showed that if $1 \leq k \leq m$ and $\delta(N, N') < \sin \varphi_k$, then $\dim(L \cap N') < k$. He also proved that this is no longer true with $\sin \varphi_k$ replaced by a smaller number.

Example 3.8 (The spherical gap between two subspaces). Given a closed subspace $K \neq \{0\}$ of \mathcal{H} , we denote by S_K the unit sphere of K , that is, $S_K = \{y \in K : \|y\| = 1\}$. If $x \in \mathcal{H}$, then

$$\text{dist}(x, S_K) = \sqrt{\|x\|^2 + 1 - 2\|P_K x\|}. \tag{35}$$

Indeed, this is obvious for $x \in K^\perp$, while if $x \notin K^\perp$, we have, for every $y \in S_K$,

$$\|x - y\|^2 = \|x\|^2 + 1 - 2 \text{Re}(x, y) = \|x\|^2 + 1 - 2 \text{Re}(P_K x, y) \geq \|x\|^2 + 1 - 2\|P_K x\|$$

and equality is attained if and only if $y = P_K x / \|P_K x\|$.

Now let $L \neq \{0\}$ and $N \neq \{0\}$ be two closed subspaces of \mathcal{H} . Recall that the gap between L and N is the number

$$\delta(L, N) = \max \left(\sup_{x \in S_L} \text{dist}(x, N), \sup_{y \in S_N} \text{dist}(y, L) \right). \tag{36}$$

The spherical gap between L and N was introduced by Gohberg and Markus [52] and is defined by

$$\tilde{\delta}(L, N) = \max \left(\sup_{x \in S_L} \text{dist}(x, S_N), \sup_{y \in S_N} \text{dist}(y, S_L) \right). \tag{37}$$

The connection between (36) and (37) is

$$\tilde{\delta}(L, N) = \sqrt{2 - 2\sqrt{1 - \delta(L, N)^2}}. \tag{38}$$

To see this, note that $\|P_N x\| = \sqrt{1 - \text{dist}(x, N)^2}$ for $x \in S_L$, which in conjunction with (35) implies that

$$\text{dist}(x, S_N) = \sqrt{2 - 2\|P_N x\|} = \sqrt{2 - 2\sqrt{1 - \text{dist}(x, N)^2}}. \tag{39}$$

Since $\sup\{\text{dist}(x, N)^2 : x \in S_L\} = \sin^2 \theta_{\max}(L, N)$ by definition, equality (39) immediately gives

$$\sup_{x \in S_L} \text{dist}(x, S_N) = \sqrt{2 - 2\sqrt{1 - \sin^2 \theta_{\max}(L, N)}}.$$

Switching the roles of L and N we obtain

$$\sup_{y \in S_N} \text{dist}(y, S_L) = \sqrt{2 - 2\sqrt{1 - \sin^2 \theta_{\max}(N, L)}}.$$

Inserting the last two equalities in (37) and taking into account that the gap is defined by (32) we arrive at (38).

Why do we need the spherical gap? It turns out that both (36) and (37) are metrics on the set of all closed subspaces of a Hilbert space, but that (36) is in general no longer a metric on the set of all closed subspaces of a Banach space (because the triangle inequality need not hold), whereas (37) remains a metric in the Banach space setting; see [51,52,65, IV. § 2. Section 1]). Note that in order to speak of metrics, we have to extend (36) and (37) to the case where $L = \{0\}$ or $N = \{0\}$. For the gap (36), this was done in Example 3.7 via the equality $\delta(L, N) = \|P_L - P_N\|$, which is equivalent to letting

$$\delta(L, \{0\}) = \delta(\{0\}, N) = 1 \quad (L \neq \{0\}, N \neq \{0\}), \quad \delta(\{0\}, \{0\}) = 0.$$

In the case of the spherical gap one may put

$$\tilde{\delta}(L, \{0\}) = \tilde{\delta}(\{0\}, N) = \delta_0 \quad (L \neq \{0\}, N \neq \{0\}), \quad \tilde{\delta}(\{0\}, \{0\}) = 0,$$

where δ_0 is any positive real number.

Formula (38) was proved by Nakamoto [78] under the assumption that L and N are graphs of operators A and B in $\mathcal{B}(\mathcal{H})$. This proof is based on explicit expressions of $\delta(L, N)$ (derived in [77]) and $\tilde{\delta}(L, N)$ (obtained in [78]) in terms of A and B . We have not seen (38) for general closed subspaces L and N in the literature.

Example 3.9 (von Neumann’s formula). Consider the orthogonal projection $P_{L \cap N}$ of \mathcal{H} onto $L \cap N$. Evidently,

$$P_{L \cap N} = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} U.$$

On the other hand, von Neumann’s formula [105] says that

$$P_{L \cap N} v = \lim_{n \rightarrow \infty} (PQ)^n v \quad \text{for every } v \in \mathcal{H}.$$

Theorem 1.2 implies that

$$(PQ)^n = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} K^n & K^{n-1}W \\ 0 & 0 \end{pmatrix} U,$$

with $K = I - H$. Since K is selfadjoint with $\sigma(K) \subset [0, 1]$ and 1 not in the point spectrum, the powers K^n converge strongly (= pointwise) to zero. This follows easily from the spectral decomposition of K . A proof avoiding the spectral decomposition is due to Prager [84] and can also be found in [45, Theorem 7.119]. Von Neumann’s formula is clearly a straightforward consequence of the fact that K^n converges strongly to zero. Inserting the above representations in $(P_{L \cap N} - (PQ)^n)(P_{L \cap N} - (PQ)^n)^*$ we get

$$\|P_{L \cap N} - (PQ)^n\|^2 = \left\| \begin{pmatrix} K^{2n-1} & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|K^{2n-1}\|.$$

Thus, if $\max \sigma(K) = 1 - \min \sigma(H) < 1$ (which is always the case for $\dim \mathcal{H} < \infty$), then the norm $\|P_{L \cap N} - (PQ)^n\|$ goes to zero exponentially fast. This was probably first observed by Aronszajn [5]. We refer to the papers [31,32] and the book [33] by Deutsch and to Galantai’s book [45] for more on this issue.

Notice also that if P and Q commute, then $(PQ)^n = PQ$ coincides with $P_{L \cap N}$ for all $n \geq 1$.

Example 3.10 (The Friedrichs angle between two subspaces). The Friedrichs angle between L and N , introduced in [44], is the angle $\theta_F(L, N) \in [0, \pi/2]$ whose cosine is

$$\sup\{|(x, y)| : x \in L \cap (L \cap N)^\perp, y \in N \cap (L \cap N)^\perp, \|x\| = 1, \|y\| = 1\}.$$

It is easily seen that this is equal to

$$\sup\{|(P_{L \cap (L \cap N)^\perp} u, P_{N \cap (L \cap N)^\perp} v)| : u, v \in \mathcal{H}, \|u\| \leq 1, \|v\| \leq 1\} = \|P_{L \cap (L \cap N)^\perp} P_{N \cap (L \cap N)^\perp}\|.$$

We obviously have

$$P_{L \cap N} = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} U, \quad I = (1, 1, 1, 1) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} U,$$

$$P_{(L \cap N)^\perp} = I - P_{L \cap N} = (0, 1, 1, 1) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} U.$$

This implies that $P = P_L$ and $Q = P_N$ commute with $P_{(L \cap N)^\perp}$ and that therefore

$$P_{L \cap (L \cap N)^\perp} = P_L P_{(L \cap N)^\perp} = (0, 1, 0, 0) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U,$$

$$P_{N \cap (L \cap N)^\perp} = P_N P_{(L \cap N)^\perp} = (0, 0, 1, 0) \oplus U^* \begin{pmatrix} I - H & W \\ W & H \end{pmatrix} U,$$

which gives

$$P_{L \cap (L \cap N)^\perp} P_{N \cap (L \cap N)^\perp} = (0, 0, 0, 0) \oplus U^* \begin{pmatrix} I - H & W \\ 0 & 0 \end{pmatrix} U. \tag{40}$$

Computing $PQ - P_{L \cap N}$ we obtain the same right-hand side as in (40). This proves that

$$\cos \theta_F(L, N) = \|PQ - P_{L \cap N}\| = \sqrt{\|I - H\|} = \sqrt{1 - \min \sigma(H)},$$

again with the convention to put $\min \sigma(H) = 1$ and $\|I - H\| = 0$ if $M_0 = \{0\}$.

Example 3.11 (Approximating the projection onto the sum of two subspaces). Let L and N be two closed subspaces of \mathcal{H} and suppose that $L \cap N \neq \{0\}$ and that $L + N$ is also closed. Put $P = P_L$ and $Q = P_N$. One is interested in the best approximation of the orthogonal projection P_{L+N} by a linear combination of the orthogonal projections P, Q , and $P_{L \cap N}$. In [58] (and also in [37]) it is shown that if $\alpha, \beta, \gamma \in \mathbb{C}$, then

$$\|\alpha P + \beta Q + \gamma P_{L \cap N} - P_{L+N}\| \geq \cos \theta_F(L, N)$$

and that equality is achieved for $\alpha = \beta = 1$ and $\gamma = -1$. The inequality can be shown as follows. As $\cos \theta_F(L, N) = 0$ if $M_0 = \{0\}$, we may assume that $M_0 \neq \{0\}$. Since

$$P_{L \cap N} = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} U, \quad P_{L+N} = (1, 1, 1, 0) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} U,$$

the operator $\alpha P + \beta Q + \gamma P_{L \cap N} - P_{L+N}$ is

$$(\alpha + \beta - 1, \alpha - 1, \beta - 1, 0) \oplus U^* \begin{pmatrix} (\alpha + \beta - 1)I - \beta H & \beta W \\ \beta W & \beta H - I \end{pmatrix} U.$$

The norm of the 2×2 matrix is

$$\max_{x \in \sigma(H)} \left\| \begin{pmatrix} \alpha + \beta - 1 - \beta x & \beta \sqrt{x(1-x)} \\ \beta \sqrt{x(1-x)} & \beta x - 1 \end{pmatrix} \right\| =: \max_{x \in \sigma(H)} \|A(x)\|.$$

Decomposing α and β into real and imaginary parts, we get $A(x) = B(x) + iC(x)$ with real symmetric matrices $B(x)$ and $C(x)$. It follows that $\|A(x)\| \geq \|B(x)\|$. We may therefore assume from the beginning that α and β are real and that, consequently, $A(x)$ is a real symmetric matrix. The eigenvalues of $A(x)$ are

$$\lambda_{1,2}(x) = \frac{\alpha + \beta - 2 \pm \sqrt{(\alpha - \beta)^2 + 4\alpha\beta(1-x)}}{2}$$

and the maximum of $[\lambda_1(x)]^2$ and $[\lambda_2(x)]^2$ equals

$$\frac{(\alpha + \beta - 2)^2 + 2|\alpha + \beta - 2| \sqrt{(\alpha - \beta)^2 + 4\alpha\beta(1-x)} + (\alpha - \beta)^2 + 4\alpha\beta(1-x)}{4}.$$

It is quite elementary to show that this is never smaller than $1 - x$, the only ‘‘critical’’ case being $\alpha > 0, \beta > 0, \alpha + \beta < 2$, where, however, the estimate

$$\sqrt{(\alpha - \beta)^2 + 4\alpha\beta(1-x)} = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta x} \geq (\alpha + \beta) \left(1 - \frac{4\alpha\beta x}{(\alpha + \beta)^2} \right)$$

leads to the desired result. In summary,

$$\max_{x \in \sigma(H)} \|A(x)\| \geq \max_{x \in \sigma(H)} \sqrt{1-x} = \sqrt{1 - \min \sigma(H)}$$

and from Example 3.10 we know that $1 - \min \sigma(H) = \cos^2 \theta_F(L, N)$.

Example 3.12 (The Feldman–Krupnik–Markus formulas). Let Π be a skew projection and suppose $\Pi \neq 0$ and $\Pi \neq I$. Put $L = \text{Ran } \Pi$, $N = \text{Ker } \Pi$ and $P = P_L$, $Q = P_N$. In [40], Feldman, Krupnik, and Markus computed the norms $\|f(\Pi, \Pi^*)\|$ for various polynomials f in terms of the norm of only Π . The simplest of their formulas says that if $\alpha, \beta \in \mathbf{C}$, then

$$\|\alpha\Pi + \beta(I - \Pi)\| = \frac{w_+ + w_-}{2}, \quad w_{\pm} := \sqrt{(|\alpha| \pm |\beta|)^2 + |\alpha - \beta|^2(\|\Pi\|^2 - 1)}.$$

One of the proofs goes as follows. The case $M_0 = \{0\}$ is trivial. So let $M_0 \neq \{0\}$. By Corollary 1.7,

$$\Pi = I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ 0 & 0 \end{pmatrix} U, \quad I - \Pi = I_{L^\perp \cap N} \oplus U^* \begin{pmatrix} 0 & H^{-1}W \\ 0 & I \end{pmatrix} U,$$

whence

$$\alpha\Pi + \beta(I - \Pi) = \alpha I_{L \cap N^\perp} + \beta I_{L^\perp \cap N} \oplus U^* \begin{pmatrix} \alpha I & (\beta - \alpha)H^{-1}W \\ 0 & \beta I \end{pmatrix} U.$$

It follows that $(\alpha\Pi + \beta(I - \Pi))(\alpha\Pi + \beta(I - \Pi))^*$ equals

$$|\alpha|^2 I_{L \cap N^\perp} \oplus |\beta|^2 I_{L^\perp \cap N} \oplus U^* \begin{pmatrix} |\alpha|^2 I + |\beta - \alpha|^2 X^2 & \bar{\beta}(\beta - \alpha)X \\ \beta(\bar{\beta} - \bar{\alpha})X & |\beta|^2 I \end{pmatrix} U, \tag{41}$$

with $X := H^{-1}W$. Since

$$\begin{aligned} \text{Det} \begin{pmatrix} |\alpha|^2 I + |\beta - \alpha|^2 X^2 - \lambda I & \bar{\beta}(\beta - \alpha)X \\ \beta(\bar{\beta} - \bar{\alpha})X & |\beta|^2 I - \lambda I \end{pmatrix} \\ = \lambda^2 I - \lambda(|\alpha|^2 + |\beta|^2 + |\alpha - \beta|^2 X^2) + |\alpha|^2 |\beta|^2 I, \end{aligned}$$

the norm of the 2×2 matrix in (41) is

$$\max\{|\lambda| : \lambda^2 - \lambda(|\alpha|^2 + |\beta|^2 + |\alpha - \beta|^2 X^2) + |\alpha|^2 |\beta|^2 = 0 \text{ for some } x \in \sigma(H)\}.$$

This is

$$\frac{|\alpha|^2 + |\beta|^2 + |\alpha - \beta|^2 x_0^2 + \sqrt{(|\alpha|^2 + |\beta|^2 + |\alpha - \beta|^2 x_0^2)^2 - 4|\alpha|^2 |\beta|^2}}{2}$$

with $x_0 = \max \sigma(X)$. The identity

$$\frac{b + \sqrt{c}}{2} = \left(\frac{\sqrt{b + \sqrt{b^2 - c}} + \sqrt{b - \sqrt{b^2 - c}}}{2} \right)^2,$$

therefore shows that the norm of the 2×2 matrix in (41) equals

$$g(x_0) := \frac{1}{4} \left(\sqrt{(|\alpha| + |\beta|)^2 + |\alpha - \beta|^2 x_0^2} + \sqrt{(|\alpha| - |\beta|)^2 + |\alpha - \beta|^2 x_0^2} \right)^2.$$

Clearly,

$$g(x_0) \geq \frac{1}{4} (|\alpha| + |\beta| + ||\alpha| - |\beta||)^2 = \max(|\alpha|^2, |\beta|^2).$$

Consequently, from (41) we see that $\|\alpha\Pi + \beta(I - \Pi)\|^2$ is

$$\max \left(\left\| |\alpha|^2 I_{L \cap N^\perp} \right\|^2, \left\| |\beta|^2 I_{L^\perp \cap N} \right\|^2, g(x_0) \right) = g(x_0).$$

Finally, from Example 3.4 we know that

$$x_0^2 = \max \sigma(X^2) = \max \sigma(H^{-1} - I) = \frac{1}{\min \sigma(H)} - 1 = \|\Pi\|^2 - 1.$$

This completes the proof.

Example 3.13 (Unitary equivalence of skew projections). Due to Proposition 2.4 and Theorem 2.5, one can decide whether two skew projections are unitarily equivalent by computing the principal angles. As this requires the determination of eigenvalues, one is interested in “elementary verifiable” criteria. One such criterion was given by Doković [36], who used Corollary 2.3 to prove that if Π_1 and Π_2 are two projections on a finite-dimensional Hilbert space of dimension n , then Π_1 and Π_2 are unitarily equivalent if and only if

$$\text{tr } \Pi_1 = \text{tr } \Pi_2 \quad \text{and} \quad \text{tr } (\Pi_1 \Pi_1^*)^j = \text{tr } (\Pi_2 \Pi_2^*)^j \quad \text{for } 1 \leq j \leq [n/2], \tag{42}$$

where $[n/2]$ denotes the integral part of $n/2$. With the help of Corollary 1.7, the proof is as follows. It is clear that (42) is necessary for unitary equivalence. To prove the sufficiency, put $L_i = \text{Ran } \Pi_i$, $N_i = \text{Ker } \Pi_i$, $k_i = \dim(L_i \cap N_i^\perp)$ ($i = 1, 2$), and denote by r_1 and r_2 the dimensions of the spaces M_0 on which the operators H_1 and H_2 act. From Corollary 1.7 we immediately see that $\text{tr } \Pi_i = k_i + r_i$. Thus, $k_1 + r_1 = k_2 + r_2$. Since also $k_1 + 2r_1 = n$ and $k_2 + 2r_2 = n$, it follows that $k_1 = k_2 =: k$ and $r_1 = r_2 =: r$. Corollary 1.7 also yields that

$$\Pi_i \Pi_i^* = I_{L_i \cap N_i^\perp} \oplus U^* \begin{pmatrix} H_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} U,$$

that is, $\text{tr } (\Pi_i \Pi_i^*)^j = k + \text{tr } (H_i^{-j})$. Consequently, $\text{tr } (H_1^{-j}) = \text{tr } (H_2^{-j})$ for $1 \leq j \leq r$. But if the r first power sums of the r inverse eigenvalues of H_1 and H_2 coincide, then, by Newton’s identities, so do also the symmetric functions and hence the characteristic polynomials. It follows that $\sigma(H_1) = \sigma(H_2)$, and as H_1 and H_2 are selfadjoint, this implies H_1 and H_2 and thus also Π_1 and Π_2 are unitarily equivalent.

Example 3.14 (Unitary equivalence of pairs of orthogonal projections). Again suppose that $\dim \mathcal{H} = n < \infty$ and let P_1, P_2, Q_1, Q_2 be orthogonal projections on \mathcal{H} . In [2], Al’pin and Ikramov showed that the pairs (P_1, Q_1) and (P_2, Q_2) are unitarily equivalent if and only if one of the following two equivalent conditions is satisfied:

- (i) $\text{tr } P_1 = \text{tr } P_2, \text{tr } Q_1 = \text{tr } Q_2, P_1 Q_1$ and $P_2 Q_2$ have the same singular values,
- (ii) $\text{tr } P_1 = \text{tr } P_2, \text{tr } Q_1 = \text{tr } Q_2, \text{tr } (P_1 Q_1)^j = \text{tr } (P_2 Q_2)^j$ for $1 \leq j \leq n$.

The necessity of the conditions is clear. Let us prove their sufficiency. For $i = 1, 2$, we put

$$\begin{aligned} \ell_i &= \dim(L_i \cap N_i), & k_i &= \dim(L_i \cap N_i^\perp), \\ \ell_i^\perp &= \dim(L_i^\perp \cap N_i), & k_i^\perp &= \dim(L_i^\perp \cap N_i^\perp) \end{aligned}$$

and denote by r_i the dimension of the space M_0 associated with the operator H_i . Assume first that (i) holds. Using Theorem 1.2 to represent $(P_i Q_i)(P_i Q_i)^*$ we obtain that $P_i Q_i$ has ℓ_i times the singular value 1, $n - \ell_i - r_i$ times the singular value 0, and that the remaining r_i singular values are $1 - \sigma(H_i)$. Thus, $\ell_1 = \ell_2 =: \ell, r_1 = r_2 =: r$, and $\sigma(H_1) = \sigma(H_2)$, implying that H_1 and H_2 are unitarily equivalent. The equalities $\text{tr } P_1 = \text{tr } P_2$ and $\text{tr } Q_1 = \text{tr } Q_2$ yield that $k_1 = k_2 =: k$ and $\ell_1^\perp = \ell_2^\perp =: \ell^\perp$, respectively. Since $\ell + k + \ell^\perp + k^\perp + 2r = n$, we finally obtain that $k_1^\perp = k_2^\perp$. This proves the desired unitary equivalence. Now assume (ii) is valid. From Theorem 1.2 it is readily seen that

$$\text{tr } (P_i Q_i)^j = \ell_i + \text{tr } (I - H_i)^j = \text{tr } (P_i Q_i P_i)^j = \text{tr } ((P_i Q_i)(P_i Q_i)^*)^j$$

Thus, for $1 \leq j \leq n$, the traces of $((P_i Q_i)(P_i Q_i)^*)^j$ coincide, which via Newton implies that the singular values of $P_i Q_i$ are the same. We therefore arrive at condition (ii).

4. The C^* -algebra generated by two orthogonal projections

Let P and Q be two bounded orthogonal projection on \mathcal{H} with the ranges L and N , respectively. We denote by $C^*(P, Q)$ the smallest closed subalgebra of $\mathcal{B}(\mathcal{H})$ which contains I, P, Q . Since P and Q are selfadjoint, $C^*(P, Q)$ is a C^* -algebra. Note that alternatively we may define $C^*(P, Q)$ as the closure in $\mathcal{B}(\mathcal{H})$ of the set $\{f(P, Q)\}$ where f ranges over all polynomials of the form (26). If $\dim \mathcal{H} < \infty$, we need not pass to the closure, because then $C^*(P, Q)$ is simply the set of all polynomials $f(P, Q)$.

If $M_0 = \{0\}$, then $C^*(P, Q)$ is the set of all operators $(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$ with $\alpha_{jk} \in \mathbf{C}$ and thus isometrically isomorphic to the C^* -algebra of all complex diagonal matrices of order $|\Lambda| \leq 4$. The following theorem is essentially due to Pedersen [81]. It was independently established (in exactly the form it is cited here) in [102].

Theorem 4.1. *Let $M_0 \neq \{0\}$. Then $C^*(P, Q)$ consists exactly of the operators of the form*

$$A = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) \oplus U^* \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix} U, \tag{43}$$

where $\varphi_{00}, \varphi_{01}, \varphi_{10}, \varphi_{11}$ are arbitrary continuous complex-valued functions on $\sigma(H)$ satisfying the following additional constraints:

- if $0 \in \sigma(H)$ then $\varphi_{01}(0) = \varphi_{10}(0) = 0$,
- if $0 \in \sigma(H)$ and $M_{00} \neq \{0\}$ then $\varphi_{00}(0) = \alpha_{00}$,
- if $0 \in \sigma(H)$ and $M_{11} \neq \{0\}$ then $\varphi_{11}(0) = \alpha_{11}$,
- if $1 \in \sigma(H)$ then $\varphi_{01}(1) = \varphi_{10}(1) = 0$,
- if $1 \in \sigma(H)$ and $M_{01} \neq \{0\}$ then $\varphi_{00}(1) = \alpha_{01}$,
- if $1 \in \sigma(H)$ and $M_{10} \neq \{0\}$ then $\varphi_{11}(1) = \alpha_{10}$.

Example 4.2. The projection $P_{L \cap N}$ belongs to $C^*(P, Q)$ if and only if one of the following conditions is satisfied:

- (a) $L \cap N = \{0\}$,
- (b) $(L \cap N) \oplus (L \cap N^\perp) = L$,
- (c) $(L \cap N) \oplus (L \cap N^\perp) \neq L$ and H is invertible.

This can be seen as follows. If (a) holds then $P_{L \cap N} = 0 \in C^*(P, Q)$. If (b) is valid, we have $M_0 = \{0\}$ and hence

$$P_{L \cap N} = (1, 0, 0, 0) = (1, 1, 0, 0) \cdot (1, 0, 1, 0) = PQ \in C^*(P, Q).$$

In case (c) is true, Theorem 4.1 shows that

$$P_{L \cap N} = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} U \tag{44}$$

is in $C^*(P, Q)$. Conversely, assume $P_{L \cap N}$ belongs to $C^*(P, Q)$ but neither (a) nor (b) are in force. Then $M_{00} \neq \{0\}$ and $M_0 \neq \{0\}$. We have again (44), and if H would not be invertible, $0 \in \sigma(H)$, Theorem 4.1 would imply that $1 = \alpha_{00} = \varphi_{00}(0) = 0$, which is impossible. Thus, H must be invertible and therefore (c) must be true.

Since $C^*(P, Q)$ is a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, the invertibility of an operator $A \in C^*(P, Q)$ in $\mathcal{B}(\mathcal{H})$ is equivalent to its invertibility in $C^*(P, Q)$. For A of the form (43), we define

$$\Phi_A(x) = \begin{pmatrix} \varphi_{00}(x) & \varphi_{01}(x) \\ \varphi_{10}(x) & \varphi_{11}(x) \end{pmatrix}, \quad x \in \sigma(H).$$

Proposition 4.3. *An operator A in the C^* -algebra $C^*(P, Q)$ of the form (43) is invertible if and only if $\det \Phi_A(x) \neq 0$ for all $x \in \sigma(H)$ and $\alpha_{jk} \neq 0$ whenever $M_{jk} \neq \{0\}$.*

This follows from the fact that $\Phi_A(H)$ is invertible if and only if so is its operator determinant (note that the entries of $\Phi_A(H)$ commute), which in turn happens if and only if $\det \Phi_A(x) \neq 0$ for all $x \in \sigma(H)$.

Our next concern is to rephrase Proposition 4.3 in a language that avoids the use of H . This language will allow us to pass from invertibility criteria in $C^*(P, Q)$ to the description of the C^* -algebra generated by two selfadjoint idempotents (Theorems 4.6 and 4.7) and afterwards even to an invertibility criterion in the Banach algebra generated by two arbitrary idempotents (Section 6). The following example is a first step towards this objective. It reveals that the spectrum of the operator $P + 2Q$ is able to distinguish the nontrivial subspaces among M_{jk} .

Example 4.4. We have

$$\begin{aligned} 0 \in \sigma(P + 2Q) &\iff M_{11} \neq \{0\} \text{ or } 0 \in \sigma(H), \\ 1 \in \sigma(P + 2Q) &\iff M_{01} \neq \{0\} \text{ or } 1 \in \sigma(H), \\ 2 \in \sigma(P + 2Q) &\iff M_{10} \neq \{0\} \text{ or } 1 \in \sigma(H), \\ 3 \in \sigma(P + 2Q) &\iff M_{00} \neq \{0\} \text{ or } 0 \in \sigma(H). \end{aligned}$$

Indeed, from Theorem 1.2 we obtain that

$$P + 2Q = (3, 1, 2, 0) \oplus U^* \begin{pmatrix} 3I - 2H & 2\sqrt{H(I-H)} \\ 2\sqrt{H(I-H)} & 2H \end{pmatrix} U$$

and since

$$\det \begin{pmatrix} 3 - 2x - \lambda & 2\sqrt{x(1-x)} \\ 2\sqrt{x(1-x)} & 2x - \lambda \end{pmatrix} = 2x - 3\lambda + \lambda^2,$$

the assertion is almost immediate from Proposition 4.3.

Example 4.5. Put

$$t = (P - Q)^2, \quad s = PQP + (I - P)(I - Q)(I - P).$$

Once again by Theorem 1.2,

$$t = (0, 1, 1, 0) \oplus U^* \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} U, \quad s = (1, 0, 0, 1) \oplus U^* \begin{pmatrix} I - H & 0 \\ 0 & I - H \end{pmatrix} U$$

and hence

$$\sigma(t) = \sigma((0, 1, 1, 0)) \cup \sigma(H), \quad \sigma(s) = \sigma((1, 0, 0, 1)) \cup (1 - \sigma(H)).$$

We may therefore replace $\sigma(H) \setminus \{0, 1\}$ (which equals $\sigma(H)$ if $\dim \mathcal{H} < \infty$) by $\sigma(t) \setminus \{0, 1\}$ or $(1 - \sigma(s)) \setminus \{0, 1\}$ and thus by the spectra of objects that no longer involve H explicitly. For example, the condition $\det \Phi_A(x) \neq 0$ for $x \in \sigma(H) \setminus \{0, 1\}$ is equivalent to the condition

$$\det \Phi_A(x) \neq 0 \text{ for all } x \in \sigma(t) \setminus \{0, 1\} \tag{45}$$

and also equivalent to the condition

$$\det \Phi_A(1 - x) \neq 0 \text{ for all } x \in \sigma(s) \setminus \{0, 1\}. \tag{46}$$

The operator H has disappeared in (45) and (46). Finally, since 0 and 1 cannot be isolated points of $\sigma(H)$ (as otherwise they were eigenvalues), we arrive at the conclusion that $0 \in \sigma(H) \iff 0$ is a cluster point of $\sigma(t) \iff 1$ is a cluster point of $\sigma(s)$. Analogously, $1 \in \sigma(H) \iff 1$ is a cluster point of $\sigma(t) \iff 0$ is a cluster point of $\sigma(s)$. Note that a point $z \in \mathbf{C}$ is referred to as a cluster point of a set $E \subset \mathbf{C}$ if for every $\varepsilon > 0$ the disk $\{\zeta \in \mathbf{C} : |\zeta - z| < \varepsilon\}$ contains infinitely many points of E .

We now pass to abstract C^* -algebras. Suppose \mathcal{A} is a complex C^* -algebra with unit element e and $p, q \in \mathcal{A}$ are two selfadjoint idempotents, $p^2 = p = p^*$ and $q^2 = q = q^*$. We define $C^*(p, q)$ as

the smallest closed subalgebra of \mathcal{A} which contains e, p, q . Since $C^*(p, q)$ is a C^* -subalgebra of \mathcal{A} , the spectrum of an element $a \in C^*(p, q)$ in $C^*(p, q)$ is the same as the spectrum in \mathcal{A} . We therefore simply write $\sigma(a)$ for the spectrum of a . Put $t = (p - q)^2$.

Theorem 4.6.

- (a) The spectrum $\sigma(t)$ is a subset of $[0, 1]$.
- (b) For each point $x \in \sigma(t) \setminus \{0, 1\}$ the map $F_x : \{e, p, q\} \rightarrow \mathbf{C}^{2 \times 2}$ given by

$$F_x(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_x(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_x(q) = \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & x \end{pmatrix},$$

where $\sqrt{x(1-x)}$ denotes the positive square root of $x(1-x)$, extends to a continuous C^* -algebra homomorphism of $C^*(p, q)$ to $\mathbf{C}^{2 \times 2}$.

- (c) For each $\lambda \in \sigma(p + 2q) \cap \{0, 1, 2, 3\}$ the map $G_\lambda : \{e, p, q\} \rightarrow \mathbf{C}$ given by $G_\lambda(e) = 1$ and

$$\begin{aligned} G_0(p) = 0, \quad G_0(q) = 0, \quad G_1(p) = 1, \quad G_1(q) = 0, \\ G_2(p) = 0, \quad G_2(q) = 1, \quad G_3(p) = 0, \quad G_3(q) = 1 \end{aligned}$$

extends to a continuous algebra homomorphism of $C^*(p, q)$ to \mathbf{C} .

- (d) An element $a \in C^*(p, q)$ is invertible if and only if $\det F_x(A) \neq 0$ for all $x \in \sigma(t) \setminus \{0, 1\}$ and $G_\lambda(a) \neq 0$ for all $\lambda \in \sigma(p + 2q) \cap \{0, 1, 2, 3\}$.

Using the Gelfand–Naimark theorem, one can derive this theorem from Theorem 4.1, Proposition 4.3, and Examples 4.4 and 4.5.

Given a set $K \subset \mathbf{C}$ with the usual topology, we let $C(K)$ and $\mathbf{C}^{2 \times 2}(K)$ be the C^* -algebra of all continuous functions $f : K \rightarrow \mathbf{C}$ and $f : K \rightarrow \mathbf{C}^{2 \times 2}$, respectively. For a subset M of K , we denote by $C_M^{2 \times 2}(K)$ the C^* -subalgebra of the C^* -algebra $\mathbf{C}^{2 \times 2}(K)$ that consists of all matrix functions in $\mathbf{C}^{2 \times 2}(K)$ which are diagonal matrices at the points of M .

Theorem 4.7. The C^* -algebra $C^*(p, q)$ is (isometrically) isomorphic to

$$\mathbf{C}^{2 \times 2}(\sigma(t) \setminus \{0, 1\}) \oplus C(\sigma(p + 2q) \cap \{0, 1, 2, 3\})$$

if neither 0 nor 1 is a cluster point of $\sigma(t)$, to

$$C_{\{0\}}^{2 \times 2}(\sigma(t) \setminus \{1\}) \oplus C(\sigma(p + 2q) \cap \{1, 2\})$$

if 0 is a cluster point of $\sigma(t)$ but 1 is not, to

$$C_{\{1\}}^{2 \times 2}(\sigma(t) \setminus \{0\}) \oplus C(\sigma(p + 2q) \cap \{0, 3\})$$

if 1 is a cluster point of $\sigma(t)$ but 0 is not, and to

$$C_{\{0,1\}}^{2 \times 2}(\sigma(t))$$

if both 0 and 1 are cluster points of $\sigma(t)$. The C^* -algebra isomorphism is the corresponding restriction of the map $\Psi = \Psi_2 \oplus \Psi_1$ given by

$$(\Psi_2(a))(x) = \begin{cases} F_x(a) & \text{if } x \in \sigma(t) \setminus \{0, 1\}, \\ \text{diag}(G_3(a), G_0(a)) & \text{if } x = 0, \\ \text{diag}(G_1(a), G_2(a)) & \text{if } x = 1 \end{cases}$$

and

$$(\Psi_1(a))(\lambda) = G_\lambda(a) \quad \text{if } \lambda \in \sigma(p + 2q) \cap \{0, 1, 2, 3\}.$$

Theorems 4.6 and 4.7 are essentially already in [81,102]. In the form they are stated here, we learned them from [74]. Full proofs are also contained in [91]. Here is a simple application of Theorem 4.7 to a linear algebra problem.

Corollary 4.8. *Suppose $\dim \mathcal{H} < \infty$. Then $\dim C^*(P, Q) = 4d + \eta$ where d is the number of distinct principal angles among $\theta_1, \dots, \theta_r$ and η is the number of nontrivial subspaces among $L \cap N, L \cap N^\perp, L^\perp \cap N, L^\perp \cap N^\perp$.*

Indeed, if $\dim \mathcal{H} < \infty$, then $\sigma(t)$ is a finite subset of $[0, 1]$ and hence Theorem 4.7 implies that $C^*(P, Q)$ is isomorphic to

$$C^{2 \times 2}(\sigma(t) \setminus \{0, 1\}) \oplus C(\sigma(P + 2Q) \cap \{0, 1, 2, 3\}).$$

We know from (21) and Example 4.4 that

$$\sigma(t) \setminus \{0, 1\} = \sigma(H) \setminus \{0, 1\} = \sigma(H) = \{\mu_1, \dots, \mu_r\}.$$

Thus, $\sigma(t) \setminus \{0, 1\}$ contains exactly d distinct points and hence $\dim C^{2 \times 2}(\sigma(t) \setminus \{0, 1\}) = 4d$. Since 0 and 1 are not in $\sigma(H)$, Example 4.4 tells us that

$$0 \in \sigma(P + 2Q) \iff M_{11} \neq \{0\}, \quad 1 \in \sigma(P + 2Q) \iff M_{01} \neq \{0\},$$

$$2 \in \sigma(P + 2Q) \iff M_{10} \neq \{0\}, \quad 3 \in \sigma(P + 2Q) \iff M_{00} \neq \{0\}.$$

This shows that $\dim C(\sigma(P + 2Q) \cap \{0, 1, 2, 3\}) = \eta$.

5. The C^* -algebra generated by one skew projection

Let $\Pi \in \mathcal{B}(\mathcal{H})$ be a projection, $\Pi^2 = \Pi$. As the cases $\Pi = 0$ and $\Pi = I$ are trivial, we assume throughout this section that $\Pi \neq 0$ and $\Pi \neq I$. (Note that, for example, the equality $\|\Pi\| = \|I - \Pi\|$ is true if and only if $\Pi \notin \{0, I\}$.) We put $L = \text{Ran } \Pi, N = \text{Ker } \Pi, P = P_L, Q = P_N$. Note that L and N are complementary closed subspaces and that in particular $L \cap N = L^\perp \cap N^\perp = \{0\}$. By Corollary 1.7, the operator H is invertible if $M_0 \neq \{0\}$. We denote by $C^*(\Pi)$ the smallest closed subalgebra of $\mathcal{B}(\mathcal{H})$ which contains I, Π, Π^* . Equivalently, $C^*(\Pi)$ is the smallest C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ which contains I and Π . Clearly, $C^*(\Pi)$ coincides with the closure in $\mathcal{B}(\mathcal{H})$ of the set of all polynomials $f(\Pi, \Pi^*)$ where f is as in (26). If $\dim \mathcal{H} < \infty$, the set of these polynomials is already closed and hence is $C^*(\Pi)$. The following theorem is from [95] (and was also discovered in [74]).

Theorem 5.1. *We have $C^*(\Pi) = C^*(P, Q)$.*

This can be seen as follows. Proposition 1.6 implies that $C^*(\Pi) \subset C^*(P, Q)$. To get equality, we must show that P and Q are in $C^*(\Pi)$. By Corollary 1.7,

$$\Pi = I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ 0 & 0 \end{pmatrix} U.$$

The operator

$$\Pi^\dagger := I_{L \cap N^\perp} \oplus U^* \begin{pmatrix} H & 0 \\ -W & 0 \end{pmatrix} U$$

is readily seen to satisfy

$$\Pi \Pi^\dagger \Pi = \Pi, \quad \Pi^\dagger \Pi \Pi^\dagger = \Pi^\dagger, \quad (\Pi \Pi^\dagger)^* = \Pi \Pi^\dagger, \quad (\Pi^\dagger \Pi)^* = \Pi^\dagger \Pi$$

and hence to be the Moore–Penrose inverse of Π . But if an operator in a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ is Moore–Penrose invertible, then the Moore–Penrose inverse automatically belongs to the

C^* -subalgebra (see, for example, [20, Corollary 4.22]). Thus, $\Pi^\dagger \in C^*(\Pi)$. It follows that $P = \Pi \Pi^\dagger$ is also in $C^*(\Pi)$. Analogously, $Q = (I - \Pi)(I - \Pi)^\dagger \in C^*(\Pi)$.

Corollary 5.2 (Doković). *If $\dim \mathcal{H} < \infty$, then $\dim C^*(\Pi) = 4d + \varepsilon$ where d is the number of distinct principal angles among $\theta_1, \dots, \theta_r$ and ε is the number of nontrivial subspaces among $L \cap N^\perp$ and $L^\perp \cap N$.*

This follows from Theorem 5.1 and Corollary 4.8. In [36], Doković proved Corollary 5.2 in a straightforward way, using Corollary 2.3. (Note that the algebra considered in [36] is the algebra generated by solely Π and Π^* , that is, I is not included in the generating elements. Therefore the ε in [36] is 1 if $L^\perp \cap N \neq \{0\}$ and 0 if $L^\perp \cap N = \{0\}$.)

Theorem 5.1 in conjunction with Corollary 1.7 and Proposition 4.3 allows us to study invertibility and norms of operators in $C^*(\Pi)$. We already demonstrated this in Example 3.12. Here is another result from [40] that can be derived in this way.

Example 5.3 (Feldman, Krupnik, Markus). *If $\alpha, \beta, \gamma \in \mathbf{C}$, then*

$$\|\alpha I + \beta \Pi + \gamma \Pi^*\| = \frac{\sqrt{r+s} - \sqrt{r-s}}{2},$$

where

$$\begin{aligned} r &= |\alpha|^2 + |\alpha + \beta + \gamma|^2 + (|\beta|^2 + |\gamma|^2)(\|\Pi\|^2 - 1), \\ s &= 2|\alpha(\alpha + \beta + \gamma) - \beta\gamma(\|\Pi\|^2 - 1)|. \end{aligned}$$

To tackle more complicated cases one has to employ more heavy machinery. Let $f(p, q)$ be a polynomial of the form (26). Put

$$\Pi_z = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \quad \Pi_z^* = \begin{pmatrix} 1 & 0 \\ \bar{z} & 0 \end{pmatrix} \quad (z \in \mathbf{C}).$$

It is not difficult to verify that

$$f(\Pi_z, \Pi_z^*) = \begin{pmatrix} A_{11}(|z|^2) & A_{12}(|z|^2)z \\ A_{21}(|z|^2)\bar{z} & A_{22}(|z|^2) \end{pmatrix},$$

where the A_{jk} 's are polynomials in one variable. We then define

$$\begin{aligned} r(x) &= |A_{11}(x)|^2 + |A_{22}(x)|^2 + (|A_{12}(x)|^2 + |A_{21}(x)|^2)x, \\ s(x) &= 2|A_{11}(x)A_{22}(x) - xA_{12}(x)A_{21}(x)|, \\ \psi(x) &= \frac{\sqrt{r(x) + s(x)} + \sqrt{r(x) - s(x)}}{2}; \end{aligned}$$

note that, obviously, $r(x) \geq s(x)$ for all $x \geq 0$.

Theorem 5.4 (Feldman, Krupnik, Markus). *If $\dim \mathcal{H} = 2$ or if $\psi : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing, then*

$$\|f(\Pi, \Pi^*)\| = \psi(\|\Pi\|^2 - 1).$$

A proof is in [40]. The following examples are also from this paper.

Example 5.5. Let us do Example 5.3 using Theorem 5.4. Thus, $f(p, q) = \alpha + \beta p + \gamma q$ and hence

$$f(\Pi_z, \Pi_z^*) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \beta \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ \bar{z} & 0 \end{pmatrix} = \begin{pmatrix} \alpha + \beta + \gamma & \beta z \\ \gamma \bar{z} & \alpha \end{pmatrix},$$

$$\begin{aligned}
 A_{11}(x) &= \alpha + \beta + \gamma, \quad A_{12}(x) = \beta, \quad A_{21}(x) = \gamma, \quad A_{22}(x) = \alpha, \\
 r(x) &= |\alpha + \beta + \gamma|^2 + |\alpha|^2 + (|\beta|^2 + |\gamma|^2)x, \\
 s(x) &= 2|(\alpha + \beta + \gamma)\alpha - \beta\gamma x|, \\
 \psi(x) &= \frac{\sqrt{r(x) + s(x)} + \sqrt{r(x) - s(x)}}{2}.
 \end{aligned}$$

One can show that ψ is non-decreasing. Theorem 5.4 is therefore applicable and the result coincides with that of Example 5.3.

Example 5.6. We have

$$\begin{aligned}
 \|\Pi \Pi^* \Pi + \Pi^* \Pi \Pi^*\| &= \|\Pi\|^2(1 + \|\Pi\|), \\
 \|\Pi \Pi^* \Pi - \Pi^* \Pi \Pi^*\| &= \|\Pi\|^2 \sqrt{\|\Pi\|^2 - 1}.
 \end{aligned}$$

In the first case, $f(p, q) = pqp + qpq$,

$$\begin{aligned}
 f(\Pi_z, \Pi_z^*) &= \begin{pmatrix} 2(1 + |z|^2) & (1 + |z|^2)z \\ (1 + |z|^2)\bar{z} & 0 \end{pmatrix}, \\
 A_{11}(x) &= 2(1 + x), \quad A_{12}(x) = 1 + x, \quad A_{21}(x) = 1 + x, \quad A_{22}(x) = 0, \\
 r(x) &= (4 + 2x)(1 + x)^2, \quad s(x) = 2x(1 + x)^2, \\
 \psi(x) &= (1 + x)(1 + \sqrt{1 + x})
 \end{aligned}$$

and in the second case, $f(p, q) = pqp - qpq$,

$$\begin{aligned}
 f(\Pi_z, \Pi_z^*) &= \begin{pmatrix} 0 & (1 + |z|^2)z \\ -(1 + |z|^2)\bar{z} & 0 \end{pmatrix}, \\
 A_{11}(x) &= 0, \quad A_{12}(x) = 1 + x, \quad A_{21}(x) = -1 - x, \quad A_{22}(x) = 0, \\
 r(x) &= 2x(1 + x)^2, \quad s(x) = 2x(1 + x)^2, \quad \psi(x) = (1 + x)\sqrt{x}.
 \end{aligned}$$

In both cases the function ψ is non-decreasing and Theorem 5.4 yields the asserted formulas.

Example 5.7. Let $f(p, q) = (p - q)^2 + 1$. Then

$$f(\Pi_z, \Pi_z^*) = \begin{pmatrix} 1 - |z|^2 & 0 \\ 0 & 1 - |z|^2 \end{pmatrix}$$

and hence

$$\begin{aligned}
 A_{11}(x) &= 1 - x, \quad A_{12}(x) = 0, \quad A_{21}(x) = 0, \quad A_{22}(x) = 1 - x, \\
 r(x) &= 2|1 - x|^2, \quad s(x) = 2|1 - x|^2, \quad \psi(x) = |1 - x|.
 \end{aligned}$$

The function ψ is not monotonous and the equality $f(\Pi, \Pi^*) = \psi(\|\Pi\|^2 - 1)$ is in general not true. Indeed, taking

$$\Pi = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbf{C}^3),$$

we get

$$f(\Pi, \Pi^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi \Pi^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $\|f(\Pi, \Pi^*)\| = 1$ although $\|\Pi\| = \sqrt{2}$, that is, $\psi(\|\Pi\|^2 - 1) = \psi(1) = 0$.

Example 5.8. We already observed in Example 3.4 that $\|\Pi\| = \|I - \Pi\|$. Clearly, Example 3.12 provides us with another proof of this equality. We refer the reader to Szyld’s article [101] for the history and many more proofs of the identity $\|\Pi\| = \|I - \Pi\|$. We learned in particular from [101] that the identity was within a five-year period independently discovered by three authors: Del Pasqua [28], Ljance [76], Kato [64]. Feldman, Krupnik, and Markus paper [40] contains a very elementary proof. It is based on the orthogonal decomposition $\mathcal{H} = \text{Ran } \Pi \oplus (\text{Ran } \Pi)^\perp$, in which the projections Π and $I - \Pi$ are given by operator matrices of the form

$$\Pi = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}, \quad I - \Pi = \begin{pmatrix} 0 & -A \\ 0 & -I \end{pmatrix}.$$

It follows that

$$\Pi \Pi^* = \begin{pmatrix} I + AA^* & 0 \\ 0 & 0 \end{pmatrix}, \quad (I - \Pi)^*(I - \Pi) = \begin{pmatrix} 0 & 0 \\ 0 & I + A^*A \end{pmatrix}$$

and thus

$$\begin{aligned} \|\Pi\|^2 &= \|\Pi \Pi^*\| = \|I + AA^*\| = 1 + \max \sigma(AA^*) = 1 + \|A\|^2, \\ \|I - \Pi\|^2 &= \|(I - \Pi)^*(I - \Pi)\| = \|I + A^*A\| = 1 + \max \sigma(A^*A) = 1 + \|A\|^2, \end{aligned}$$

which immediately gives the equality $\|\Pi\| = \|I - \Pi\|$. This proof was also communicated to Szyld by G. Corach and is one of the many proofs listed in [101].

Note that Examples 3.12 and 5.3 imply the equalities

$$\begin{aligned} \|\alpha \Pi + \beta(I - \Pi)\| &= \|\beta \Pi + \alpha(I - \Pi)\|, \\ \|\alpha I + \beta \Pi + \gamma \Pi^*\| &= \|\alpha I + \gamma \Pi + \beta \Pi^*\| = \|\alpha I + \beta(I - \Pi) + \gamma(I - \Pi)^*\| \\ &= \|\alpha I + \gamma(I - \Pi) + \beta(I - \Pi)^*\|, \end{aligned}$$

which generalize the identity $\|\Pi\| = \|I - \Pi\|$ significantly. Finally, in [40] it is also shown that $\|f(\Pi, \Pi^*)\| = \|f(\Pi^*, \Pi)\|$ for every polynomial $f(p, q)$

Example 5.9 (Gerisch). Years before Feldman, Krupnik, and Markus [40], nice formulas for the norms of certain operators in $C^*(\Pi)$ were established by Gerisch [47]. Let Π be a skew projection on \mathcal{H} with range L and kernel N , and put $P = P_L$ and $Q = P_N$. Let further $\text{Re } \Pi = (\Pi + \Pi^*)/2$ and $\text{Im } \Pi = (\Pi - \Pi^*)/(2i)$ be the real and imaginary parts (= Hermitian components) of Π . Suppose $\Pi \neq 0$. Gerisch proved that

$$\|\text{Re } \Pi\| = \frac{\|\Pi\| + 1}{2}, \quad \|\text{Im } \Pi\| = \frac{\sqrt{\|\Pi\|^2 - 1}}{2}.$$

Replacing in the second identity Π by $I - \Pi$ and taking into account that $\text{Im}(I - \Pi) = -\text{Im } \Pi$, we get one more proof of the formula $\|\Pi\| = \|I - \Pi\|$. Here are examples of other identities derived by Gerisch:

$$\begin{aligned} \|2\Pi - I\| &= \|\text{Re}(2\Pi - I)\| + \|\text{Im}(2\Pi - I)\| = \|\Pi\| + \sqrt{\|\Pi\|^2 - 1}, \\ \|\Pi^* \Pi - \Pi \Pi^*\| &= \|\Pi\| \sqrt{\|\Pi\|^2 - 1}, \quad \|\Pi^* \Pi + \Pi \Pi^*\| = \|\Pi\|(\|\Pi\| + 1) \end{aligned}$$

and denoting by $\delta := \|P - Q\|$ the gap between L and N , one also has

$$\|\operatorname{Re} \pi\| = \frac{1 + \sqrt{1 - \delta^2}}{2\sqrt{1 - \delta^2}}, \quad \|\operatorname{Im} \pi\| = \frac{\delta}{2\sqrt{1 - \delta^2}}.$$

The equality $\|2\pi - I\| = \|\pi\| + \sqrt{\|\pi\|^2 - 1}$ was earlier obtained in [94, Lemma 2 on p. 236]. We learned from [47] that for $\dim \mathcal{H} < \infty$ it is actually due to Householder and Carpenter [60].

6. Roch, Silbermann, Gohberg, and Krupnik

The undoubtedly greatest achievement in the two projections business since Halmos' two projections theorem is the extension of that theorem to the case of two idempotents in Banach algebras. This was done by Roch and Silbermann [90] and Gohberg and Krupnik [49,50].

Let \mathcal{A} be a complex Banach algebra with unit e and let p and q be two idempotents of \mathcal{A} , that is, elements satisfying $p^2 = p$ and $q^2 = q$. We denote by $B(p, q)$ the smallest closed subalgebra of \mathcal{A} which contains e, p, q . Equivalently, $B(p, q)$ is the closure in \mathcal{A} of the set $\{f(p, q)\}$ where f ranges over all polynomials of the form (26). Given $a \in B(p, q)$, we denote by $\sigma_{\mathcal{A}}(a)$ the spectrum of a in \mathcal{A} . As usual, we put $t = (p - q)^2$.

Theorem 6.1 (Roch, Silbermann, Gohberg, Krupnik).

(a) For each point $x \in \sigma_{\mathcal{A}}(t)$ the map $F_x : \{e, p, q\} \rightarrow \mathbf{C}^{2 \times 2}$ given by

$$F_x(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_x(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_x(q) = \begin{pmatrix} 1 - x & \sqrt{x(1 - x)} \\ \sqrt{x(1 - x)} & x \end{pmatrix},$$

where $\sqrt{x(1 - x)}$ denotes any number the square of which equals $x(1 - x)$, extends to a continuous algebra homomorphism of $B(p, q)$ to $\mathbf{C}^{2 \times 2}$.

(b) For each $\lambda \in \sigma_{\mathcal{A}}(p + 2q) \cap \{0, 1, 2, 3\}$ the map $G_\lambda : \{e, p, q\} \rightarrow \mathbf{C}$ given by $G_\lambda(e) = 1$ and

$$G_0(p) = 0, \quad G_0(q) = 0, \quad G_1(p) = 1, \quad G_1(q) = 0, \\ G_2(p) = 0, \quad G_2(q) = 1, \quad G_3(p) = 1, \quad G_3(q) = 1$$

extends to a continuous algebra homomorphism of $B(p, q)$ to \mathbf{C} .

(c) An element $a \in B(p, q)$ is invertible in \mathcal{A} if and only if $\det F_x(a) \neq 0$ for all $x \in \sigma_{\mathcal{A}}(t) \setminus \{0, 1\}$ and $G_\lambda(a) \neq 0$ for all $\lambda \in \sigma_{\mathcal{A}}(p + 2q) \cap \{0, 1, 2, 3\}$.

We remark that the theorem remains literally true after replacing the matrix for $F_x(q)$ by

$$\begin{pmatrix} 1 - x & x \\ 1 - x & x \end{pmatrix}.$$

To see this, note that if $x \in \mathbf{C} \setminus \{0, 1\}$, then

$$\begin{pmatrix} 1 - x & \sqrt{x(1 - x)} \\ \sqrt{x(1 - x)} & x \end{pmatrix} = D \begin{pmatrix} 1 - x & x \\ 1 - x & x \end{pmatrix} D^{-1}$$

with

$$D = \operatorname{diag} \left(\sqrt[4]{\frac{1 - x}{x}}, \sqrt[4]{\frac{x}{1 - x}} \right).$$

Secondly, the theorem holds with $t = (p - q)^2$ replaced by $s = pqp + (e - p)(e - q)(e - p)$ and the matrix for $F_x(q)$ replaced by

$$\begin{pmatrix} x & \sqrt{x(1 - x)} \\ \sqrt{x(1 - x)} & 1 - x \end{pmatrix}.$$

This follows from the identity $s = e - t$. We also emphasize that the theorem is of course valid in the case where $\mathcal{A} = B(p, q)$. In that minimal case of a surrounding algebra spectra become maximal. In particular, we always have $\sigma_{\mathcal{A}}(t) \subset \sigma_{B(p,q)}(t)$ and $\sigma_{\mathcal{A}}(p + 2q) \subset \sigma_{B(p,q)}(p + 2q)$. The following supplement to Theorem 6.1 is frequently useful.

Theorem 6.2 (Roch and Silbermann).

(a) If 0 and 1 do not belong to $\sigma_{\mathcal{A}}(t)$, then

$$\sigma_{\mathcal{A}}(p + 2q) \cap \{0, 1, 2, 3\} = \emptyset.$$

(b) If 0 and 1 are cluster points of $\sigma_{\mathcal{A}}(t)$ then (bad message)

$$\sigma_{\mathcal{A}}(p + 2q) \cap \{0, 1, 2, 3\} = \{0, 1, 2, 3\},$$

but (good message) the maps F_x introduced in Theorem 6.1 extend to continuous algebra homomorphisms of $B(p, q)$ to $\mathbf{C}^{2 \times 2}$ for all $x \in \sigma_{\mathcal{A}}(t)$, and an element $a \in B(p, q)$ is invertible in \mathcal{A} if and only if $\det F_x(a) \neq 0$ for all $x \in \sigma_{\mathcal{A}}(t)$.

Theorem 6.1 was essentially established in [90] and then completed in [49,50]. In fact, [90] contains exactly Theorem 6.2. Full proofs can also be found in [18,19,91].

The main motivation for the search for theorems like Theorems 6.1 and 6.2 came from singular integral operators, and the applications of the theorems to algebras of singular integral operators are dominating in [90,49,50,18,19,91]. Here are a few very simple applications of Theorem 6.1 which are mainly motivated by the recent linear algebra literature. In the following examples, p and q are idempotents in a complex Banach algebra \mathcal{A} and invertibility always means invertibility in \mathcal{A} .

Example 6.3. Let

$$\mathcal{L}_0(p, q) = \{\alpha p + \beta q : \alpha \in \mathbf{C}, \beta \in \mathbf{C}, \alpha \neq 0, \beta \neq 0, \alpha + \beta \neq 0\}.$$

Then either all elements of $\mathcal{L}_0(p, q)$ are invertible or none of them are invertible.

This can be proved as follows. Let $a = \alpha p + \beta q \in \mathcal{L}_0(p, q)$. For $x \in \sigma_{\mathcal{A}}(a) \setminus \{0, 1\}$ we have

$$\det F_x(a) = \det \begin{pmatrix} \alpha + \beta(1-x) & \beta\sqrt{x(1-x)} \\ \beta\sqrt{x(1-x)} & \beta x \end{pmatrix} = \alpha\beta x \neq 0$$

and for $\lambda \in \sigma_{\mathcal{A}}(p + 2q) \cap \{1, 2, 3\}$ we get

$$G_1(a) = \alpha \neq 0, \quad G_2(a) = \beta \neq 0, \quad G_3(a) = \alpha + \beta \neq 0.$$

Thus, the matter is decided by solely G_0 . If $0 \notin \sigma_{\mathcal{A}}(p + 2q)$, then every a in $\mathcal{L}_0(p, q)$ is invertible, while if $0 \in \sigma_{\mathcal{A}}(p + 2q)$, we obtain that $G_0(a) = 0$ for all $a \in \mathcal{L}_0(p, q)$, which means that no element of $\mathcal{L}_0(p, q)$ is invertible. That's it.

The result of this example was first established in [7] in the case where p and q are idempotents in $\mathcal{A} = \mathcal{B}(\mathbf{C}^n)$, then proved in [38] under the assumption that p and q are skew projections on Hilbert space, that is, $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and in [69] for skew projections on Banach spaces, $\mathcal{A} = \mathcal{B}(X)$.

Example 6.4. Put $b = p + 2q$. Then

- $p + q$ is invertible $\iff b$ is invertible,
- $p - q$ is invertible $\iff b, b - 3e$ are invertible,
- $e - p + q$ is invertible $\iff b - e$ is invertible,
- $e - p - q$ is invertible $\iff b - e, b - 2e$ are invertible,
- $e - pq$ is invertible $\iff b - 3e$ is invertible,
- $pq + qp$ is invertible $\iff b, b - e, b - 2e$ are invertible,
- $pq - qp$ is invertible $\iff b, b - e, b - 2e, b - 3e$ are invertible.

Furthermore, the elements $p + q + pq$, $p + q - pq$, $p - q + pq$, $p - q - pq$, b are all simultaneously invertible or simultaneously not invertible. Finally,

$$\begin{aligned}
 p + pq + qp \text{ is invertible} &\iff p + pq - qp \text{ is invertible} \\
 &\iff b, b - 2e \text{ are invertible.}
 \end{aligned}$$

To see this notice that

$$\det F_x(p - q) = \det \begin{pmatrix} x & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & -x \end{pmatrix} = -x,$$

which is nonzero for $x \in \sigma_{\mathcal{A}}(t) \setminus \{0, 1\}$. The values of $G_\lambda(p - q)$ are $0, 1, -1, 0$, respectively, if $0, 1, 2, 3$ is in $\sigma_{\mathcal{A}}(b)$. Hence, by Theorem 6.1, $p - q$ is invertible if and only if 0 and 3 are not in the spectrum of b . The remaining cases can be checked analogously: the determinant of F_x does not vanish outside $\{0, 1\}$ and therefore invertibility is determined by the four values of G_λ . This completes the proof.

Combining the above equivalences we arrive at conclusions such as

$$\begin{aligned}
 p - q \text{ is invertible} &\iff p + q, e - pq \text{ are invertible} \\
 &\iff p + q - pq, e - pq \text{ are invertible,} \\
 pq - qp \text{ is invertible} &\iff p - q, e - p - q \text{ are invertible} \\
 &\iff pq + pq, e - pq \text{ are invertible} \\
 &\iff pq + pq, p - q \text{ are invertible,} \\
 pq + qp \text{ is invertible} &\iff p + q, e - p - q \text{ are invertible} \\
 &\iff p + pq + qp, e - p + q \text{ are invertible.}
 \end{aligned}$$

These equivalences were derived by different methods in [7,55,70] for $\mathcal{A} = \mathcal{B}(C^n)$ and in [66,67] in a general ring theoretic setting.

Example 6.5. Another result along these lines is that if $\sigma_{\mathcal{A}}(t) \setminus \{0, 1\}$ is contained in the open unit disk, then

$$p - q \text{ is invertible} \iff \varrho(p + q - e) < 1,$$

where ϱ denotes the spectral radius. To see this, put $a = p + q - e$ and note first that

$$\det F_x(a - \mu e) = \mu^2 - (1 - x) \neq 0 \text{ for } x \in \sigma_{\mathcal{A}}(t) \setminus \{0, 1\} \text{ and } |\mu| \geq 1.$$

If $\lambda \in \sigma_{\mathcal{A}}(p + 2q)$, then $G_\lambda(a - \mu e)$ equals $1 + \mu, \mu, \mu, \mu - 1$ for $\lambda = 0, 1, 2, 3$, respectively. Thus, if $p - q$ is invertible and hence 0 and 3 do not belong to $\sigma_{\mathcal{A}}(p + 2q)$, then $G_\lambda(a - \mu e) \neq 0$ for $|\mu| \geq 1$, which implies that $\varrho(a) < 1$. Conversely, if $\varrho(a) < 1$, then $a - \mu e$ is invertible for $\mu = \pm 1$ and hence 0 and 3 cannot be in $\sigma_{\mathcal{A}}(p + 2q)$, implying that $p - q$ is invertible.

If \mathcal{A} is a C^* -algebra and p and q are two selfadjoint idempotents of \mathcal{A} , then $\sigma_{\mathcal{A}}(t) \setminus \{0, 1\}$ is a subset of $(0, 1)$ (because, by Example 4.5, we may assume that $\sigma_{\mathcal{A}}(t) \setminus \{0, 1\} = \sigma(H) \setminus \{0, 1\}$ for some selfadjoint operator H with spectrum in $[0, 1]$) and $\varrho(p + q - e)$ is equal to $\|p + q - e\|$. Thus, for selfadjoint idempotents in C^* -algebras we arrive at the equivalence

$$p - q \text{ is invertible} \iff \|p + q - e\| < 1.$$

For $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the last equivalence was established in [22]. In the case where \mathcal{A} is the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, it was derived in [68]. See also [45, Theorem 7.90]. These authors employed different tools.

Example 6.6. We have

$$\begin{aligned}
 \sigma(pq) \setminus \{0, 1\} &= \{(1 - \mu)^2 : \mu \in \sigma(p + q) \setminus \{0, 1, 2\}\} \\
 &= \{1 - \mu^2 : \mu \in \sigma(p - q) \setminus \{-1, 0, 1\}\}.
 \end{aligned}$$

One proof goes as follows. Let $v \in \sigma(pq) \setminus \{0, 1\}$. Then $v \notin \{0, 1\}$ and $pq - ve$ is not invertible. The possible values of $G_\lambda(pq - ve)$ are $-v$ and $1 - v$ and hence different from zero. It follows that there must be an $x \in \sigma(t) \setminus \{0, 1\}$ such that

$$\det F_x(pq - ve) = \det \begin{pmatrix} 1 - x - v & \sqrt{x(1-x)} \\ 0 & -v \end{pmatrix} = v(1-x-v) = 0. \tag{47}$$

Consequently, $v = 1 - x$ for some $x \in \sigma(t) \setminus \{0, 1\}$. Let $\mu \in \mathbf{C}$ be any number satisfying $\mu^2 - 2\mu + x = 0$. Then $v = (1 - \mu)^2$. Since $x \notin \{0, 1\}$, we necessarily have $\mu \notin \{0, 1, 2\}$. As

$$\det F_x(p + q - \mu e) = \det \begin{pmatrix} 2 - x - \mu & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & x - \mu \end{pmatrix} = \mu^2 - 2\mu + x, \tag{48}$$

we see that $F_x(p + q - \mu e)$ is not invertible. Thus, $\mu \in \sigma(p + q)$. This proves that

$$\sigma(pq) \setminus \{0, 1\} \subset \{(1 - \mu)^2 : \mu \in \sigma(p + q) \setminus \{0, 1, 2\}\}.$$

Conversely, take $\mu \in \sigma(p + q) \setminus \{0, 1, 2\}$ and put $v = (1 - \mu)^2$. Clearly, $v \notin \{0, 1\}$. The values that may be assumed by $G_\lambda(p + q - \mu e)$ are $-\mu, 1 - \mu, 2 - \mu$ and thus nonzero. From (48) we therefore obtain that there is an $x \in \sigma(t) \setminus \{0, 1\}$ such that $\mu^2 - 2\mu + x = 0$. This implies that $v = 1 - x$. Using (47) we arrive at the conclusion that $\det F_x(pq - ve) = 0$, which shows that $v \in \sigma(pq)$. In summary,

$$\{(1 - \mu)^2 : \mu \in \sigma(p + q) \setminus \{0, 1, 2\}\} \subset \sigma(pq) \setminus \{0, 1\},$$

which completes the proof of the first of the asserted equalities. The second can be proved analogously. The two equalities of this example were established by different methods in paper [12].

7. The W^* -algebra generated by two orthogonal projections

A C^* -subalgebra \mathcal{W} of $\mathcal{B}(\mathcal{H})$ is called a W^* -algebra (or a von Neumann algebra) if it is closed under strong convergence, that is, if $A_n \in \mathcal{W}$ and $A_n y \rightarrow Ay$ for all $y \in \mathcal{H}$ imply that $A \in \mathcal{W}$. Let P and Q be two orthogonal projections in $\mathcal{B}(\mathcal{H})$ with the ranges L and N , respectively. We denote by $W^*(P, Q)$ the smallest W^* -subalgebra of $\mathcal{B}(\mathcal{H})$ which contains L, P, Q . If $M_0 = \{0\}$, then $W^*(P, Q) = C^*(P, Q)$ is the algebra of all operators of the form $(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$ with $\alpha_{jk} \in \mathbf{C}$. Thus, let $M_0 \neq \{0\}$.

The selfadjoint operator H induces a spectral measure μ on the real line with values in $\mathcal{B}(\mathcal{H})$. The support of this measure is $\sigma(H)$ and thus contained in $[0, 1]$. The sets of measure zero are the sets $E \subset [0, 1]$ for which the corresponding spectral projection $\chi_E(H)$ is the zero operator. Here χ_E is the characteristic function of E , that is, $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$. We denote by $L^\infty(\sigma(H))$ the complex-valued functions φ on $\sigma(H)$ for which the preimage of every Borel subset of \mathbf{C} is μ -measurable and which are essentially bounded. Two functions in $L^\infty(\sigma(H))$ will be identified if they differ on a set of μ -measure zero only.

Theorem 7.1 (Giles and Kummer). *The W^* -algebra $W^*(P, Q)$ is the set of all operators of the form*

$$A = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) \oplus U^* \begin{pmatrix} \varphi_{00}(H) & \varphi_{01}(H) \\ \varphi_{10}(H) & \varphi_{11}(H) \end{pmatrix} U, \tag{49}$$

where α_{jk} are arbitrary complex numbers and φ_{jk} are arbitrary functions in $L^\infty(\sigma(H))$.

This theorem is proved in [48]. We confine ourselves to the following. For a subset \mathcal{A} of $\mathcal{B}(\mathcal{H})$, the commutant \mathcal{A}' is defined as the set of all operators $T \in \mathcal{B}(\mathcal{H})$ such that $TA = AT$ for all $A \in \mathcal{A}$. The commutant of \mathcal{A}' is denoted by \mathcal{A}'' . It is well known that if \mathcal{A} is invariant under passage to adjoints, then the smallest W^* -subalgebra of $\mathcal{B}(\mathcal{H})$ which contains \mathcal{A} coincides with \mathcal{A}'' . Thus, the theorem can be proved by showing that the operators (49) just constitute \mathcal{A}'' for $\mathcal{A} := \{P, Q\}$.

A function $\varphi \in L^\infty(\sigma(H))$ is said to be separated from zero on a μ -measurable set E if there is an $\varepsilon > 0$ such that $|\varphi| \geq \varepsilon$ almost everywhere on E . Throughout the rest of this section we suppose that A is given by (49). Recall that Φ_A is defined as

$$\Phi_A(x) = \begin{pmatrix} \varphi_{00}(x) & \varphi_{01}(x) \\ \varphi_{10}(x) & \varphi_{11}(x) \end{pmatrix}, \quad x \in \sigma(H). \tag{50}$$

Proposition 7.2. *An operator $A \in W^*(P, Q)$ is invertible if and only if $\det \Phi_A$ is separated from zero on $\sigma(H)$ and $\alpha_{jk} \neq 0$ whenever $M_{jk} \neq \{0\}$. In that case $\text{Det } \Phi_A(H)$ is invertible and*

$$A^{-1} = \left(\bigoplus_{M_{jk} \neq \{0\}} \alpha_{jk}^{-1} I_{M_{jk}} \right) \oplus U^* \begin{pmatrix} (\text{Det } \Phi_A(H))^{-1} & 0 \\ 0 & (\text{Det } \Phi_A(H))^{-1} \end{pmatrix} \begin{pmatrix} \varphi_{11}(H) & -\varphi_{01}(H) \\ -\varphi_{10}(H) & \varphi_{01}(H) \end{pmatrix} U.$$

Indeed, the entries of the matrix $\Phi_A(H)$ commute and hence the invertibility of $\Phi_A(H)$ is equivalent to the invertibility of $\text{Det } \Phi_A(H)$. The last operator is invertible if and only if $\det \Phi_A$ is separated from zero on $\sigma(H)$.

Example 7.3. In Example 4.2, we found necessary and sufficient conditions for $P_{L \cap N}$ to be in $C^*(P, Q)$. Theorem 7.1 with $\alpha_{00} = 1, \alpha_{01} = \alpha_{10} = \alpha_{11} = 0, \varphi_{00} = \varphi_{01} = \varphi_{10} = \varphi_{11} = 0$ reveals that $P_{L \cap N}$ is always in $W^*(P, Q)$. In fact, this also follows from von Neumann’s formula cited in Example 3.9, which identifies $P_{L \cap N}$ as the strong limit of $(PQ)^n$.

Example 7.4. Employing Theorem 7.1 it is easy to identify the idempotents in $W^*(P, Q)$. Indeed, an operator $A \in W^*(P, Q)$ satisfies $A^2 = A$ if and only if $\alpha_{jk} \in \{0, 1\}$ whenever $M_{jk} \neq \{0\}$ and $\Phi_A^2 = \Phi_A$ on $\sigma(H)$ provided $M_0 \neq \{0\}$. For $x \in \sigma(H)$, we have $\Phi_A(x)^2 = \Phi_A(x)$ if and only if one of the following is satisfied:

- (a) $\varphi_{00}(x) = \varphi_{01}(x) = \varphi_{10}(x) = \varphi_{11}(x) = 0$,
- (b) $\varphi_{00}(x) = \varphi_{11}(x) = 1$ and $\varphi_{01}(x) = \varphi_{10}(x) = 0$,
- (c) $\varphi_{00}(x) + \varphi_{11}(x) = 1$ and $\varphi_{00}(x)\varphi_{11}(x) = \varphi_{01}(x)\varphi_{10}(x)$.

In [58, Theorem 1, 37], the authors considered the operator $A = \alpha P + \beta Q + \gamma P_{L \cap N}$ under the assumption that $L \cap N \neq \emptyset$. Suppose first that $M_0 \neq \{0\}$. Then

$$A = (\alpha + \beta + \gamma, \alpha, \beta, 0) \oplus U^* \begin{pmatrix} (\alpha + \beta)I - \beta H & \beta W \\ \beta W & \beta H \end{pmatrix} U.$$

Since 0 and 1 are not in the point spectrum of H , there exists a point $x \in \sigma(H) \cap (0, 1)$. For this point x ,

$$\varphi_{00}(x) = \alpha + \beta - \beta x, \quad \varphi_{01}(x) = \varphi_{10}(x) = \sqrt{x(1-x)}, \quad \varphi_{11}(x) = \beta x$$

and hence (a) holds if and only if $(\alpha, \beta) = (0, 0)$, (b) cannot be fulfilled, and (c) is valid if and only if $(\alpha, \beta) = (0, 1)$ or $(\alpha, \beta) = (1, 0)$. Consequently, $A^2 = A$ in exactly the six cases where (α, β, γ) equals

$$(0, 0, 0), \quad (0, 0, 1), \quad (0, 1, 0), \quad (0, 1, -1), \quad (1, 0, 0), \quad (1, 0, -1).$$

If $M_0 = \{0\}$ ($\iff PQ = QP$), we obtain that $A^2 = A$ if and only if $\alpha + \beta + \gamma \in \{0, 1\}, \alpha \in \{0, 1\}$ for $L \cap N^\perp \neq \{0\}$, and $\beta \in \{0, 1\}$ for $L^\perp \cap N \neq \{0\}$. We remark that Theorem 1 of [58] is incorrect (but apparently not used in the rest of the paper).

Our next concern is the description of the kernel and range of operators in $W^*(P, Q)$.

For $r \in \{0, 1, 2\}$, let Δ_r be the set of all $x \in \sigma(H)$ for which the rank of $\Phi_A(x)$ equals r . As Φ_A is defined only almost everywhere, the sets $\Delta_0, \Delta_1, \Delta_2$ are also specified up to null sets only. We may assume that they are chosen so that they are mutually disjoint and that their union is $\sigma(H)$.

Recall that χ_E stands for the characteristic function of E . The range of the operator $\chi_{\Delta_r}(H)$ ($r = 0, 1, 2$), $\mathcal{M}_r := \chi_{\Delta_r}(H)M_0$, is called the spectral subspace of H corresponding to Δ_r . The spaces \mathcal{M}_r are invariant subspaces of A and $M_0 = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$. Let H_r be the restriction of H to \mathcal{M}_r . Thus, $H = H_0 \oplus H_1 \oplus H_2$. If $\Delta_r = \emptyset$, we define $\mathcal{M}_r = \{0\}$ and $H_r = 0$. Finally, for $r = 0, 1, 2$, put

$$A_r = \begin{pmatrix} \varphi_{00}(H_r) & \varphi_{01}(H_r) \\ \varphi_{10}(H_r) & \varphi_{11}(H_r) \end{pmatrix}.$$

The operator A_r acts on $\mathcal{M}_r \oplus \mathcal{M}_r$. Recall that Λ denotes the pairs (j, k) for which $M_{jk} \neq \{0\}$. We may now write

$$\mathcal{H} = \left(\bigoplus_{(j,k) \in \Lambda} M_{jk} \right) \oplus U^* [(\mathcal{M}_0 \oplus \mathcal{M}_0) \oplus (\mathcal{M}_1 \oplus \mathcal{M}_1) \oplus (\mathcal{M}_2 \oplus \mathcal{M}_2)]$$

and accordingly

$$A = \left(\bigoplus_{(j,k) \in \Lambda} \alpha_{jk} I_{M_{jk}} \right) \oplus U^* (A_0 \oplus A_1 \oplus A_2) U.$$

For $x \in [0, 1]$, we define

$$\varphi(x) = |\varphi_{00}(x)|^2 + |\varphi_{01}(x)|^2 + |\varphi_{10}(x)|^2 + |\varphi_{11}(x)|^2 \tag{51}$$

and for $x \in \Delta_1$, we put

$$\chi_0(x) = \sqrt{\frac{|\varphi_{00}(x)|^2 + |\varphi_{10}(x)|^2}{\varphi(x)}}, \quad \chi_1(x) = \sqrt{\frac{|\varphi_{01}(x)|^2 + |\varphi_{11}(x)|^2}{\varphi(x)}}.$$

Note that $\chi_0^2 + \chi_1^2 = 1$ on Δ_1 . Further, for $x \in [0, 1]$, let

$$\begin{aligned} \eta(x) &= \overline{\varphi_{00}(x)} \varphi_{01}(x) + \overline{\varphi_{10}(x)} \varphi_{11}(x), \\ \theta(x) &= \frac{\eta(x)}{|\eta(x)|} \text{ if } \eta(x) \neq 0, \quad \theta(x) = 1 \text{ if } \eta(x) = 0. \end{aligned}$$

Finally, for $x \in \Delta_1$, we define

$$\begin{aligned} \psi_0(x) &= \sqrt{\frac{|\varphi_{00}(x)|^2 + |\varphi_{01}(x)|^2}{\varphi(x)}}, \quad \psi_1(x) = \sqrt{\frac{|\varphi_{10}(x)|^2 + |\varphi_{11}(x)|^2}{\varphi(x)}}, \\ \zeta(x) &= \varphi_{00}(x) \overline{\varphi_{10}(x)} + \varphi_{01}(x) \overline{\varphi_{11}(x)}, \\ \tau(x) &= \frac{\zeta(x)}{|\zeta(x)|} \text{ if } \zeta(x) \neq 0, \quad \tau(x) = 1 \text{ if } \zeta(x) = 0. \end{aligned}$$

The following theorem is from [95].

Theorem 7.5. *The kernel of A equals*

$$\text{Ker } A = \left(\bigoplus_{(j,k) \in \Lambda, \alpha_{jk} \neq 0} M_{jk} \right) \oplus U^* \left[(\mathcal{M}_0 \oplus \mathcal{M}_0) \oplus \begin{pmatrix} \theta(H_1) \chi_1(H_1) \\ -\chi_0(H_1) \end{pmatrix} \mathcal{M}_1 \right]$$

and the closure of the range is

$$\overline{\text{Ran } A} = \left(\bigoplus_{(j,k) \in \Lambda, \alpha_{jk} \neq 0} M_{jk} \right) \oplus U^* \left[\begin{pmatrix} \tau(H_1) \psi_1(H_1) \\ \psi_0(H_1) \end{pmatrix} \mathcal{M}_1 \oplus (\mathcal{M}_2 \oplus \mathcal{M}_2) \right].$$

The defect numbers $\alpha(B)$ and $\beta(B)$ of an operator $B \in \mathcal{B}(\mathcal{H})$ are defined by

$$\alpha(B) = \dim \text{Ker } B, \quad \beta(B) = \dim \text{Ker } B^* = \dim(\mathcal{H}/\overline{\text{Ran } B}).$$

Theorem 7.5 and minor additional arguments imply the following.

Corollary 7.6. *We have*

$$\alpha(A) = \beta(A) = \left(\bigoplus_{(j,k) \in \Lambda, \alpha_{jk} \neq 0} \dim M_{jk} \right) + 2 \dim \mathcal{M}_0 + \dim \mathcal{M}_1.$$

An operator is said to be normally solvable if its range is closed.

Theorem 7.7. *An operator $A \in W^*(P, Q)$ is normally solvable if and only if φ is separated from zero on Δ_1 and $\det \Phi_A$ is separated from zero on Δ_2 .*

An operator $B \in \mathcal{B}(\mathcal{H})$ is called semi-Fredholm if it is normally solvable and at least one of the numbers $\alpha(B)$ and $\beta(B)$ is finite. The index of a semi-Fredholm operator is defined as $\text{Ind } B = \alpha(B) - \beta(B)$. An operator B is normally solvable and both $\alpha(B)$ and $\beta(B)$ are finite if and only if B is a Fredholm operator, that is, if and only if B is invertible modulo compact operators.

Theorem 7.8. *For A to be semi-Fredholm it is necessary and sufficient that*

- (a) $\dim M_{jk} < \infty$ whenever $M_{jk} \neq \{0\}$ and $\alpha_{jk} = 0$,
- (b) $\dim \text{Ker } \text{Det } \Phi_A(H) < \infty$,
- (c) 0 is not a cluster point of $\sigma(\text{Det } \Phi_A(H))$.

If conditions (a)–(c) are satisfied, then A is Fredholm and $\text{Ind } A = 0$.

Theorem 7.9. *Let $\varphi(x)$ and $\omega(x)$ be the squared Frobenius norm and determinant of $\Phi_A(x)$, respectively, that is, define $\varphi(x)$ by (51) and put $\omega(x) = \varphi_{00}(x)\varphi_{11}(x) - \varphi_{01}(x)\varphi_{10}(x)$. Then for every $A \in W^*(P, Q)$,*

$$\|A\| = \max \left(\max_{(j,k) \in \Lambda} |\alpha_{jk}|, \max_{x \in \sigma(H)} \sqrt{\frac{\varphi(x) + \sqrt{\varphi(x)^2 - 4|\omega(x)|^2}}{2}} \right).$$

Corollary 7.6 and Theorems 7.7 and 7.8 were established in [102] for operators in $C^*(P, Q)$ and in [95] for operators in $W^*(P, Q)$. Theorem 7.9 is from [95].

8. Moore–Penrose inversion

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be Moore–Penrose invertible if there exists an operator $B \in \mathcal{B}(\mathcal{H})$ such that

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB, \quad (BA)^* = BA.$$

Such an operator B exists if and only if A is normally solvable. In that case B is unique, denoted by A^\dagger , and called the Moore–Penrose inverse of A . Note that AA^\dagger and $A^\dagger A$ are the orthogonal projections onto $\text{Ran } A$ and $\text{Ran } A^*$, respectively.

In the case where $\mathcal{H} = \mathbf{C}^n$, we may think of A as a matrix. If $A = USV$ with $S = \text{diag}(s_1, \dots, s_n)$ is the singular value decomposition, then $A^\dagger = V^*S^\dagger U^*$ where S^\dagger is the diagonal matrix $\text{diag}(s_1^\dagger, \dots, s_n^\dagger)$ and $s_j^\dagger = 1/s_j$ for $s_j \neq 0$ and $0^\dagger = 0$.

Let $P, Q \in \mathcal{B}(\mathcal{H})$ be two orthogonal projections and A be an operator in $W^*(P, Q)$. If $M_0 \neq \{0\}$, then Theorem 7.7 tells us that A is Moore–Penrose invertible if and only if φ and $\det \Phi_A$ are separated from zero on Δ_1 and Δ_2 , respectively.

Theorem 8.1. *Let $M_0 \neq \{0\}$. If $\varphi|_{\Delta_1}$ and $\det \Phi_A|_{\Delta_2}$ are separated from zero, then*

$$A^\dagger = \left(\bigoplus_{(j,k) \in \Lambda} \alpha_{jk}^\dagger I_{M_{jk}} \right) \oplus U^*(B_0 \oplus B_1 \oplus B_2)U,$$

where α_{jk}^\dagger is $1/\alpha_{jk}$ for $\alpha_{jk} \neq 0$ and 0 for $\alpha_{jk} = 0$, B_0 is the zero operator on $\mathcal{M}_0 \oplus \mathcal{M}_0$, B_1 is the operator on $\mathcal{M}_1 \oplus \mathcal{M}_1$ that is given by

$$B_1 = \begin{pmatrix} (\varphi(H_1))^{-1} & 0 \\ 0 & (\varphi(H_1))^{-1} \end{pmatrix} \begin{pmatrix} \bar{\varphi}_{00}(H_1) & \bar{\varphi}_{10}(H_1) \\ \bar{\varphi}_{01}(H_1) & \bar{\varphi}_{11}(H_1) \end{pmatrix}$$

and B_2 acts on $\mathcal{M}_2 \oplus \mathcal{M}_2$ and is defined by

$$B_2 = \begin{pmatrix} (\text{Det } \Phi_A(H_2))^{-1} & 0 \\ 0 & (\text{Det } \Phi_A(H_2))^{-1} \end{pmatrix} \begin{pmatrix} \varphi_{11}(H_2) & -\varphi_{01}(H_2) \\ -\varphi_{10}(H_2) & \varphi_{01}(H_2) \end{pmatrix}.$$

This theorem was established in [95].

The operators P and Q themselves are obviously Moore–Penrose invertible and $P^\dagger = P, Q^\dagger = Q$. Here are some more interesting examples. Recall that $\min \sigma(H)$ is just the minimum of $\sigma(H)$ if $M_0 \neq \{0\}$ and that $\min \sigma(H) := 1$ if $M_0 = \{0\}$.

Example 8.2. *The operator $P - Q$ is Moore–Penrose invertible if and only if $\min \sigma(H) > 0$. In that case*

$$(P - Q)^\dagger = (0, 1, -1, 0) \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ -H^{-1}W & I \end{pmatrix} U.$$

This can be seen as follows. If $M_0 = \{0\}$, we have $P - Q = (0, 1, -1, 0)$ and the Moore–Penrose inverse is the operator itself. So assume $M_0 \neq \{0\}$. Then, by Theorem 1.2,

$$P - Q = (0, 1, -1, 0) \oplus U^* \begin{pmatrix} H & -W \\ -W & -H \end{pmatrix} U \tag{52}$$

and it follows that

$$\det \Phi_A(x) = -x, \quad \varphi(x) = x(1 - x), \\ \Delta_0 = \{0\} \cap \sigma(H), \quad \Delta_1 = \emptyset, \quad \Delta_2 = (0, 1] \cap \sigma(H).$$

By Theorem 7.7, $P - Q$ is Moore–Penrose invertible if and only if x is separated from zero on Δ_2 , which happens if and only if $\sigma(H) \subset \{0\} \cup [\varepsilon, 1)$ for some $\varepsilon > 0$. Since 0 cannot be an isolated point of $\sigma(H)$ (recall that 0 is not an eigenvalue of H), we arrive at the conclusion that $P - Q$ is Moore–Penrose invertible if and only if H is invertible. In that case we deduce from Theorem 8.1 that

$$(P - Q)^\dagger = (0, 1, -1, 0) \oplus U^* \begin{pmatrix} -H^{-1} & 0 \\ 0 & -H^{-1} \end{pmatrix} \begin{pmatrix} -H & W \\ W & H \end{pmatrix} U, \tag{53}$$

which is equivalent to the asserted formula.

Example 8.3. *Let $A = \alpha P + \beta Q$ with $\alpha \neq 0, \beta \neq 0, \alpha + \beta \neq 0$. Then A is Moore–Penrose invertible if and only if $\min \sigma(H) > 0$, in which case*

$$A^\dagger = \left(\frac{1}{\alpha + \beta}, \frac{1}{\alpha}, \frac{1}{\beta}, 0 \right) \oplus \frac{1}{\alpha\beta} U^* \begin{pmatrix} \beta I & -\beta H^{-1}W \\ -\beta H^{-1}W & (\alpha + \beta)H^{-1} - I \end{pmatrix} U.$$

This is evident for $M_0 = \{0\}$. Suppose $M_0 \neq \{0\}$. Then

$$A = (\alpha + \beta, \alpha, \beta, 0) \oplus U^* \begin{pmatrix} (\alpha + \beta)I - \beta H & \beta W \\ \beta W & \beta H \end{pmatrix} U \tag{54}$$

and consequently,

$$\begin{aligned} \det \Phi_A(x) &= \alpha\beta x, \quad \varphi(x) = (\alpha - \beta - \beta x)^2 + \beta^2 x^2 + 2\beta^2 x(1 - x), \\ \Delta_0 &= \emptyset, \quad \Delta_1 = \{0\} \cap \sigma(H), \quad \Delta_2 = (0, 1] \cap \sigma(H). \end{aligned}$$

We therefore obtain as in Example 8.2 that A is Moore–Penrose invertible if and only if H is invertible. Theorem 1.1 then yields that A^\dagger is

$$\left(\frac{1}{\alpha + \beta}, \frac{1}{\alpha}, \frac{1}{\beta}, 0 \right) \oplus U^* \frac{1}{\alpha\beta} \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix} \begin{pmatrix} \beta H & -\beta W \\ -\beta W & (\alpha + \beta)I - H \end{pmatrix} U \tag{55}$$

as desired.

Example 8.4. The operator $P + Q$ is Moore–Penrose invertible if and only if the space $L + N := \text{Ran } P + \text{Ran } Q$ is closed, and in that case $L + N = \text{Ran } (P + Q)$ and

$$P_{L \cap N} = 2P(P + Q)^\dagger Q, \quad P_{L+N} = (P + Q)(P + Q)^\dagger.$$

Indeed, from Example 8.3 we deduce that $P + Q$ is Moore–Penrose invertible if and only if $\min \sigma(H) > 0$, and Example 3.2 tells us that $\min \sigma(H) > 0$ if and only if $L + N$ is closed. To prove the remaining assertions, we may assume that $M_0 \neq \{0\}$. Example 8.3 yields

$$(P + Q)^\dagger = (1/2, 1, 1, 0) \oplus U^* \begin{pmatrix} I & -H^{-1}W \\ -H^{-1}W & 2H^{-1} - I \end{pmatrix} U,$$

which implies that

$$2P(P + Q)^\dagger Q = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} U = P_{L \cap N}$$

and

$$(P + Q)(P + Q)^\dagger = (1, 1, 1, 0) \oplus U^* \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} U = P_{L+N}.$$

As $(P + Q)(P + Q)^\dagger$ is the orthogonal projection onto the space $\text{Ran } (P + Q)$, we finally obtain that $L + N = \text{Ran } (P + Q)$. This completes the proof.

By means of different methods, the formula $P_{L \cap N} = 2P(P + Q)^\dagger Q$ was established by Anderson and Duffin [3] for matrices and by Anderson and Schreiber [4] for Hilbert space operators. Paper [82] contains more formulas of this type for the projections $P_{L \cap N}$ and P_{L+M} in the case where $\dim \mathcal{H} < \infty$. We also remark that

$$P + Q - P_{L+N} = (1, 0, 0, 0) \oplus U^* \begin{pmatrix} I - H & W \\ W & H - I \end{pmatrix} U$$

and that the spectrum of the operator matrix on the right is $\{\pm\sqrt{1 - x} : x \in \sigma(H)\}$. This shows that

$$\|P + Q - P_{L+N}\| = \begin{cases} 1 & \text{if } L \cap N \neq \{0\}, \\ \sqrt{1 - \min \sigma(H)} & \text{if } L \cap N = \{0\}. \end{cases}$$

Comparing the last formula with the formula for $\|PQ\|$ established in Example 3.1, we obtain that $\|P + Q - P_{L+N}\| = \|PQ\|$, which was derived in [11] for $\dim \mathcal{H} < \infty$ in a different way.

The following theorem is also proved in [95].

Theorem 8.5. *Let $f(p, q)$ be a polynomial of the form (26). The operator $A = f(P, Q)$ is Moore–Penrose invertible if and only if $M_0 = \{0\}$ or if $M_0 \neq \{0\}$ and one of the following conditions is satisfied:*

- (a) $\det \Phi_A$ does not vanish at the cluster points of $\sigma(H)$,
- (b) $\det \Phi_A$ is identically zero and φ is nonzero at the cluster points of $\sigma(H)$,
- (c) both $\det \Phi_A$ and φ are identically zero.

Example 8.6. Consider the operator

$$E = P_{L \cap N} + P_{L^\perp \cap N^\perp} + (P - Q)^2.$$

This operator is the identity operator for $M_0 = \{0\}$ and has the representation

$$\begin{aligned} E &= (1, 1, 1, 1) \oplus U^* \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I - H & W \\ W & H \end{pmatrix} \right)^2 U \\ &= (1, 1, 1, 1) \oplus U^* \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} U \end{aligned}$$

for $M_0 \neq \{0\}$. From Theorem 7.1 it easily follows $E \in W^*(P, Q)$. Suppose $M_0 \neq \{0\}$. From the representation of E we infer that E is invertible if and only if H is invertible, in which case

$$E^{-1} = (1, 1, 1, 1) \oplus U^* \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix} U.$$

Combining this insight with Example 8.2 we obtain that $P - Q$ is Moore–Penrose invertible if and only if E is invertible and that then, by (52) and (53),

$$(P - Q)^\dagger = E^{-1}(P - Q). \tag{56}$$

Analogously, for A as in Example 8.3, we see that A is Moore–Penrose invertible exactly if E is invertible. Since

$$\begin{aligned} &\frac{1}{\alpha\beta} E^{-1} [\alpha(I - P) + \beta(I - Q)] \\ &= \left(0, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha} + \frac{1}{\beta} \right) \oplus \frac{1}{\alpha\beta} U^* \begin{pmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix} \begin{pmatrix} \beta H & -\beta W \\ -\beta W & (\alpha + \beta)I - H \end{pmatrix} U, \end{aligned}$$

we obtain from (54) and (55) that if E is invertible, then

$$A^\dagger = \frac{1}{\alpha + \beta} P_{L \cap N} - \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) P_{L^\perp \cap N^\perp} + \frac{1}{\alpha\beta} E^{-1} [\alpha(I - P) + \beta(I - Q)]. \tag{57}$$

Formulas (56) and (57) express the Moore–Penrose inverses in terms of explicit operators in $W^*(P, Q)$. The operator H is no longer present in these formulas.

9. Drazin inversion

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be Drazin invertible if the sequences $\{\text{Ker } A^j\}_{j=0}^\infty$ and $\{\text{Ran } A^j\}_{j=0}^\infty$ stabilize. In that case there is a smallest non-negative integer k such that $\text{Ker } A^k = \text{Ker } A^{k+1}$ and $\text{Ran } A^k = \text{Ran } A^{k+1}$, and the Drazin inverse of A is the uniquely determined operator $B \in \mathcal{B}(\mathcal{H})$ satisfying

$$A^{k+1}B = A^k, \quad BAB = B, \quad AB = BA. \tag{58}$$

We denote the Drazin inverse of A by A^D and refer to k as the Drazin index of A (which should not be confused with the index $\text{Ind } A := \alpha(A) - \beta(A)$ of a semi-Fredholm operator A).

Let $\mathcal{H} = \mathbf{C}^n$ and accordingly A be a matrix and let $A = CJC^{-1}$ be the Jordan canonical form of A . Then $J = \text{diag } (J_1, \dots, J_m)$ with Jordan blocks J_ℓ . The Drazin inverse is $A^D = CJ^D C^{-1}$ where $J^D = \text{diag } (J_1^D, \dots, J_m^D)$ and $J_\ell^D = J_\ell^{-1}$ if J_ℓ is nonsingular and $J_\ell^D = 0$ if J_ℓ is singular. The Drazin index of A is the maximal k for which there is a singular $k \times k$ Jordan block.

Now let $A \in W^*(P, Q)$ be as in Theorem 7.1, define Φ_A by (50), and let φ be the function (51). Recall the definitions of the sets $\Delta_0, \Delta_1, \Delta_2$ and of the associated spectral spaces \mathcal{M}_r and operators $H_r = H|_{\mathcal{M}_r}$ ($r = 0, 1, 2$) in Section 7. We now have to stratify the spectrum $\sigma(H)$ further. We put

$$\Delta_{10} = \{x \in \Delta_1 : \text{tr } \Phi_A(x) = 0\}, \quad \Delta_{11} = \{x \in \Delta_1 : \text{tr } \Phi_A(x) \neq 0\}$$

denote by \mathcal{M}_{10} and \mathcal{M}_{11} the corresponding spectral subspaces of H , and let H_{10} and H_{11} stand for the restrictions of H to \mathcal{M}_{10} and \mathcal{M}_{11} , respectively.

If $M_0 = \{0\}$ and accordingly $A = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$, then the Drazin index is 0 or 1 and $A^D = (\alpha_{00}^\dagger, \alpha_{01}^\dagger, \alpha_{10}^\dagger, \alpha_{11}^\dagger)$. So suppose $M_0 \neq \{0\}$.

Theorem 9.1. *An operator $A \in W^*(P, Q)$ is Drazin invertible if and only if $\det \Phi_A$ is separated from zero on Δ_2 and $\text{tr } \Phi_A$ is separated from zero on Δ_{11} . In that case the Drazin index of A is at most 2 and*

$$A^D = \left(\bigoplus_{(j,k) \in \Lambda} \alpha_{jk}^\dagger I_{M_{jk}} \right) \oplus U^* (B_0 \oplus C_{10} \oplus C_{11} \oplus B_2) U,$$

where $\alpha_{jk}^\dagger, B_0, B_1$ are as in Theorem 8.1, C_{10} is the zero operator on $\mathcal{M}_{10} \oplus \mathcal{M}_{10}$, and C_{11} is the operator on $\mathcal{M}_{11} \oplus \mathcal{M}_{11}$ given by

$$C_{11} = \begin{pmatrix} (\text{tr } \Phi_A(H_{11}))^{-1} & 0 \\ 0 & (\text{tr } \Phi_A(H_{11}))^{-1} \end{pmatrix} \begin{pmatrix} \varphi_{00}(H_{11}) & \varphi_{01}(H_{11}) \\ \varphi_{10}(H_{11}) & \varphi_{11}(H_{11}) \end{pmatrix}.$$

A more careful analysis reveals that the Drazin index of A is 2 exactly if $\mathcal{M}_{10} \neq \{0\}$ and that it equals 0 (which is equivalent to usual invertibility) if and only if $\alpha_{jk} \neq 0$ whenever $M_{jk} \neq \{0\}$ and $\mathcal{M}_0 = \mathcal{M}_1 = \{0\}$. In all other cases when A^D exists, it does so with Drazin index 1.

Example 9.2. *There exist operators in $C^*(P, Q)$ which are Moore–Penrose invertible but not Drazin invertible.*

To see this, suppose the point $1/2$ is a cluster point of $\sigma(H)$ and put $A = 2PQ - P$. Then

$$A = (1, -1, 0, 0) \oplus U^* \begin{pmatrix} I - 2H & 2W \\ 0 & 0 \end{pmatrix} U$$

and we have

$$\Phi_A(x) = \begin{pmatrix} 1 - 2x & 2\sqrt{x(1-x)} \\ 0 & 0 \end{pmatrix}, \quad \det \Phi_A(x) = 0, \quad \Delta_2 = \emptyset,$$

$$\varphi(x) = 1, \quad \Delta_1 = \sigma(H), \quad \Delta_0 = \emptyset,$$

$$\text{tr } \Phi_A(x) = 1 - 2x, \quad \Delta_{10} = \{1/2\} \cap \sigma(H), \quad \Delta_{11} = \sigma(H) \setminus \{1/2\}.$$

Since $\Delta_2 = \emptyset$ and $\varphi|_{\Delta_1}$ is separated from zero, the operator A is Moore–Penrose invertible and Theorem 8.1 yields that

$$A^\dagger = (1, -1, 0, 0) \oplus U^* \begin{pmatrix} I - 2H & 0 \\ 2W & 0 \end{pmatrix} U = 2QP - P.$$

On the other hand, as $\text{tr } \Phi_A$ is not separated from zero on Δ_{11} , the operator A is not Drazin invertible.

Example 9.3. There exist operators in $C^*(P, Q)$ which are Drazin invertible but not Moore–Penrose invertible.

Indeed, suppose 0 or 1 is a cluster point of $\sigma(H)$ and let A be the operator

$$\sqrt{PQP(I-Q)P} - \sqrt{(I-P)Q(I-P)(I-Q)(I-P)} - PQ(I-P) + (I-P)QP.$$

From Theorem 1.2 we infer that

$$A = (0, 0, 0, 0) \oplus U^* \begin{pmatrix} W & -W \\ W & -W \end{pmatrix} U.$$

The operator is clearly in $C^*(P, Q)$. We have

$$\begin{aligned} \Phi_A(x) &= \begin{pmatrix} \sqrt{x(1-x)} & -\sqrt{x(1-x)} \\ \sqrt{x(1-x)} & -\sqrt{x(1-x)} \end{pmatrix}, \quad \det \Phi_A(x) = 0, \quad \Delta_2 = \emptyset, \\ \varphi(x) &= 4x(1-x), \quad \Delta_1 = (0, 1) \cap \sigma(H), \quad \Delta_0 = \{0, 1\} \cap \sigma(H) \\ \text{tr } \Phi_A(x) &= 0, \quad \Delta_{11} = \emptyset, \quad \Delta_{10} = (0, 1) \cap \sigma(H). \end{aligned}$$

Since $\varphi|_{\Delta_1}$ is not separated from zero, the operator is not Moore–Penrose invertible. However, because $\Delta_2 = \emptyset$ and $\Delta_{11} = \emptyset$, the operator is Drazin invertible due to Theorem 9.1. The same theorem shows that A^D is the zero operator and that the Drazin index is 1.

Theorem 9.4. An operator $A \in W^*(P, Q)$ is both Moore–Penrose and Drazin invertible if and only if $\det \Phi_A|_{\Delta_2}$, $\varphi|_{\Delta_1}$, and $\text{tr } \Phi_A|_{\Delta_{11}}$ are separated from zero. In that case $A^\dagger = A^D$ if and only if $\mathcal{M}_{10} = \{0\}$ and $\Phi_A|_{\Delta_1}$ is normal, that is, $\Phi_A \Phi_A^* = \Phi_A^* \Phi_A$ almost everywhere on Δ_1 .

Example 9.5. A linear combination $A = \alpha P + \beta Q$ ($\alpha, \beta \in \mathbf{C}$) is Drazin invertible if and only if it is Moore–Penrose invertible, and in this case $A^\dagger = A^D$.

This can be proved as follows. The Drazin and Moore–Penrose inverses of $0, P, Q$ are the operators themselves. If $A = P - Q$, then, by Example 8.2, $\Delta_1 = \Delta_{10} = \Delta_{11} = \emptyset$. It follows that both Drazin and Moore–Penrose invertibility are equivalent to the requirement that $\det \Phi_A|_{\Delta_2}$ is separated from zero. Theorem 9.4 implies that then $A^\dagger = A^D$. Now let $A = \alpha P + \beta Q$ with $\alpha\beta(\alpha + \beta) \neq 0$. From Example 8.3 we know that

$$\Phi_A(x) = \begin{pmatrix} \alpha + \beta - \beta x & \beta \sqrt{x(1-x)} \\ \beta \sqrt{x(1-x)} & \beta x \end{pmatrix},$$

whence $\Delta_{10} = \emptyset$ and $\Delta_{11} = \{0\} \cap \sigma(H)$. Since $\text{tr } \Phi_A(0) = \alpha + \beta \neq 0$ is separated from zero on $\{0\}$, Example 8.3 and Theorem 9.1 show that Moore–Penrose and Drazin invertibility of A are equivalent. Finally, as $\mathcal{M}_{10} = \{0\}$ and $\Phi_A(0)$ is normal, we deduce from Theorem 9.4 that $A^\dagger = A^D$.

Example 9.6. For the operator PQ , both Drazin and Moore–Penrose invertibility are equivalent to the condition that either $M_0 = \{0\}$ or $M_0 \neq \{0\}$ and $I - H$ is invertible. If $M_0 = \{0\}$, then $(PQ)^\dagger = (PQ)^D = PQ$, while if $M_0 \neq \{0\}$ and $I - H$ is invertible, we have

$$\begin{aligned} (PQ)^\dagger &= (1, 0, 0, 0) \oplus U^* \begin{pmatrix} (I-H)^{-1} & 0 \\ 0 & (I-H)^{-1} \end{pmatrix} \begin{pmatrix} I-H & 0 \\ W & 0 \end{pmatrix} U, \\ (PQ)^D &= (1, 0, 0, 0) \oplus U^* \begin{pmatrix} (I-H)^{-2} & 0 \\ 0 & (I-H)^{-2} \end{pmatrix} \begin{pmatrix} I-H & W \\ 0 & 0 \end{pmatrix} U. \end{aligned}$$

In particular, $(PQ)^\dagger \neq (PQ)^D$ whenever $M_0 \neq \{0\}$ and $1 \notin \sigma(H)$.

Here is a proof. The case $M_0 = \{0\}$ is trivial. So let $M_0 \neq \{0\}$. We have

$$\Phi_{PQ}(x) = \begin{pmatrix} 1-x & \sqrt{x(1-x)} \\ 0 & 0 \end{pmatrix}, \quad \det \Phi_{PQ}(x) = 0, \quad \Delta_2 = \emptyset,$$

$$\varphi(x) = 1-x, \quad \Delta_1 = \sigma(H) \setminus \{1\}, \quad \Delta_0 = \{1\} \cap \sigma(H),$$

$$\text{tr } \Phi_{PQ}(x) = 1-x, \quad \Delta_{11} = \sigma(H) \setminus \{1\}, \quad \Delta_{10} = \emptyset.$$

By Theorems 7.7 and 9.1, PQ is Moore–Penrose and Drazin invertible, respectively, if and only if $1-x$ is separated from zero on Δ_1 and Δ_{11} . Thus, both Moore–Penrose and Drazin invertibility are equivalent to the condition that 1 is not a cluster point of H . As 1 is not in the point spectrum of H , it is not a cluster point of $\sigma(H)$ if and only if $I-H$ is invertible. Theorems 8.1 and 9.1 now yield the explicit expressions for $(PQ)^\dagger$ and $(PQ)^D$ quoted above. From these expressions we see that $(PQ)^\dagger \neq (PQ)^D$. Incidentally, the last conclusion can also be drawn from Theorem 9.4 because $\Delta_{11} \setminus \{0, 1\} = \sigma(H) \setminus \{0, 1\}$ cannot be empty and Φ_{PQ} is not normal for $x \notin \{0, 1\}$.

Theorem 9.7. *Let $f(p, q)$ be a polynomial of the form (26). The operator $A = f(P, Q)$ is Drazin invertible if and only if $M_0 = \{0\}$ or if $M_0 \neq \{0\}$ and one of the following holds:*

- (a) $\det \Phi_A$ does not vanish at the cluster points of $\sigma(H)$,
- (b) $\det \Phi_A$ is identically zero and $\text{tr } \Phi_A$ is nonzero at the cluster points of $\sigma(H)$,
- (c) both $\det \Phi_A$ and $\text{tr } \Phi_A$ are identically zero.

Clearly, the criteria established in Examples 9.5 and 9.6 can also be derived using Theorem 9.7 instead of Theorem 9.1.

Example 9.8. Suppose $M_0 \neq \{0\}$. The operator

$$F = P_{L \cap N^\perp} + P_{L^\perp \cap N} + PQP + (I-P)(I-Q)(I-P)$$

has the representation

$$F = (1, 1, 1, 1) \oplus U^* \begin{pmatrix} I-H & 0 \\ 0 & I-H \end{pmatrix} U$$

and is invertible if and only if so is $I-H$. A straightforward computation using the representations for $(PQ)^\dagger$ and $(PQ)^D$ in Example 9.6 gives

$$(PQ)^\dagger = F^{-1}QP, \quad (PQ)^D = F^{-2}PQ.$$

In these two formulas, the operator H has disappeared and been replaced by an explicit operator $F \in W^*(P, Q)$.

Proceeding as in Example 9.6, one can show without difficulty that if A is one of the operators $PQ, PQP, PQPQ, PQPQP, \dots$, then A is Moore–Penrose invertible if and only if it is Drazin invertible and that this is in turn equivalent to the invertibility of F . If F is invertible, then for every integer $m \geq 1$,

$$((PQ)^m)^\dagger = F^{-m}QP, \quad ((PQ)^m)^D = F^{-m-1}PQ,$$

$$((PQ)^m P)^\dagger = ((PQ)^m P)^D = F^{-m-1}PQP.$$

In this section we follow [21]. Drazin invertibility and Drazin inverses of several special operators were previously studied and constructed in Deng’s papers [29,30]. In fact, the papers by Deng motivated us to look for a single theorem (which eventually became Theorem 9.1) that implied all the special results known so far.

10. Commuting idempotents

Let \mathcal{A} be an algebra with unit e over a field \mathbf{K} and let $p_1, \dots, p_n \in \mathcal{A}$ be commuting idempotents. Thus $p_j^2 = p_j$ and $p_j p_k = p_k p_j$ for all j and k . We denote by \mathcal{B} (which should not be confused with $\mathcal{B}(\mathcal{H})$) the smallest subalgebra of \mathcal{A} which contains the unit e and the idempotents p_1, \dots, p_n . Clearly, \mathcal{B} is the set of all linear combinations

$$\sum \gamma_{\varepsilon_1, \dots, \varepsilon_n} p_1^{\varepsilon_1} \dots p_n^{\varepsilon_n} \quad (\varepsilon_j \in \{0, 1\}, \gamma_{\varepsilon_1, \dots, \varepsilon_n} \in \mathbf{K}).$$

Consider the 2^n products $b_1 \dots b_n$ in which each b_j is either p_j or $e - p_j$ and denote these products (in any order) by $\pi_0, \pi_1, \dots, \pi_N$. Thus, $N = 2^n - 1$. For example, if $n = 2$ and $p := p_1, q := p_2$, then \mathcal{B} is just the set of all linear combinations

$$\gamma_{00}e + \gamma_{10}p + \gamma_{01}q + \gamma_{11}pq \tag{59}$$

and we may put

$$\pi_0 = (e - p)(e - q), \quad \pi_1 = p(e - q), \quad \pi_2 = q(e - p), \quad \pi_3 = pq \tag{60}$$

(the labeling of the π_k 's being unessential). It is easily seen that $\pi_j^2 = \pi_j$ for all j and $\pi_j \pi_k = 0$ for $j \neq k$. Moreover, we have $e = \pi_0 + \pi_1 + \dots + \pi_N$ and hence it is impossible that $\pi_k = 0$ for all k unless $\mathcal{A} = \{0\}$, which trivial case may be excluded. Put

$$D = \{k \in \{0, 1, \dots, N\} : \pi_k \neq 0\},$$

let d be the number of elements in D , and denote by $C(D)$ the algebra of all functions $\alpha : D \rightarrow \mathbf{K}$ with pointwise operations. We write the elements of $C(D)$ in the form $\{\alpha_k\}_{k \in D}$. The algebra $C(D)$ is obviously isomorphic to the algebra of all diagonal matrices in $\mathbf{K}^{d \times d}$. The following proposition reveals the simple structure of \mathcal{B} .

Proposition 10.1. *The map $\Psi : C(D) \rightarrow \mathcal{B}, \{\alpha_k\}_{k \in D} \mapsto \sum_{k \in D} \alpha_k \pi_k$ is an algebra isomorphism.*

Indeed, it is clear that Ψ is linear, and since also

$$\sum_{k \in D} \alpha_k \beta_k \pi_k = \left(\sum_{k \in D} \alpha_k \pi_k \right) \left(\sum_{j \in D} \beta_j \pi_j \right),$$

it follows that Ψ is an algebra homomorphism. The map Ψ is injective because if $\sum_{k \in D} \alpha_k \pi_k = 0$, multiplication of this equality by π_j ($j \in D$) gives $\alpha_j \pi_j = 0$ and $\alpha_j = 0$. Finally, Ψ is surjective since every $a \in \mathcal{B}$ may be written as

$$a = a(\pi_0 + \pi_1 + \dots + \pi_N) = \sum_{k \in D} a \pi_k$$

and $a \pi_k$ is easily seen to be a scalar multiple of π_k .

The next proposition, which can be proved by standard arguments, shows that nothing spectacular happens when passing from the “non-closed” setting to Banach algebras.

Proposition 10.2. *If \mathcal{A} is a Banach algebra over $\mathbf{K} = \mathbf{C}$, then the following hold.*

- (a) *The closure of \mathcal{B} in \mathcal{A} is \mathcal{B} itself.*
- (b) *The maximal ideal space of the commutative Banach algebra \mathcal{B} may be identified with D with the discrete topology and the Gelfand map $\Gamma : \mathcal{B} \rightarrow C(D)$ is the inverse of the algebraic isomorphism Ψ introduced in Proposition 10.1,*

$$\Gamma \left(\sum_{k \in D} \alpha_k \pi_k \right) = \{\alpha_k\}_{k \in D}.$$

The Gelfand transform is in particular bijective.

(c) An element $a \in \mathcal{B}$ is invertible in \mathcal{A} if and only if it is invertible in \mathcal{B} , which is in turn equivalent to the invertibility of Γa in $C(D)$.

Example 10.3. This example is motivated by results of [10]. An element $a \in \mathcal{B}$ is said to be generalized invertible in \mathcal{B} if there exists a $b \in \mathcal{B}$ such that $a^2b = a$. Writing $a = \sum_{k \in D} \alpha_k \pi_k$ and $b = \sum_{k \in D} \beta_k \pi_k$, the equation $a^2b = a$ is equivalent to the scalar equations $\alpha_k(\alpha_k \beta_k - 1) = 0$ for $k \in D$. This shows that every $a \in \mathcal{B}$ is generalized invertible in \mathcal{B} . If, for instance, $n = 2$ and notation is as in (59) and (60), then

$$p + q = \pi_1 + \pi_2 + 2\pi_3$$

and $b = \sum_{k \in D} \beta_k \pi_k$ is a generalized inverse of $p + q$ if and only if one of the following eight conditions is satisfied:

- (1) $\beta_1 = 1, \beta_2 = 1, \beta_3 = 1/2,$ (2) $\beta_1 = 1, \beta_2 = 1, \pi_3 = 0,$
- (3) $\beta_1 = 1, \beta_3 = 1/2, \pi_2 = 0$ (4) $\beta_1 = 1, \pi_2 = 0, \pi_3 = 0,$
- (5) $\beta_2 = 1, \beta_3 = 1/2, \pi_1 = 0$ (6) $\beta_2 = 1, \pi_1 = 0, \pi_3 = 0,$
- (7) $\beta_3 = 1/2, \pi_1 = 0, \pi_2 = 0$ (8) $\pi_1 = 0, \pi_2 = 0, \pi_3 = 0.$

If $p \neq 0$ and $q \neq 0$, then the three conditions (4), (6), (8) are never satisfied and we are therefore left with the remaining five conditions. Transforming these into the coordinates γ_{jk} in (59) we get the result that was established in [10] for two commuting projections on \mathbb{C}^n .

Example 10.4. The following result was established in [6].

Let \mathcal{A} be an algebra with identity element $e \neq 0$ over a field \mathbf{K} , let p and q be two different and nonzero (not necessarily commuting) idempotents in \mathcal{A} , and let $\alpha \neq 0$ and $\beta \neq 0$ be two scalars in \mathbf{K} . Define $\pi_0, \pi_1, \pi_2, \pi_3$ by (60). Then the element $\alpha p + \beta q$ is idempotent if and only if one of the following holds:

- (i) $pq \neq qp, (p - q)^2 = 0, \alpha + \beta = 1,$
- (ii) $pq = qp, \pi_1 = 0, \alpha = -1, \beta = 1,$
- (iii) $pq = qp, \pi_2 = 0, \alpha = 1, \beta = -1,$
- (iv) $pq = qp, \pi_3 = 0, \alpha = 1, \beta = 1.$

This can be proved as follows. The equation $(\alpha p + \beta q)^2 = \alpha p + \beta q$ is equivalent to the equation

$$\alpha(\alpha - 1)p + \alpha\beta(pq + qp) + \beta(\beta - 1)q = 0. \tag{61}$$

Multiplying (61) by p from the left and from the right and taking the difference of the two results we obtain $\beta(\alpha + \beta - 1)(pq - qp) = 0$. Thus, if $\alpha p + \beta q$ is an idempotent and $pq \neq qp$, then necessarily $\alpha + \beta = 1$. Inserting $\beta = 1 - \alpha$ in (61) we get $\alpha(\alpha - 1)(p + q - pq - qp) = 0$, which is the same as $(p - q)^2 = 0$ (note that $\alpha \neq 0$ and $\alpha = 1 - \beta \neq 1$). This proves that if $pq \neq qp$, then $\alpha p + \beta q$ is an idempotent if and only if $(p - q)^2 = 0$ and $\alpha + \beta = 1$.

So suppose $pq = qp$. Since $p = \pi_1 + \pi_3, q = \pi_2 + \pi_3, pq = qp = \pi_3$, Eq. (61) can be written in the equivalent form

$$\alpha(\alpha - 1)\pi_1 + \beta(\beta - 1)\pi_2 + (\alpha + \beta)(\alpha + \beta - 1)\pi_3 = 0 \tag{62}$$

and the rest is done by Proposition 10.1. If $\pi_3 = 0$, then $\pi_1 = p \neq 0$ and $\pi_2 = q \neq 0$ and hence (62) holds if and only if $\alpha = 1$ and $\beta = 1$. Thus, let $\pi_3 \neq 0$. Since $\pi_1 = p - pq$ and $\pi_2 = q - pq$, we have $\pi_1 \neq \pi_2$. Consequently, if $\pi_1 = 0$, then $\pi_2 \neq 0$ and it follows that (62) is true if and only if $\beta = 1$ and $\alpha + \beta = 0$, while if $\pi_2 = 0$ and therefore $\pi_1 \neq 0$, we conclude that (62) is valid if and only if $\alpha = 1$ and $\alpha + \beta = 0$. Finally, (62) never holds with nonzero α, β if $\pi_1 \neq 0, \pi_2 \neq 0, \pi_3 \neq 0$.

We remark that case (i) cannot occur if p and q are selfadjoint idempotents (or orthogonal projections on some Hilbert space).

Now assume that \mathcal{H} is a complex separable Hilbert space of dimension at least 2. Let $f(p, q)$ be a polynomial of the form (26). There exist such polynomials with the property that if P and Q are two orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $f(P, Q) = 0$, then necessarily $PQ = QP$. For instance

$$pq - qp, \quad p + q - 1, \quad p - q, \quad 1$$

are such polynomials. (In the last case the equality $f(P, Q) = 0$ cannot be fulfilled, but we use the convention that a void set of operators is commutative.) Less trivial examples are the polynomials

$$(p - q)^2 = p + q - pq - qp, \quad pq + qp, \quad pqpqp - pqp.$$

We say that a polynomial f is enforcing commutativity on \mathcal{H} if it has the following property: if $P, Q \in \mathcal{B}(\mathcal{H})$ are any orthogonal projections and $f(P, Q) = 0$, then $PQ = QP$. Equivalently, f is enforcing commutativity if and only if $f(P, Q) \neq 0$ for every pair (P, Q) of non-commuting orthogonal projections in $\mathcal{B}(\mathcal{H})$. And in still other terms, f is not enforcing commutativity if and only if there exist two orthogonal projections $P, Q \in \mathcal{B}(\mathcal{H})$ such that $f(P, Q) = 0$ and $PQ \neq QP$.

We associate with the polynomial $f(p, q)$ the four polynomials

$$\begin{aligned} \varphi_1(x) &= f_{11} + f_{31}x + f_{51}x^2 + \dots, & \varphi_2(x) &= f_{21} + f_{41}x + f_{61}x^2 + \dots, \\ \varphi_3(x) &= f_{12} + f_{32}x + f_{52}x^2 + \dots, & \varphi_4(x) &= f_{22} + f_{42}x + f_{62}x^2 + \dots \end{aligned}$$

and then define another set of four polynomials by

$$\begin{aligned} \psi_{00}(x) &= f_{00} + \varphi_1(x) + x[\varphi_2(x) + \varphi_3(x) + \varphi_4(x)], \\ \psi_{01}(x) &= \varphi_2(x) + \varphi_3(x), \\ \psi_{10}(x) &= \varphi_3(x) + \varphi_4(x), \\ \psi_{11}(x) &= f_{00} + (1 - x)\varphi_3(x). \end{aligned}$$

The following theorem is the main result of [96].

Theorem 10.5. *The following are equivalent:*

- (i) f is enforcing commutativity,
 - (ii) the polynomials $\psi_{00}, \psi_{01}, \psi_{10}, \psi_{11}$ have no common zero in $(0, 1)$.
- If $f_{00} = 0$, then (ii) is equivalent to the condition
- (iii) the polynomials $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ have no common zero in $(0, 1)$.

Example 10.6. *If $P, Q \in \mathcal{B}(\mathcal{H})$ are two orthogonal projections and $PQ + QP = 0$, then $PQ = QP$.*

Indeed, we have $f(p, q) = pq + qp, f_{00} = 0$, and since $\varphi_1(x) = 0, \varphi_2(x) = 1, \varphi_3(x) = 0, \varphi_4(x) = 1$ have no common zero in $(0, 1)$, the polynomial f enforces commutativity.

Example 10.7. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be two orthogonal projections. If two of the operators*

$$I, P, Q, PQ, QP, PQP, QPQ, PQPQ, QPQP, PQPQP, QPQPQ, \dots$$

coincide, then

$$PQ = QP = PQP = QPQ = PQPQ = QPQP = PQPQP = \dots$$

To see this, let $\mathcal{S} = \{p, q, pq, qp, pqp, qpq, \dots\}$. If f is a polynomial in \mathcal{S} , then three of the associated polynomials $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ are identically zero and one is x^n for some $n \geq 0$. Thus, if $f_1, f_2 \in \mathcal{S}$, then one of the polynomials $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ associated with $f_1 - f_2$ is $\pm x^n$ or $x^n - x^m$ with $n \neq m$. This polynomial has no zeros in $(0, 1)$ and hence $f_1 - f_2$ enforces commutativity. We are left with $1 - f$ for $f \in \mathcal{S}$. Let $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ be the polynomials that are associated with f . If φ_3 is identically zero, then the polynomial $\psi_{11}(x)$ is identically 1 and thus has no zeros in $(0, 1)$.

If φ_3 is not identically zero, then $\varphi_3(x) = x^n$ and $\varphi_1 = \varphi_2 = \varphi_4 = 0$ identically. This implies that $\psi_{01}(x)$ equals $-x^n$ and is therefore nonzero in $(0, 1)$. In either case it follows that $1 - f$ is enforcing commutativity.

Remark 10.8 (Quantum-mechanical interpretation). Let P and Q be orthogonal projections. Example 10.7 shows in particular that

$$PQP = QPQ \implies PQ = QP. \tag{63}$$

We may think of the selfadjoint operators P and Q as observables in a quantum-mechanical system. The equality $PQ = QP$ means that P and Q are commensurable, that is, they can be measured simultaneously. The selfadjoint operator PQP is also an observable. The expected value of PQP when the system is in state φ is $(\varphi, PQP\varphi) = (P\varphi, QP\varphi)$. Thus, PQP determines the conditional probability of Q under the condition that P is given. Clearly, QPQ may be interpreted in an analogous fashion as the probability of P , given Q . Rehder [86] writes that in this light “it comes as no surprise that $PQP = QPQ$ should imply $PQ = QP$. Mathematically speaking, however, the implication seems curious: (63) means that for $PQ = QP$ it is sufficient that PQ has the same value for Px as QP has for Qx , for all $x \in \mathcal{H}$. In other words, (63) permits an implication from the equality of positive selfadjoint operators PQP and QPQ to the equality of *prima facie* more general operators PQ and QP . Putting $A = PQ, A^* = QP$, (63) may be restated as: $A = A^*$ is equivalent to $AA^* = A^*A$, i.e., for $A = PQ$ selfadjointness and normality are the same”.

It was moreover pointed out in [86,87] that the implication (63) is a special case of the Fuglede–Putnam theorem. This theorem says that if A, B, T are in $\mathcal{B}(\mathcal{H})$ and A and B are normal, then $AT = TB$ implies $A^*T = TB^*$. Taking $A = PQ, B = QP, T = P$ yields (63).

We also learned from [86,87] that the selfadjoint operator

$$J(P, Q) := QP(I - Q) + (I - Q)PQ = PQ + QP - 2QPQ$$

is the observable which defines Mittelstaedt’s probability of interference: the probability of interference of P and Q for the system in state φ is $(\varphi, J(P, Q)\varphi)$. For $f(p, q) = pq + qp - 2qpq$ we have $f_{00} = 0, \varphi_1(x) = 0, \varphi_2(x) = 1, \varphi_3(x) = -2x, \varphi_4(x) = 1$, and hence Theorem 10.5 gives the implication

$$J(P, Q) = 0 \implies PQ = QP, \tag{64}$$

which was by different methods already proved in [86,87], too. Physically speaking this means that absence of interference implies commensurability, which is again not a surprise.

To our knowledge, the implications (63) and (64) are due to Rehder [86,87]. The statement of Example 10.7 along with a very short purely C^* -algebraic proof is in the one-pager [24]. In [88], Rehder proved the following generalization: if A and B are selfadjoint and $A \geq 0$ or $B \geq 0$, then

$$AB^2A = BA^2B \implies AB = BA.$$

In other words, if AB is normal, it is automatically selfadjoint. It is also shown in [88] that this is not true if the positivity hypothesis is dropped.

Papers [8–10] contain Examples 10.6 and 10.7 and some more complicated particular commutativity enforcing polynomials in the case where $\mathcal{H} = \mathbb{C}^n$.

11. Concluding remarks

There are many more topics on two projections we could embark on. We leave the matter with a few remarks on the problem of what happens if we have more than two idempotents. As shown in Section 10, things are trivial in case the idempotents commute pairwise. Already in 1955, Davis [25] discovered that there exist three orthogonal projections on \mathcal{H} such that the smallest W^* -subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity and these three projections is all of $\mathcal{B}(\mathcal{H})$. In other words, $\mathcal{B}(\mathcal{H})$ is always generated by three projections in the sense of W^* -algebras. Different proofs of this result (and another proof of Halmos’ theorem) are in Behncke’s papers [14,15]. It is clear that a Banach algebra that is generated in the sense of Banach algebras by a finite number of elements must be

separable, that is, must have a countable dense subset. In [18], it is shown that every separable Banach algebra is isomorphic to a subalgebra of an algebra that is generated in the sense of Banach algebras by three idempotents. Consequently, a theory for C^* -algebras or W^* -algebras generated by at least three orthogonal projections or of Banach algebras generated by at least three idempotents would be a theory of everything and thus a hopeless venture.

However, if further axioms are imposed on the generating idempotents, results like Theorems 4.7 or 6.1 are available. Such additional axioms may, for example, come from the theory of singular integral operators. In that connection one has, for instance, to deal with Banach algebras generated by e, p, q, j where $p^2 = p, q^2 = q, j^2 = e$ (which means that $(e + j)/2$ is an idempotent) and either $jpj = e - p$ and $jqj = e - q$ or $ppj = p$ and $jqj = e - q$. The reader is referred to Roch, Santos, and Silbermann's book [91] for an exhaustive treatment of this subject. Original works on the topic include Finck, Roch, and Silbermann [43], Krupnik and Spigel [75], and Power [83]. The N -projections theorem proved in [18,19] is based on still another set of additional axioms but also motivated by the theory of singular integral operators.

We take up the opportunity to mention that Fillmore [41] showed that every bounded linear operator A on a separable infinite-dimensional Hilbert space can be written as a linear combination of 257 orthogonal projections. (This result has meanwhile been improved considerably, for instance, by Pearcy and Topping [80].) The case where A is a scalar multiple of the identity is very well understood. Let Σ_n denote the set of all $\lambda \in \mathbf{R}$ for which λI is the sum (sic!) of n orthogonal projections. Kruglyak, Rabanovich, and Samoilenko [73] refer to the equalities

$$\Sigma_1 = \{0, 1\}, \quad \Sigma_2 = \{0, 1, 2\}, \quad \Sigma_3 = \left\{0, 1, \frac{3}{2}, 2, 3\right\},$$

$$\Sigma_4 = \left\{0, 1, 1 + \frac{k}{k+2} (k \in \mathbf{N}), 2, 3 - \frac{k}{k+2} (k \in \mathbf{N}), 3, 4\right\}$$

as mathematical folklore and completely describe Σ_n for general n , showing that if $n \geq 5$, then Σ_n is the union of a segment $[\alpha_n, \beta_n]$ and of two sequences S_n^1 and S_n^2 converging to α_n and β_n , respectively.

Bart, Ehrhardt, and Silbermann [13] studied the following problem. Let \mathcal{A} be a Banach algebra and let $p_1, \dots, p_n \in \mathcal{A}$ be idempotents such that $p_1 + \dots + p_n = 0$. Does it follow that $p_1 = \dots = p_n = 0$? They showed that the answer is “yes” for $n \leq 4$ or if \mathcal{A} is a Banach algebra that satisfies a polynomial identity (which is e.g. the case for $\mathcal{A} = \mathbf{C}^{N \times N}$) but that for $n \geq 5$ there exist \mathcal{A} for which the answer is “no”. These latter Banach algebras are far away from being commutative.

To quote another result in this vein, we take the liberty to cite Holland, who begins his article [59] as follows. “Noncommutativity and infinite dimensionality seem to lie at the source of the mysteries of Hilbert space. Consider a theorem of Fillmore [42]: given any bounded selfadjoint operator T on a separable infinite-dimensional complex Hilbert space, there exists a positive number α such that $\alpha T + 4I$ equals the sum of eight or fewer orthogonal projections. Not a linear combination: simply a sum. Put another way, Fillmore's theorem says that *after scaling, any bounded selfadjoint operator equals the sum of eight or fewer orthogonal projections.*” That is a nice end of a guided tour, isn't it?

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