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Counting Real Conjugacy Classes in Some Finite Classical Groups

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Counting Real Conjugacy Classes in Some Finite Classical Groups

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Abstract

An element g in a group G is real if there exists $x \in G$ such that $xgx^{-1} = g^{-1}$. If g is real then all elements in the conjugacy class of g are real. In [GS11b] and [GS11a], Gill and Singh showed that the number of real $\mathrm{GL}_n(q)$-conjugacy classes contained in $\mathrm{SL}_n(q)$ equals the number of real $\mathrm{PGL}_n(q)$-conjugacy classes when q is even or n is odd. In this paper, we use generating functions to show that the result is also true for odd q. We then follow the methods of [GS11b] and [GS11a] to count the number of real $\mathrm{U}_n(q)$ conjugacy classes contained in $\mathrm{SU}_n(q)$ and the number of real conjugacy classes in $\mathrm{PGU}_n(q)$, and we show that these are equal to the analogous quantities for $\mathrm{GL}_n(q)$ and $\mathrm{PGL}_n(q)$. Thus, we show that these four sets of conjugacy classes have equal size for all n, q.

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Contents

Chapter 1

Introduction

Let G be a group. For $g, h \in G$, we say g is conjugate to h if there exists $x \in G$ such that $xgx^{-1} = h$. Conjugacy is an equivalence relation, so we refer to conjugacy classes. We say $g \in G$ is real if g is conjugate to its inverse g^{-1}. If g is real, then every element in the conjugacy class of g is real, as we now prove. Thus we can refer to real conjugacy classes.

Lemma 1.1. *Let G be a group. If $g \in G$ is real, then every element in the conjugacy class of g is also real.*

Proof. An arbitrary element of the conjugacy class of g is of the form xgx^{-1} for some $x \in G$. If g is real, then there is some $y \in G$ such that $ygy^{-1} = g^{-1}$. Then $(xyx^{-1})(xgx^{-1})(xyx^{-1})^{-1} = xygy^{-1}x^{-1} = xg^{-1}x^{-1} = (xgx^{-1})^{-1}$, so xgx^{-1} is real. $\square$

The study of real conjugacy classes is motivated by representation theory:

Theorem 1.2 ([JL93, Theorem 23.1])**.** *Let G be a finite group. The number of real-valued irreducible characters of G is equal to the number of real conjugacy classes of G.*

The general linear group of degree n over the finite field $\mathbb{F}_q$ is the set of invertible $n \times n$ matrices with entries from $\mathbb{F}_q$. We denote this group by $\mathrm{GL}_n(q)$. The unitary group $\mathrm{U}_n(q)$ is the subgroup of $\mathrm{GL}_n(q^2)$ which is fixed by the map $g \mapsto {}^t(Fg)^{-1}$, where $(Fg)_{ij} = g_{ij}^q$. The special linear group $\mathrm{SL}_n(q)$ and special unitary group $\mathrm{SU}_n(q)$ are the subgroups of $\mathrm{GL}_n(q)$

and $\mathrm{U}_n(q)$, respectively, consisting of matrices with determinant one. The projective linear group $\mathrm{PGL}_n(q)$ and projective unitary group $\mathrm{PGU}_n(q)$ are the quotients of $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$ by their respective centers.

In [Leh75, Corollary A″], Lehrer proved the following theorem:

Theorem 1.3. *For all n, q, the number of $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$ equals the number of $\mathrm{PGL}_n(q)$ conjugacy classes.*

In [Mac81, Section 6], Macdonald showed that a similar result holds for unitary groups:

Theorem 1.4. *For all n, q, the number of $\mathrm{U}_n(q)$ conjugacy classes contained in $\mathrm{SU}_n(q)$ equals the number of $\mathrm{PGU}_n(q)$ conjugacy classes.*

In this paper, we show that these results still hold when we specialize to real conjugacy classes. In [GS11b] and [GS11a], Gill and Singh showed that the number of real $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$ equals the number of real $\mathrm{PGL}_n(q)$ conjugacy classes when q is even or n is odd. We show that this result also holds for q odd. We also show that the number of real $\mathrm{U}_n(q)$ conjugacy classes contained in $\mathrm{SU}_n(q)$ and the number of real $\mathrm{PGU}_n(q)$ conjugacy classes are equal to the analogous quantities for the general linear group, and thus all four sets have equal size. Our results are summarized in the following Theorem:

Theorem 1.5. *For all n, q, the following sets have equal size:*

1. *real $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$*
2. *real $\mathrm{PGL}_n(q)$ conjugacy classes*
3. *real $\mathrm{U}_n(q)$ conjugacy classes contained in $\mathrm{SU}_n(q)$*
4. *real $\mathrm{PGU}_n(q)$ conjugacy classes.*

Macdonald [Mac81] has shown how to parametrize the conjugacy classes of these groups using sequences of polynomials $(u_1(t), u_2(t), \ldots)$ which are derived from the elementary divisors. We outline these parametrizations in Chapter 2. If ν is the partition where each

$i \in \mathbb{N}$ has multiplicity equal to the degree of u_i, then the conjugacy class parametrized by $(u_1(t), u_2(t), \ldots)$ is said to be of type ν.

A polynomial is said to be self-reciprocal if its set of roots is invariant under the map $\alpha \mapsto \alpha^{-1}$. In [GS11b], Gill and Singh showed that a conjugacy class of $\mathrm{GL}_n(q)$ is real if and only if it is parametrized by a sequence of self-reciprocal polynomials, and used this fact to to calculate gl_ν, the number of real conjugacy classes in $\mathrm{GL}_n(q)$ of type ν, and sl_ν, the number of real $\mathrm{GL}_n(q)$ conjugacy classes of type ν contained in $\mathrm{SL}_n(q)$.

For a fixed non-square scalar $\zeta \in \mathbb{F}_q$, $g \in \mathrm{GL}_n(q)$ is said of be ζ-real if it is conjugate to ζg^{-1}. By substituting ζg^{-1} for g^{-1} in the proof of Lemma 1.1, it can be shown that if g is ζ-real, then all elements in the conjugacy class of g are ζ-real. A polynomial is said to be ζ-self-reciprocal if its set of roots is invariant under the map $\alpha \mapsto \zeta\alpha^{-1}$. In [GS11b], Gill and Singh showed that a conjugacy class of $\mathrm{GL}_n(q)$ is real if and only if it is parametrized by a sequence of ζ-self-reciprocal polynomials. In [GS11a], Gill and Singh showed that real classes of $\mathrm{PGL}_n(q)$ lift to real or ζ-real classes of $\mathrm{GL}_n(q)$ and calculated pgl_ν, the number of real conjugacy classes in $\mathrm{PGL}_n(q)$ of type ν [GS11a].

We describe self-reciprocal and ζ-self-reciprocal polynomials in Chapter 3. In Chapter 4, we use the results from [GS11b] and [GS11a] to calculate generating functions for the sums over all $|\nu| = n$ of sl_ν and pgl_ν and we show that these are equal. In Chapter 5, we follow the methods of [GS11b] and [GS11a] to calculate gu_ν, su_ν, and pgu_ν for $\mathrm{U}_n(q)$ and $\mathrm{PGU}_n(q)$, and we show that these are equal to the analogous quantities for $\mathrm{GL}_n(q)$ and $\mathrm{PGL}_n(q)$.

Chapter 2

Conjugacy Classes in $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$

Let k be a field, and let V be an n-dimensional vector space over k. By choosing a basis in V, any invertible linear transformation $V \to V$ can be expressed as a matrix in $\mathrm{GL}_n(k)$. Elements in $\mathrm{GL}_n(k)$ are conjugate if and only if they can be defined by the same linear transformation on the vector space V. Linear transformations on a k-vector space define $k[t]$-modules. To count the conjugacy classes of $\mathrm{GL}_n(k)$, we will count the isomorphism classes of n-dimensional $k[t]$-modules which can be defined using invertible linear transformations. This will correspond to counting sequences of polynomials in $k[t]$ which are derived from the elementary divisors.

To count the conjugacy classes of $\mathrm{U}_n(q)$, we will count the isomorphism classes of n-dimensional $k[t]$-modules which can be defined using unitary transformations. These correspond to a subset of the sequences of polynomials which parametrize the $\mathrm{GL}_n(q^2)$ conjugacy classes. In particular, they correspond to sequences of polynomials which are products of u-irreducible polynomials.

2.1 Module Theory

Definition 2.1. Let R be a ring. A left R-module is a set M together with:

1. a binary operation $+$ on M under which M is an abelian group;

2. an action of R on M (i.e. a map $R \times M \to M$) denoted by rm which for all $r, s \in R$ and all $m, n \in M$ satisfies:

   (a) $(r + s)m = rm + sm$,

   (b) $(rs)m = r(sm)$,

   (c) $r(m + n) = rm + rn$,

   and if R has a 1:

   (d) $1m = m$.

When R is a field k, the definition of an R-module coincides with the definition of a vector space over k.

2.1.1 $k[t]$-modules

Let k be a field, and let t be an indeterminate. Let V be a vector space over k. Then V is a k-module. We will now show how to make V into a $k[t]$-module.

Let T be a fixed linear transformation from $V \to V$. Define $T^0 = I$ to be the identity map. For $n \in \mathbb{N}$, define $T^n = T \circ T \circ \cdots \circ T$ to be T composed with itself n times. Then for a polynomial $p(t) = \sum_{0 \le i \le n} a_i t^i \in k[t]$ and a vector $v \in V$, define the action of $p(t)$ on v by

$$p(t)v = \sum_{0 \le i \le n} a_i T^i v.$$

This action makes V into a $k[t]$-module.

An R-module M is a *torsion module* if for each $m \in M$, there is some $r \in R$, $r \neq 0$, such that $rm = 0$.

Lemma 2.2. *If V is finite dimensional, then V is a torsion $k[t]$-module.*

Proof. Let V be a vector space of finite dimension n, and let T be a linear transformation defining a $k[t]$-module structure on V. Then the set $\{v, Tv, T^2v, \ldots, T^nv\}$ is linearly dependent, so there is some choice of scalars $a_0, \ldots, a_n \in k$, not all zero, such that $a_0v + a_1Tv + \cdots + a_nT^nv = 0$. $\square$

A polynomial ring $k[t]$ over a field k is a principal ideal domain (PID), and a finite-dimensional vector space over a finite field is finitely generated. Thus, we can apply the following theorem.

Theorem 2.3. (Fundamental Theorem of Finitely Generated Modules over a PID, Elementary Divisor Form.) *Let R be a PID and let M be a finitely generated R-module. Then M is the direct sum of a finite number of cyclic modules whose annihilators are either (0) or generated by powers of primes in R, i.e.*

$$M \cong R^r \oplus R/\left(p_1^{\alpha_1}\right) \oplus R/\left(p_2^{\alpha_2}\right) \oplus \cdots \oplus R/\left(p_t^{\alpha_t}\right) \tag{2.1}$$

where $r \geq 0$ and $p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ are positive powers of (not necessarily distinct) primes in R. Further, M is a torsion module if and only if $r = 0$.

The prime powers $p_t^{\alpha_t}$ are called elementary divisors and are unique up to multiplication by units. A proof of Theorem 2.3 can be found in [DF03, Theorem 12.1.6].

2.2 Conjugacy Classes in $\mathrm{GL}_n(q)$

In [Mac81, Section 1], Macdonald constructed a parametrization of the conjugacy classes of $\mathrm{GL}_n(q)$. We summarize those results in this section.

Let $k = \mathbb{F}_q$ be a finite field. For $g \in \mathrm{GL}_n(k)$, let V_g denote the $k[t]$-module defined by g. Two elements $g, h \in \mathrm{GL}_n(k)$ are conjugate if and only if the $k[t]$-modules V_g and V_h are isomorphic. Thus the conjugacy classes of $\mathrm{GL}_n(k)$ are in one-to-one correspondence with the isomorphism classes of $k[t]$-modules V such that

$$\dim_k V = n \tag{2.2}$$

and

$$t \cdot v = 0 \text{ implies } v = 0. \tag{2.3}$$

We showed that a finite-dimensional $k[t]$-module is a torsion module, so we must have $r = 0$ in the decomposition in (2.1). The primes in $k[t]$ are irreducible polynomials. We can uniquely specify the elementary divisors of a $k[t]$-module if we require them to be monic. For $k[t]$-modules satisfying (2.3), powers of t cannot be elementary divisors. Let Φ denote the set of irreducible polynomials in $k[t]$ excluding the polynomial t.

The dimension of each cyclic module is given by $\dim_k (k[t]/(f)^a) = a\deg(f)$. Thus every partition-valued function λ on Φ satisfying

$$\sum_{f \in \Phi} |\lambda(f)|\deg(f) = n \tag{2.4}$$

defines a $k[t]$-module satisfying (2.2) and (2.3) whose factorization into elementary divisors is given by

$$V \cong \bigoplus_{f \in \Phi} \oplus_{l \in \lambda(f)} k[t]/\left(f^l\right), \tag{2.5}$$

where the direct sum is over all multiplicities in the multiset $\lambda(f)$.

Take any $g \in \mathrm{GL}_n(k)$. Then g defines an n-dimensional $k[t]$-module with a unique multiset of monic elementary divisors which are powers of irreducibles in Φ. Define λ_g to be the partition-valued function on Φ such that for all $i \in \mathbb{N}$, the multiplicity of i in $\lambda_g(f)$ equals the number of times f^i appears in the list of elementary divisors of V_g. If no powers of f appear in the list of elementary divisors, then define $\lambda_g(f) = \emptyset$. Then λ_g satisfies (2.4) and λ_g is the unique partition-valued function on Φ such that

$$V_g \cong \bigoplus_{f\in\Phi} \underset{l\in\lambda_g(f)}{\oplus} k[t]/\left(f^l\right). \tag{2.6}$$

Thus the set of partition-valued functions which satisfy (2.4) are in bijection with the isomorphism classes of $k[t]$-modules which satisfy (2.2) and (2.3), which are in bijection with the conjugacy classes of $\mathrm{GL}_n(k)$.

We will now modify this parametrization in order to obtain something that can be counted directly. Let $\bar{k}$ denote the algebraic closure of k and let F denote the Frobenius automorphism $\alpha \mapsto \alpha^q$. Irreducible polynomials in $k[t]$ correspond to F-orbits in $M = \bar{k}^\times$. That is, each irreducible polynomial in $k[t]$ is of the form $(t-\alpha)(t-\alpha^q)(t-\alpha^{q^2})\cdots(t-\alpha^{q^d})$ for some $\alpha \in M$ such that $\alpha^{q^{d+1}} = \alpha$ and $\alpha^{q^k} \neq \alpha$ for all $1 \leq k < d+1$.

Let μ be the partition-valued function on M defined by $\mu(y) = \lambda(f)$, where f is the minimal polynomial of y in $k[t]$. Then

$$\mu(Fy) = \mu(y) \tag{2.7}$$

for all $y \in M$ and the condition (2.4) becomes

$$\sum_{y\in M} |\mu(y)| = n. \tag{2.8}$$

The conjugacy classes of $\mathrm{GL}_n(k)$ are thus in one-to-one correspondence with the partition-valued functions μ satisfying (2.7) and (2.8). An element $g \in \mathrm{GL}_n(k)$ defines a $\bar{k}[t]$-module $\bar{V}_g$ when k is identified with its isomorphic subfield in $\bar{k}$. The function μ appears in the decomposition of $\bar{V}_g$,

$$\bar{V}_g \cong \bigoplus_{y\in M} \underset{l\in\mu_g(y)}{\oplus} k[t]/(t-y)^l. \tag{2.9}$$

For a partition-valued function μ, define a sequence of polynomials $(u_i)_{i\in\mathbb{N}}$ where

$$u_i(t) = \prod_{y\in M}(t-y)^{d_i(\mu(y))} \tag{2.10}$$

where d_i is the multiplicity of the part i in $\mu(y)$. If $i \notin \mu(y)$ then $d_i = 0$. The condition (2.7) ensures that each $u_i(t) \in k[t]$. Note that the characteristic polynomial of g is given by

$$\det(t-g) = \prod_{f\in\Phi} f(t)^{|\lambda(f)|} = \prod_{y\in M}(t-y)^{|\mu(y)|} = \prod_{i\in\mathbb{N}} u_i(t)^i. \tag{2.11}$$

Each sequence of polynomials $(u_i)_{i\in\mathbb{N}}$ corresponds to a unique partition-valued function μ on M. Thus the conjugacy classes of $\mathrm{GL}_n(k)$ are in one-to-one correspondence with sequences of monic polynomials $(u_i)_{i\in\mathbb{N}}$ satisfying

$$u_i \in k[t], \quad u_i(0) \neq 0 \tag{2.12}$$

for all $i \in \mathbb{N}$, and

$$\sum_{i\in\mathbb{N}} i\deg(u_i) = n. \tag{2.13}$$

If $\deg(u_i) = m_i$, then $\nu = 1^{m_1}2^{m_2}\cdots$ is a partition of n, and we say the conjugacy class parametrized by u is of type ν.

The number p_{q,m_i} of monic polynomials u_i of degree m_i is

$$p_{q,m_i} = \begin{cases} q^{m_i} - q^{m_i-1} & m_i > 0 \\ 1 & m_i = 0. \end{cases} \tag{2.14}$$

Thus the number of conjugacy classes of type ν in $\mathrm{GL}_n(q)$ is given by

$$c_\nu = \prod_{i\in\mathbb{N}} p_{q,m_i} \tag{2.15}$$

and the total number of conjugacy classes in $\mathrm{GL}_n(q)$ is given by

$$c_n = \sum_{|\nu|=n} c_\nu. \tag{2.16}$$

A generating function for the partition function $p(n)$ is given by

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^i)^k. \tag{2.17}$$

In $P(x)$, each partition of x adds 1 to the coefficient of x^n. To create a generating function for c_n, we replace the $(x^i)^k$ in $P(x)$ with $(x^i)^k p_{q,k}$ so that each partition ν of n adds c_ν to the coefficient of x^n. Thus a generating function for c_n is given by

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \prod_{i=1}^{\infty}\sum_{k=0}^{\infty} x^{ik}p_{q,k} = \prod_{i=1}^{\infty}\left(\sum_{k=0}^{\infty} x^{ik}q^k - \sum_{k=1}^{\infty} x^{ik}q^{k-1}\right) = \prod_{i=1}^{\infty}\frac{1-x^i}{1-qx^i}. \tag{2.18}$$

2.3 Conjugacy Classes in $\mathrm{U}_n(q)$

Recall that the unitary group $\mathrm{U}_n(q)$ is the subgroup of $g \in \mathrm{GL}_n(q^2)$ fixed by the map $\sigma(g) = {}^t(Fg)^{-1}$ where $(Fg)_{ij} = g_{ij}^q$.

To prove the next lemma, we require the Lang-Steinberg Theorem [DM91, Theorem 3.10].

Theorem 2.4 (Lang-Steinberg)**.** *Let G be a connected affine algebraic group and let F be a surjective endomorphism of G with a finite number of fixed points. Then the map $\mathcal{L} : g \mapsto g^{-1}F(g)$ from G to itself is surjective.*

Lemma 2.5 ([Mac81, Lemma 6.1])**.** *Every conjugacy class of $G = \mathrm{GL}_n(q^2)$ which intersects $U = \mathrm{U}_n(q)$ does so in a single conjugacy class of U.*

Proof. Suppose $u \in U$ and $g \in G$ such that $gug^{-1} \in U$. We must show that u and gug^{-1} are conjugate in U. Since $gug^{-1} \in U$, we have $gug^{-1} = g^\sigma ug^{-\sigma}$, so $g^{-1}g^\sigma \in G_u$, the centralizer of u in G. Since $u^\sigma = u$ we have $\sigma(G_u) \subset G_u$. The kernel of σ is the identity because $\gcd(q, q^2-1) = 1$ so the identity is the only element with order dividing q. Thus $\sigma|G_u$ is surjective. Centralizers in G are connected, so by the Lang-Steinberg theorem the map $h \mapsto h^{-1}h^{\sigma}$ from $G_u \mapsto G_u$ is surjective. Thus $g^{-1}g^{\sigma} = h^{-1}h^{\sigma}$ for some $h \in G_u$ so $gh^{-1} = g^{\sigma}h^{-1\sigma}$ and $gh^{-1} \in U$. Then $gug^{-1} = gh^{-1}uhg^{-1}$ is conjugate to u in U. $\square$

Thus, counting the conjugacy classes of $\mathrm{U}_n(q)$ is equivalent to counting the conjugacy classes of $\mathrm{GL}_n(q^2)$ which intersect $\mathrm{U}_n(q)$. These correspond to the partition-valued functions μ on $M = \bar{k}^{\times}$ satisfying (2.8) and $\mu(x^{-q}) = \mu(x)$ instead of (2.7) [Mac81, Section 6]. Let $\bar{F}$ denote the map $\alpha \mapsto \alpha^{-q}$. We say a polynomial is u-irreducible if its roots form a single $\bar{F}$-orbit. That is, a u-irreducible polynomial is of the form $(t-\alpha)(t-\alpha^{-q})(t-\alpha^{q^2})\cdots(t-\alpha^{(-q)^d})$ where $\alpha \in M$ and $d \geq 0$ is the smallest number such that $\alpha^{(-q)^{d+1}} = \alpha$. Polynomials whose sets of roots are stable under $\bar{F}$ are products of u-irreducible polynomials.

Given $f(t) = \sum_{0 \leq i \leq n} a_i t^i \in \mathbb{F}_{q^2}[t]$, we define the u-reciprocal polynomial $\tilde{f}(t)$ as

$$\tilde{f}(t) = \sum_{0 \leq i \leq n} \left(\frac{a_{n-i}}{a_0}\right)^q t^i = \frac{1}{a_0^q} t^n f^q\left(\frac{1}{t}\right) \tag{2.19}$$

where we define the action

$$f^q(t) = \sum_{0 \leq i \leq n} a_i^q t^i. \tag{2.20}$$

Lemma 2.6 ([FNP05, Lemma 1.3.11])**.** *The set of roots of a monic polynomial $f(t)$ is fixed by $\bar{F}$ if and only if $f(t) = \tilde{f}(t)$.*

Proof. Let $M = \overline{\mathbb{F}}_{q^2}^{\times}$. Then each monic $f \in \mathbb{F}_{q^2}[t]$ of degree n with nonzero constant term is of the form

$$f(t) = \prod_{0 \leq i \leq n} (t + b_i) = \sum_{0 \leq i \leq n} a_i t^i \tag{2.21}$$

where each $b_i \in M$ and these are the additive inverses of the roots of f, and the a_i are related to the b_i by

$$a_i = \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J| = n-i}} \prod_{j \in J} b_j. \tag{2.22}$$

Note that

$$a_0 = \prod_{0 \le i \le n} b_i. \tag{2.23}$$

Then we can also write

$$a_i = a_0 \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{1}{b_j}. \tag{2.24}$$

Now we have

$$\left(\frac{a_i}{a_0}\right)^q = \left(\sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{1}{b_j}\right)^q = \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{1}{b_j^q}$$

Thus if $\{b_i : 1 \le i \le n\} = \{\frac{1}{b_i^q} : 1 \le i \le n\}$ then $a_i = \left(\frac{a_{n-i}}{a_0}\right)^q$ for each i. So if the set of roots of $f(t)$ is fixed by $\bar{F}$, then $f(t) = \tilde{f}(t)$.

Now suppose $f(t) = \tilde{f}(t)$. Then for each i,

$$\left(\frac{a_{n-i}}{a_0}\right)^q = a_i = \sum_{\substack{J \subseteq \{1,2,\ldots,n\}, \\ |J|=n-i}} \prod_{j \in J} \frac{1}{b_j^q} = \sum_{\substack{J \subseteq \{1,2,\ldots,n\}, \\ |J|=n-i}} \prod_{j \in J} b_j$$

so we have

$$f(t) = \prod_{1 \le i \le n} \left(t - \frac{1}{b_i^q}\right)$$

so the set of roots of $f(t)$ is fixed by $\bar{F}$. □

Lemma 2.7. *The number $p(u)_{q,k}$ of monic polynomials of degree k in $\mathbb{F}_{q^2}[t]$ that are products of powers of u-irreducible polynomials is given by $p(u)_{q,k} = q^k + q^{k-1}$.*

Proof. The coefficients of a u-self-reciprocal polynomial must satisfy $a_i = \left(\frac{a_{k-i}}{a_0}\right)^q$. Thus the coefficient a_0 must satisfy $a_0^{q+1} = 1$. There are $q+1$ possibilities for a_0 in $\mathbb{F}_{q^2}$, corresponding to the unique subgroup of order $q+1$.

If k is odd, the coefficients $a_0, \ldots, a_{\frac{k-1}{2}}$ determine the remaining coefficients. The coefficients $a_1, \ldots, a_{\frac{k-1}{2}}$ may be chosen arbitrarily from $\mathbb{F}_{q^2}$, so the total number of possible polynomials is $(q^2)^{\frac{k-1}{2}}(q+1) = q^k + q^{k-1}$.

If k is even, the coefficients $a_0, \ldots, a_{\frac{k}{2}}$ determine the remaining coefficients and the coefficients $a_1, \ldots, a_{\frac{k-1}{2}}$ may be chosen arbitrarily. The coefficient $a_{\frac{k}{2}}$ must satisfy $a_{\frac{k}{2}} = a_{\frac{k}{2}}^q a_0^{-q}$ so either $a_{\frac{k}{2}} = 0$ or $a_{\frac{k}{2}}^{q-1} = a_0^q$. There are $q-1$ distinct solutions to this equation, so for a fixed a_0 there are q total possibilities for $a_{\frac{k}{2}}$. So the total number of self-reciprocal polynomials of degree k is $(q^2)^{\frac{k}{2}-1}(q+1)q = q^k + q^{k-1}$. □

The number $p(u)_{q,k}$ is used to count the conjugacy classes of $\mathrm{U}_n(q)$ in the same way that $p_{q,k}$ was used to count the conjugacy classes of $\mathrm{GL}_n(q)$ in Section 2.2. The number $c(u)_\nu$ of $\mathrm{U}_n(q)$ conjugacy classes of type ν is given by

$$c(u)_\nu = \prod_{i \in \mathbb{N}} p(u)_{q,m_i} \tag{2.25}$$

and the number $c(u)_n$ of conjugacy classes in $\mathrm{U}_n(q)$ is given by the generating function

$$C_u(x) = \sum_{n=0}^{\infty} c(u)_n x^n = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} x^{ik} p(u)_{q,k} = \prod_{i=1}^{\infty} \left(\sum_{k=0}^{\infty} x^{ik} q^k + \sum_{k=1}^{\infty} x^{ik} q^{k-1} \right) = \prod_{i=1}^{\infty} \frac{1+x^i}{1-qx^i}. \tag{2.26}$$

Chapter 3

Self-Reciprocal and ζ-Self-Reciprocal Polynomials

In this chapter we introduce the concepts of self-reciprocal and ζ-self-reciprocal polynomials. In Chapter 2 we saw that the conjugacy classes of $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$ are parametrized by sequences of polynomials which are products of irreducibles and products of u-irreducibles, respectively. In Chapter 4, we will see that the sequences of polynomials corresponding to real classes of $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$ are precisely the sequences of self-reciprocal polynomials. In Chapter 5, we will see that the sequences of polynomials corresponding to real classes of $\mathrm{PGL}_n(q)$ and $\mathrm{PGU}_n(q)$ are the sequences of self-reciprocal polynomials and sequences of ζ-self-reciprocal polynomials.

3.1 Self-Reciprocal Polynomials

Let k be a field. Given $f(t) = \sum_{0 \le i \le n} a_i t^i \in k[t]$, we define the reciprocal polynomial $\bar{f}(t)$ as

$$\bar{f}(t) = \sum_{1 \le i \le n} \frac{a_{n-i}}{a_0} t^i = \frac{1}{a_0} t^n f\left(\frac{1}{t}\right). \tag{3.1}$$

We say $f(t)$ is *self-reciprocal* if $f(t) = \bar{f}(t)$.

Lemma 3.1. *The set of roots of a monic polynomial $f(t)$ is invariant under the map $\alpha \mapsto \alpha^{-1}$ if and only if $f(t) = \bar{f}(t)$.*

Proof. Let $M = \overline{k}^{\times}$. Then each monic $f \in k[t]$ is described by Eqs. 2.21-2.24 where each $b_i \in M$ and these are the additive inverses of the roots of f.

Suppose $\{b_i : 1 \leq i \leq n\} = \{\frac{1}{b_i} : 1 \leq i \leq n\}$. Then applying (2.24) we have

$$\frac{a_i}{a_0} = \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{1}{b_j} = \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} b_j = a_{n-i}$$

for each $1 \leq i \leq n$. So if the set of roots of $f(t)$ is invariant under the inverse map, then $f(t) = \bar{f}(t)$.

Now suppose $f(t) = \bar{f}(t) = \frac{1}{a_0} t^n f\left(\frac{1}{t}\right)$. Then $\bar{f}(\alpha) = f(\alpha) = 0$ if and only if $f(\alpha^{-1}) = 0$, so the set of roots of $f(t)$ is invariant under the map $\alpha \mapsto \alpha^{-1}$. □

Lemma 3.2 ([GS11b, Lemma 2.1])**.** *The number $n_{q,d}$ of monic self-reciprocal polynomials in $\mathbb{F}_q[t]$ of degree $d \geq 1$ is given in the following table:*

	q is odd	q is even
d is odd	$2q^{\frac{d-1}{2}}$	$q^{\frac{d-1}{2}}$
d is even	$q^{\frac{d}{2}} + q^{\frac{d}{2}-1}$	$q^{\frac{d}{2}}$

Proof. Let $f(t) = \sum_{0 \leq i \leq d} a_i t^i$ be monic and self-reciprocal. Then by (3.1) we must have $a_0^2 = 1$ and so $a_0 \in \{-1, 1\}$. If q is even then $a_0 = 1 = -1$.

Case 1: d is odd. Then $a_0, a_1, \ldots, a_{\frac{d-1}{2}}$ determine the remaining coefficients and $a_1, \ldots, a_{\frac{d-1}{2}}$ can be chosen arbitrarily from $\mathbb{F}_q$.

Case 2: d is even. Then $a_0, a_1, \ldots, a_{\frac{d}{2}}$ determine the remaining coefficients. The coefficients $a_1, \ldots, a_{\frac{d}{2}-1}$ can be chosen arbitrarily from $\mathbb{F}_q$. By (3.1) we must have $a_{\frac{d}{2}} = a_0^{-1} a_{\frac{d}{2}}$. If $a_0 = 1$ then $a_{\frac{d}{2}}$ is arbitrary. If $a_0 \neq 1$ then we must have $a_{\frac{d}{2}} = 0$. □

Note that $n_{q,0} = 1$ because the constant polynomial $f(t) = 1$ is self-reciprocal.

Lemma 3.3. *The set of self-reciprocal polynomials in $\mathbb{F}_{q^2}[t]$ that are products of u-irreducible polynomials is precisely the set of self-reciprocal polynomials in $\mathbb{F}_q[t]$.*

Proof. Suppose $f(t) \in \mathbb{F}_{q^2}[t]$ is self-reciprocal and is a product of u-irreducible polynomials. Then the set of roots of f is invariant under the maps $\alpha \mapsto \alpha^{-q}$ and $\alpha \mapsto \alpha^{-1}$, so it is invariant under $\alpha \mapsto \alpha^q$. So $f(t) \in \mathbb{F}_q[t]$. We claim that every self-reciprocal polynomial in $\mathbb{F}_q[t]$ is a product of u-irreducible polynomials. Let $g(t)$ be a self-reciprocal polynomial in $\mathbb{F}_q[t]$. Then the set of roots of g is invariant under the maps $\alpha \mapsto \alpha^{-1}$, $\alpha \mapsto \alpha^q$ so it is invariant under the map $\alpha \mapsto \alpha^{-q}$. So g is a product of u-irreducible polynomials. □

Corollary 3.4. *Let $n(u)_{q,d}$ denote the number of self-reciprocal polynomials in $\mathbb{F}_{q^2}[t]$ of degree d that are products of u-irreducible polynomials. Then*

$$n(u)_{q,d} = n_{q,d}.$$

3.2 ζ-Self-Reciprocal Polynomials

Fix a non-square $\zeta \in k^\times$. Given $f(t) = \sum_{0 \le i \le n} a_i t^i \in k[t]$, we define the ζ-reciprocal polynomial $\breve{f}(t)$ as

$$\breve{f}(t) = \sum_{0 \le i \le n} \left(\frac{a_0 a_{n-i}}{\zeta^i} \right) t^i = a_0 \left(\frac{t}{\zeta} \right)^n f\left(\frac{\zeta}{t} \right). \tag{3.2}$$

We say that $f(t)$ is ζ-self reciprocal if $f(t) = \breve{f}(t)$.

Lemma 3.5. *The set of roots of a monic polynomial $f(t)$ is invariant under the map $\alpha \mapsto \zeta\alpha^{-1}$ if and only if $f(t) = \breve{f}(t)$.*

Proof. Let $M = \overline{k}^\times$. Then each monic $f \in k[t]$ is described by Eqs. 2.21-2.24 where each $b_i \in M$ and these are the additive inverses of the roots of f. Suppose the roots of f are invariant under $\alpha \mapsto \zeta\alpha^{-1}$. Then applying (2.24) we have

$$a_i = a_0 \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{1}{b_j} = a_0 \sum_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J|=i}} \prod_{j \in J} \frac{b_j}{\zeta} = \frac{a_0 a_{n-i}}{\zeta^i}$$

so $f(t) = \breve{f}(t)$.

Now suppose $f(t) = \breve{f}(t) = a_0 \left(\frac{t}{\zeta}\right)^n f\left(\frac{\zeta}{t}\right)$. Then $\breve{f}(\alpha) = f(\alpha) = 0$ if and only if $f(\frac{\zeta}{\alpha}) = 0$, so the set of roots is fixed by the map $\alpha \mapsto \zeta\alpha^{-1}$. □

Lemma 3.6 ([GS11b, Lemma 2.2])**.** *The number $n_{q,d}^{\zeta}$ of monic ζ-self-reciprocal polynomials in $\mathbb{F}_q[t]$ of degree d is given by*

$$n_{q,d}^{\zeta} = \begin{cases} n_{q,d} & d \text{ is even} \\ 0 & d \text{ is odd.} \end{cases} \tag{3.3}$$

Proof. Let $f(t) = \sum_{0 \le i \le d} a_i t^i$ be monic and ζ-self-reciprocal. Then by (3.2) we must have $a_0^2 = \zeta^d$. By construction ζ is a non-square, so if d is odd this is a contradiction and we must have $n_{q,d}^{\zeta} = 0$. If d is even then $a_0 \in \{-\zeta^{\frac{d}{2}}, \zeta^{\frac{d}{2}}\}$, and if q is even then $a_0 = \zeta^{\frac{d}{2}} = -\zeta^{\frac{d}{2}}$.

The coefficients $a_0, a_1, \ldots, a_{\frac{d}{2}}$ determine the remaining coefficients. The coefficients $a_1, \ldots, a_{\frac{d}{2}-1}$ can be chosen arbitrarily from $\mathbb{F}_q$. By (3.2) we must have $a_{\frac{d}{2}} = a_0 a_{\frac{d}{2}} \zeta^{-\frac{d}{2}}$. If $a_0 = \zeta^{\frac{d}{2}}$ then $a_{\frac{d}{2}}$ is arbitrary. If $a_0 \neq \zeta^{\frac{d}{2}}$ then we must have $a_{\frac{d}{2}} = 0$. □

Lemma 3.7. *Let $n(u)_{q,d}^{\zeta}$ denote the number of monic ζ-self reciprocal polynomials of degree d in $\mathbb{F}_{q^2}[t]$ that are products of u-irreducible polynomials. Then*

$$n(u)_{q,d}^{\zeta} = n_{q,d}^{\zeta}.$$

Proof. We showed in Lemma 3.6 that there are no ζ-self-reciprocal polynomials of odd degree. So $n(u)_{q,d}^{\zeta} = n_{q,d}^{\zeta} = 0$ for d odd.

Suppose d is even. A monic polynomial $f(t) \in \mathbb{F}_{q^2}[t]$ is a product of u-irreducible polynomials and is ζ-self reciprocal if and only if $f(t) = \tilde{f}(t) = \breve{f}(t)$. So we must have

$$\left(\frac{a_i}{a_0}\right)^q = \frac{a_0 a_i}{\zeta^{d-i}} \tag{3.4}$$

and

$$a_i = \frac{a_0 a_{d-i}}{\zeta^i} \tag{3.5}$$

for all $1 \le i \le d$. From (3.5) we must have $a_0 \in \{-\zeta^{\frac{d}{2}}, \zeta^{\frac{d}{2}}\}$. The coefficients $1 \le i \le \frac{d}{2}$ must be chosen to satisfy (3.4), and the remaining coefficients are then determined by (3.5).

For all $0 < i < d$, $a_i = 0$ satisfies (3.4). If $a_i \neq 0$, then (3.4) gives

$$a_i^{q-1} = \frac{a_0^{q+1}}{\zeta^{d-i}} \tag{3.6}$$

which has $q - 1$ distinct solutions.

From (3.5) we must have

$$a_{\frac{d}{2}} = \frac{a_0 a_{\frac{d}{2}}}{\zeta^{\frac{d}{2}}} \tag{3.7}$$

so if $a_0 \neq \zeta^{\frac{d}{2}}$, then $a_{\frac{d}{2}} = 0$.

For $a_0 = \zeta^{\frac{d}{2}}$, there are q possible choices for each of the coefficients $1 \le i \le \frac{d}{2}$ so there are $q^{\frac{d}{2}}$ possible polynomials. Thus $n(u)^{\zeta}_{q,d} = q^{\frac{d}{2}}$ for even q. For $a_0 \neq \zeta^{\frac{d}{2}}$, there are q possible choices for each of the coefficients $1 \le i < \frac{d}{2}$ and we must have $a_{\frac{d}{2}} = 0$, so there are $q^{\frac{d}{2}-1}$ possible polynomials. Thus $n(u)^{\zeta}_{q,d} = q^{\frac{d}{2}} + q^{\frac{d}{2}-1}$ for odd q. So $n(u)^{\zeta}_{q,d} = n^{\zeta}_{q,d}$ for all q, d. $\square$

Chapter 4

Real Conjugacy Classes in $\mathrm{GL}_n(q)$ and $\mathrm{PGL}_n(q)$

In this chapter, we prove that the number of real $\mathrm{GL}_n(q)$-conjugacy classes contained in $\mathrm{SL}_n(q)$ equals the number of real $\mathrm{PGL}_n(q)$-conjugacy classes. In [GS11b] and [GS11a], Gill and Singh proved this result for q even or n odd. We show that the result holds for q odd by calculating generating functions for the two quantities and showing that they are equal.

4.1 Real Conjugacy Classes in $\mathrm{GL}_n(q)$

In [GS11b], Gill and Singh showed how to count the real conjugacy classes of $\mathrm{GL}_n(q)$. In this section, we outline their results. Recall that $n_{q,k}$ is the number of monic self-reciprocal polynomials of degree k.

Lemma 4.1. *A generating function for $n_{q,k}$ is*

$$\sum_{k=0}^{\infty} n_{q,k} x^k = \frac{(1+x)^{(2,q-1)}}{1-qx^2}. \tag{4.1}$$

Proof. For $k > 0$, $n_{q,k}$ is given by Lemma 3.2. For $k = 0$, $n_{q,0} = 1$.

Case 1: q is odd. Then

$$\sum_{k=0}^{\infty} n_{q,k} x^k = 1 + \sum_{k=0}^{\infty} 2q^k x^{2k+1} + \sum_{k=1}^{\infty} (q^k + q^{k-1}) x^{2k} = \frac{(1+x)^2}{1-qx^2}. \tag{4.2}$$

Case 2: q is even. Then

$$\sum_{k=0}^{\infty} n_{q,k} x^k = 1 + \sum_{k=0}^{\infty} q^k x^{2k+1} + \sum_{k=1}^{\infty} q^k x^{2k} = \frac{1+x}{1-qx^2}. \tag{4.3}$$

□

Lemma 4.2 ([GS11b, Proposition 3.7])**.** *An element $g \in \mathrm{GL}_n(q)$ is real (resp. ζ-real) if and only if each polynomial u_i in the sequence parametrizing the conjugacy class of g is self-reciprocal (resp. ζ-self-reciprocal).*

Thus, to count the real conjugacy classes of $\mathrm{GL}_n(q)$, we must replace p_{q,m_i} with n_{q,m_i} in our calculations from Sec. 2.2.

Corollary 4.3. *The number of real $GL_n(q)$ conjugacy classes of type ν is given by*

$$gl_\nu = \prod_{i \in \mathbb{N}} n_{q,m_i}. \tag{4.4}$$

Theorem 4.4. *A generating function for the number of $GL_n(q)$-real conjugacy classes is*

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} gl_\nu = \prod_{i=1}^{\infty} \frac{(1+x^i)^{(2,q-1)}}{1-qx^{2i}}. \tag{4.5}$$

Proof. We obtain a generating function for the total number of $\mathrm{GL}_n(q)$-real conjugacy classes by inserting the factors of $n_{q,k}$ into the expansion of the partition function (2.17) and replacing x with x^i in (4.1):

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} gl_\nu = \sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} \prod_{i \in \mathbb{N}} n_{q,m_i} = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} (x^i)^k n_{q,k} = \prod_{i=1}^{\infty} \frac{(1+x^i)^{(2,q-1)}}{1-qx^{2i}}. \tag{4.6}$$

$\square$

4.2 Real $\mathrm{GL}_n(q)$ Conjugacy Classes Contained in $\mathrm{SL}_n(q)$

Lemma 4.5 ([GS11b, Proposition 4.1]). *Let ν be a partition of n. Then the total number of real $\mathrm{GL}_n(q)$ conjugacy classes of type ν contained in $\mathrm{SL}_n(q)$ is given by*

$$sl_\nu = \begin{cases} \frac{1}{2}\left(\prod_{i \text{ odd}} n_{q,m_i} + \prod_{i \text{ odd}} n'_{q,m_i}\right) \prod_{i \text{ even}} n_{q,m_i} & q \text{ is odd and } m_i \text{ is even for all } i \text{ odd} \\ \frac{1}{2} \prod_{i \in \mathbb{N}} n_{q,m_i} & q \text{ is odd and there exists } i \text{ odd with } m_i \text{ odd} \\ \prod_{i \in \mathbb{N}} n_{q,m_i} & q \text{ is even} \end{cases} \tag{4.7}$$

where

$$n'_{q,2k} = \begin{cases} 1 & k = 0 \\ q^k - q^{k-1} & k > 0. \end{cases} \tag{4.8}$$

Proof. Let C be a conjugacy class of type ν given by $u = (u_1(t), u_2(t), \ldots)$ where each u_i is monic and $c_i = u_i(0)$ denotes the constant term of u_i. Suppose that C is real and so u_i is self-reciprocal for all i. Then $c_i \in \{-1, 1\}$ for all i. Note that if $m_i = 0$ then $u_i(t) = 1$ and so $c_i = 1$.

The determinant of a matrix is the product of the eigenvalues. The characteristic polynomial of the conjugacy class C is given by $\prod_{i \in \mathbb{N}} u_i(t)^i$. The zeroes of the characteristic polynomial are the eigenvalues, so the constant term of the monic characteristic polynomial is the product of the negative eigenvalues. Then $\det(g) = (-1)^n \prod_{i \in \mathbb{N}} c_i^i \in \{-1, 1\}$ for all $g \in C$. If q is even, then $\mathbb{F}_q$ has characteristic 2 and so all real $\mathrm{GL}_n(q)$-conjugacy classes lie in $\mathrm{SL}_n(q)$.

Suppose q is odd and there is some part j with jm_j odd. There are $\prod_{i \neq j} n_{q,m_i}$ possibilities for $\frac{u}{u_j} = \prod_{i \neq j} u_i$, all of which have $(-1)^n \prod_{i \neq j} c_i^i \in \{-1, 1\}$. There are $n_{q,m_j} = 2q^{\frac{m_j - 1}{2}}$

possibilities for u_j, half with $c_j = 1$ and half with $c_j = -1$. So for any possible choice of $\frac{u}{u_j}$, exactly half of the possible choices for u_j will give determinant 1. So the number of possible sequences of u_i with determinant 1 is given by $\frac{1}{2} n_{q,m_j} \prod_{i \neq j} n_{q,m_i} = \frac{1}{2} \prod_{i \in \mathbb{N}} n_{q,m_i}$.

Suppose q is odd and all odd parts have even multiplicity. Then n must be even, so we must have $\det(g) = \prod_{i \text{ odd}} c_i^i = 1$. The number of possibilities for the subsequence of u_i corresponding to odd i is given by

$$\prod_{i \text{ odd}} n_{q,m_i} = (q+1)^r \prod_{\substack{i \text{ odd} \\ m_i > 0}} q^{\frac{m_i}{2} - 1}$$

where r is the number of odd i with $m_i > 0$. The factor of $q+1$ in n_{q,m_i} corresponds to the number of choices for the coefficient $a_{\frac{m_i}{2}}$. The q corresponds to the case where $a_0 = 1$ and $a_{\frac{m_i}{2}}$ is arbitrary. The 1 corresponds to the case where $a_0 = -1$ and $a_{\frac{m_i}{2}}$ must be zero. We must count the number $f_\nu(q)$ of subsequences with an even number of constant terms equal to -1. This is equivalent to counting the number of terms in the expansion of $(q+1)^r$ in which 1 occurs an even number of times. Then $f_\nu(q) = \frac{1}{2}\left((q+1)^r + (q-1)^r\right)$ and the number of possible choices of u with determinant 1 is given by

$$f_\nu(q) \prod_{\substack{i \text{ odd} \\ m_i > 0}} q^{\frac{m_i}{2} - 1} \prod_{i \text{ even}} n_{q,m_i} = \frac{1}{2} \left(\prod_{i \text{ odd}} n_{q,m_i} + \prod_{i \text{ odd}} n'_{q,m_i} \right) \prod_{i \text{ even}} n_{q,m_i}.$$

□

Lemma 4.6. *When q is odd, a generating function for the number of real $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$ is*

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu| = n} sl_\nu = \frac{1}{2} \left(\prod_{i=1}^{\infty} (1 + x^{2i}) + \prod_{i=1}^{\infty} (1 + x^i)^2 \right) \prod_{i=1}^{\infty} \frac{1}{1 - qx^{2i}}. \tag{4.9}$$

Proof. We will calculate a generating function for each of the cases in Lemma 4.5, and then combine these to obtain the result.

Case 1:

Let $p_1(n)$ denote the number of partitions of n with even multiplicity for all odd parts. A generating function for $p_1(n)$ is

$$\sum_{n=0}^{\infty} p_1(n)x^n = \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i})^k \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i-1})^{2k}. \tag{4.10}$$

Inserting the factors of $n_{q,k}$ into the expansion of (4.10) and using (4.6) we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ im_i \text{ even } \forall i}} sl_\nu = \frac{1}{2}\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ im_i \text{ even } \forall i}} \left(\prod_{i \text{ odd}} n_{q,m_i} + \prod_{i \text{ odd}} n'_{q,m_i}\right)\prod_{i \text{ even}} n_{q,m_i}$$

$$= \frac{1}{2}\left(\prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i-1})^{2k} n_{q,2k} + \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i-1})^{2k} n'_{q,2k}\right)\prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i})^k n_{q,k}. \tag{4.11}$$

We have

$$\sum_{k=0}^{\infty}(x^{2i-1})^{2k} n_{q,2k} = \sum_{k=0}^{\infty} q^k (x^{2i-1})^{2k} + \sum_{k=1}^{\infty} q^{k-1}(x^{2i-1})^{2k} = \frac{1+x^{4i-2}}{1-qx^{4i-2}} \tag{4.12}$$

and

$$\sum_{k=0}^{\infty}(x^{2i-1})^{2k} n'_{q,2k} = \sum_{k=0}^{\infty} q^k (x^{2i-1})^{2k} - \sum_{k=1}^{\infty} q^{k-1}(x^{2i-1})^{2k} = \frac{1-x^{4i-2}}{1-qx^{4i-2}}. \tag{4.13}$$

Inserting (4.6), (4.12), and (4.13) into (4.11) we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ im_i \text{ even } \forall i}} sl_\nu = \frac{1}{2}\left(\prod_{i=1}^{\infty}\frac{1+x^{4i-2}}{1-qx^{4i-2}} + \prod_{i=1}^{\infty}\frac{1-x^{4i-2}}{1-qx^{4i-2}}\right)\prod_{i=1}^{\infty}\frac{(1+x^{2i})^2}{1-qx^{4i}}$$

$$= \frac{1}{2}\left(\prod_{i=1}^{\infty}(1+x^{4i-2}) + \prod_{i=1}^{\infty}(1-x^{4i-2})\right)\prod_{i=1}^{\infty}\frac{(1+x^{2i})^2}{1-qx^{2i}}. \tag{4.14}$$

Case 2:

Let $p_2(n)$ denote the number of partitions of n with im_i odd for at least one part i. A generating function for $p_2(n)$ is

$$\sum_{n=0}^{\infty} p_2(n)x^n = \sum_{n=0}^{\infty} p(n)x^n - \sum_{n=0}^{\infty} p_1(n)x^n = \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^i)^k - \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i})^k \sum_{k=0}^{\infty}(x^{2i-1})^{2k}. \quad (4.15)$$

Inserting the factors of $n_{q,k}$ into the expansion of (4.15) and using (4.6) we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ \exists\ i \text{ with } im_i \text{ odd}}} sl_\nu = \frac{1}{2}\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ \exists\ i \text{ with } im_i \text{ odd}}} \prod_{i\in\mathbb{N}} n_{q,m_i}$$

$$= \frac{1}{2}\left(\prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^i)^k n_{q,k} - \prod_{i=1}^{\infty}\sum_{k=0}^{\infty}(x^{2i})^k n_{q,k}\sum_{k=0}^{\infty}(x^{2i-1})^{2k} n_{q,2k}\right)$$

$$= \frac{1}{2}\left(\prod_{i=1}^{\infty}\frac{(1+x^i)^2}{1-qx^{2i}} - \prod_{i=1}^{\infty}\frac{(1+x^{2i})^2}{1-qx^{4i}}\frac{1+x^{4i-2}}{1-qx^{4i-2}}\right)$$

$$= \frac{1}{2}\left(\prod_{i=1}^{\infty}(1+x^i)^2 - \prod_{i=1}^{\infty}(1+x^{2i})^2(1+x^{4i-2})\right)\prod_{i=1}^{\infty}\frac{1}{1-qx^{2i}}. \quad (4.16)$$

Combining (4.14) and (4.16) we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} sl_\nu = \frac{1}{2}\left(\prod_{i=1}^{\infty}(1+x^{2i})^2(1-x^{4i-2}) + \prod_{i=1}^{\infty}(1+x^i)^2\right)\prod_{i=1}^{\infty}\frac{1}{1-qx^{2i}}. \quad (4.17)$$

Using the identity

$$\prod_{i=1}^{\infty}(1+x^{2i})(1-x^{4i-2}) = \prod_{i=1}^{\infty}\frac{(1-x^{4i})(1-x^{4i-2})}{1-x^{2i}} = \prod_{i=1}^{\infty}\frac{1-x^{2i}}{1-x^{2i}} = 1. \quad (4.18)$$

we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} sl_\nu = \frac{1}{2}\left(\prod_{i=1}^{\infty}(1+x^{2i}) + \prod_{i=1}^{\infty}(1+x^i)^2\right)\prod_{i=1}^{\infty}\frac{1}{1-qx^{2i}}. \tag{4.9}$$

□

4.3 Real Conjugacy Classes in $\mathrm{PGL}_n(q)$

Theorem 4.7. ([GS11a] Theorem 2.8) *The number of real conjugacy classes of type ν in $\mathrm{PGL}_n(q)$ is given by*

$$pgl_\nu = \frac{1}{2^{\sigma_\nu}} \prod_{i \in \mathbb{N}} n_{q,m_i} \tag{4.19}$$

where

$$\sigma_\nu = \begin{cases} 0 & \text{if } |q \gcd(m_1, m_2, \ldots)|_2 > 1 \\ 1 & \text{if } |q \gcd(m_1, m_2, \ldots)|_2 = 1. \end{cases} \tag{4.20}$$

When q is even, Lemma 4.5 and Theorem 4.7 give $sl_\nu = pgl_\nu = gl_\nu$. Thus we have the following corollary:

Corollary 4.8. *When q is even or n is odd, $pgl_\nu = sl_\nu$ for all $|\nu| = n$, and the number of real conjugacy classes in $\mathrm{PGL}_n(q)$ is equal to the number of real $\mathrm{GL}_n(q)$-conjugacy classes contained in $\mathrm{SL}_n(q)$.*

Corollary 4.8 was observed by Gill and Singh in [GS11a].

Lemma 4.9. *When q is odd, a generating function for the number of real conjugacy classes contained in $\mathrm{PGL}_n(q)$ is*

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} pgl_\nu = \frac{1}{2}\left(\prod_{i=1}^{\infty}(1+x^{2i}) + \prod_{i=1}^{\infty}(1+x^i)^2\right)\prod_{i=1}^{\infty}\frac{1}{1-qx^{2i}}. \tag{4.21}$$

Proof. We wish to calculate a generating function for

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} pgl_\nu = \sum_{n=0}^{\infty} x^n \left(\sum_{\substack{|\nu|=n \\ \text{all } m_i \text{ are even}}} \prod_{i\in\mathbb{N}} n_{q,m_i} + \frac{1}{2} \sum_{\substack{|\nu|=n \\ \exists\, m_i \text{ odd}}} \prod_{i\in\mathbb{N}} n_{q,m_i} \right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} x^n \left(\sum_{\substack{|\nu|=n \\ \text{all } m_i \text{ are even}}} \prod_{i\in\mathbb{N}} n_{q,m_i} + \sum_{|\nu|=n} \prod_{i\in\mathbb{N}} n_{q,m_i} \right).$$

Let $p_{\text{even}}(n)$ denote the number of partitions of n with all even multiplicities. A generating function for $p_{\text{even}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\text{even}}(n) x^n = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} (x^i)^{2k}. \tag{4.22}$$

Inserting the factors of $n_{q,k}$ into (4.22) and using (4.6) we obtain

$$\sum_{n=0}^{\infty} x^n \sum_{\substack{|\nu|=n \\ \text{all } m_i \text{ are even}}} \prod_{i\in\mathbb{N}} n_{q,m_i} = \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} (x^i)^{2k} n_{q,2k} = \prod_{i=1}^{\infty} \frac{1+x^{2i}}{1-qx^{2i}}. \tag{4.23}$$

Combining (4.23) with (4.6) we obtain the result

$$\sum_{n=0}^{\infty} x^n \sum_{|\nu|=n} pgl_\nu = \frac{1}{2} \left(\prod_{i=1}^{\infty} (1+x^{2i}) + \prod_{i=1}^{\infty} (1+x^i)^2 \right) \prod_{i=1}^{\infty} \frac{1}{1-qx^{2i}}. \tag{4.21}$$

$\square$

Theorem 4.10. *For all n, q the number of real $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$ is equal to the number of real $\mathrm{PGL}_n(q)$ conjugacy classes.*

Proof. The case for even q is given by Corollary 4.8. The case for odd q follows from the equality of the generating functions in Lemmas 4.6 and 4.9. $\square$

Chapter 5

Real Conjugacy Classes in $\mathrm{U}_n(q)$ and $\mathrm{PGU}_n(q)$

In this chapter, we show that the number of real $\mathrm{U}_n(q)$-conjugacy classes contained in $\mathrm{SU}_n(q)$ equals the number of real $\mathrm{GL}_n(q)$-conjugacy classes contained in $\mathrm{SL}_n(q)$, and the number of real $\mathrm{PGU}_n(q)$ conjugacy classes equals the number of real $\mathrm{PGL}_n(q)$ conjugacy classes. Combining these results with Theorem 4.10, we show that all four sets of conjugacy classes have equal size.

5.1 Real Conjugacy Classes in $\mathrm{U}_n(q)$

Let gu_ν denote the number of real conjugacy classes of type ν in $\mathrm{U}_n(q)$ and let su_ν denote the number of real conjugacy classes of type ν in $\mathrm{U}_n(q)$ that are contained in $\mathrm{SU}_n(q)$. In Lemma 3.3, we showed that the set of self-reciprocal polynomials in $\mathbb{F}_{q^2}[t]$ that are products of u-irreducible polynomials is exactly the set of self-reciprocal polynomials in $\mathbb{F}_q[t]$. Thus we have the following corollary.

Corollary 5.1. *For any partition ν, $gu_\nu = gl_\nu$ and $su_\nu = sl_\nu$.*

5.2 Real Conjugacy Classes in $\mathrm{PGU}_n(q)$

In [GS11a], Gill and Singh counted the real conjugacy classes of $\mathrm{PGL}_n(q)$. In this section, we use identical methods to count the real conjugacy classes of $\mathrm{PGU}_n(q)$.

5.2.1 Reality in $\mathrm{PGU}_n(q)$

For a group G with center Z, we say that $gZ \in G/Z$ lifts to $h \in G$ if $hZ = gZ$. We say that h projects onto gZ if $hZ = gZ$. For the remainder of this section, Z will refer to the center of the unitary group, $Z = Z(\mathrm{U}_n(q))$.

Let $\mathcal{U}_q = \{\eta \in \mathbb{F}_{q^2} : \eta^{q+1} = 1\} = \{\eta \in \mathbb{F}_{q^2} : \eta I_n \in \mathrm{U}_n(q)\}$ denote the set of unitary scalars. We will abuse notation and write η to mean ηI_n.

In [GS11b], Gill and Singh introduced the concept of ζ-reality. For a fixed non-square $\zeta \in \mathcal{U}_q$, we say $g \in \mathrm{U}_n(q)$ is ζ-real if g is conjugate to ζg^{-1} in $\mathrm{U}_n(q)$.

Lemma 5.2. *Fix a non-square $\zeta \in \mathcal{U}_q$. If $gZ \in \mathrm{PGU}_n(q)$ is real, then gZ lifts to a real or ζ-real element in $\mathrm{U}_n(q)$.*

Proof. If gZ is real, then there is some $x \in \mathrm{PGU}_n(q)$ such that $xgx^{-1}Z = g^{-1}Z$, so $xgx^{-1}g = \eta$ for some $\eta \in Z$. If $\eta = \lambda^2$ is a square, then $\lambda^{-1}xgx^{-1} = \lambda g^{-1}$ and gZ lifts to the real element $\lambda^{-1}g$. If η is not a square then $\eta = \zeta\lambda^2$ for some $\lambda \in \mathcal{U}_q$. Then $x\lambda^{-1}gx^{-1} = \zeta\lambda g^{-1}$ and gZ lifts to the ζ-real element $\lambda^{-1}g$. $\square$

If the conjugacy class of $g \in \mathrm{U}_n(q)$ is parametrized by the sequence of polynomials $(u_1(t), u_2(t), \ldots)$, then the conjugacy class of $\eta^{-1}g$ is parametrized by the monic scalar multiples of $(u_1(\eta t), u_2(\eta t), \ldots)$. Thus the conjugacy class of $gZ \in \mathrm{PGU}_n(q)$ is the union of the conjugacy classes parametrized by the sequences in the set $\{(\frac{1}{\eta^{n_1}}u_1(\eta t), \frac{1}{\eta^{n_2}}u_2(\eta t), \ldots) : \eta \in \mathcal{U}_q\}$ where n_i is the degree of u_i, and counting the real conjugacy classes of $\mathrm{PGU}_n(q)$ is equivalent to counting the distinct orbits of sequences $(u_1(t), u_2(t), \ldots)$ which parametrize real or ζ-real classes in $\mathrm{U}_n(q)$.

5.2.2 Equivalence Classes of Real and ζ-Real Polynomials

Let T_n denote the set of self-reciprocal polynomials in $\mathbb{F}_{q^2}[t]$ that are products of u-irreducible polynomials. Let S_n denote the set of ζ-self-reciprocal polynomials in $\mathbb{F}_{q^2}[t]$ that are products of u-irreducible polynomials. Recall from Lemma 4.2 that an element $g \in \mathrm{U}_n(q) \leq \mathrm{GL}_n(q^2)$ is real (resp. ζ-real) if and only if each polynomial u_i in the sequence parametrizing the conjugacy class of g is self-reciprocal (resp. ζ-self-reciprocal).

We say $f(t)$ and $g(t)$ are equivalent if $f(t) = \frac{1}{\eta^n} g(\eta t)$ for $\eta \in \mathcal{U}_q$, where n is the degree of f and g. Let $[f] = \{\frac{1}{\eta^n} f(\eta t) : \eta \in \mathcal{U}_q\}$ denote the equivalence class of f. Following the notation in [GS11a], let $[f]_T = [f] \cap T_n$ and $[f]_S = [f] \cap S_n$.

For $k \in \mathbb{Z}$, let $|k|_2$ denote the largest power of 2 which divides k.

Lemma 5.3. *If $f(t) = \sum_{0 \leq i \leq n} a_i t^i$ and $\{f(t), \frac{1}{\eta^n} f(\eta t)\} \subset T_n$, then $|\eta| \Big| 2k$ for all k such that $a_k \neq 0$.*

Proof. Let $g(t) = \frac{1}{\eta^n} f(\eta t) = \sum_{0 \leq i \leq n} \frac{a_i t^i}{\eta^{n-i}} \in T_n$. Then $g(t)$ has constant term $g(0) = \frac{a_0}{\eta^n}$ so we must have $\eta^n \in \{-1, 1\}$ because f and g are self-reciprocal. Then we must have $\frac{a_i}{\eta^{n-i}} = \frac{a_{n-i}}{g(0)\eta^i} = \frac{\eta^{n-i} a_{n-i}}{a_0} = a_i \eta^{n-i}$ for all $0 \leq i \leq n$. Then we must have $\eta^{2k} = \eta^{2n} = 1$ for all k such that $a_k \neq 0$. So $|\eta| \Big| 2k$ for all such k. □

Lemma 5.4. *If $f(t) = \sum_{0 \leq i \leq n} a_i t^i$ and $\{f(t), \frac{1}{\eta^n} f(\eta t)\} \subset S_n$, then $|\eta| \Big| 2k$ for all k such that $a_k \neq 0$.*

Proof. Let $g(t) = \frac{1}{\eta^n} f(\eta t) = \sum_{0 \leq i \leq n} \frac{a_i t^i}{\eta^{n-i}} \in S_n$. Then $g(t)$ has constant term $g(0) = \frac{a_0}{\eta^n}$ and g is ζ-self-reciprocal so $\frac{a_i}{\eta^{n-i}} = \frac{a_{n-i} g(0)}{\eta^i \zeta^i} = \frac{a_{n-i} a_0}{\eta^{n+i} \zeta^i} = \frac{a_i}{\eta^{n+i}}$ for all $0 \leq i \leq n$. Then we must have $\eta^{2k} = 1$ for all k such that $a_k \neq 0$. So $|\eta| \Big| 2k$ for all such k. □

Lemma 5.5. *If q is even then $[f]_T$ and $[f]_S$ contain at most one element. If q is odd then $[f]_T$ and $[f]_S$ contain at most two elements.*

Proof. Let $X \in \{T, S\}$. Take $f(t) \in X_n$ and $\eta \in \mathcal{U}_q$ such that $\frac{1}{\eta^n} f(\eta t) \in X_n$. Then by Lemmas 5.3 and 5.4, $|\eta| \Big| 2k$ for all k such that $a_k \neq 0$.

If q is even, then $|\eta|$ is odd because $|\eta|\Big|q+1$. Then $|\eta|\Big|k$ for all k such that $a_k \neq 0$ so $f(t) \in \mathbb{F}_{q^2}[t^{|\eta|}]$ and $\frac{1}{\eta^n}f(\eta t) = f(t)$ as required.

Suppose q is odd. If $|\eta|$ is odd then $f(t) \in \mathbb{F}_{q^2}[t^{|\eta|}]$ and $\frac{1}{\eta^n}f(\eta t) = f(t)$ as above. Suppose $|\eta|$ is even. Then $\frac{|\eta|}{2}\Big|k$ so $f(t) \in \mathbb{F}_{q^2}[t^{\frac{|\eta|}{2}}]$ and so $\frac{1}{\eta^{2n}}f(\eta^2 t) = f(t)$.

Suppose $\frac{1}{\eta^n}f(\eta t), \frac{1}{\epsilon^n}f(\epsilon t) \in X_n$ and $\frac{1}{\eta^n}f(\eta t), \frac{1}{\epsilon^n}f(\epsilon t) \neq f(t)$ for some $\eta, \epsilon \in \mathcal{U}_q$. We will show that $\frac{1}{\eta^n}f(\eta t) = \frac{1}{\epsilon^n}f(\epsilon t)$. We must have $f(t) \in \mathbb{F}_{q^2}[t^{\frac{|\eta|}{2}}] \cap \mathbb{F}_{q^2}[t^{\frac{|\epsilon|}{2}}] = \mathbb{F}_{q^2}[t^{\mathrm{lcm}\left(\frac{|\eta|}{2}, \frac{|\epsilon|}{2}\right)}]$ and $f(t) \notin \mathbb{F}_{q^2}[t^{|\eta|}] \cup \mathbb{F}_{q^2}[t^{|\epsilon|}]$. If $\big||\eta|\big|_2 < \big||\epsilon|\big|_2$ then $|\eta|\Big|\mathrm{lcm}\left(\frac{|\eta|}{2}, \frac{|\epsilon|}{2}\right)$, and vice versa. So we must have $\big||\eta|\big|_2 = \big||\epsilon|\big|_2$. Then $a\frac{|\eta|}{2} = b\frac{|\epsilon|}{2}$ for $a, b \in \mathbb{Z}$ if and only if $|a|_2 = |b|_2$. Then η and ϵ flip the sign of coefficients of $t^{a\frac{|\eta|}{2}}$ for odd a and leave the other coefficients unchanged. So $\frac{1}{\eta^n}f(\eta t) = \frac{1}{\epsilon^n}f(\epsilon t)$. □

Lemmas 5.3-5.5 are the unitary analogs of [GS11a, Lemma 2.2].

Lemma 5.6. *Let q be odd and n be even.*

1. *If $f(t) \in S_n$, then $\frac{1}{\lambda^n}f(\lambda t) \in T_n$ if and only if $f(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$ and $\lambda^2 = \eta\zeta$ for some $\eta \in \mathcal{U}_q$ with $\big||\eta|\big|_2 = |q+1|_2$.*

2. *If $f(t) \in T_n$, then $\frac{1}{\lambda^n}f(\lambda t) \in S_n$ if and only if $f(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$ and $\lambda^2 = \frac{\eta}{\zeta}$ for some $\eta \in \mathcal{U}_q$ with $\big||\eta|\big|_2 = |q+1|_2$.*

Proof. Suppose that $f(t) \in S_n$. Then

$$f(t) = a_{\frac{n}{2}}t^{\frac{n}{2}} + \sum_{i=0}^{\frac{n}{2}-1}\left[a_i t^i + \frac{a_0 a_i}{\zeta^{n-i}}t^{n-i}\right]$$

with $a_0 \in \{-\zeta^{\frac{n}{2}}, \zeta^{\frac{n}{2}}\}$ and $a_{\frac{n}{2}} = a_{\frac{n}{2}}\frac{a_0}{\zeta^{\frac{n}{2}}}$. Then

$$g(t) = \frac{1}{\lambda^n}f(\lambda t) = a_{\frac{n}{2}}\frac{t^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}} + \sum_{i=0}^{\frac{n}{2}-1}\left[\frac{a_i}{\lambda^{n-i}}t^i + \frac{a_0 a_i}{\zeta^{n-i}\lambda^i}t^{n-i}\right].$$

Then $g(0) = \frac{a_0}{\lambda^n}$. Then $g \in T_n$ if and only if

$$\frac{a_i}{\lambda^{n-i}} = \frac{a_0 a_i}{\zeta^{n-i}\lambda^i g(0)} = \frac{a_i \lambda^{n-i}}{\zeta^{n-i}}$$

for all $0 \leq i \leq n$. So for all $0 \leq i \leq n$ such that $a_i \neq 0$ we must have $(\lambda^2)^{n-i} = \zeta^{n-i}$. We have $a_i \neq 0$ if and only if $a_{n-i} \neq 0$ so we must have $\lambda^{2i} = \zeta^i$ for all i such that $a_i \neq 0$. Then $\lambda^2 = \eta\zeta$ for some η such that $\eta^i = 1$ for all i such that $a_i \neq 0$. So $|\eta| \Big| \gcd\{i : a_i \neq 0\}$. ζ is not a square so η must not be a square, so we must have $|q+1|_2 \Big| |\eta|$. So we must have $|q+1|_2 \Big| i$ for all $a_i \neq 0$ and thus $f(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$.

Suppose that $f(t) \in T_n$. Then

$$f(t) = a_{\frac{n}{2}} t^{\frac{n}{2}} + \sum_{i=0}^{\frac{n}{2}-1} \left[a_i t^i + \frac{a_i}{a_0} t^{n-i} \right]$$

with $a_0 \in \{-1, 1\}$ and $a_{\frac{n}{2}} = a_0 a_{\frac{n}{2}}$. Then

$$g(t) = \frac{1}{\lambda^n} f(\lambda t) = a_{\frac{n}{2}} \frac{t^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}} + \sum_{i=0}^{\frac{n}{2}-1} \left[\frac{a_i}{\lambda^{n-i}} t^i + \frac{a_i}{a_0 \lambda^i} t^{n-i} \right].$$

Then $g(0) = \frac{a_0}{\lambda^n}$. Then $g \in S_n$ if and only if

$$\frac{a_i}{\lambda^{n-i}} = \frac{a_i g(0)}{a_0 \lambda^i \zeta^i} = \frac{a_i}{\lambda^{i+n}\zeta^i}$$

for all $0 \leq i \leq n$. So for all $0 \leq i \leq n$ such that $a_i \neq 0$ we must have $\lambda^{2i} = \frac{1}{\zeta^i}$. Then $\lambda^2 = \frac{\eta}{\zeta}$ for some η such that $\eta^i = 1$ for all i such that $a_i \neq 0$. So $|\eta| \Big| \gcd\{i : a_i \neq 0\}$. By construction ζ is not a square so η must not be a square, so we must have $|q+1|_2 \Big| |\eta|$. So we must have $|q+1|_2 \Big| i$ for all $a_i \neq 0$ and thus $f(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$. $\square$

Lemma 5.7. *Let q be odd.*

1. *If n is odd then S_n is empty and if $f(t) \in T_n$, then $\big|[f]_T\big| = 2$.*

2. *Suppose n is even. Let $X \in \{T, S\}$.*

   (a) *If $f(t) \notin \mathbb{F}_{q^2}[t^{|q+1|_2}]$ and $f(t) \in X_n$, then $\big|[f]_X\big| = 2$ and $[f]_{X^c} = \emptyset$.*

   (b) *If $f(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$ and $f(t) \in X_n$, then $\big|[f]_X\big| = \big|[f]_{X^c}\big| = 1$.*

Proof. Let n be odd. There are no ζ-self-reciprocal polynomials of odd order, so $S_n = \emptyset$. We can partition T_n into the sets F_n and G_n defined by

$$F_n = \{f(t) \in T_n : f(0) = 1\} \qquad G_n = \{f(t) \in T_n : f(0) = -1\}. \tag{5.1}$$

For odd n, there is a bijection between F_n and G_n given by $f(t) \mapsto -f(-t)$. So for $f \in T_n$, $[f]_T = \{f(t), f(-t)\}$.

Let n be even. Let $f \in X_n$. Then $[f]_X = \{f(t), \frac{1}{\eta^n} f(\eta t)\}$ where these are not necessarily distinct. From the proof of Lemma 5.5, we may assume without loss of generality that the order of η is a power of 2. If $f(t) \in F_{q^2}[t^{|q+1|_2}]$ then $\frac{1}{\eta^n} f(\eta t) = f(t)$ so $\big|[f]_X\big| = 1$, and $\big|[f]_{X^c}\big| = 1$ by Lemma 5.6.

If $f(t) \notin \mathbb{F}_{q^2}[t^{|q+1|_2}]$ let e be the smallest power of 2 such that $f(t) \notin \mathbb{F}_{q^2}[t^e]$. There is some $\eta \in \mathcal{U}_q$ of order e. Then η acts on f by flipping the sign of the coefficients of $t^{ke/2}$ for odd k. We have $e \nmid n$ so $\frac{n}{e/2}$ is odd. So the sign of exactly one coefficient in each nonzero pair a_i, a_{n-i} is changed. Thus $\frac{1}{\eta^n} f(\eta t) \in X_n$ and $\frac{1}{\eta^n} f(\eta t) \neq f(t)$. Lemma 5.6 implies $[f]_{X^c} = \emptyset$. □

Lemmas 5.6 and 5.7 are the unitary analogs of [GS11a, Lemma 2.3].

5.2.3 Counting Real Conjugacy Classes in $\mathrm{PGU}_n(q)$

Let pgu_ν denote the number of real conjugacy classes of type ν contained in $\mathrm{PGU}_n(q)$. In this section we calculate pgu_ν for different possible cases of q and ν .

5.2.4 q is even

Every element in a finite field of characteristic 2 is a square, so there are no ζ-real conjugacy classes. By Lemma 5.5, each $\mathrm{PGU}_n(q)$-real class lifts to only one $\mathrm{U}_n(q)$-real class. So $pgu_\nu = gu_\nu$.

5.2.5 q is odd

Lemma 5.8. *If C is a real (resp. ζ-real) class in $\mathrm{U}_n(q)$ then C is equivalent to at most one other real (resp. ζ-real) class in $\mathrm{U}_n(q)$.*

Proof. Let C be a real (resp. ζ-real) class in $\mathrm{U}_n(q)$ parametrized by the sequence of polynomials $u(t) = (u_1(t), u_2(t), \ldots)$. We would like to determine the size of the set $\{(\frac{1}{\eta^{n_1}}u_1(\eta t), \frac{1}{\eta^{n_2}}u_2(\eta t), \ldots) : \eta \in \mathcal{U}_q,\ u_i(t) \in T_{n_i} \text{ (resp. } S_{n_i}) \ \forall i\}$.

Suppose the class C_η parametrized by $\frac{1}{\eta^n}u(\eta t) = (\frac{1}{\eta^n}u_1(\eta t), \frac{1}{\eta^n}u_2(\eta t), \ldots)$ is real (resp. ζ-real), so $u_i(t) \in T_{n_i}$ (resp. S_{n_i}) for all i. Let e be the largest power of 2 such that each $u_i(t) \in \mathbb{F}_{q^2}[t^{e/2}]$. If $\big||\eta|\big|_2 < e$ then $C_\eta = C$. If $\big||\eta|\big|_2 > e$ then C_η is not real by Lemmas 5.3 and 5.4. So if $C_\eta \neq C$ and C_η is real then we must have $\big||\eta|\big|_2 = e$, and from the proof of Lemma 5.5 we have $\frac{1}{\epsilon^n}u(\epsilon t) = \frac{1}{\eta^n}u(\eta t)$ for all $\epsilon \in \mathcal{U}_q$ with $\big||\epsilon|\big|_2 = \big||\eta|\big|_2$. So there is at most one distinct real (resp. ζ-real) class in $\mathrm{U}_n(q)$ which is equivalent to C. $\square$

Lemma 5.8 is the unitary analog of [GS11a, Lemma 2.5].

Lemma 5.9. *Let C be a conjugacy class of type ν in $\mathrm{U}_n(q)$ parametrized by $u(t) = (u_1(t), u_2(t), \ldots)$. Let $d(\nu) = |(n_1, n_2, \ldots)|_2$.*

1. *If $d(\nu) = 1$, then there are no ζ-real classes of type ν, and the real classes of type ν are partitioned into equivalence classes of size 2.*

2. *Suppose $d(\nu) \geq 2$.*

(a) *If C is real (resp. ζ-real) and there is some $u_i(t) \notin \mathbb{F}_{q^2}[t^{|q+1|_2}]$, then C is equivalent to one other real (resp. ζ-real) class and is not equivalent to any ζ-real (resp. real) class.*

(b) *If C is real (resp. ζ-real) and each $u_i(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$, then C is equivalent to one other ζ-real (resp. real) class and C is not equivalent to any other real (resp. ζ-real) class.*

Proof. Suppose $d(\nu) = 1$. Then n_i is odd for some i, so there can be no ζ-real classes of type ν. Suppose C is real. Since there is an odd n_i, $-u(-t) \neq u(t)$ is a distinct equivalent real class. So the real classes are partitioned into equivalence classes of size 2.

Suppose $d(\nu) \geq 2$ and let C be real (resp. ζ-real). If each $u_i(t) \in \mathbb{F}_{q^2}[t^{|q+1|_2}]$, then Lemma 5.7 implies that C is not equivalent to any other real (resp. ζ-real) class. Lemma 5.6 implies that C is equivalent to exactly one ζ-real (resp. real) class parametrized by $\frac{1}{\lambda^n} u(\lambda t)$, for λ given by Lemma 5.6.

If there is at least one $u_i \notin \mathbb{F}_{q^2}[t^{|q+1|_2}]$, then Lemma 5.6 implies that C is not equivalent to any ζ-real (resp. real) class. Let e be the largest power of 2 such that each $u_i \in \mathbb{F}_{q^2}[t^{e/2}]$ and take $\eta \in \mathcal{U}_q$ of order e. Then C is equivalent to the the distinct real (resp. ζ-real) class parametrized by $\frac{1}{\eta^n} u(\eta t)$. □

Lemma 5.9 is the unitary analog of [GS11a, Lemma 2.6].

Corollary 5.10. *Let q be odd and let $\nu = 1^{n_1} 2^{n_2} \cdots$ be a partition of n. If $d = 1$ then $pgu_\nu = \frac{1}{2} gu_\nu$. If $d > 1$ then $pgu_\nu = gu_\nu$.*

Corollary 5.11. *For any partition ν, $pgu_\nu = pgl_\nu$.*

Combining Theorem 4.10, Corollary 5.1, and Corollary 5.11, we arrive at the main theorem, which generalizes Corollary 4.8.

Theorem 5.12. *For all n, q, the following sets have equal size:*

1. *real $\mathrm{GL}_n(q)$ conjugacy classes contained in $\mathrm{SL}_n(q)$*

2. *real* $\mathrm{PGL}_n(q)$ *conjugacy classes*

3. *real* $\mathrm{U}_n(q)$ *conjugacy classes contained in* $\mathrm{SU}_n(q)$

4. *real* $\mathrm{PGU}_n(q)$ *conjugacy classes.*

Chapter 6

Further Work

Let G be a group. We say $g \in G$ is *strongly real* if there exists $x \in G$ such that $xgx^{-1} = g^{-1}$ and $x^2 = 1$. If g is strongly real, then all elements in the conjugacy class of g are strongly real, so we may refer to strongly real conjugacy classes.

Theorem 5.12 specializes Theorems 1.3 and 1.4 to the case of real conjugacy classes. All real conjugacy classes of $\mathrm{GL}_n(q)$ and $\mathrm{PGL}_n(q)$ are strongly real, so Theorem 1.3 also specializes to the strongly real case. In the case of the unitary groups, it is not true that all real conjugacy classes are strongly real. It is natural to ask whether Theorem 1.4 specializes to the strongly real case. The strongly real classes of $\mathrm{U}_n(q)$ were classified for odd q by Gates, Singh, and Vinroot in [GSV14]. This classification could be used to attempt to specialize Theorem 1.4 to the strongly real case when q is odd. The strongly real classes of $\mathrm{U}_n(q)$ have not been completely classified for even q.

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