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Minimal Principal Series Representations of SL(3,R)

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Minimal Principal Series Representations of $\mathbf{SL}(3, \mathbb{R})$

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Abstract

We discuss the properties of principal series representations of $\mathrm{SL}(3,\mathbb{R})$ induced from a minimal parabolic subgroup. We present the general theory of induced representations in the language of fiber bundles, and outline the construction of principal series from structure theory of semisimple Lie groups. For $\mathrm{SL}(3,\mathbb{R})$, we show the explicit realization a novel picture of principal series based on the nonstandard picture introduced by Kobayashi, Ørsted, and Pevzner for symplectic groups. We conclude by studying the K-types of $\mathrm{SL}(3,\mathbb{R})$ through Frobenius reciprocity, and evaluate prospects in developing simple intertwiners between principal series representations.

Chapter 1

Introduction

Groups are the natural mathematical objects to describe transformations. On any set, groups can act by switching elements around in a reversible manner. In particular, they can act on a vector space, like the collection of possible physical states of a system. When a physical process transforms one state to another, this transformation is described by an element of a group. In the presence of a symmetry, the states of a system do not change when they are transformed by a group. As a result, the language of groups allows us to classify and describe symmetries of the real world.

The way in which a group acts on a vector space is called a representation, and therefore the study of transformations and symmetries becomes a study of different kinds of representations. In this work, we will describe a particular class of representations of the the special linear group $\mathrm{SL}(3, \mathbb{R})$ called principal series. This is the group of matrix transformations in three-dimensional Euclidean space that preserve volume, like rotations.

Representations of a group can be decomposed into sums and integrals of smaller representations. The smallest building blocks of a representation are called irreducible representations. In a physical system, irreducible representations correspond to sets of physical states that are related by some sort of symmetry. Often it is difficult to study how a group acts on a large space, and we can gain insight by looking at the decomposition into irreducible representations.

Every group has a representation in which it acts on itself called the regular representation. In compact groups, the regular representation decomposes into a direct sum of all the irreducible representations, with multiplicities given by their dimension. This is a consequence of the Peter-Weyl theorem, which further implies that all representations of a compact group decompose into a direct sum of irreducibles.

In noncompact groups like $\mathrm{SL}(3, \mathbb{R})$, decomposition is more complicated. The set of irreducible representations of a noncompact group can be described by a subset called the tempered dual. For noncompact groups that commute, like the real numbers $\mathbb{R}$, decomposition into irreducible representations gives the theory of Fourier transforms. Representations in the tempered dual are those that arise in the generalization of the Plancherel theorem, which relates the integral of a function and of its Fourier transform. Our objects of study, principal series, are families of representations that form the continuous part of the tempered dual.

The two fundamental objects in representation theory are the group being represented, G, and the linear space that it acts on, V. When we decompose a representation into irreducibles, we are in effect breaking up the space V into a direct sum of subspaces. Conversely, we could imagine breaking up the group G into subgroups and leaving the space V unchanged. This is a process called restriction, and for each representation of a group we can construct a representation of any of its subgroups. The inverse process, going from a representation of a subgroup to a representation of a larger group, is called induction. Principal series representations of $\mathrm{SL}(3, \mathbb{R})$ are produced by induction from a parabolic subgroup.

Almost all principal series of $\mathrm{SL}(3, \mathbb{R})$ are irreducible and act like fundamental building blocks for larger representations of the group. However, when we restrict principal series to a subgroup, they no longer are fundamental and themselves break down into smaller representations. The way in which a basic unit of representation theory breaks down even further when the group is changed is called a branching problem.

In physics, this is analogous to symmetry breaking. When a system has a large symmetry group G, there are many states with the same properties grouped together by an irreducible representation. If some external process breaks the symmetry so that the symmetry group is reduced to a subgroup H, then the irreducible representation is further decomposed. As a result, the states that once shared physical properties become distinct. For example, this occurs when degenerate energy states of a quantum system "split" due to an external perturbation.

In the context of principal series, we study this symmetry-breaking when our group is reduced to SO(3), the group of rotations in three dimensions. This branching problem is of interest because SO(3) is the maximal compact subgroup of $\mathrm{SL}(3, \mathbb{R})$, and therefore principal series will automatically decompose into a discrete sum of irreducibles when they are restricted to this subgroup. The existence of this decomposition is guaranteed by the Peter-Weyl theorem and the units in the direct sum are called K-types.

We will begin by introducing the process of induction, through which a group representation is constructed from a representation of a subgroup. Following van den Ban [1], we review the geometrical basis for induction in fiber bundles and the normalization procedure to maintain unitarity. Motivated by structure theory of semisimple Lie groups, with a particular focus on the special case of $\mathrm{SL}(3, \mathbb{R})$ we introduce principal series by inducing from a minimal parabolic subgroup. We present three different pictures in which these representations are realized, emphasizing their comparative advantages.

Following this exposition, we consider the extension to a fourth picture for our principal series: the non-standard picture. This picture arises from the partial Fourier transform technique developed by Kobayashi, Ørsted, and Pevzner in [2]. We then study the branching problem posed by restricting the principal series for $\mathrm{SL}(n, \mathbb{R})$ to its maximal compact subgroup. We conclude by formulating an approach to calculate the multiplicities in the resulting K-type decomposition using Frobenius reciprocity. Outlining future directions, we discuss the possibility of a geometrical interpretation for the intertwiners of $\mathrm{SL}(n, \mathbb{R})$ using the approach of [2] and Olafssón and Pasquale [3].

Chapter 2

Background

Before discussing the details of principal series, we outline a few mathematical preliminaries for representation theory.

Definition 2.0.1. Let G be a group and k be a field. A *k-representation* of G is a pair (π, V), where V is a linear space over k and

$$\pi : G \to \mathrm{GL}(V)$$

is a group homomorphism. The dimensionality of the representation is said to be $\dim(V)$.

Depending on the context, we denote a representation (π, V) simply by the space V or by the homomorphism π. We also omit the field k when it is implied from the context. As a homomorphism, a representation must send each group element to an invertible linear operator on V and preserve the group action:

$$\pi(g_1)\pi(g_2)v = \pi(g_1 g_2)v \quad \forall g_1, g_2 \in G, \forall v \in V. \tag{2.1}$$

Consequently, we say that G has an *action* on V, and sometimes we omit the homomorphism in our notation:

$$\pi(g)v = g \cdot v \in V.$$

Example 2.0.1. For every group there exists the *trivial representation*, in which each element is mapped to the identity:

$$\pi(g)v = v \text{ for all } v \in V.$$

Example 2.0.2. Another example is the *regular representation*, in which group elements act on themselves by translation. It can be realized on the space of functions

$$f : G \to V,$$

with V a linear space. The action of G is defined on the left as

$$[\pi_l(g)f](x) = f(g^{-1}x),$$

where $x \in G$ is the point at which we are evaluating the transformed function, or equivalently on the right as $[\pi_r(g)]f(x) = f(xg)$.

In classifying representations, we are interested in determining whether two different representations of the same group are related. Treating representations as objects in a category like [4], we define morphisms between these objects.

Definition 2.0.2. A morphism, or *intertwiner*, between two k-representations of G, (π_1, V_1) and (π_2, V_2), is a k-linear map $\Phi : V_1 \to V_2$ that intertwines the two different actions of G:

$$\Phi(\pi_1(g)v) = \pi_2(g)(\Phi(v)) \quad \forall g \in G, \forall v \in V_1.$$

Definition 2.0.3. If an intertwiner Φ is a bijection, then it gives an *isomorphism of representations*:

$$\pi_1 \simeq \pi_2.$$

Starting from a representation of G on a space V, we can alter both the group and the space to get new representations. For example, if $H \leq G$ is a subgroup and (π, V) is a representation of G, then the *restriction*

$$\mathrm{Res}_H^G(\pi) = \pi|_H \tag{2.2}$$

is automatically a representation of H on the same vector space by Definition 2.0.1. On the other hand, we can also restrict to a subspace $W \subseteq V$ and have a representation if the action of G leaves W invariant.

Definition 2.0.4. Let G be a group with representation (π, V) and $W \subseteq V$. If $\pi(g)w \in W$ for all $g \in G$ and $w \in W$, then W is a *subrepresentation* of V.

Definition 2.0.5. A representation (π, V) is said to be *irreducible* if its only subrepresentations are $\{0\}$ and V. A representation that is not irreducible is called *reducible*.

A large class of representations are *semisimple*, meaning that they can be decomposed as a direct sum of irreducible representations. In this sense, irreducible representations function as the atomic building blocks of representations. Furthermore, these blocks are "orthogonal" in the sense that an intertwiner between two irreducible representations of a group G is either an isomorphism or zero. This result is known as Schur's lemma, and a proof of the lemma can be found in [4].

Lemma 2.0.1 (Schur's Lemma)**.** *Let G be a group and π_1, π_2 be two finite-dimensional irreducible representations. The space of intertwiners from π_1 to π_2 is defined by*

$$\dim \mathrm{Hom}_G(\pi_1, \pi_2) = \begin{cases} 1 & \textit{if } \pi_1 \simeq \pi_2 \\ 0 & \textit{if } \pi_1 \not\simeq \pi_2 \end{cases}. \tag{2.3}$$

Additionally, the self-intertwiners of an irreducible representation π must be scalar multiples of the identity:

$$\mathrm{Hom}_G(\pi, \pi) = \lambda\,\mathrm{Id}.$$

The converse also holds:

$$\dim \mathrm{Hom}_G(\pi, \pi) = 1 \;\longrightarrow\; \pi \textit{ irreducible},$$

for a finite-dimensional, semisimple representation π.

The lemma above is only true if we are working over an algebraically closed field like $\mathbb{C}$. An immediate consequence of Schur's Lemma is the following:

Corollary 2.0.1.1. *If G is an Abelian group, then any finite-dimensional irreducible representation has dimension 1.*

Proof. Let G be an Abelian group and (π, V) a finite-dimensional irreducible representation of G. Because G is Abelian, for any $g \in G$ the operator $\Phi = \pi(g)$ will commute with every element of $\pi(G)$. As a result, Φ is an intertwiner and by Schur's lemma

$$\Phi = \pi(g) = \lambda_g \, \mathrm{Id}.$$

If π were not one-dimensional, each one-dimensional subspace of V would provide a nontrivial subrepresentation, and π could not be irreducible. Therefore π has dimension 1. $\square$

If a representation space is endowed with additional structure, we can further constrain the representations by stipulating that they preserve the structure. In the case of a Hilbert space complete under an inner product the resulting representations are unitary operators.

Definition 2.0.6. A continuous representation of a topological group G on a Hilbert space $\mathcal{H}$, $(\pi, \mathcal{H})$, is called *unitary* if it preserves the complex inner product:

$$\langle \pi(g)v | \pi(g)w \rangle = \langle v | w \rangle \quad \forall g \in G, \forall v, w \in \mathcal{H}.$$

Here, a *topological group* is one in which group multiplication and inversion are continuous maps with respect to the chosen topology. Lie groups admit a topology because they are smooth manifolds, yet all groups are topological using the discrete topology. Unitary representations are useful because, in the finite-dimensional case, they are always semisimple and thus can be decomposed into a direct sum of irreducible representations.

If π is a unitary representation and $v, w \in \mathcal{H}$, then any function of the form

$$g \in G \mapsto \langle v| \, \phi(x) \, |w\rangle \in \mathbb{C} \tag{2.4}$$

is a *matrix coefficient* of π. On a finite-dimensional representation space, the matrix coefficient given by

$$\chi_\pi(g) = \mathrm{Tr}[\pi(g)] \tag{2.5}$$

is called the *character* of representation π. Characters encode much of the information associated with a representation. For example, character is invariant under conjugacy transformations $x \mapsto g^{-1}xg$. In finite groups, where the number of conjugacy classes is the number of irreducible representations, character theory then serves to classify irreducibles.

Definition 2.0.7. The *unitary dual* of a group G is the set of equivalence classes of unitary irreducible representations of G, denoted by $\widehat{G}$.

A central theme in the study of representations is the decomposition of representations into irreducibles. The unitary dual acts as a basis for the decomposition of the regular representation. For compact groups, this decomposition is described by the Peter-Weyl theorem.

Theorem 2.0.2 (Peter-Weyl)**.** *Let G be a compact group with unitary dual $\widehat{G}$. The regular representation of G on the space $L^2(G)$ decomposes as:*

$$L^2(G) = \widehat{\bigoplus_{\rho \in \widehat{G}}} \mu_\rho V_\rho, \tag{2.6}$$

where $\widehat{G}$ is the unitary dual, V_ρ is the Hilbert space of the unitary irreducible subrepresentation ρ, and we have taken the closure over the direct sum. The multiplicity of each space V_ρ is

$$\mu = \dim V_\rho,$$

and all $\rho \in \widehat{G}$ *are finite-dimensional in the decomposition. Furthermore, the matrix coefficients of the* V_ρ *form a topological basis of* $L^2(G)$*, and any unitary representation of* G *decomposes into a direct sum of finite-dimensional unitary irreducibles.*

A proof of the Peter-Weyl theorem can be found in [4]. For locally compact semisimple groups the decomposition is more complicated, involving several families of unitary irreducibles called *tempered* representations, a category that includes principal series [5].

In all cases, we see that unitary irreducible representations are akin to building blocks of larger representations. In constructing principal series, it is precisely these representations that we will begin with.

Chapter 3

Induced Representations

3.1 Fiber Bundles

The process of induction creates a representation of a group G starting from a representation of a subgroup H. For a Lie group, this construction can be described geometrically through the theory of fiber bundles. Much like a manifold is a topological space that locally looks like Euclidean space, a fiber bundle is a topological space that locally looks like a product space.

Definition 3.1.1. A *fiber bundle* is a triple (E, p, M), where E is a manifold called the *total space*, M is a manifold called the *base space*, and the *projection* is a surjective map

$$p : E \to M.$$

The inverse image of a point $x \in M$ under the projection, $p^{-1}(x) = F_x$, is a manifold called the *fiber* at x.

The base manifold has an open covering $\{U_i\}$ equipped with maps

$$\begin{aligned} \phi_i : U_i \times F \to p^{-1}(U_i), \\ p \circ \phi_i(x, f) = x. \end{aligned} \tag{3.1}$$

Here ϕ_i are diffeomorphisms and $F \simeq F_x$ is a typical fiber. These maps are called *local*

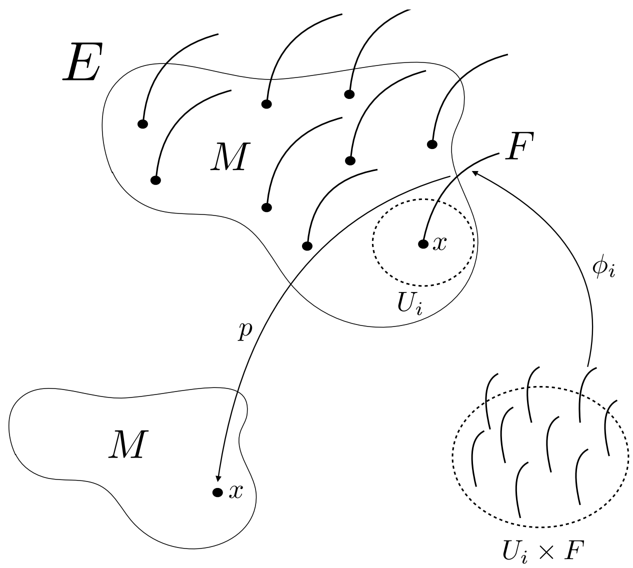

Figure 3.1: Fiber bundle E shown with base manifold M and fiber F labeled. The maps ϕ_i and p are a local trivialization into coordinate charts and projection, respectively.

trivializations [6] because for each $y \in E$, there is a neighborhood U_i of $p(y) \in M$ such that

$$\phi_i^{-1} : p^{-1}(U_i) \to U_i \times F$$

is onto, giving the fiber bundle the local structure of a trivial product space.

Two different local trivializations are related by smooth transition functions:

$$t_{ij} = \phi_i^{-1} \circ \phi_j : F \to F.$$

Such transformations live in a group G called the *structure group*, which acts on the fiber F. A diagram of a fiber bundle, with local trivialization and projection maps displayed, is shown in Fig.3.1.

A *section* of a fiber bundle is a smooth map

$$s : M \to E$$

such that $p \circ s = \text{id}$ is the identity map on M. The space of smooth sections on a bundle

E is denoted by

$$\mathcal{C}(E) = \{s : M \to E,\ p \circ s = \text{id}\}.$$

A vector bundle is a fiber bundle in which all the fibers are isomorphic to a vector space. We can imagine this as a family of vector spaces parametrized by a manifold M. With this in mind, a smooth section on a vector bundle is akin to a smooth vector-valued function on M.

3.2 Induced Representations on Vector Bundles

Let G be a Lie group and H a closed subgroup. It follows that H is a Lie subgroup and G/H is a manifold [6]. Given a finite-dimensional continuous representation (ξ, V) of H, we construct the product space $G \times V$, equipped with an action of H:

$$h \cdot (g, v) = (gh^{-1},\ \xi(h)v) \in G \times V. \tag{3.2}$$

This action defines a representation of H on the space $G \times V$. If we take the quotient over the equivalence relation

$$h \cdot (g, v) \sim (g, v), \tag{3.3}$$

we can construct the smooth manifold $G \times_H V$, in which the points (gh, v) and $(g, \xi(h)v)$ are identified for all $h \in H$.

Following this identification, we have a well-defined projection

$$\begin{aligned} p : G \times_H V &\to G/H, \\ [g, h] &\mapsto gH. \end{aligned} \tag{3.4}$$

Conversely, the fiber can be identified with the vector space V,

$$p^{-1}(gH) \simeq V, \tag{3.5}$$

as there is a canonical bijection $[g, v] \mapsto v$. Our manifold $G \times_H V$, which we will also denote as $\mathcal{V}$, is thus endowed with the structure of a vector bundle over G/H. The fibers of $\mathcal{V}$ are interwoven by the structure group H, and thus for each representation V we can construct a corresponding $\mathcal{V}$.

The left-action of G on the first coordinate in $G \times V$ preserves the equivalence relation (3.3). Consequently

$$g \cdot [\tilde{g}, v] = [g\tilde{g}, v]$$

is a homogeneous group action on $\mathcal{V}$, which is in turn a *homogeneous vector bundle*. In general, a homogeneous bundle $(\mathcal{W}, p, G/H)$ is a bundle equipped with a group action $G \curvearrowright \mathcal{W}$ satisfying two conditions [1] : G acts linearly on the fibers of $\mathcal{W}$, and

$$g\cdot : p^{-1}(xH) \mapsto p^{-1}(gxH) \text{ for all } xH \in G/H. \tag{3.6}$$

While we have shown that we can construct a homogeneous vector bundle starting from a representation of H, the converse is also true: for each homogeneous vector bundle over G/H there is an associated representation of H. One can see this by considering the restriction of the homogeneous G action on the fiber $p^{-1}(eH) \in \mathcal{W}$ to its subgroup H. For all $h \in H$,

$$h \cdot p^{-1}(eH) = p^{-1}(eH),$$

indicating that the fiber at the identity, which is isomorphic to a vector space, preserves an action of H and thus acts as a representation space. Thus, there is a correspondence between G-homogeneous vector bundles over G/H and representations of H.

From representations of H, we would like to construct representations of G. To this end, let $\mathcal{V}$ be the bundle constructed from a representation ξ of H. The group G already has a (homogeneous) action on $\mathcal{V}$, yet it translates between different fibers. Instead, we should construct an action that maps a fiber to itself, turning it into a linear representation space. Recall that a section maps each base point $x \in G/H$ to a vector in the fiber F_x, and that the regular action of G on the space of sections $\mathcal{C}(\mathcal{V})$ also translates between fibers. Combining these two actions, we construct a representation of G that acts on $\mathcal{C}(\mathcal{V})$.

Definition 3.2.1 (Geometric picture)**.** Let G be a Lie group, H a closed subgroup, and $\mathcal{V} = G \times_H V$, where V is a representation space of H. We can construct a G-representation:

$$\begin{aligned} \pi(g) &: \mathcal{C}(\mathcal{V}) \to \mathcal{C}(\mathcal{V}), \\ [\pi(g)s](xH) &= g \cdot s(g^{-1}xH), \end{aligned} \tag{3.7}$$

where $s \in \mathcal{C}(\mathcal{V})$ and $x, g \in G$. This is called the representation of G induced from (ξ, V):

$$\pi = \mathrm{ind}_H^G(\xi). \tag{3.8}$$

We see that homogeneity guarantees that the right hand side of (3.7) is an element of $p^{-1}(xH)$, and thus this representation is well-defined.

It is useful to have an equivalent construction in terms of function spaces. There is a canonical quotient map

$$G \to G/H,$$

and $p^{-1}(eH) \simeq V$, motivating the identification of sections on the homogeneous vector bundle with vector-valued functions on G. Given a section $s \in \mathcal{C}(\mathcal{V})$, we construct a continuous function:

$$f_s(g) = g^{-1} \cdot s(gH) \in V, \tag{3.9}$$

where $f_s \in \mathcal{C}(G, V)$, the space of continuous functions from G to V. As $s(gH)$ is already a map from g to $\mathcal{V}$, the target space $V \simeq p^{-1}(eH)$ is reached by a left translation $l_{g^{-1}}$. Likewise, $f \in \mathcal{C}(G, V)$ defines a section

$$s_f : gH \mapsto [g, f(g)]. \tag{3.10}$$

We notice that a function defined by (3.9) has a particular transformation property under the right action of H:

$$f_s(gh) = h^{-1}g^{-1} \cdot s(gH) = h^{-1} \cdot f_s(g).$$

Because $f_s(g) \in V$ and the homogeneous left action of H on V defines a representation ξ, we can alternatively define induced representations on the space of continuous V-valued functions on G that are H equivariant. Using (3.9) to transform the action of G from sections to the functions, we also arrive at the action in this new picture.

Definition 3.2.2 (Functional picture)**.** Let (ξ, V) be a representation of a closed subgroup $H \leq G$. The induced representation of G may be expressed in the function space

$$\mathrm{ind}_H^G(V) = \{f \in \mathcal{C}(G, V) : f(gh) = \xi(h)^{-1} f(g),\ g \in G, h \in H\}, \tag{3.11}$$

with a group action

$$[\pi(g)f](x) = f(g^{-1}x). \tag{3.12}$$

As a result, the induced representation of G has a very simple action (left-translation) on a complicated space (continuous functions obeying H-equivariance).

3.3 Density Bundles

The induced representations constructed above have a major fault: the mapping from $\xi \mapsto \mathrm{ind}_H^G(\xi)$ does not preserve unitarity. To do so, we must twist the representations by half-densities that allow us to define a G-invariant inner product.

Definition 3.3.1. Let V be an n-dimensional vector space and $T \in \mathrm{End}(V)$. A *density* is a map $\omega : V^n \to \mathbb{C}$ that transforms as

$$T^*\omega = |\det T|\, \omega$$

under the action of the pullback $T^*w := w \circ T^n$.

The space of densities on V, denoted by $\mathcal{D}V$, is one-dimensional because any element $(v_1, \cdots, v_n) \in V^n$ is mapped to a scalar multiple of $1 \in \mathbb{C}$. We can generalize our discussion to α-densities $\mathcal{D}^\alpha V$ that transform with a factor of $|\det T|^\alpha$. Multiplication of an α_1-density with an α_2-density gives rise to the isomorphism

$$\mathcal{D}^{\alpha_1} V \otimes \mathcal{D}^{\alpha_2} V \simeq \mathcal{D}^{\alpha_1+\alpha_2} V.$$

In particular, two half-densities combine to give an ordinary density.

On an n-dimensional differentiable manifold M, any point x is associated with a natural n-dimensional vector space $T_x M$, called the tangent space. Differential one-forms are maps $T_x M \to \mathbb{C}$, and in general r-forms are the antisymmetrized tensor product of r one-forms. The space of n-forms at $x \in M$ is thus one-dimensional by the antisymmetry condition, allowing an identification with the space of densities $\mathcal{D}T_x M$. More accurately, on orientable manifolds there is a nowhere vanishing n-form called a volume form. The absolute value of a volume form is a density. We can then define the density bundle $\mathcal{D}TM$ as a vector bundle with base space M and fibers $p^{-1}(x) = \mathcal{D}T_x M$. A density on M is a continuous section of the density bundle.

Let $\phi : M \to N$ be a diffeomorphisms of manifolds. The *pushforward* of ϕ is the induced map between tangent spaces

$$D\phi(x) : T_x M \to T_{\phi(x)} N. \tag{3.13}$$

This map "pushes forward" vectors from the tangent space of the original manifold to the tangent space of the target manifold. We can also define the *pullback* of ϕ, denoted by ϕ^*, which "pulls back" differential forms from N to M. In particular, we can pull back densities:

$$[\phi^*\omega](x) = (D\phi)^*\omega(\phi(x)) = \omega(\phi(x)) \circ (D\phi)^n = |\det D\phi|\, \omega(\phi(x)), \tag{3.14}$$

where $x \in M$ and the transformation factor is the Jacobian determinant. The presence of the Jacobian is precisely what allows for the integration of densities over a manifold. The general procedure for integrating densities on a manifold relies on partitioning the manifold into coordinate charts. By the transformation property of densities (3.14), the change of variables formula for integration naturally follows.

Now that we have defined densities on a manifold, we turn our attention to densities on G/H. The tangent bundle $T(G/H)$ is itself a homogeneous bundle, with a G-action on the fibers

$$g \cdot v = (Dl_g)v, \;\; v \in T_{xH}(G/H).$$

Here $l_g : G/H \to G/H$ is left translation by $g \in G$, and the pushforward is a linear map between fibers $T_{xH}(G/H) \to T_{gxH}(G/H)$. As a homogeneous bundle over G/H, the tangent bundle defines a representation of H on the fiber at the identity.

Definition 3.3.2. A *Lie algebra* is a vector space $\mathfrak{g}$ closed under a bilinear, antisymmetric map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket that satisfies the Jacobi identity:

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0, \quad \forall a,b,c \in \mathfrak{g}.$$

Definition 3.3.3. The Lie algebra of a Lie group G is the tangent space at the identity, $\mathfrak{g} = T_eG$, or equivalently the space of sections $s \in \mathcal{C}(TG)$ that are invariant under left-translations Dl_g for all $g \in G$.

Any Lie group G has a unique associated Lie algebra $\mathfrak{g}$. Conversely, the exponentiation map is a diffeomorphism from Lie algebra to Lie group:

$$\begin{aligned} \exp : \mathfrak{g} &\to G, \\ tX &\mapsto \sigma(t), \end{aligned} \tag{3.15}$$

where $\sigma : \mathbb{R} \to G$ is a curve whose tangent vector at $t = 0$ is X. For matrix groups, this map is simply the matrix exponential. Given a Lie algebra $\mathfrak{g}$, exponentiation maps onto the connected component of G containing the identity, but different Lie groups can have the same Lie algebra.

Definition 3.3.4. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider the conjugation map $\Psi(g) : x \mapsto g^{-1}xg$ for $g, x \in G$. The pushforward $D\Psi(g) : T_xG \to T_xG$ evaluated at the identity is a linear map $\mathrm{Ad} : \mathfrak{g} \to \mathfrak{g}$ called the *adjoint representation* of G.

The fiber at the identity of $T(G/H)$ is isomorphic to the quotient of Lie algebras:

$$T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h}.$$

Because left-translation and conjugation of a coset gH by $h \in H$ give the same result, the homogeneous action of H on $\mathfrak{g}/\mathfrak{h}$ is through the adjoint representation of G on $\mathfrak{g}$, restricted to H. The representation associated with the tangent bundle is then $(\xi, V) = (\mathrm{Ad}, \mathfrak{g}/\mathfrak{h})$:

$$\xi(h)(\gamma + \mathfrak{h}) = \mathrm{Ad}(h)\gamma + \mathfrak{h}, \text{ for } \gamma \in \mathfrak{g}. \tag{3.16}$$

Similarly, the density bundle $\mathcal{D}T(G/H)$ is a homogeneous vector bundle with a G-action

$$g \cdot \omega = l^*_{g^{-1}}\omega, \;\; \omega \in \mathcal{C}(\mathcal{D}T(G/H)).$$

One can see that because $l_g^{-1} : gxH \mapsto xH$, the pullback will send a density on $T_{xH}(G/H)$ to a density on $T_{gxH}(G/H)$, thus satisfying the homogeneity conditions.

Again, the fiber at the identity, $\mathcal{D}T_{eH}(G/H)$, acts as a representation space for H. We recall from (3.14) that for $\omega \in \mathcal{D}T_{eH}(G/H)$, the pullback can be written as

$$[l^*_{h^{-1}}\omega](eH) = |\det Dl_{h^{-1}}|\, \omega(eH).$$

However, we already established that when acting on $\mathfrak{g}/\mathfrak{h}$, the map $Dl_{h^{-1}} = Dl_h^{-1}$ is the adjoint representation of H. Consequently, the representation associated with the density bundle $\mathcal{D}T(G/H)$ is

$$\delta(h)\omega = |\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)|^{-1}\omega, \tag{3.17}$$

for $\omega \in \mathcal{D}T(G/H)$. Here we have swapped inversion and determinant for clarity, and we notice that this representation of H is one-dimensional. For half-densities, the representation becomes $\delta(h)^{1/2}$, which we will exploit shortly.

3.4 Normalized Induction

We are now ready to construct unitary representations of G induced from unitary representations of H. Recall from Def. 2.0.6 that a unitary representation acts on a Hilbert space endowed with a G-invariant inner product. Our aim is to construct such an inner product.

Let (ξ, V) be a unitary representation of $H \leq G$ and $(\delta^{1/2}, \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}))$ be the one-dimensional representation of H on the space of half-densities, where we have made use of the isomorphism $\mathfrak{g}/\mathfrak{h} \simeq T_{eH}(G/H)$. We can construct a tensor product representation of H,

$$\xi \otimes \delta^{1/2},$$

which in turn we induce to to a representation of G. Instead of working with sections of the complicated bundle

$$G \times_H (V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})),$$

we study the induced space in the functional picture of (3.9):

$$\mathrm{ind}_H^G(\xi \otimes \delta^{1/2}) = \{f \in \mathcal{C}(G, V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})) : f(gh) = \delta(h)^{-1/2}\xi(h)^{-1}f(g)\}. \tag{3.18}$$

We must construct an inner product between functions in this space. For $v_1, v_2 \in V$, there already exists an inner product such that

$$\langle \xi(h)v_1 | \xi(h)v_2 \rangle = \langle v_1 | v_2 \rangle \in \mathbb{C}\ \forall h \in H.$$

Furthermore, we can identify the product of two half-densities as a density in $\mathcal{D}(\mathfrak{g}/\mathfrak{h})$. We define the pairing

$$\begin{gathered}(\,\cdot\,,\,\cdot\,) : V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \times V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \to \mathcal{D}(\mathfrak{g}/\mathfrak{h}), \\ (v_1 \otimes \rho_1, v_2 \otimes \rho_2) = \langle v_1 | v_2 \rangle\, \bar{\rho}_1 \rho_2 = \langle v_1 | v_2 \rangle\, \omega.\end{gathered} \tag{3.19}$$

This form is linear in the second coordinate and conjugate linear in the first. Note that the combination of two half-densities is a density:

$$\bar{\rho}_1 \rho_2 = \omega \in \mathcal{D}(\mathfrak{g}/\mathfrak{h}),$$

and $\langle v_1|v_2\rangle$ is simply an overall complex scaling factor. Given two elements

$$f_1, f_2 \in \mathrm{ind}_H^G(\xi \otimes \delta^{1/2}),$$

the pairing (f_1, f_2) corresponds to an element of the function space $\mathcal{C}(G, \mathcal{D}(\mathfrak{g}/\mathfrak{h}))$. In turn, we may use (3.7) to identify this with a section on the homogeneous density bundle $\mathcal{D}(T(G/H))$, which we can then integrate over the manifold. We thus propose the following inner product:

$$\langle f_1|f_2\rangle = \int_{G/H} (f_1, f_2), \quad f_1, f_2 \in \mathrm{ind}_H^G(\xi \otimes \delta^{1/2}). \tag{3.20}$$

Now it remains to be checked that this preserves the action of G in the induced representation. We recall that G acts on $\mathrm{ind}_H^G(\xi \otimes \delta^{1/2})$ by the left-regular representation:

$$\pi(g)f(x) = f(g^{-1}x).$$

In the language of pullbacks, we can say that f is precomposed with a left-translation l_g^{-1}, or that

$$\pi(g)f = l_g^{-1*}f.$$

A straightforward calculation then shows

$$\langle \pi(g)f_1|\pi(g)f_2\rangle = \int_{G/H} (l_g^{-1*}f_1, l_g^{-1*}f_2) = \int_{G/H} l_g^{-1*}(f_1, f_2) = \int_{G/H} (f_1, f_2) = \langle f_1|f_2\rangle,$$

where we have first used the fact that the precomposing f_1 and f_2 individually is the same as precomposing the density (f_1, f_2) in the first step, and the change of variables formula in the second. The key detail here is that

$$l_g \cdot G/H = G/H,$$

so the manifold we are integrating over is unchanged.

Theorem 3.4.1. *Let G be a Lie group and H a closed subgroup with unitary representation (ξ, V). Let*

$$\delta^{1/2} = |\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)|^{-1/2}$$

be a one-dimensional representation of H on the space of half-densities $\mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \simeq \mathcal{D}^{1/2}_{eH}(T(G/H))$. Then, the completion of the pre-Hilbert space

$$\mathrm{Ind}_H^G(\xi) = \mathrm{ind}_H^G(\xi \otimes \delta^{1/2})$$

under the inner product (3.20) *provides a unitary representation of G, whose action is*

$$[\pi(g)f](x) = [\mathrm{Ind}_H^G(\xi)(g)f](x) = f(g^{-1}x).$$

Chapter 4

Principal Series

4.1 Structure Theory

For compact groups, the Peter-Weyl theorem gives an explicit decomposition of the regular representation in terms of unitary irreducible representations. For the larger locally compact groups, this is no longer the case [5]. Principal series arise in the decomposition of the regular representation for such noncompact groups, and are constructed through the process of parabolic induction. Their construction relies on decompositions of G and P into subgroups, which we briefly review before outlining parabolic induction for general connected semisimple Lie groups.

Let G be a connected semisimple Lie group. A *Cartan involution* is a map

$$\begin{aligned} \Theta : G &\to G, \\ g &\mapsto \left(g^t\right)^{-1}, \end{aligned}$$

The set of fixed points of Θ is a *maximal compact subgroup*

$$K = \{g \in G : \Theta(g) = g\}. \tag{4.1}$$

The differential $\theta = D\Theta|_e$ is a Cartan involution on the Lie algebra $\mathfrak{g}$, which decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{4.2}$$

where $\mathfrak{k}$, and $\mathfrak{p}$ is are the eigenspaces of θ with eigenvalues 1 and -1, respectively. Because the Lie subgroup K is connected and compact, exponentiation of the Lie algebra gives the full subgroup: $K = \exp \mathfrak{k}$.

Differentiating the adjoint representation of group G in Def. 3.3.4, we get the adjoint representation of the Lie algebra $\mathfrak{g}$:

$$\begin{aligned} \mathrm{ad}(\gamma) : \mathfrak{g} &\to \mathfrak{g}, \\ x &\mapsto [\gamma, x]. \end{aligned} \tag{4.3}$$

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. While $\mathfrak{p}$ is not itself closed under the Lie bracket, the subspace $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$. As such, it has an adjoint action on the representation space $\mathfrak{g}$. All $H \in \mathfrak{a}$ commute by definition, so $\mathrm{ad}(\mathfrak{a})$ is a family of commuting operators, implying that there exits a basis of shared eigenvectors:

$$\mathrm{ad}(H)X_i = [H, X_i] = \lambda_i(H)X_i, \tag{4.4}$$

where $X_i \in \mathfrak{g}$ form the basis and the eigenvalues $\lambda_i(H)$ are real by the symmetry of the operators. Keeping X_i fixed, we see that

$$H \mapsto \lambda_i(H) \in R$$

is a linear map, and thus $\lambda_i \in \mathfrak{a}^*$ is a linear functional.

Definition 4.1.1. The subset of eigenvectors of $\mathrm{ad}(\mathfrak{a})$ with joint eigenvalue $\lambda \in \mathfrak{a}^*$ is called the *root space* of $\mathfrak{g}$ relative to $\mathfrak{a}$:

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \forall H \in \mathfrak{a}, [H, X] = \lambda(H)X\}, \tag{4.5}$$

and the corresponding functional λ is called the *root*. The set of roots is denoted by Σ or $\Delta(\mathfrak{g}; \mathfrak{a})$.

If $\lambda = 0$ we do not call it a root, yet note that the subspace $\mathfrak{g}_0$ contains the maximal abelian subspace $\mathfrak{a}$. Due to the Jacobi identity, the roots are additive under brackets:

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}. \tag{4.6}$$

Furthermore, we can decompose the entire Lie algebra into root spaces:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda, \tag{4.7}$$

which is simply a reflection of the fact that the joint eigenvectors in $\mathfrak{g}$ form a basis. We can define an order on $\Sigma \subseteq \mathfrak{a}^*$ by fixing an ordered basis for $\mathfrak{a}^*$, writing any $\lambda \in \mathfrak{a}^*$ in this basis, and taking the sign of λ to be the sign of the first nonzero coefficient. Having done so, it makes sense to talk about the set of positive roots, Σ^+. Under the Cartan involution,

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}, \tag{4.8}$$

implying a one to one correspondence between positive and negative roots.

Definition 4.1.2. A Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $[\mathfrak{g}, [\mathfrak{g}, [\ldots, [\mathfrak{g}, \mathfrak{g}] \ldots]]] = 0$ for some finite number of commutators.

Let

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda. \tag{4.9}$$

Each element in $\mathfrak{n} \in \mathfrak{g}_\lambda$ for some positive root λ, so taking commutators within $\mathfrak{n}$ will give elements in root spaces with even larger roots by (4.6). Because the set of roots is finite, repeating this process will eventually yield a commutator of zero. Consequently, the subalgebra $\mathfrak{n}$ is nilpotent.

Theorem 4.1.1 (Lie algebra Iwasawa decomposition)**.** *The Lie algebra $\mathfrak{g}$ of a connected semisimple Lie group G decomposes as a direct sum:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \tag{4.10}$$

where $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, and $\mathfrak{k}$ is stable under a Cartan involution.

Clearly $\mathfrak{k} \cap \mathfrak{a} = \{0\}$ because the subalgebras belong to different eigenspaces of θ, and since $\mathfrak{n}$ contains only positive root spaces, there is no overlap with $\mathfrak{a}$. Exponentiating Theorem 4.1.1 we arrive at a Lie group decomposition.

Theorem 4.1.2 (Iwasawa decomposition)**.** *Let G be a connected semisimple Lie group. Then there is a surjective diffeomorphism*

$$\begin{aligned} K \times A \times N &\to G, \\ (k, a, n) &\mapsto kan, \end{aligned} \tag{4.11}$$

where K is the maximal compact subgroup of G, $A = \exp \mathfrak{a}$ is abelian, and $N = \exp \mathfrak{n}$ is nilpotent.

Furthermore, $A \subseteq N_G(N)$, where N_G denotes the normalizer, and the Lie algebra Iwasawa decomposition (4.10) implies that $K \cap AN = e$. Suppose an element $g \in G$ has two possible Iwasawa decompositions:

$$g = kan = k'a'n'.$$

We can manipulate this to arrive at

$$k^{-1}k = a'n'n^{-1}a^{-1},$$

where the left hand side is an element of K and, because $aN = Na$ for all $a \in A$, the right hand side is an element of AN. These two subgroups only overlap at the identity, requiring $k = k'$, $a = a'$, and $n = n'$. The Iwasawa decomposition is therefore unique, allowing us to write

$$g = k(g)a(k)n(g)\,; \qquad k : G \to K, \;\; a : G \to A, \;\; n : G \to N \tag{4.12}$$

for any $g \in G$. We have now developed the background necessary to define the parabolic subgroups used to construct principal series [5].

Definition 4.1.3. A *Borel subgroup* of a connected semisimple Lie group G is

$$P = MAN, \tag{4.13}$$

where $M = Z_K(A)$ is the centralizer of A in K.

Definition 4.1.4. A *parabolic subgroup* is any closed proper subgroup of G that contains a conjugate of a Borel subgroup.

The decomposition in (4.13) is a diffeomorphism $M \times A \times N \to MAN$. This is a special case of the *Langlands decomposition*, which exists for any parabolic subgroup [5].

Definition 4.1.5 (Langlands decomposition)**.** A general parabolic subgroup $\mathcal{P}$ can be decomposed as

$$\mathcal{P} = \mathcal{MAN}, \tag{4.14}$$

where $\mathcal{M}$ is a reductive subgroup, $\mathcal{A}$ is abelian, and $\mathcal{N}$ is nilpotent.

Motivated by Def. 4.1.4, we can also call Borel subgroups *minimal* parabolic subgroups.

Before embarking on the construction of principal series, we review one more Lie group decomposition that will be useful.

Definition 4.1.6. Let G be a connected semisimple Lie group and A a maximal abelian subgroup. The *Weyl group* is

$$W = N_K(A)/Z_K(A) = N_K(A)/M. \tag{4.15}$$

The set of roots $\Delta(\mathfrak{g}; \mathfrak{a})$ is equivalent to a collection of vectors in Euclidean space. In this context, W is the group generated by reflections across an orthogonal axis. Instead of (4.11), we can use the Weyl group to decompose G into disjoint double cosets of Borel subgroup P:

$$G = \bigsqcup_{w \in W} PwP \tag{4.16}$$

This is known as the *Bruhat decomposition*, and it can be show that it leads to the following result [5].

Theorem 4.1.3. *Let $\mathcal{P} \leq G$ be a parabolic subgroup with Langlands decomposition $\mathcal{P} = \mathcal{MAN}$ and define*

$$\overline{\mathcal{N}} = \Theta(\mathcal{N}).$$

Then the set

$$\overline{\mathcal{N}}\mathcal{MAN} \subseteq G, \tag{4.17}$$

is open and $G \setminus \left(\overline{\mathcal{N}}\mathcal{MAN}\right)$ has Haar measure zero. In particular, this holds for the Borel subgroup with subset $\overline{N}MAN$.

Much like the previous Iwasawa decomposition in (4.12), for nearly every $g \in G$ we can write

$$g = \bar{n}(g)\, m(g)\alpha(g)\eta(g), \tag{4.18}$$

where $\alpha(g) \in A$ and $\eta(g) \in N$. Although the the subset in (4.17) does not cover our entire group G, when working in a space of square-integrable functions this distinction will not matter. Because G differs from integration over $\overline{N}MAN$ by a set of measure zero, integration over these two domains is identical.

Different choices for the parabolic subgroup $\mathcal{P}$ give rise to principal series with different properties. For example, when inducing from a maximal parabolic subgroup, the base manifold $G/\mathcal{P}$ on which sections are defined is "small", allowing for a geometric approach in the study of intertwiners between principal series [2, 3]. We will instead restrict our attention to the case in which $\mathcal{P} = P$ is minimal.

4.2 Parabolic Induction

Let $G \subseteq \mathrm{GL}(n, \mathbb{R})$ be a connected semisimple matrix Lie group, and $P = MAN$ a minimal parabolic subgroup. We begin by building a representation of P from unitary irreducible representations of its components.

By Schur's lemma, all irreducible representations of the Lie algebra $\mathfrak{a}$ are one-dimensional. We therefore write a generic such representation as a map

$$\lambda : \mathfrak{a} \to \mathbb{C}. \tag{4.19}$$

The inverse of exponentiation is the log map, which sends Lie group to Lie algebra. Using this correspondence, unitary irreducible representations $\nu \in \widehat{A}$ take the form

$$\nu_{i\lambda}(a) = e^{i\lambda \log a} := a^{i\lambda}, \quad \lambda \in \mathfrak{a}^*, \tag{4.20}$$

where the imaginary exponent ensures unitarity on the representation space $\mathbb{C}$. On the other hand, M is not automatically abelian, so the representation space V_σ of $\sigma \in \widehat{M}$ is not necessarily one-dimensional. We construct the tensor product representation of $P = MAN$:

$$\begin{aligned} \xi &: MAN \to V_\sigma, \\ & man \mapsto \sigma(m) \otimes \nu_{i\lambda}(a) \otimes 1_N = \sigma(m) e^{i\lambda \log a}, \qquad \sigma \in \widehat{M},\, \nu \in \widehat{A} \end{aligned} \tag{4.21}$$

where $1_N \in \widehat{N}$ is the trivial character. Normalized induction on $\xi = \sigma \otimes \nu_{i\lambda} \otimes 1_N$ then yields unitary principal series representations of G:

$$\pi_{\sigma,\lambda} \simeq \mathrm{Ind}_{MAN}^{G}(\sigma \otimes \nu_{i\lambda} \otimes 1_N). \tag{4.22}$$

Recall that normalization requires twisting by half-densities, so the representation that we are really inducing to G is

$$\xi \otimes \delta^{1/2},$$

where δ is a density on the tangent bundle of G/P. Explicitly, the character δ of P is given by

$$\delta(man) = |\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}}(man)|^{-1}, \tag{4.23}$$

where we have used $\mathfrak{q}$ to denote the Lie algebra of P. As it is a representation, we can decompose it into

$$|\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}}(man)| = |\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}}(m)||\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}}(a)||\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}}(n)|. \tag{4.24}$$

The modular function associated with the Haar measure on G is one when restricted to a compact subgroup like M [7]. For Lie groups, the modular function coincides with the determinant of the adjoint representation. As a result, $|\det \mathrm{Ad}_{\mathfrak{g}}(m)| = 1$. The same is true for the modular function of P when restricted to M, so the first factor in (4.24) is one.

Now we consider the adjoint representation of N. We write any element $n \in N$ as the exponential of a vector in the Lie algebra $\mathfrak{n}$:

$$n = \exp X, \;\; X \in \mathfrak{n}.$$

With this substitution, we can write the factor in (4.24) in terms of the adjoint representation of $\mathfrak{n}$:

$$\det \mathrm{Ad}(\exp X) = \det \exp[\mathrm{ad}(X)] = \exp[\mathrm{Tr}\,(\mathrm{ad}(X))], \tag{4.25}$$

where we have used an identity of matrix exponentials. However, $\mathfrak{n}$ is nilpotent, so all eigenvalues of the operator $\mathrm{ad}(X)$ must be zero. Consequently, the trace is zero and the third factor in (4.24) is also one.

We turn our attention now to the remaining factor, the adjoint representation of A acting on $\mathfrak{g}/\mathfrak{q}$. Again we can rewrite the action in terms of the Lie algebra adjoint:

$$\det \mathrm{Ad}(a) = \det \exp[\mathrm{ad}(H)] := \det e^{\mathrm{ad}(H)}, \tag{4.26}$$

where $a = \exp H$ with $H \in \mathfrak{a}$. We recall that $\mathrm{ad}(\mathfrak{a})$ acts on root spaces $\mathfrak{g}_\lambda \subseteq \mathfrak{g}$ by linear functionals $\lambda \in \mathfrak{a}^*$. Furthermore, we can rewrite the decomposition of $\mathfrak{g}$ in (4.7) as [1]

$$\mathfrak{g} = \bigoplus_{\lambda\in\Sigma^-} \mathfrak{g}_\lambda \oplus \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda = \bigoplus_{\lambda\in\Sigma^-} \mathfrak{g}_\lambda \oplus \mathfrak{q}, \tag{4.27}$$

where Σ^- denotes the negative roots of $\mathfrak{g}$. We are interested in the action of $\mathrm{ad}(\mathfrak{a})$ on

$$\mathfrak{g}/\mathfrak{q} \simeq \bigoplus_{\lambda\in\Sigma^-} \mathfrak{g}_\lambda = \theta(\mathfrak{n}), \tag{4.28}$$

Writing the determinant as the product of eigenvalues, we find

$$\det e^{\mathrm{ad}(H)} = \prod_{\lambda\in\Sigma^-} e^{\lambda(H)} = \prod_{\lambda\in\Sigma^+} e^{-\lambda(H)} = e^{-\sum_{\lambda\in\Sigma^+}\lambda(H)}, \tag{4.29}$$

where we have taken advantage of the correspondence between positive and negative roots. Since we must twist by *half*-densities in normalized induction, we define the half sum of positive roots:

$$\rho = \frac{1}{2}\sum_{\lambda\in\Sigma^+} \lambda. \tag{4.30}$$

Combining (4.29) and (4.30) and writing $H = \log a$, we finally arrive at an expression for the half-density representation of P:

$$\delta^{1/2}(man) = e^{\rho \log a}, \quad \rho \in \mathfrak{a}^* \tag{4.31}$$

Definition 4.2.1 (Principal series)**.** Principal series representations of G are induced representations

$$\pi_{\sigma,\lambda} = \mathrm{ind}_{MAN}^{G}(\sigma \otimes \nu_{i\lambda+\rho} \otimes 1_N), \tag{4.32}$$

where $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}^*$, and $\rho \in \mathfrak{a}^*$ is the half-sum of positive roots.

There are three different pictures of $\pi_{\sigma,\lambda}$: the induced picture, the compact picture, and the noncompact picture. In each of these pictures, the representation space and action are different, providing certain advantages and disadvantages.

Induced Picture

The functional picture introduced in (3.11) and (3.12), when applied to principal series, is known as the *induced picture.* The space of the induced picture is

$$\mathcal{H}_{\sigma,\lambda} = \{f \in \mathcal{C}(G, V_\sigma) : f(gman) = \sigma(m)^{-1} e^{-(i\lambda+\rho)\log a} f(g)\}, \tag{4.33}$$

and the action of G is

$$[\pi_{\sigma,\lambda}(g)f](x) = f(g^{-1}x). \tag{4.34}$$

The group action is very simple in this picture. In return, the space is quite complicated, as it is a function space obeying a specific equivariance property parametrized by two representations σ and λ. One way to simplify the space considerably is to consider only the principal series induced from $\sigma = 1_M$, called *spherical principal series.* For the moment, we retain the freedom to choose any $\sigma \in \widehat{M}$.

This representation is unitary, and in particular we must define an inner product in the Hilbert space $\mathcal{H}_{\sigma,\lambda}$. Our general theory of normalized induction indicates that we must integrate the inner product in V_σ over the space G/P. From the Iwasawa decomposition $G = KAN$ and the Langlands decomposition $P = MAN$, we recognize that

$$G/P = KAN/MAN \simeq K/M, \tag{4.35}$$

as an isomorphism of manifolds. Shifting the integral to the compact space K, we define an inner product between $f_1, f_2 \in \mathcal{H}_{\sigma,\lambda}$ [1]:

$$\langle f_1 | f_2 \rangle = \int_K (f_1(k), f_2(k))_\sigma \, dk, \tag{4.36}$$

where dk is the Haar measure on K and $(\,\cdot\,,\,\cdot\,)_\sigma$ is the inner product on V_σ.

Compact Picture

Any G with an Iwasawa decomposition can be written $G = KMAN$, motivating the decomposition of $g \in G$:

$$g = k(g)m(g)e^{H(g)}n(g), \tag{4.37}$$

where $H(g) \in \mathfrak{a}$. Here $k(g)$ and $m(g)$ are not unique due to the overlap between K and M. We notice that a function in the induced picture is entirely determined by its restriction to K:

$$f(g) = f(k(g)m(g)e^{H(g)}n(g)) = \sigma(m(g))^{-1}e^{-(i\lambda+\rho)H(g)}f(k(g)) \tag{4.38}$$

The restriction of $\mathcal{H}_{\sigma,\lambda}$ to K yields an isometry between the space of the induced picture and the space of the *compact picture*:

$$L^2(K;\sigma) = \{f \in L^2(K, V_\sigma) : f(km) = \sigma(m)^{-1}f(k)\}, \tag{4.39}$$

where $L^2(K, V_\sigma)$ is the space of V_σ-valued square integrable functions on K. Functions in this space are square integrable because (4.36) defines a norm, which is finite because K is compact. The action of G can be derived by applying the Iwasawa decomposition to the action in the induced picture (4.34):

$$f(x^{-1}k') = f(k(x^{-1}k')m(x^{-1}k')e^{H(x^{-1}k')}n(x^{-1}k')), \tag{4.40}$$

where $x \in G$ and $k' \in K$. We then apply the transformation properties of the induced picture and restrict f to the compact picture to get the action

$$[\pi^{\mathrm{C}}_{\sigma,\lambda}(x)f](k') = \sigma(m(x^{-1}k'))^{-1}e^{-(i\lambda+\rho)H((x^{-1}k')}f(k(x^{-1}k')). \tag{4.41}$$

Without having to resort to spherical principal series, we notice that $L^2(K;\sigma)$ does not depend on λ, while the action of G does. Consequently, the compact picture allows us to study the dependence of the representation on λ. Furthermore, as K is compact, it is easier to access geometric approaches in this picture.

On the other hand, the action of G is significantly more complicated than in the induced picture. The uniqueness of the Iwasawa decomposition, however, ensures that it is always possible to write out such an action. Lastly, restriction to K preserves the inner product because it is already defined in the compact picture in (4.36).

Noncompact Picture

The *noncompact picture* is similarly derived from the induced picture using a group decomposition. Nearly everywhere, $G = \overline{N}MAN$, allowing us to write all $g \in G$ except for a set of measure zero as

$$g = \bar{n}(g)\, m(g)\alpha(g)\eta(g), \tag{4.42}$$

in the manner of (4.18). The functions in $\mathcal{H}_{\sigma,\lambda}$ are nearly determined by their restriction to $\overline{N}$, as the transformation properties of the functions under right multiplication by $man \in P$ automatically constrain them.

The inner product on the induced and compact spaces can be recast in terms of an integral over $\overline{N}$ by a manipulation of measures outlined by van den Ban in [1]:

$$\langle f_1 | f_2 \rangle = \int_{\overline{N}} \left(f_1(\bar{n}), f_2(\bar{n}) \right)_\sigma d\bar{n}. \tag{4.43}$$

Because the decomposition we are using is valid almost everywhere, the points $g \in G$ where $g \notin \overline{N}MAN$ do not affect the value of the above integral. As we can see from the inner product in the noncompact picture, the space is formed by restriction to $\overline{N}$:

$$L^2(\overline{N}). \tag{4.44}$$

There is no additional equivariance condition because $\bar{N}$ does not overlap with P, unlike K in the compact picture. If we had not induced from a unitary representation $\nu_{i\lambda}$ of A, there would be an additional factor in the measure that is missing here.

The action of G in the noncompact picture is given by

$$[\pi^{\mathrm{NC}}_{\sigma,\lambda}(x)f](\bar{n}') = \sigma(m(x^{-1}\bar{n}'))^{-1}e^{-(i\lambda+\rho)\log\alpha(x^{-1}\bar{n}')}f(\bar{n}(x^{-1}\bar{n}')), \tag{4.45}$$

where m, α, and $\bar{n}$ are the functions in (4.42). This is a complicated action, but in return the space is simply an L^2 space with no additional equivariance to account for. The noncompact picture is useful for analytic study of principal series [1] and is the starting point for the construction of a fourth picture: the nonstandard picture, that exists for a certain parabolic subgroup of G.

4.3 Principal Series of $\mathrm{SL}(3,\mathbb{R})$

We now focus our attention on the particular example of minimal principal series representations of the group,

$$G = \mathrm{SL}(3,\mathbb{R}) = \{A \in M_3(\mathbb{R}) : \det A = 1\}, \tag{4.46}$$

The minimal parabolic subgroup $P \leq G$ is the closed subgroup of upper triangular matrices:

$$P = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{R},\ a_{11}a_{22}a_{33} = 1 \right\}. \tag{4.47}$$

Parabolic induction rests on the Iwasawa decomposition $G = KAN$ and the Langlands decomposition $P = MAN$. For $G = \mathrm{SL}(3,\mathbb{R})$, these subgroups take the form:

$$K = \mathrm{SO}(3,\mathbb{R}) = \{A \in \mathrm{SL}(3,\mathbb{R}) : A^tA = 1\}, \tag{4.48}$$

for the maximal compact subgroup,

$$A = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & (a_1a_2)^{-1} \end{pmatrix} : a_1, a_2 > 0 \right\}, \tag{4.49}$$

for the abelian subgroup, and

$$N = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \tag{4.50}$$

for the nilpotent subgroup. Collecting the matrices in $K = \mathrm{SO}(3,\mathbb{R})$ that commute with diagonal matrices, we find the form of $M = Z_K(A)$:

$$M = \left\{ \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix} : \epsilon_i = \pm 1 \right\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{4.51}$$

Not only is M compact, it is *finite* group of order 4, greatly simplifying its representation theory.

To construct principal series of $\mathrm{SL}(3,\mathbb{R})$, we start from unitary irreducible representations of M and A. As both subgroups are abelian, $\widehat{M}$ and $\widehat{A}$ will only contain one-dimensional representations. Recasting the generalities of (4.20) and (4.21) for our concrete groups, we find:

$$\widehat{M} = \left\{ \sigma : M \to \mathbb{C} \;\middle|\; \sigma\left[\begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2 \end{pmatrix}\right] = \epsilon_1^{\sigma_1}\epsilon_2^{\sigma_2};\ \text{where } \sigma_i = 0, 1 \right\}, \tag{4.52}$$

$$\widehat{A} = \left\{ \nu_{i\lambda} : A \to \mathbb{C} \;\middle|\; \nu_{i\lambda}\left[\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & (a_1a_2)^{-1} \end{pmatrix}\right] = a_1^{i\lambda_1}a_2^{i\lambda_2};\ \text{where } \lambda_i \in \mathbb{R} \right\}. \tag{4.53}$$

The representations of M are parametrized by

$$(\sigma_1, \sigma_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{4.54}$$

indicating that there are four possible representations: the trivial $(0,0)$ representation, the symmetric $(0,1)$ and $(1,0)$ representations, and the sign representation $(1,1)$ that maps from $m \in M$ to the product of its first two diagonal entries.

The representations of A are parametrized by

$$(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}, \tag{4.55}$$

indicating a continuous family of representations indexed by two parameters. We induce from these representations to principal series $\pi_{\sigma,\lambda}$ of $\mathrm{SL}(3,\mathbb{R})$:

$$\pi_{\sigma,\lambda} = \mathrm{ind}_{MAN}^{G}(\sigma \otimes \nu_{i\lambda+\rho} \otimes 1_N), \tag{4.56}$$

where $\sigma = (\sigma_1, \sigma_2)$ and $\lambda = (\lambda_1, \lambda_2)$. Topologically, the family of principal series representations is homeomorphic to four copies of the plane $\mathbb{R}^2$, as there are four representations of M and for each we can choose two continuous indices to parametrize a representation of A.

In order to write the representation space and action of G in the induced picture, we need to twist by the half-density ρ in (4.56). To do this, we sum the positive roots on $\mathfrak{sl}(3, \mathbb{R})$, the Lie algebra of 3×3 traceless matrices. Recall that a root is a linear functional

$$\mathfrak{a} \to \mathbb{R}$$

that acts as a joint eigenvector for an eigenspace of $\mathfrak{sl}(3, \mathbb{R})$ under the action of $\mathrm{ad}(\mathfrak{a})$. For any $a = \mathrm{diag}\,(a_1, a_2, a_3) \in A$, there is a corresponding Lie algebra element $\log a$:

$$\log a = \begin{pmatrix} \log a_1 & & \\ & \log a_2 & \\ & & \log a_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix}, \tag{4.57}$$

where $\alpha_3 = -\alpha_1 - \alpha_2$ by the traceless condition. Let us choose a basis for $\mathfrak{a}^*$,

$$\begin{aligned} E_i : \mathfrak{a} &\to \mathbb{R}, \\ \log a &\mapsto \alpha_i. \end{aligned} \tag{4.58}$$

There are three positive roots in $\mathfrak{a}^*$, which in this basis take the form

$$\Sigma^+ = \{(E_1 - E_2),\ (E_2 - E_3),\ (E_1 - E_3)\} \tag{4.59}$$

The half-density that we must tensor with our representation of $P = MAN$ is

$$\delta^{1/2}(man) = \exp\left[\frac{1}{2} \sum_{\eta \in \Sigma^+} \eta(\log a)\right], \tag{4.60}$$

where we have used η to denote roots in order to distinguish from our existing representation of A indexed by λ. Summing up the positive roots, we find

$$\delta^{1/2}(man) = \exp[2\alpha_1 + \alpha_2] = e^{2\log a_1 + \log a_2} = a_1^2 a_2. \tag{4.61}$$

In matrix form, the representation of P needed for normalized induction to G is therefore

$$\xi_{\sigma,\lambda}[man] = \xi_{\sigma,\lambda}\left[\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2\end{pmatrix}\begin{pmatrix}a_1 & & \\ & a_2 & \\ & & (a_1a_2)^{-1}\end{pmatrix}\begin{pmatrix}1 & x & t \\ & 1 & y \\ & & 1\end{pmatrix}\right] = \epsilon_1^{\sigma_1}\,\epsilon_2^{\sigma_2}\,a_1^{i\lambda_1+2}\,a_2^{i\lambda_2+1}, \tag{4.62}$$

where our representation space is simply the complex numbers $\mathbb{C}$.

In the induced picture, the space and group action of the principal series representations of $\mathrm{SL}(3,\mathbb{R})$ are

$$\begin{gathered}\mathcal{H}_{\sigma,\lambda} = \{f \in \mathcal{C}(\mathrm{SL}(3,\mathbb{R}),\mathbb{C}) : f(gman) = \epsilon_1^{-\sigma_1}\,\epsilon_2^{-\sigma_2}\,a_1^{-i\lambda_1-2}\,a_2^{-i\lambda_2-1}f(g)\}, \\ [\pi_{\sigma,\lambda}(g)f](x) = f(g^{-1}x),\end{gathered} \tag{4.63}$$

with man given by the matrices in (4.62). Because we are inducing from the space $\mathbb{C}$, which carries the standard inner product, (3.20) gives rise to the L^2 norm on functions $f \in \mathcal{H}_{\sigma,\lambda}$:

$$\|f\|_2{}^2 = \int_{\mathrm{SO}(3)} |f(k)|^2 dk, \tag{4.64}$$

where dk is the Haar measure on $\mathrm{SO}(3)$.

Turning to the compact picture, the representation space becomes that of functions on $\mathrm{SO}(3,\mathbb{R})$ that transform according to the parity of $\sigma \in \widehat{M}$:

$$L^2(\mathrm{SO}(3,\mathbb{R}))_\sigma = \{f \in L^2(\mathrm{SO}(3,\mathbb{R})) : f(km) = \pm f(k) \text{ for } m \in M\}, \tag{4.65}$$

where M acts on the right of K to transform rotations $k = R(\boldsymbol{n},\theta) \in K$, and the particular representations (σ_1,σ_2) determines the parity for each m-transformation. For example, consider the element

$$m = \mathrm{diag}\,(-1,-1,1).$$

We notice that for a rotation of θ about z,

$$k = R(\boldsymbol{z},\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}, \tag{4.66}$$

the action of the map $k \mapsto km$ is an additional rotation of π about the z-axis

$$R(\boldsymbol{z}, \theta) \mapsto R(\boldsymbol{z}, \theta + \pi). \tag{4.67}$$

As a result, if we induce from the trivial or sign representations of M (such that $\sigma(m) = +1$), then the space of the compact picture will be *even* functions under π-rotation in the xy-plane. On the other hand, if we induce from the $(0,1)$ or $(1,0)$ representations, such that $\sigma(m) = -1$, then the representation space will be *odd* functions under this rotation.

Using (4.40), we see that the action of G in the compact picture comes from an explicit computation of the Iwasawa decomposition of a generic element of $\mathrm{SL}(3,\mathbb{R})$. We do not show this calculation, as the key steps have already been captured in the discussion on general G. The main advantage of the compact picture is not the group action, but rather the space, which is diffeomorphic to a ball in $\mathbb{R}^3$ with antipodal surface points identified [6]

$$\mathrm{SO}(3,\mathbb{R}) \simeq \mathbb{RP}^3. \tag{4.68}$$

This geometric analogy with a projective space allows for the study of principal series, and in particular intertwiners of principal series [2], using geometric tools.

In the noncompact picture, the decomposition $G = \overline{N}MAN$, which is valid almost everywhere, is used to recast the inner product in $\mathcal{H}_{\sigma,\lambda}$ as an integral over $\overline{N}$. The representation space thus is

$$L^2(\overline{N}), \tag{4.69}$$

where

$$\overline{N} = \Theta N = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ t & y & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \tag{4.70}$$

This space of functions on lower-triangular matrices is particularly simple because $N \simeq \overline{N}$ is a Heisenberg group.

Definition 4.3.1. The *Heisenberg group* of dimension $(2n-1)$ over a field $\mathbb{K}$ is a lower (upper) triangular matrix group of the form:

$$H_{2n-1}(\mathbb{K}) = \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{x} & \mathbb{1}_n & 0 \\ t & \boldsymbol{y} & 1 \end{pmatrix}, \tag{4.71}$$

where $t \in \mathbb{K}$, $\mathbb{1}_n$ is the identity matrix, and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n$ are column and row vectors, respectively.

The name Heisenberg group originates from physics, as the Lie algebra of a Heisenberg group represents quantum mechanical phase space. The Heisenberg uncertainty principle, which sets a lower bound on the precision with which one can simultaneously measure position and momentum, arises from the canonical commutation relations encoded in the Lie algebra.

For example, the Lie algebra of $H_1(\mathbb{R}) = \overline{N}$ is spanned by three basis vectors:

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i\hbar z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.72}$$

where we have judiciously scaled the third vector by the imaginary unit and Planck's constant to make contact with the physical scale of quantum effects. The commutation relations are

$$[x, p] = i\hbar z, \;\; [x, z] = [p, z] = 0. \tag{4.73}$$

These are the canonical relations describing the noncommutativity between position x and momentum p in quantum mechanics.

Returning to the Heisenberg group, its structure affords the noncompact picture a major advantage in studying principal series. Topologically,

$$H_{2n-1} \simeq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n, \tag{4.74}$$

and in particular we can identify the Heisenberg group of interest, $\overline{N} = H_1(\mathbb{R})$, with $\mathbb{R}^3$ by homeomorphism. Furthermore, this identification also holds as a correspondence between measure spaces: the Haar measure on $\overline{N}$, which we integrate over in the noncompact space $L^2(\overline{N})$, is the same as the Lebesgue measure in $\mathbb{R}^3$. As a result, the noncompact space is akin to an ordinary space of square-integrable functions in three dimensional Euclidean space.

This fact simplifies the analytic aspect of principal series greatly, and much of the theory of intertwiners between principal series [5, 8, 9] involves integral transforms that act as operators in the noncompact picture. Once again, the action of the group is complicated, yet computable starting from an explicit decomposition (4.42).

The parallel between the noncompact space and $\mathbb{R}^3$ suggests that we may perform similar operations on functions in the noncompact picture. In particular, in any Heisenberg group one can extend the notion of a Fourier transform:

Definition 4.3.2. The *Partial Fourier Transform* is an operator between functions on the Heisenberg group

$$\begin{aligned} \mathcal{T} : L^2(H_{2n-1}(\mathbb{R})) \simeq L^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) &\longrightarrow L^2(\widehat{\mathbb{R}}^n \times \widehat{\mathbb{R}} \times \mathbb{R}^n), \\ f(x,t,y) &\longmapsto [\mathcal{T} f](\chi, \tau, y) = \int_{\mathbb{R}\times\mathbb{R}^n} f(x,t,y) e^{-i(t\tau + \langle x|\chi\rangle)}. \end{aligned} \tag{4.75}$$

Transforming the functions in the compact picture using a partial Fourier transform in two of the variables, we arrive at a new picture.

Definition 4.3.3 (Nonstandard picture)**.** For principal series in which N is a Heisenberg group, the partial Fourier transform of the noncompact picture gives a new representation space:

$$\mathcal{T}[L^2(\overline{N})] = L^2(\widehat{\mathbb{R}}^n \times \widehat{\mathbb{R}} \times \mathbb{R}^n). \tag{4.76}$$

This picture was developed by Kobayashi, Ørsted, and Pevzner in [2] for principal series induced from a maximal parabolic subgroup of a symplectic group. They found that Knapp-Stein intertwiners between principal series in the nonstandard picture became algebraic, as opposed to the complicated integral transforms of the other pictures [8, 5].

Although the partial Fourier transform was critical in this simplification, it is difficult to decouple it from the role that starting from a maximal instead of minimal parabolic played. Any semisimple Lie group has a parabolic subgroup P with a nilpotent part that is of Heisenberg type. Consequently, one can construct nonstandard picture for any principal series, provided that the right parabolic starting point is chosen.

In [2] this subgroup is maximal, allowing for a smaller space G/P and more geometric tools in studying intertwiners. As a result, the nonstandard picture does not immediately afford us the same advantages in $\mathrm{SL}(3, \mathbb{R})$, yet nevertheless presents another complementary way of realizing principal series.

Chapter 5

K-Types

5.1 K-Type Decomposition

Many questions in representation theory are based on the paradigm of decomposition, whereby larger representations are broken down into elemental ones. In general, irreducible representations play the role of fundamental building blocks in the representation of a group. When an irreducible representation is restricted to a subgroup, however, it can cease to be irreducible.

Branching problems are concerned with how irreducible representations decompose when they are restricted to a subgroup [2]. Such questions are particularly relevant in physics, where symmetries are broken by perturbations to a system. Quantum systems in physics are characterized by a Hamiltonian operator H, which describes the total energy of a system. A quantum state is encoded by an eigenvector of the Hamiltonian, and its energy by the corresponding eigenvalue.

Often the system obeys some kind of symmetry, which in practice means that certain transformations P leave the Hamiltonian unchanged,

$$PHP^{-1} = H. \tag{5.1}$$

The set of such transformations forms a group G, which we call the symmetry group of the system. We see from (5.1) that operators in the symmetry group commute with the Hamiltonian and therefore share the same quantum states. States associated with an eigenvalue of degeneracy k then form a basis for a unique k-dimensional (projective unitary) irreducible representation of G.

Perturbations to the quantum system can break the symmetry, reducing the symmetry group from G to a subgroup H. A once irreducible k-dimensional representation of G now decomposes into irreducible representations of H. Physically, this corresponds to the breaking of degeneracy due to the perturbation, as the once degenerate state is split into several states associated new irreducible representations. Returning to a mathematical angle, various subgroups are critical in constructing principal series representations, giving rise to a wealth of branching problems.

Let π be a unitary irreducible representation of a group G. In general, if we restrict π to a maximal compact subgroup K it is no longer irreducible. However, as $\pi|_K$ is a unitary representation of a compact group, we can take advantage of the Peter-Weyl theorem to decompose it as:

$$\pi|_K \simeq \bigoplus_{\rho \in \widehat{K}} \mu_\rho V_\rho, \tag{5.2}$$

where the sum is over unitary irreducible representations of K. This decomposition is the *branching law* for subgroup K. A vector v in the Hilbert space of π is called *K-finite* if $\pi(K)v$ spans a finite dimensional subspace. The Peter-Weyl theorem stipulates that each of the spaces V_ρ is finite, thus all vectors in the decomposition are K-finite.

Definition 5.1.1. The *K-types* of an irreducible representation π of G are the irreducible subrepresentations

$$(\rho, V_\rho) \tag{5.3}$$

that appear in the decomposition of $\pi|_K$ (5.2) with multiplicity μ_ρ, where K is a maximal compact subgroup of G.

A representation is said K-*admissible* if all the multiplicities μ_ρ are finite.

Theorem 5.1.1. *Let G be a connected semisimple group. If π is a unitary irreducible representation of G, then π is K-admissible.*

The proof of this theorem can be found in [5]. K-type decompositions can also characterize intertwiners between representations in $\widehat{G}$:

$$T : \pi \to \pi'. \tag{5.4}$$

If the multiplicities

$$\mu_\rho, \mu_{\rho'} \leq 1$$

for all K-types in the decompositions, Schur's lemma ensures that T factors into K-morphisms that act as scalar multiples of the identity between K-types. These scalars are said to form the K-*spectrum* of the intertwiner.

Principal series representations of G are K-admissible by Theorem 5.1.1. As a result, we have a K-type decomposition for (4.32):

$$\operatorname{Res}^G_H \pi_{\sigma,\lambda} \simeq \bigoplus_{\rho \in \widehat{K}} \mu_\rho V_\rho, \quad \mu_\rho < \infty \, \forall \rho \in \widehat{K}, \tag{5.5}$$

where we have used the notation introduced in (2.2) for the restriction of a representation of G to a subgroup K.

5.2 K-Type Multiplicities

We now develop a technique to compute the multiplicities of the principal series K-types above. Our approach rests on the duality of restricted and induced representations.

Theorem 5.2.1 (Frobenius reciprocity)**.** *Let G be a group with representation (π, V) and let $H \leq G$ be a subgroup with representation (ρ, W). The following is a linear isomorphism:*

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}^G_H \rho) \simeq \operatorname{Hom}_H(\operatorname{Res}^G_H \pi, \rho), \tag{5.6}$$

where $\mathrm{Hom}_G(\pi_1, \pi_2)$ *is the space of G-intertwiners between representations* π_1 *and* π_2.

We begin by considering the compact picture of principal series $\pi_{\sigma,\lambda}$ of G. Recall that the representation space in this picture takes the form

$$L^2(K;\sigma) = \{f \in L^2(K, V_\sigma) : f(km) = \sigma(m)^{-1} f(k) \text{ for } m \in M\} \tag{5.7}$$

Restricting the representation to K leaves the space unchanged, as $M \subseteq K$, yet the action (4.41) simplifies considerably:

$$[\pi^C_{\sigma,\lambda}(x)f](k) = f(x^{-1}k), \tag{5.8}$$

where $x, k \in K$. This is simply left-translation, the action of the induced picture. In fact, the space (5.7) is that of continuous functions on K which transform equivariantly under right action by M. Together, these features suggest an identification

$$\mathrm{Res}^G_K\, \pi^C_{\sigma,\lambda} \simeq \mathrm{Ind}^K_M \sigma, \tag{5.9}$$

where the restriction $\pi_{\sigma,\lambda}|_K$ in the *compact* picture can be seen as a principal series representation of K induced from $\sigma \in \widehat{M}$ in the *induced* picture. This relationship implies that the multiplicities in the K-type decomposition of $\pi_{\sigma,\lambda}$ are given by

$$\mu_\rho = \dim \mathrm{Hom}_K(\rho, \mathrm{Red}^G_K \pi_{\sigma,\lambda}) = \dim \mathrm{Hom}_K(\rho, \mathrm{Ind}^K_M \sigma). \tag{5.10}$$

For each copy of a V_ρ that appears in the K-type decomposition, there is an K-intertwiner mapping

$$\mathcal{H}_{\sigma,\lambda} \to V_\rho,$$

thus counting the number of distinct intertwiners gives the number of copies of V_ρ present for a particular irreducible representation ρ. We also notice that the multiplicities are independent of $\lambda \in \widehat{A}$.

Having expressed our restricted representations of G as induced representations from M, we apply Frobenius reciprocity:

$$\mathrm{Hom}_K(\rho, \mathrm{Ind}_M^K \sigma) = \mathrm{Hom}_M(\mathrm{Res}_M^K \rho, \sigma). \tag{5.11}$$

By taking advantage of the duality between induction and restriction, we have reformulated the problem in terms of the representation theory of M, which is a smaller group. The K-type multiplicities of $\pi_{\sigma,\lambda}$ then take the form:

$$\mu_\rho(\pi_{\sigma,\lambda}) = \dim \mathrm{Hom}_M(\mathrm{Res}_M^K \rho, \sigma), \tag{5.12}$$

where $\sigma \in \widehat{M}$ parametrizes the principal series in question, and

$$\mathrm{Res}_M^K \rho = \rho|_M$$

is the restriction of an irreducible representation of K to the subgroup M.

The procedure for finding the multiplicity of a K-type for a principal series is as follows. Identify σ, the unitary irreducible representation of M that is used to construct the principal series of G. Chose a unitary irreducible representation of K (a K-type) and restrict it to M:

$$(\rho|_M, V_\rho).$$

Decompose this representation into unitary irreducibles $\sigma_i \in \widehat{M}$, and find the multiplicity of σ in this direct sum. This is the multiplicity of V_ρ.

We now apply this technique to find the K-type decomposition of principal series representations of $\mathrm{SL}(3,\mathbb{R})$. In this case, M is a finite group and our dual approach will thus simplify the problem. Recall that for $G = \mathrm{SL}(3,\mathbb{R})$, we have $K = \mathrm{SO}(3,\mathbb{R})$ and $M \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Any matrix $k \in \mathrm{SO}(3)$ can be parametrized by three rotations called *Euler* *angles*:

$$k = R(\boldsymbol{z},\alpha)R(\boldsymbol{y},\beta)R(\boldsymbol{z},\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.13}$$

Similarly, any matrix $m \in M$ can be labeled by the signs of the first two diagonal elements:

$$m = m_{\epsilon_1,\epsilon_2} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix}, \quad M = \{m_{1,1}, m_{1,-1}, m_{-1,1}, m_{-1,-1}\}. \tag{5.14}$$

We must first construct the unitary irreducible representations of SO(3), as they are the possible K-types in the branching law for principal series. The rotation group $SO(3)$ shares a Lie algebra with SU(2):

$$\mathfrak{so}(3) \simeq \mathfrak{su}(2).$$

Exponentiating Lie algebra representations gives all the unitary irreducible representations of a simply connected Lie group. Here SO(3) is not simply connected but its universal cover SU(2) is by definition. Specifically, SU(2) is a double cover of SO(3), meaning that there is a 2-to-1 continuous map

$$p : \mathrm{SU}(2) \to \mathrm{SO}(3)$$

as topological spaces. As a result, only half of the irreducible representations of the special unitary group are also representations of the special orthogonal group. The unitary irreducible representations that are trivial on $\{\pm\mathbb{1}\} \in \mathrm{SU}(2)$ are then the desired K-types. In the following, we use the physical language of spin to describe these representations.

The Lie algebra $\mathfrak{so}(3)$ of skew-symmetric, traceless matrices has a basis of three vectors

$$\mathcal{L}_x = -iL_x, \mathcal{L}_y = -iL_y, \mathcal{L}_z = -iL_z, \tag{5.15}$$

where L_x, L_y and L_z are hermitian operators that obey commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \tag{5.16}$$

in which ϵ_{ijk} is the Levi-Civita symbol. Let us consider the universal envelopping algebra of $\mathfrak{so}(3)$, which is an algebra in which the Lie bracket becomes the commutator for the operations of the algebra:

$$[a, b] = a \cdot b - b \cdot a.$$

In this space, we can construct the *Casimir element*:

$$L^2 = L_x^2 + L_y^2 + L_z^2, \quad [L^2, L_i] = 0.$$

Regardless of what representation we choose, we see that the Casimir element will commute with all elements in the Lie algebra. For irreducible representations, Schur's lemma requires that

$$d(L^2) = \lambda \, d(\mathbb{1}),$$

where d is the representation. We therefore label the irreducible representations by the scalar λ with which the Casimir operator acts on their space.

We work in a basis in which one of the three basis vectors is diagonal; here we choose L_z. Simultaneously diagonalizing L_z and L^2, we can construct a basis to span each representation space. As L_z is a generator of rotations around the z-axis, it can be shown that it will act on its eigenstates to give integer or half-integer angular momentum in the z-direction, which we label by m_z. Furthermore, from a physical perspective the scalar λ that labels the irreducible representations is related to the total angular momentum of a state l:

$$\begin{aligned} L^2 \left| l, m \right\rangle &= l(l+1) \left| l, m \right\rangle , \\ L_z \left| l, m \right\rangle &= m \left| l, m \right\rangle , \end{aligned} \tag{5.17}$$

where we have labeled the states by the two eigenvalues. For each integer or half-integer l, it can be shown that the states with

$$-l \leq m \leq l$$

form a basis for a $(2l+1)$-dimensional irreducible representation of the Lie algebra. Each irreducible representation is a collection of the possible states with a given total angular momentum. Because we are only interested in the representations that will become group representations of SO(3), we consider only those with integer angular momentum.

The representation space of an irreducible representation indexed by l is a Hilbert space

$$\mathcal{H}_l = \{|l, m\rangle : -l \leq m \leq l,\ l, m \in \mathbb{Z}\}. \tag{5.18}$$

To write it concretely, we can make an identification with the standard basis in $\mathbb{C}^n$:

$$|l, -l\rangle \sim e_1,\ \ldots,\ |l, l\rangle \sim e_{2l+1},$$

In an l-representation, the diagonal generator then takes the form

$$d_l(L_z) = \mathrm{diag}(-l, -l+1, \cdots, l-1, l).$$

To find the form of the other two generators we define raising and lowering operators:

$$\begin{aligned} L_\pm &= L_x \pm i L_y, \\ d_l(L_\pm)\,|l, m\rangle &= C^\pm_{l,m}\,|l, m \pm 1\rangle\,. \end{aligned} \tag{5.19}$$

Raising (lowering) a state of maximal (minimal) angular momentum m_z yields zero, and the coefficients of (5.19) are given by

$$C^\pm_{l,m} = \sqrt{l(l+1) - m(m \pm 1)}. \tag{5.20}$$

Given an irreducible representation of $\mathfrak{so}(3)$ with integer angular momentum l,

$$(d_l, \mathcal{H}_l),$$

we construct a unitary irreducible representation of SO(3) by exponentiation. Because we are dealing with matrix groups, the rotation matrices are simply given by matrix exponentials of the Lie algebra:

$$D_l(R(\boldsymbol{x}, \theta)) = e^{-i d_l(L_x)\theta},\ D_l(R(\boldsymbol{y}, \theta)) = e^{-i d_l(L_y)\theta},\ D_l(R(\boldsymbol{z}, \theta)) = e^{-i d_l(L_z)\theta}, \tag{5.21}$$

where

$$D_l : \mathrm{SO}(3) \to \mathcal{H}_l$$

is the resulting Lie group representation. For low dimensions, these matrices can be computed directly from the matrix exponentiation, yet in general their matrix elements are given by Wigner functions [10].

Definition 5.2.1. Parametrize an element $k \in \mathrm{SO}(3)$ by its Euler angles given by (5.13):

$$k(\alpha, \beta, \gamma) = R(\boldsymbol{z}, \alpha) R(\boldsymbol{y}, \beta) R(\boldsymbol{z}, \alpha).$$

A *Wigner function* of k is given by:

$$D^l_{m_1,m_2}(k(\alpha,\beta,\gamma)) = e^{im_1\alpha} \Delta^l_{m_1,m_2}(\cos\beta) e^{im_2\gamma}, \tag{5.22}$$

where $-l \le m_1, m_2 \le l$ and

$$\Delta^l_{m_1,m_2}(x) = (-1)^{l-m_1} 2^{-l} \sqrt{\frac{(l+m_2)!(1-x)^{m_1-m_2}}{(l-m_2)!(l+m_1)!(l-m_1)!(1+x)^{m_1+m_2}}} \left(\frac{d}{dx}\right)^{l-m_2} (1-x)^{l-m_1}(1+x)^{l+m_1}. \tag{5.23}$$

In a Wigner function $D^l_{m_1,m_2}$, m_1 and m_2 function as matrix indices such that each individual Wigner function is a matrix element of a $(2l+1)$ by $(2l+1)$ representation matrix (previously labeled D_l) acting on $\mathcal{H}_l$. Wigner functions are orthogonal,

$$\int_{\mathrm{SO}(3)} D^l_{m_1,m_2}(k) \overline{D^{l'}_{m_1',m_2'}(k)}\, dk = \frac{1}{2l+1} \delta_{l,l'} \delta_{m_1,m_1'} \delta_{m_2,m_2'}, \tag{5.24}$$

and by the Peter-Weyl theorem are a topological basis the space $L^2(K)$. In fact, the regular representation of $K = \mathrm{SO}(3)$ decomposes as

$$L^2(K) = \bigoplus_l \mathcal{H}_l^* \otimes \mathcal{H}_l, \tag{5.25}$$

where for each l the set of Wigner functions are matrix elements in the space $\mathcal{H}_l^* \otimes \mathcal{H}_l$. Fixing a row m_1 in a particular Wigner matrix gives $(2l+1)$ functions

$$f : K \to \mathbb{C},$$

which form an irreducible representation of K. Each row has its own such representation, and thus there are $(2l+1)$ copies of a $(2l+1)$-dimensional irreducible representation of K [10]. The decomposition into irreducibles can then be written as

$$L^2(K) = \bigoplus_l \dim(\mathcal{H}_l)\,\mathcal{H}_l, \tag{5.26}$$

in agreement with the Peter-Weyl theorem. For our K-decomposition, however, we must decompose the irreducible representations $(D_l, \mathcal{H}_l)$ given by the Wigner functions themselves when they are restricted to M.

We begin with a K-type of dimension $(2l+1)$

$$(D_l, \mathcal{H}_l)$$

that we have constructed from the Wigner function representation in . We restrict its action to action on the four elements of M, which are π-rotations in SO(3):

$$M = \{I = m_{1,1},\ R(\boldsymbol{x}, \pi) = m_{1,-1},\ R(\boldsymbol{y}, \pi) = m_{-1,1},\ R(\boldsymbol{z}, \pi) = m_{-1,-1}\}\,. \tag{5.27}$$

We then decompose the representation space $\mathcal{H}_l$ into $(2l+1)$ basis vectors on which M acts by one of four irreducible representations in (4.52):

$$\widehat{M} = \{1_M = (0,0),\, \mathrm{sgn} = (1,1),\, q_+ = (1,0),\, q_- = (0,1)\} \tag{5.28}$$

Collecting basis vectors that belong to the same irreducible representation, we form subspaces of $\mathcal{H}_l$ of various dimensions. For each K-type, the multiplicity in a principal series representation $\pi_{\sigma,\lambda}$ is then given by the dimension of the subspace associated with $\sigma \in \widehat{M}$. As a result, at fixed σ, each continuous family of principal series indexed by λ has the same K-type decomposition.

The multiplicities of K-types behave differently for l odd and l even. When l is odd, the representation $D_l|_M$ acts on the following subspaces:

$$\begin{aligned} D_l(m)\,|l,0\rangle &= \mathrm{sgn}(m)\,|l,0\rangle\,, \\ D_l(m)\,(|l,n\rangle \pm |l,-n\rangle) &= q_{\mp}(m)\,(|l,n\rangle \pm |l,-n\rangle)\,, \quad n \text{ odd}, \\ D_l(m)\,(|l,n\rangle \pm |l,-n\rangle) &= {}^{\mathrm{sgn}}_{1_M}(m)\,(|l,n\rangle \pm |l,-n\rangle)\,, \quad n \text{ even}. \end{aligned} \tag{5.29}$$

On the other hand, if l is even then it acts on the following subspaces:

$$\begin{aligned} D_l(m)\,|l,0\rangle &= 1_M(m)\,|l,0\rangle\,, \\ D_l(m)\,(|l,n\rangle \pm |l,-n\rangle) &= q_{\pm}(m)\,(|l,n\rangle \pm |l,-n\rangle)\,, \quad n \text{ odd}, \\ D_l(m)\,(|l,n\rangle \pm |l,-n\rangle) &= {}^{1_M}_{\mathrm{sgn}}(m)\,(|l,n\rangle \pm |l,-n\rangle)\,, \quad n \text{ even}. \end{aligned} \tag{5.30}$$

Counting the dimensions of subspaces spanned by the basis vectors in (5.29) and (5.30), we arrive at a formula for K-type multiplicities in principal series of $\mathrm{SL}(3,\mathbb{R})$:

$$\mu_{l=2n}(\pi_{\sigma,\lambda}) = \begin{cases} n+1 & \sigma = 1_M \\ n & \text{otherwise,} \end{cases} \qquad \mu_{l=2n+1}(\pi_{\sigma,\lambda}) = \begin{cases} n & \sigma = 1_M \\ n+1 & \text{otherwise.} \end{cases} \tag{5.31}$$

We see a symmetry between cases of even and odd l, in agreement with [10]. The approach outlined above hinges on the general property of Frobenius reciprocity and therefore can be applied to any K-type decomposition. Through this method, the representations of a very large group, like $\mathrm{SL}(3,\mathbb{R})$ in our example, can be characterized under symmetry-breaking by the representations of a much smaller subgroup M.

In the particular case we are working with, M is a finite group, thus simplifying the calculations of K-types. For general principal series representations of a semisimple group, M is the reductive subgroup int the Langlands decomposition, and this K-type technique will still apply, albeit with possibly more involved calculations.

Chapter 6

Conclusion

In this work we have discussed principal series representations of a semisimple Lie group, with a focus on $\mathrm{SL}(3, \mathbb{R})$. After presenting the geometrical foundation of the subject in the theory of fiber bundles, we showed how such representations arise at the Lie algebra level from infinitesimal decompositions. For $\mathrm{SL}(3, \mathbb{R})$, we restricted our attention to principal series induced from a *minimal* parabolic subgroup in contrast to the previous literature, which focuses on the maximal case [2, 3].

The nilpotent part of the minimal parabolic in $\mathrm{SL}(3, \mathbb{R})$ is isomorphic to the Heisenberg group, allowing for an additional picture of principal series beyond the three classical ones. This is called the *nonstandard* picture in [2], where it was developed for maximal parabolic induction. In this picture, it was found that intertwiners between principal series of the symplectic group are algebraic operators instead of the complicated Knapp-Stein integrals that arise in the classical pictures. However, intertwiners between minimal principal series of $\mathrm{SL}(3, \mathbb{R})$ did not undergo the same simplification.

For any semisimple Lie group G, there is a parabolic subgroup P with a nilpotent part whose Lie algebra is isomorphic to the Heisenberg algebra. Consequently, the nonstandard picture can be developed for the principal series of any group given the right choice of parabolic subgroup. For $\mathrm{SL}(3, \mathbb{R})$, the lack of simple intertwiners could be a result of the fact that this subgroup is minimal. Because induced representations are associated with a fiber bundle over G/P, a larger P implies a smaller quotient space, simplifying the problem from a geometric standpoint.

As geometric considerations have been used to simplify intertwiners in [2, 3], one could attempt a similar approach for minimal principal series of $\mathrm{SL}(3, \mathbb{R})$ without using the non-standard picture. One possible avenue is through the compact picture, which can be identified with the space of functions on the flag manifold $\mathcal{F} \simeq K/M$. The task of finding an intertwiner between principal series becomes that of finding a G-equivariant integral kernel on $\mathcal{F}$. The geometry of the flag manifold, and in particular its inner product, could then guide the search for simple intertwiners.

Lastly, the intertwiners between minimal principal series of $\mathrm{SL}(3, \mathbb{R})$ can be studied through the simpler intertwiners of $\mathrm{SL}(2, \mathbb{R})$ by a process known as *rank-one reduction*. Here rank refers to the dimension of the abelian subgroup A in the Langlands decomposition of the minimal parabolic subgroup. For $\mathrm{SL}(3, \mathbb{R})$, A has dimension 2, yet $\mathrm{SL}(2, \mathbb{R})$ has rank one. Knapp and Stein outline this procedure in [8, 9]: decomposing principal series intertwiners of a semisimple group into those of a subgroup. In this vein, we may use this approach coupled with geometric tools to find a simpler form for minimal principal series intertwiners of $\mathrm{SL}(3, \mathbb{R})$.

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