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Exploring Stochastic Quantum Mechanics and Emergent Gravity

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Exploring Stochastic Quantum Mechanics and Emergent Gravity

A thesis submitted in partial fulfillment of the requirement
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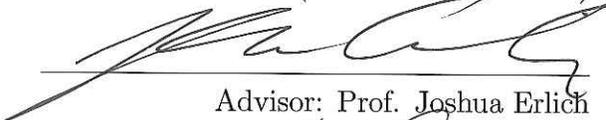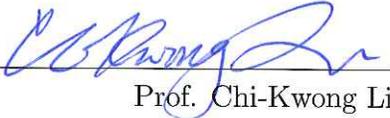Physics from the College of William and Mary in Virginia,

by

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Abstract

It has been proposed that gravity is not a fundamental force, but rather a force that emerges at long distances as a result of quantum phenomena. Recent work has attempted to combine the ideas of stochastic quantization and the quantum field theories in which gravity emerges at these distances. In this project, we extend previous work done on this subject and describe a model for stochastic quantization compatible with general relativity. Numerical simulations of these relativistic stochastic systems were created to test for certain quantum mechanical properties expected to arise from this description, and it was shown that classical equations of motion can accurately produce quantum mechanical behavior. A perturbation theory for field theories is under development to further strengthen the validity of the stochastic interpretation.

Chapter 1

Introduction

It has been proposed that gravity and quantum mechanics (QM) share a common origin. Certain stochastic theories can adequately describe quantum mechanics at the microscopic level, and induce a gravitational interaction at long distances by analogy to Sakharov's induced gravity [1]. The microscopic description of these stochastic theories provides a physical covariant regulator for a stochastic quantum field theory [2]. That is, by "zooming-in" to an unobservably small scale where infinities appear and instead describing the physics at that scale with the theory of stochastic quantization, the infinities will vanish. The most widely read theory of stochastic quantum mechanics was initially developed by Nelson [3] and further developed by Guerra [4]. An interesting result of Nelson's stochastic description of quantum phenomena is the classical description of motion, with particles exhibiting Brownian-like randomness, which suggests a realist interpretation of QM. Furthermore, unlike true Brownian motion, Nelson's mechanics are time-reversal invariant. Unfortunately, the idea of stochastic quantization is no longer being explored with the same fervor as it was during Nelson's time, due to various criticisms of the theory. For instance, where is the nonlocality of QM (such as the collapse of the wavefunction) introduced into the theory? Many similar criticisms came from Nelson himself, and he eventually stopped developing the theory after he failed to see its potential. However, the field of

stochastic mechanics has the enormous potential to resolve fundamental philosophical questions about QM, and its ability to serve as a regulator in field theories cannot be overlooked. By better understanding the generalization of the stochastic description of quantum mechanics to quantum field theory, the role stochastic mechanics plays in a theory of emergent gravity will become more clear, and hopefully the theory of stochastic mechanics will gain some merit.

In this thesis, we aim to extend this theory of stochastic quantum mechanics to form a larger theory compatible with general relativity. Specifically, we will treat the stochastic description as a perturbation to quantum field theory. As the theory is being developed, we numerically simulate the stochastic processes to verify that they are in line with the predictions of quantum mechanics. This thesis begins with the theory behind a stochastic formulation of quantum mechanics. In section 2, the kinematics of a particle undergoing stochastic motion are presented, and an analogy to quantum mechanics is made. Perturbation theory with stochastic processes is then introduced. In section 3, the motion is numerically simulated, and the perturbations for certain systems are found. In particular, the case of the harmonic oscillator is considered for its simplicity. The results from this project are presented in Section 4.

Chapter 2

Theory

Since quantum mechanics is intrinsically random, it seems natural to describe it by a statistical theory such as stochastic mechanics. In this section, we overview the stochastic equations of motion for a quantum stochastic particle, following the presentation given by de la Peña [5] of Nelson's mechanics, and show that the behavior predicted by quantum mechanics can be reproduced in this way. This approach is phenomenological and aims to replicate the predictions of quantum mechanics, beginning with the basic principles of classical mechanics.

Consider an individual particle undergoing a stochastic process. We define two different velocities, u and v, to describe the motion of the particle. The stochastic velocity, u, describes the microscopic movements of the particle, and the particle's convective velocity, v, is analogous to a drift velocity of the particle. The particle's change in position at time t is then given by

$$dx = (u + v)dt + \sqrt{2D}\Delta W(t), \tag{2.1}$$

where D is a diffusion constant and $\Delta W(t)$ is the Wiener process, the mathematical description of Brownian motion. In Nelson's formalism, this is taken to be

$$\Delta W(t) = W(t + dt) - W(t), \tag{2.2}$$

with the stipulation that $\langle \Delta W(t) \rangle = 0$ and $\langle (\Delta W(t))^2 \rangle = dt$, since ΔW is taken to be

Gaussian white noise. The equation of motion given by Eq. (2.1) can be numerically integrated to find the total distance traveled by the particle.

Nelson showed that it was possible to obtain the Schrodinger equation from stochastic motion by proper choice of u, v, and D, so we will follow this derivation here. Let $\rho(x, t)$ be the probability density of particles on x and v_f, v_b be the forward and backward velocities of the stochastic particle, respectively. Then ρ satisfies the forward Fokker-Planck Equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v_f - D\nabla^2 \rho = 0. \tag{2.3}$$

By performing the substitution $t - \Delta t \to t + \Delta t$, which also transforms v_f into $-v_b$, we obtain the backward Fokker-Planck Equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v_b + D\nabla^2 \rho = 0. \tag{2.4}$$

Adding Eqs. (2.3) and (2.4), we find that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v_b = 0. \tag{2.5}$$

Eq. (2.5) is the continuity equation, which represents the local conservation of particles. From [5] we see that $u = \frac{1}{2}(v_f - v_b)$, so we can also combine Eqs. (2.3) and (2.4) to write

$$\nabla \cdot \rho u = D\nabla^2 \rho. \tag{2.6}$$

Eq. (2.6) can be rewritten as $\nabla \cdot (\rho u - D\nabla\rho) = 0$, and integrated to find the equation $\rho u = D\nabla\rho + \nabla \times G$, with G being an arbitrary vector. According to [5], it can be seen that G should be taken as 0 in general, which leads to the formula

$$u = D\nabla \ln\rho, \tag{2.7}$$

which confirms that u describes a diffusive motion. Next, we define the drift velocity v in terms of the action S as

$$v = \frac{a}{m}\nabla S, \tag{2.8}$$

with mass m, and a chosen to cancel the dimensions of the action S. From ρ, we change variables and define the complex pair

$$\psi = \sqrt{\rho}e^{iS} \tag{2.9}$$

$$\psi^* = \sqrt{\rho}e^{-iS}. \tag{2.10}$$

This gives

$$\rho = \psi^*\psi, \tag{2.11}$$

and

$$v = \frac{ia}{2m}\nabla(\ln\psi^* - \ln\psi). \tag{2.12}$$

Introducing a "potential-like" function U, it follows from Eq. (2.5) that

$$\psi^*\left(i\frac{\partial\psi}{\partial t} + \frac{a}{2m}\nabla^2\psi\right) = \psi\left(-i\frac{\partial\psi^*}{\partial t} + \frac{a}{2m}\nabla^2\psi^*\right) = U\psi^*\psi. \tag{2.13}$$

Factoring the conjugate of ψ from the LHS, we obtain

$$i\frac{\partial\psi}{\partial t} = -\frac{a}{2m}\nabla^2\psi + U\psi, \tag{2.14}$$

which is clearly related to the Schrodinger Equation. For brevity, we refer to [5] for finding the function U and for parameterizing a. The result is the Schrodinger Equation,

$$ia\frac{\partial\psi}{\partial t} = -\frac{a^2}{2m}\nabla^2\psi + V\psi, \tag{2.15}$$

where the substitution $a = \hbar$ completes the transformation.

The fact that we can derive the Schrodinger Equation from stochastic principles is in itself a profound statement, as it demonstrates that classical equations of motion can accurately describe a quantum system. This seems to suggest a realist interpretation of QM, and opens the door for the existence of an underlying theory which manifests itself as QM at larger scales. To complete the picture, we wish to further

develop the theory of stochastic quantization by studying perturbation theory. By treating stochastic mechanics as a perturbation in quantum field theory, this description gains new significance. We begin this endeavor in Chapter 3 with the case of the quartic anharmonic oscillator. In the following sections we hope to further validate the choice of stochastic mechanics as a description of quantum mechanics.

Chapter 3

Numerical and Theoretical Methods

The equation of motion for a single stochastic particle given by Eq. (2.1) can be integrated numerically using Itô Calculus (for stochastic integration) to plot trajectories. For example, de La Peña [5] recreated motion in the presence of a physical barrier using this equation of motion. The trajectories of 20 free particles of fixed energy were plotted with and without a barrier; they are shown in Figs. (3.1) and (3.2). Note that in Fig. (3.2), some particles are deflected away from the barrier. An interesting result is that some particles seem to anticipate the barrier, and are reflected before any contact is made. Interference patterns appear due to the particles being deflected off the front and back of the finite-width potential.

In an effort to recreate some of these results, Eq. (2.1) was integrated numerically using Python. At each timestep, dx was calculated and summed with the current position of the particle $x(t)$. For the random source, a random number generator was employed to choose either a forward motion by $\sqrt{dt}$, or a backward motion by $\sqrt{dt}$ (since $\langle (\Delta W) \rangle^2 = dt$) with equal probability. This choice was then multiplied by a factor of $\sqrt{2D}$ to determine the stochastic contribution to the particle's motion. At every point, u and v were numerically evaluated for the chosen wavefunction. Fortunately, the derivation of Schrodinger's Equation in Section 2.1 provided us with the
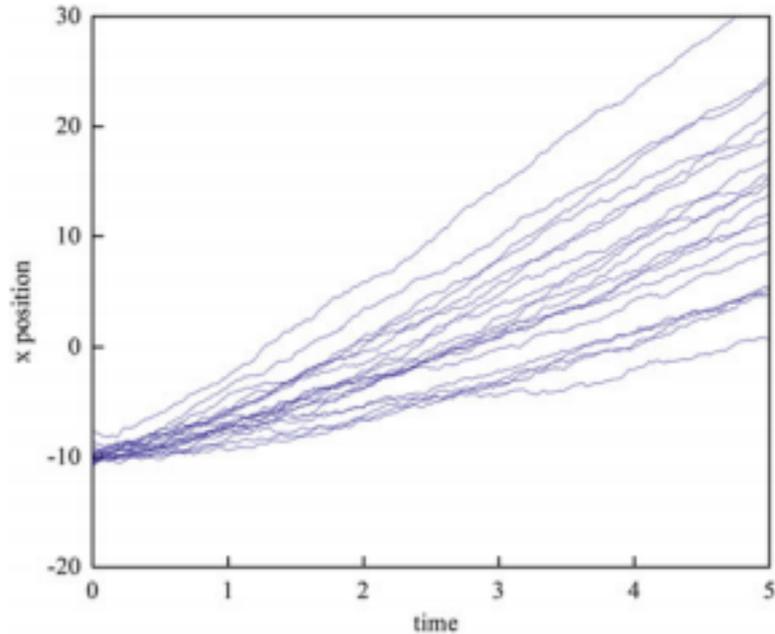
Figure 3.1: Free Particles with Fixed Energy - 20 free particles of fixed energy governed by Eq. 2.1, without a barrier. They are free to travel with positive velocity. Figure reproduced from [5].

useful definitions for u and v given by Eqs. (2.7) and (2.12). Only the wavefunction needed to be specified to the simulation, and it was be recreated using stochastic motion.

In order for the stochastic description of quantum mechanics to have any merit, the theory must be developed further. Thus, the main goal of the project is not just to numerically simulate the basic stochastic equations of motion, but also to develop a stochastic perturbation theory compatible with quantum field theory. In the next section we take a further look at the development of a perturbation theory to complete the stochastic description of QM. To support this, we will consider here the case of the quartic anharmonic oscillator and find the necessary first-order corrections. Suppose

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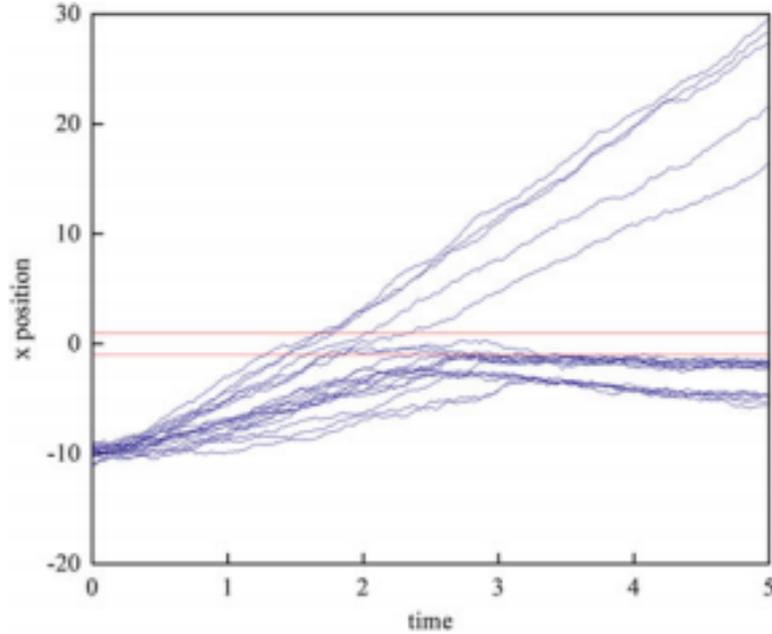
Figure 3.2: Particles of Fixed Energy in the Presence of a Finite Width Barrier - 20 particles of fixed energy, similar to those in Figure 1, but this time in the presence of a finite-width barrier. Particles are deflected from the front and back of the barrier. Figure adapted from [5].

we have the Hamiltonian for the quantum harmonic oscillator given by

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{3.1}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, respectively. We now introduce a quartic perturbation of the form

$$H' = \lambda\hat{x}^4, \tag{3.2}$$

where λ is an arbitrary coefficient used to group similar powers in the expansion.

Expressing $\hat{x}$ in terms of the raising and lowering operators $a_\pm$, we obtain equations

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a_+} + \hat{a_-}) \tag{3.3}$$

$$\hat{a_\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x}). \tag{3.4}$$

A straightforward exercise is to calculate the first-order correction to the nth state, which we can write as

$$E_n^1 = \langle \psi_n^0 | \left(\frac{\hbar}{2m\omega}\right)^2 (\hat{a_-} + \hat{a_+})^4 | \psi_n^0 \rangle . \tag{3.5}$$

Here, the notation E_n^1 describes the first-order correction to the nth state. Since $\langle \psi_m | \psi_n \rangle = \delta_{m,n}$, the expression will vanish when $m \neq n$. We only consider combinations of $\hat{a_\pm}$ that do not cause the expression to vanish. In the $\hat{x}^4$ case, this means that only the permutations of two raising and two lowering operators will contribute to the expression. Dropping the hats from operators for simplicity, we express the raising and lowering operators in terms of n and ψ as

$$a_- |\psi_n^0\rangle = \sqrt{n} |\psi_{n-1}\rangle \tag{3.6}$$

$$a_+ |\psi_n^0\rangle = \sqrt{n+1} |\psi_{n+1}\rangle . \tag{3.7}$$

We then calculate the value of each permutation below:

$$\langle \psi_n^0 | a_- a_- a_+ a_+ | \psi_n^0 \rangle = (n+1)(n+2) \tag{3.8}$$

$$\langle \psi_n^0 | a_+ a_+ a_- a_- | \psi_n^0 \rangle = n(n-1) \tag{3.9}$$

$$\langle \psi_n^0 | a_- a_+ a_- a_+ | \psi_n^0 \rangle = (n+1)(n+1) \tag{3.10}$$

$$\langle \psi_n^0 | a_+ a_- a_+ a_- | \psi_n^0 \rangle = n^2 \tag{3.11}$$

$$\langle \psi_n^0 | a_- a_+ a_+ a_- | \psi_n^0 \rangle = n(n+1) \tag{3.12}$$

$$\langle \psi_n^0 | a_+ a_- a_- a_+ | \psi_n^0 \rangle = (n+1)n. \tag{3.13}$$

Summing each contribution, we find the expression for the first-order correction to the nth energy level:

$$E_n^1 = \left(\frac{\hbar}{2m\omega}\right)^2 \lambda(6n^2 + 6n + 3). \tag{3.14}$$

It would be beneficial if we could also find the first order correction to the nth energy level of the wavefunction for the quartic anharmonic oscillator, which is a slightly more complex endeavor. The first-order correction is given neatly by

$$\left|\psi_n^1\right\rangle = \sum_{m \neq n} \frac{\langle \psi_m^0| H' |\psi_n^0\rangle}{E_n^0 - E_m^0} \left|\psi_m^0\right\rangle. \tag{3.15}$$

To find the analytical solution for $|\psi_n^1\rangle$, we first find the matrix elements $\langle \psi_m^0| x |\psi_n^0\rangle$. These elements are given by the recurrence relation

$$(x)_{mn} = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{m+1}\delta_{m+1,n} + \sqrt{n+1}\delta_{m,n+1}). \tag{3.16}$$

Squaring, we find the matrix elements for x^2 to be

$$(x^2)_{mn} = \frac{\hbar}{2m\omega}(\sqrt{m(m+1)}\delta_{n,m-2} + (2m+1)\delta_{mn} + \sqrt{(m+1)(m+2)}\delta_{n,m+2}), \tag{3.17}$$

and squaring again we find the elements $(x^4)_{mn}$, which for brevity we choose to omit. Using Eq. (3.15) we can then express the first-order correction to the wavefunction as the rather ugly equation

$$\begin{aligned}
\left|\psi_n^1\right\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \Bigg(&\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{-4\hbar\omega} \left|\psi_{n+4}^0\right\rangle \\
&+ \frac{\sqrt{(n+1)(n+2)}(4n+6)}{-2\hbar\omega} \left|\psi_{n+2}^0\right\rangle \\
&+ \frac{\sqrt{n(n-1)}(4n-2)}{2\hbar\omega} \left|\psi_{n-2}^0\right\rangle \\
&+ \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4\hbar\omega} \left|\psi_{n-4}^0\right\rangle \Bigg).
\end{aligned} \tag{3.18}$$

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Chapter 4

Stochastic Interpretation

The derivation of Schrodinger's Equation from stochastic principles gave us useful equations for the diffusion and drift velocities (Eqs. 2.7 and 2.12) that allow us to simulate the motion of a stochastic particle given only the wavefunction and arbitrary initial conditions. With no particular wavefunction specified, we see the "random walk" of the stochastic particle as shown in Fig. (4.1). The particle jumps either forward or backward with a magnitude of $\sqrt{dt}$ at every time step, resulting in jittery Brownian motion.
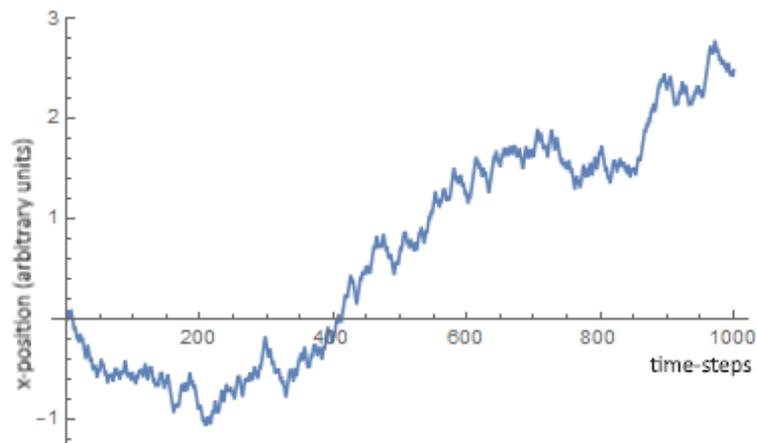

Figure 4.1: Random Walk - The trajectory of a stochastic particle. Integration was performed over 1,000 steps from time $t = 0$ to $t = 1$ second. The horizontal axis represents the number of time steps, while the vertical axis is the particle's position in arbitrary units of distance.

We now introduce the wavefunction to provide motivation for this formalism of quantum mechanics. In this project we used the wavefunction for the quantum harmonic oscillator, which has an exact analytical form given by

$$\psi(x) = \sqrt{\frac{1}{n!2^n a\sqrt{\pi}}} H_n\left(\frac{x}{a}\right) e^{\frac{-x^2}{2a^2}}, \tag{4.1}$$

where units such as $\hbar, m$ are taken to be 1 for the purpose of our simulations, and $H_n\left(\frac{x}{a}\right)$ is the nth Hermite polynomial evaluated at $\frac{x}{a}$. Eq. (2.1) was numerically simulated in Python with this choice of ψ. Fig. (4.2) shows the results of this numerical simulation with a harmonic oscillator at energy level $n = 2$. We note that x and t have arbitrary units and do not need to be exact for our purposes. The simulation was reproduced in 100 trials, with each trial running in 1000 increments of comparatively small width $dt = 1 \times 10^{-2}$. The initial velocities were chosen arbitrarily to be $u = 1$ and $v = 0$. Different initial velocities were tested but did not cause significant changes in the outcome. That is, the contribution due to the initial velocities were diluted over the course of thousands of time steps. In Fig. (4.2a) we see the interesting result that the particles all had final positions constrained between the two central nodes. This is due to the initial position, which was $x = 0$ for each particle. In Fig. (4.2b) we choose the initial particle position to be somewhere in the negative regime; we arbitrarily choose $x = -2$.

To understand why this happens, we examine u which relies on the gradient of the natural log of the probability density. This creates an infinitely high repulsive force where $\rho(x,t) = 0$, that is, at the harmonic oscillator nodes. This acts to create areas where the particles are constrained to bounce back and forth. An interesting result is that this infinite repulsive force is an artifact of the simulation, and the container can be escaped by taking a somewhat larger timestep. In Fig. (4.3) we increase the timestep to from $dt = 1 \times 10^{-2}$ to $dt = 1 \times 10^{-1}$. The observed behavior is

(a) Initial position of $x = 0$
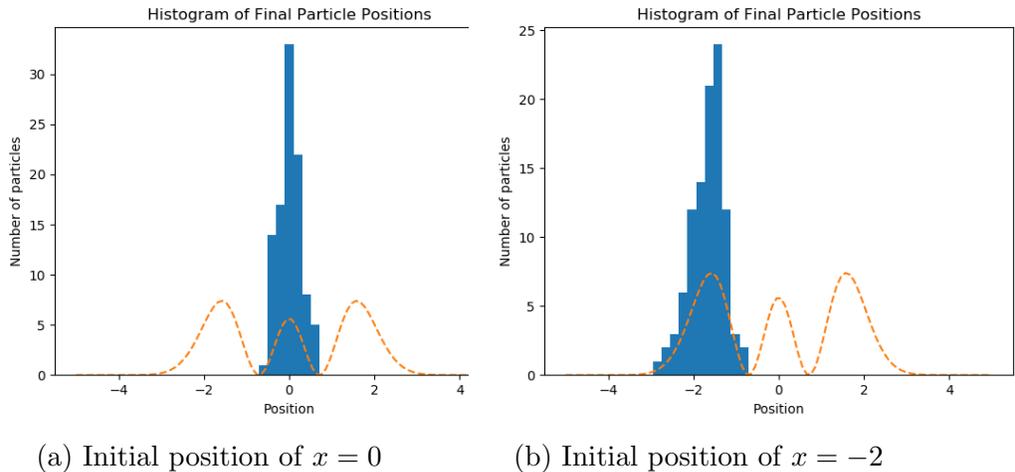(b) Initial position of $x = -2$

Figure 4.2: Effect of Initial Particle Positions on the Harmonic Oscillator - Histogram of final particle positions for 100 trials. The wavefunction for the harmonic oscillator with $n = 2$ was used, and its probability density is overlayed as a dotted line. Note that the magnitude of the overlayed wavefunction is arbitrary and can be adjusted by choice of the parameter a.

incredibly interesting, as we see that a particle is free to move around once it escapes the barriers, propagating through free space. One cannot help but to draw parallels to quantum tunneling, and more work should be done in the future on reconciling this feature of the simulation with the stochastic description of quantum mechanics.

Based on this behavior, we take an ensemble of particles, each with random initial positions taken from a Gaussian distribution, and expect this to nicely reproduce the probability density of the quantum harmonic oscillator. From this starting ensemble we obtain Fig. (4.4), where the overlayed wavefunction is parameterized to account for the number of trials. We see that the stochastic equations of motion for an ensemble of particles with initial positions randomly chosen from a Gaussian distribution accurately produce the probability density for the quantum harmonic oscillator. Since initial positions are selected from a normal distribution centered around $x = 0$, it is interesting that so many particles in the ensemble end up at the outer fringes, when
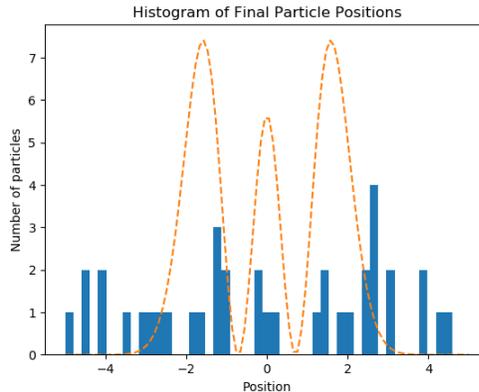
Figure 4.3: Harmonic Oscillator with Large Timestep - Histogram of final particle positions for 50 trials. The wavefunction for the harmonic oscillator with $n = 2$ was used, and its probability density is overlaid as a dotted line. A larger timestep of $dt = 1e - 1$ was used, allowing the particles to escape infinite repulsive forces and travel freely about.

more should be trapped in the center. It is possible that this is due to a fortunate coincidence resulting from either the chosen initial conditions or the widths of the nodes. However, we believe that this is an artifact of the timestep as shown in Fig. (4.3). A large enough timestep allows particles to occasionally escape into other areas, but too large of a timestep will encourage particles to escape from the picture completely. Fine-tuning the timestep to accurately reproduce experimental results and relating the timestep to some physical phenomenon is an area which deserves further investigation. If it were to be found that a certain value is optimal for reproducing the probability distribution, this would be good motivation for considering a discrete, finite timestep in the theory. See Appendix A for the python code used to perform the numerical integration.

In Sec. 3 we derived the first-order correction to the nth state of the quartic anharmonic oscillator given by Eq. (3.18). From Eq. (2.7), we know the diffusion velocity is given by $u = \frac{\partial \ln\rho}{\partial x}$. Since the full wavefunction of a perturbed system is
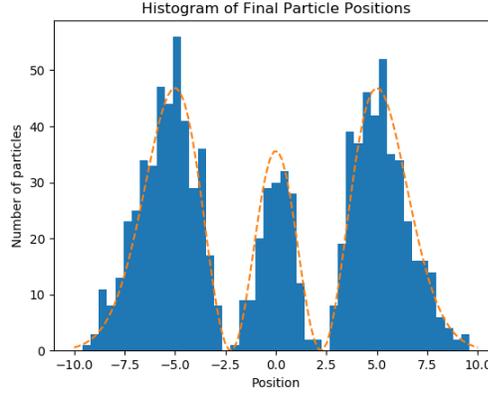
Figure 4.4: Harmonic Oscillator with an Initial Ensemble of Particles - Histogram of final positions for an ensemble of 1000 particles. The wavefunction for the harmonic oscillator with $n = 2$ was used, and its probability density is overlayed as a dotted line. A Gaussian distribution was used to determine the initial distribution of the particles.

given by

$$\psi_n(x) = \psi_n^0(x) + \lambda\psi_n^1(x), \tag{4.2}$$

we can write

$$\ln\psi_n(x) = \ln\psi_n^0(x) + \lambda\left(\frac{\psi_n^1(x)}{\psi_n^0(x)}\right) + \mathcal{O}(\lambda^2). \tag{4.3}$$

Thus, the diffusion velocity for a perturbation is given by

$$u \propto \frac{\partial\ln\psi}{\partial x} = u_0 + \lambda\frac{\partial}{\partial x}\left(\frac{\psi_n^1}{\psi_n^0}\right) + \mathcal{O}(\lambda^2), \tag{4.4}$$

where u_0 is the diffusion velocity in the unperturbed system. We note that λ now serves the purpose of adjusting the strength of the perturbation, and should be taken as $\lambda = 1$ for the full perturbation to take effect. Fig. (4.5) shows the final positions of an ensemble of 1000 particles with the quartic anharmonic oscillator probability distribution as a dashed line overlay.

Now that we have shown the stochastic equation of motions accurately recreate the predictions of quantum mechanics, it is time to further develop the theory beyond
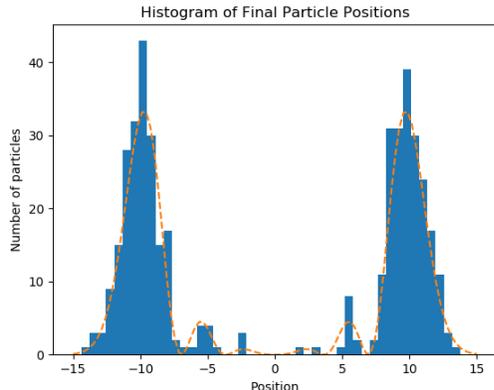
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Figure 4.5: Perturbation of the Harmonic Oscillator with an Initial Ensemble of Particles - Histogram of final positions for an ensemble of 1000 particles. The wavefunction for the quartic anharmonic oscillator with $n = 5$ was used, and its probability density is overlayed as a dotted line.

the scope of Nelson [3] and de La Peña [5]. Nelson stopped working on his theory of stochastic mechanics when he no longer believed it was the underlying theory for quantum mechanics. However, we believe the microscopic description of the stochastic processes described in the previous sections could provide a physical covariant regulator for the quantum field theory under development in the present chapter. Under these circumstances, Nelson's stochastic mechanics could be the basis for a theory of quantum gravity, where gravity emerges from the stochastic description at large enough distances. This motivates us to achieve a better understanding of the generalization of the stochastic description of quantum mechanics to quantum field theory. Guerra and Ruggiero [7] showed that there is a straightforward process to extend stochastic mechanics to field theory, however it has yet to be shown that there is a similar generalization for perturbative stochastic mechanics.

Suppose we have a free scalar field $\phi(x)$. The action is given in terms of the free Lagrangian by

$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \Delta V(\phi) \right). \tag{4.5}$$

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Using the Euler-Lagrange equations, we find

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \frac{\partial L}{\partial \phi} = -m^2 \phi - \frac{\partial \Delta V}{\partial \phi}, \tag{4.6}$$

allowing us to write

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{4.7}$$

for the free theory where $\Delta V = 0$. This is the Klein-Gordon Equation, a relativistic wave equation related to Schrodinger's Equation which admits solutions $\phi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$ with momentum modes $\mathbf{k}$. The solution can be expanded as

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}} \right) \tag{4.8}$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ become the raising and lowering operators due to quantum mechanical conditions. We note that the dot product is $\mathbf{k} \cdot \mathbf{x} = k_0 x^0 + k_i x^i \omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x}$. The action is then given by

$$S = \int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \frac{1}{2} \left(|\dot{\phi}_{\mathbf{k}}| - \omega_{\mathbf{k}}^2 |\phi_{\mathbf{k}}|^2 \right). \tag{4.9}$$

This is precisely the expansion one can do for a harmonic oscillator, and one can think of Eq. (4.9) as representing the free field as a sum of independent harmonic oscillators at each momentum mode with frequencies $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Guerra and Ruggiero [7] generalized stochastic mechanics to field theory by using Nelson's mechanics for each harmonic oscillator mode $\mathbf{k}$.

For example, let us consider the ground state. With the Schrödinger equation written as

$$i\frac{\partial \psi[\phi(\mathbf{x}, t)]}{\partial t} = H\psi, \tag{4.10}$$

where H is the Hamiltonian of the system, we can write the Schrödinger-representation wavefunction as

$$i\frac{\partial \Psi[\phi(\mathbf{x}, t)]}{\partial t} = \frac{1}{2} \int d^3\mathbf{x} \left(-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + (\nabla\phi)^2 + m^2\phi^2 \right) \psi[\phi(\mathbf{x}), t]. \tag{4.11}$$

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By decomposing 4.11 into its momentum modes, the ground state for each mode becomes the ground state harmonic oscillator wavefunction

$$\psi_{\mathbf{k}}(\phi_{\mathbf{k}}, t) = \left(\frac{\omega_{\mathbf{k}}}{\pi}\right)^{\frac{1}{4}} e^{-(\omega_{\mathbf{k}}|\phi_{\mathbf{k}}|^2 + i\omega_{\mathbf{k}}t)/2}. \tag{4.12}$$

We now have the capability to find the stochastic equations of motion in the field theory. Summarizing the derivation followed by Erlich [2], we use Eqs. (2.7) and (2.8) to define

$$u_{\mathbf{k}} = 2\frac{\partial \ln|\psi_{\mathbf{k}}|}{\partial \phi_{\mathbf{k}}^*} \tag{4.13}$$

and

$$u_{\mathbf{k}} = 2\frac{\partial S_{\mathbf{k}}}{\partial \phi_{\mathbf{k}}^*} \tag{4.14}$$

where $S_{\mathbf{k}}$ is defined as $S_{\mathbf{k}} = -\frac{1}{2}\omega_{\mathbf{k}}t$. Using the velocities $b_{\mathbf{k}} = v_{\mathbf{k}} + u_{\mathbf{k}}$ and $b_{*\mathbf{k}} = v_{\mathbf{k}} - u_{\mathbf{k}}$, the behavior of the ground state of the harmonic oscillator can be described by the Ornstein-Uhlenbeck process [2] with

$$d\phi_{\mathbf{k}} = -\omega_{\mathbf{k}}\phi_{\mathbf{k}}dt + d\xi_{\mathbf{k}}, \tag{4.15}$$

which in terms of $\phi(x)$ becomes

$$d\phi = -\sqrt{-\nabla^2 + m^2}\phi dt + d\xi. \tag{4.16}$$

Here, $\xi(\mathbf{x}; t)$ is the Brownian process, and Eq. (4.16) is the field theory analogy to Nelson's equation of motion, Eq. (2.1). Note that the way to think about the square root term is through its Taylor expansion.

Now that the ground state process for the field theory is defined, it should be relatively straightforward to develop a perturbation theory around it. It should also be clear at this point why we chose to study the harmonic oscillator in the earlier sections. Since the free field reduces to an infinite collection of harmonic oscillators, with one at each momentum mode, and the ground state for each mode is the ground state

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harmonic oscillator, the perturbation theory can be developed around the stochastic process for the harmonic oscillator.

We will use the quartic anharmonic oscillator studied in Chapter 3 as a starting point. Returning to Eq. (4.5), we write the perturbed Lagrangian (analogous to the λx^4 perturbed Hamiltonian in the nonrelativistic case) as

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{4.17}$$

In Chapter 3, we found the permutations of raising and lowering operators that contributed to the final sum. For the field theory case, we express the operator $\phi(\mathbf{x})$ in terms of raising and lowering operators (similar to Eq. 3.5) as

$$\phi(x)^4 = \left[\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega_\mathbf{k}}} \left(a_\mathbf{k} e^{-i(\omega_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})} + a_\mathbf{k}^\dagger e^{i(\omega_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})} \right) \right]^4. \tag{4.18}$$

Unfortunately, this is where we ran out of time this semester. However, it is easy to see from Eq. (4.18) how one could follow a similar treatment to the one provided in Chapter 3, and develop a stochastic perturbation theory around the free field theory developed earlier in the chapter.

Chapter 5

Conclusion/Outlook

The Schrodinger Equation was derived from principles of stochastic motion following the treatment given by Nelson [3], and we have shown that quantum mechanical behavior can be reproduced using these equations through numerical simulation. We investigated the case of the harmonic oscillator, and generalized its stochastic description to a quantum field theory. Also currently under development is a perturbation of quantum field theory that aims to reproduce the behaviour described by relativistic quantum field theory using stochastic equations of motion. We have shown here that it will be a relatively straightforward endeavor to develop a perturbation theory around the free theory presented in Chapter 4. Since the free field expansion reduces to a collection of independent harmonic oscillators existing at each momentum node, the theory can be developed around the case of the harmonic oscillator and generalized easily. It is curious that this type of analysis has not been performed before, especially considering the potential consequences: should a full stochastic field theory ever be developed, a theory of quantum gravity could arise for free by treating the microscopic description as a regulator.

Certain observations made during the project raised more questions than they answered. In particular, the timestep problem described in Chapter 4 is an area which deserves further thorough exploration. The problem is seemingly arbitrary at first

glance - why should a simulation timestep have physical significance? We recall that the numerical integration program (graciously provided in Appendix A) is an approximation to Itô calculus, which is actually a continuous process. If the simulation requires discrete timesteps of a certain size in order to accurately replicate probability distributions, it could be that the continuous description of Nelson's mechanics actually breaks down at a certain scale. In this regime, the stochastic processes would look more like a particle receiving kicks at random points in time. Furthermore, is there any physical meaning to the precise value of the timestep? If the step is too small, the particle has an extremely low probability of escaping its initial container (to the point where a particle with continuous motion would not be able to), but if the step is too large, then it will escape from the system entirely. This is the type of problem that will keep one awake at night, and it is for this reason that the code is provided in the Appendix.

The next course of action in this project would be to complete the stochastic perturbation theory developed around the free theory presented in Chapter 4. This process would be roughly analogous to the process of finding the first-order corrections to the nth state harmonic oscillator wavefunction in Chapter 3. The various permutations of the raising and lowering operators would be found in a similar manner to that of Chapter 3. When this process is completed, a more complete stochastic field theory will result.

There is still much more work to be done in the field of stochastic quantum mechanics. The intricacies of the stochastic theory need to be fleshed out; for example, how exactly do entangled multi-particle states manifest themselves in this theory? Nelson actually incorporated particle spins into his theory, whereas in this paper we considered only spinless particles. Another criticism of stochastic mechanics is the nonlocal description of motion. That is, in Nelson's formalism, the velocity of a par-

ticle here depends on the gradient of the probability distribution of the wavefunction over there. This is an interesting criticism, since modern physicists seem to be fine with the nonlocality of quantum mechanics! Whether or not nonlocal theories should be acceptable descriptions of the universe is a philosophical matter which we will not delve into. Hopefully, by developing a complete stochastic field theory, something that looks like emergent gravity will appear and breathe new life into Nelson's stochastic mechanics.

Appendix A

Stochastic Numerical Integration

The problem of fine-tuning the timestep in Chapter 4 deserves more exploration. Here, we have provided the script used to generate Figs. 4.2 - 4.5. All python scripts were run in Sublime text. There was initially a compatibility issue with matplotlib and the Windows operating system where plots would not persist. This was resolved by creating a custom build program.

A.1   oscillator.py

The following is the Python code which numerically integrates Nelson's stochastic equation of motion given by Eq. (2.1). In this example, the positions for an ensemble of 1000 particles are calculated over 1000 time steps. The final positions are recorded and histogrammed against a plot of the probability distribution for the harmonic oscillator.

```
'''************************************************************************
oscillator.py
John Mitchell
NB: To get matplotlib figures to show on windows in Sublime, run using custom
build system "python"
'''

import numpy
import scipy
import matplotlib.pyplot as plt
import math
```

```python
def hermite(x, n):
# hardcoded Hermite polynomials
herm = 0
if (n == 0):
herm = 1
elif (n == 1):
herm = 2 * x
elif (n == 2):
herm = 4 * (x ** 2) - 2
elif (n == 3):
herm = 8 * (x ** 3) - 12 * x
elif (n == 4):
herm = 16 * (x ** 4) - 48 * (x ** 2) + 12
elif (n == 5):
herm = 32 * (x ** 5) - 160 * (x ** 3) + 120 * x
elif (n == 6):
herm = 64 * (x ** 6) - 480 * (x ** 4) + 720 * (x ** 2) - 120
elif (n == 7):
herm = 128 * (x ** 7) - 1344 * (x ** 5) + 3360 * (x ** 3) - 1680 * x
elif (n == 8):
herm = 256 * (x ** 8) - 3584 * (x ** 6) + 13440 * (x ** 4) - 13444 * \
(x ** 2) + 1680
elif (n == 9):
herm = 512 * (x ** 9) - 9216 * (x ** 7) + 48384 * (x ** 5) - 80640 * \
(x ** 3) + 30240 * x
elif (n == 10):
herm = 1024 * (x ** 10) - 23040 * (x ** 8) + 161280 * (x ** 6) - 403200 * \
(x ** 4) + 302400 * (x ** 2) - 30240
else:
print("n out of range")
return herm

def particle_box(x, n):
# Particle in a box wavefunction
a = 10
return (2 / a) * (math.sin(n * math.pi * x / a) ** 2)

def get_harmonic_oscillator(x, n):
# Get probability distribution of harmonic oscillator
alpha = 1e-1 # (m * omega / hbar)
y = numpy.sqrt(alpha) * x
herm = hermite(y, n)
coeff = math.pow((alpha / numpy.pi), 1/4)
fac = 1 / numpy.sqrt((2 ** n) * math.factorial(n))
exp = numpy.exp((-y ** 2) / 2)
psi = coeff * fac * herm * exp
return psi

def get_ln_harmonic_oscillator(x, n):
# Get the natural logarithm of rho
```

```python
    psi = get_harmonic_oscillator(x, n)
    rho = psi * psi # create numpy array
    ln = numpy.log(rho) # divided by reference rho
    return ln

def get_deriv_ln_harmonic_oscillator(x, n):
    # Numerically approximate the derivative at a point
    h = 1e-1
    f_low = get_ln_harmonic_oscillator(x - (h/2), n)
    f_high = get_ln_harmonic_oscillator(x + (h/2), n)
    return (f_high - f_low) / h

def get_quartic_harmonic_oscillator(x, n):
    # Anharmonic oscillator with quartic perturbation term
    lam = .01 # How much we want to "turn on" the perturbation
    if (n - 4 < 0) or (n + 4 > 10):
    # Return 0 if the hermite polynomial does not exist for the given value of n
    print("Try again with a different n")
    return 0
    hbar = 1
    omega = 1
    alpha = 1
    coeff_1 = numpy.sqrt((n + 1) * (n + 2) * (n + 3) * (n + 4)) / (-4 * hbar * omega)
    coeff_2 = (4*n + 6) * numpy.sqrt((n + 1) * (n + 2)) / (-2 * hbar * omega)
    coeff_3 = (4*n - 2) * numpy.sqrt(n * (n - 1)) / (2 * hbar * omega)
    coeff_4 = numpy.sqrt(n * (n - 1) * (n - 2) * (n - 3)) / (4 * hbar * omega)
    term_1 = coeff_1 * get_harmonic_oscillator(x, n + 4)
    term_2 = coeff_2 * get_harmonic_oscillator(x, n + 2)
    term_3 = coeff_3 * get_harmonic_oscillator(x, n - 2)
    term_4 = coeff_4 * get_harmonic_oscillator(x, n - 4)
    psi = (alpha ** 2) * (coeff_1 * term_1 + coeff_2 * term_2 + coeff_3 * term_3 + \
    coeff_4 * term_4)
    return get_harmonic_oscillator(x, n) + (lam ** 2) * psi


n = 2
 # energy level
dt = 1e-2 # size of a time step
sqrtdt = numpy.sqrt(dt) # magnitude (x-dim) of each stochastic step
tmax = 10 # time at which simulation is stopped (time begins at -tmax)
tsteps = int(tmax / dt)
posinit = 0 # initial value of particle's position
trials = 1000 # number of trials to run for histogram
final_pos = [] # list of final positions after each trial is completed
bin_num = 50

xmin = -15
xmax = 15
xsteps = 100
dx = (xmax - xmin) / xsteps

tspace = numpy.linspace(-tmax, tmax, num=int(tmax/dt)) # linear space containing
```

```python
# timesteps
xspace = numpy.linspace(xmin, xmax, xsteps)
dW = [] # change in W

print("Beginning simulation")
for j in range(0, trials):
print("Trial " + str(j + 1) + " / " + str(trials))
posinit = numpy.random.normal(loc=0, scale=15, size=None) # randomly choose
# an initial position according to a Gaussian dist.
pos = [posinit]
u = 1 # stochastic velocity
v = 0 # drift velocity
d = 1 # (hbar / 2m)
sqrt2d = numpy.sqrt(2 * d)

for i in range(1, tsteps):
randn = numpy.random.randint(0, high=2) # choose 0 or 1 randomly
dW = sqrtdt * ((-1) ** randn) # Move forward if 0, backward if 1
dx = (u + v) * dt + (sqrt2d * dW)
# print("t: " + str(tspace[i]) + ", dt: " + str(dt) + ", step: " + #
# str(i) + " | previous " + str(W[i - 1]) + " : next " + str(W[i-1] + dW))
x_prev = pos[i - 1]
x_cur = x_prev + dx
pos.append(x_cur)
u = get_deriv_ln_harmonic_oscillator(x_cur, n) # Change for perturbation
#u = numpy.gradient(get_ln_harmonic_oscillator(x_cur, n), dx)
final_pos.append(pos[tsteps - 1]) # Append the list with the particle's final
# position

# Histogram data
bins = numpy.linspace(xmin, xmax, bin_num)

plt.hist(final_pos, bins)

plt.xlabel('Position')
plt.ylabel('Number of particles')
plt.title(r'Histogram of Final Particle Positions')
plt.plot(xspace, 500 * (get_harmonic_oscillator(xspace, n) ** 2), '--')

plt.show()
```

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