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Partial Difference Sets in Nonabelian Groups and Strongly Regular Cayley Graphs

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Abstract

A regular graph Γ with v vertices and valency k is said to be a (v, k, λ, μ)-strongly regular graph if any two adjacent vertices are both joined to exactly λ other vertices and two nonadjacent vertices are both joined to exactly μ other vertices. Let G be a group of order v and D a k-element subset of G. Then D is called a (v, k, λ, μ)-partial difference set if for every nonidentity element g of D, the equation $d_1 d_2^{-1} = g$ has exactly λ solutions $(d_1, d_2) \in D \times D$; and for every nonidentity element g' of G not in D, the equation $d_1 d_2^{-1} = g'$ has exactly μ solutions. It is known that a subset D of G with $e \notin D$ and $\{d^{-1} | d \in D\} = D$ is a partial difference set if and only if the Cayley graph generated by D is strongly regular. Yoshiara [9] has given two lemmas that describe the conditions needed for an automorphism group to act regularly on a finite generalized quadrangle. De Winter, Kamischke, and Wang [11] build upon the work of Benson to construct partial difference sets in abelian groups. In this work, we confirm Yoshiara's results, and use De Winter, Kamischke, and Wang's result in place of Benson's to generalize Yoshiara's results to nonabelian groups. In the process, we are able to rule out the existence of many partial difference sets in nonabelian groups.

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Chapter 1

Introduction

Yoshiara [9] studies the regular automorphism groups acting on generalized quadrangles. This is the basic model from which we will build to study different structures throughout this paper. In this section, we will present the background and definitions of some of these structures, which will then be further analyzed later in the paper.

Definition 1.1. An **automorphism** is a structure-preserving map that sends a structure to itself. For a graph, it is a permutation of the vertex set which preserves the adjacency and nonadjacency of vertices.

Definition 1.2. The set of all automorphisms of an object forms the **automorphism group.**

Definition 1.3. An automorphism group is considered **regular** if it is transitive and no nonidentity elements of the group fix any elements of the set being permuted. Therefore, for all i, j in the vertex set of a graph G, $\phi(i) = j$ where $i \neq j$ and ϕ represents the nonidentity automorphism.

Definition 1.4. An **incidence structure** has two distinct types of objects that are connected by a single relationship.

Definition 1.5. A **generalized quadrangle** (GQ) is an incidence structure described as $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint, nonempty sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$), and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$), and two distinct lines are incident with at most one point;

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \text{ I } M \text{ I } y \text{ I } L$.

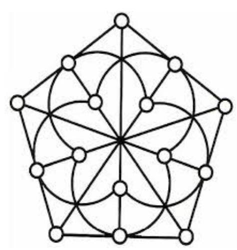


Figure 1.1: Generalized Quadrangle

The integers s and t are the parameters of the generalized quadrangle, and $\mathcal{S}$ is said to have order (s, t); if $s = t$, then $\mathcal{S}$ is said to have order s.

Definition 1.6. The **valency** of a graph is the number of neighbors of any vertex. If all vertices in a graph have the same valency k, then the graph is said to be regular with valency k. (Note the distinction between "regular group" and "regular graph.")

Definition 1.7. A **strongly regular graph** with parameters (v, k, λ, μ) (denoted by $(v, k, \lambda, \mu) - SRG$) is an undirected graph, without loops or multiple edges, on v vertices which is regular with valency k, and which has the following two properties:

(i) for each pair (x, y) of adjacent vertices there are exactly λ vertices mutually adjacent to x and to y, and

(ii) for each pair (x, y) of nonadjacent vertices there are exactly μ vertices mutually adjacent to x and to y.

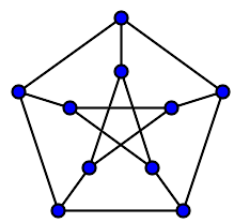


Figure 1.2: Petersen Graph: (10,3,0,1)-SRG

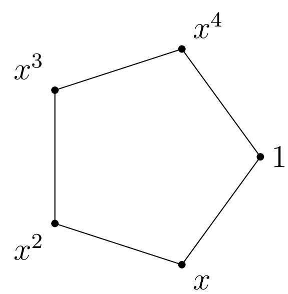
A strongly regular graph, therefore, has the property that the number of common neighbors of two distinct vertices depends only on whether they are adjacent or nonadjacent.

Definition 1.8. An **adjacency matrix** is a matrix with the rows and columns labelled as the graph vertices with a 1 in position (n_i, n_j) if n_i and n_j are adjacent and a 0 if n_i and n_j are not adjacent.

Definition 1.9. A **conference graph** is a strongly regular graph with parameters $v, k = \frac{v-1}{2}, \lambda = \frac{v-5}{4}$, and $\mu = \frac{v-1}{4}$. A conference graph is unique in that the eigenvalues of its adjacency matrix need not be integers. In fact, if G is a conference graph and $2k + (v - 1)(\lambda - \mu) = 0$, then the eigenvalues are not integers.

Definition 1.10. Let G be a group, and $D \subseteq G$ such that $D^{-1} = D$ and $1 \notin G$. The **Cayley graph** $\text{Cay}(G, D)$ is defined to be the undirected graph with vertex set G, which has no loops, and $g, h \in G$ are adjacent if and only if $gh^{-1} \in D$.

Example: Let $G = \langle x \rangle = C_5 = \{1, x, x^2, x^3, x^4\}$ and $D = \{x, x^4\}$. Then $x^2, x^3 \in G$ are adjacent since $x^3x^{-2} = x \in D$. However, $x, x^3 \in G$ are not adjacent since $x^3x^{-1} = x^2 \notin D$. For the reverse direction, since $x^4x^{-3} = x \in D$, then $x^3, x^4 \in G$ are adjacent.



Definition 1.11. We say that S is a (v, k, λ, μ)-PDS (**partial difference set**) of a group G if $|G| = v$, $|S| = k$, and each nonidentity element $g \in G$ can be written either λ or μ different ways as $g = ab^{-1}$, where $a, b \in S$, depending on whether or not g is in S.

Partial difference sets serve as a tool to construct strongly regular Cayley graphs. We prove this connection in the proposition below.

Proposition 1.12. *A (v, k, λ, μ) partial difference set D, with $1 \notin D$ and $D^{-1} = D$, is equivalent to a (v, k, λ, μ) strongly regular Cayley graph, Cay(G, D) arising from G.*

Proof. Let Γ be a strongly regular Cayley graph. Then by Definition 1.10, Γ is an undirected graph with $D \subseteq G$ where $D = D^{-1}$, $1 \notin D$, and $a, b \in G$ are adjacent if and only if $ab^{-1} \in D$. Let $d \in D$. Since d is adjacent to 1, there exist λ elements of D that are adjacent to 1 and d. Let a be one of these λ elements. Then there is $c \in D$ such that $ac = d$. Since $D = D^{-1}$, $c = b^{-1}$ for some $b \in D$, so d can be written as ab^{-1} for $a, b \in D$ in λ different ways. Now let $g \notin D$, so g is not adjacent to 1. Then there exist μ elements of D that are adjacent to 1 and g. Let d be one of these μ elements. Then there is an $a \in D$ such that $gd^{-1} = a$ so $g = ad$. Since $D = D^{-1}$, $d = b^{-1}$ for some $b \in D$, so g can be written as ab^{-1} for $a, b \in D$ in μ different ways. This satisfies the requirements for a partial difference set given by Definition 1.11. Now assume D is a $(v, k, \lambda, \mu) - PDS$ with $D^{-1} = D$ and $1 \notin D$. Let $d \in D$. Since d can be written in λ ways as ab^{-1}, with $a, b \in D$, then $d(b^{-1})^{-1} = a \in D$, and d has exactly λ neighbors in D. Now let $g \notin D$. Since g can be written in μ ways as ab^{-1}, with $a, b \in D$, then $g(b^{-1})^{-1} = a \in D$, and g has exactly μ neighbors in D. □

Definition 1.13. A **partial geometry** is an incidence geometry said to be of order (s, t, α) and described as $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint, nonempty sets of objects called points and lines, respectively. $\mathcal{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$), and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$), and two distinct lines are incident with at most one point;

(iii) If x is a point and L is a line not incident with x, then there are α pairs $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \text{ I } M \text{ I } y \text{ I } L$.

Definition 1.14. Given a partial geometry G where two points determine at most one line, a **collinearity graph**, or **point graph**, of G is a graph whose vertices are the points of G. The vertices are considered to be adjacent if and only if they are collinear in G.

Partial geometries relate to our other definitions because their points graphs are strongly regular graphs with the parameters $((s+1)(st+\alpha)/\alpha), s(t+1), s-1+t(\alpha-1), \alpha(t+1)) - SRG$. Also, partial geometries for which $\alpha = 1$ are known as generalized quadrangles. This is because of axiom (iii) of Definition 1.5 stating that there is a unique pair satisfying the incidence relation. This unique pair means that $\alpha = 1$.

Chapter 2

Generalized Quadrangles

A generalized quadrangle gives rise to a specific type of strongly regular graph; therefore, it is necessary to better understand their properties.

Lemma 2.1. *[7] The point graph associated with a generalized quadrangle of order (s, t) has parameters $((s + 1)(st + 1), s(t + 1), s - 1, t + 1) - SRG$.*

Proof. Consider a line l in the generalized quadrangle. Each point not on l is collinear with a unique point on l. Therefore, there are st points off of l which are collinear with a fixed point of l. This gives $st(s + 1)$ points off of l, and $(s + 1) + st(s + 1) = (s + 1)(st + 1)$ points in total. Now, each point of the generalized quadrangle lies on $t + 1$ lines of size $s + 1$, and none of these lines can have another point in common. Therefore, the point graph is regular of degree $s(t + 1)$. Now, by (ii) of Definition 1.5, two distinct lines are incident with at most one point. So if two points are collinear, no points off of the line can be collinear with both. Therefore, the only points in the graph adjacent to both are the other points on the line, so $\lambda = s - 1$. If two points, P and Q are not collinear, then Q must be collinear with exactly one point on each line through P, so $\mu = t + 1$. □

2.1 Benson's Lemma

De Winter, Kamischke, and Wang [11] generalize a theorem of Benson for generalized quadrangles. We present the original theorem of Benson below.

Lemma 2.2. *[5, Benson's Lemma]: Let $\mathcal{P}_1(a)$ be the number of fixed points of an automorphism a of a generalized quadrangle $\mathcal{Q}$. Let $\mathcal{P}_2(a)$ be the number of points P such that P is collinear with its image, P^a, under a. Then*

$$((t+1)|\mathcal{P}_1(a)| + |\mathcal{P}_2(a)| - (1+s)(1+t))/(s+t)$$

is an integer.

Proof. Let $\mathcal{Q}$ be a generalized quadrangle, and index the points of $\mathcal{Q}$. Define Q to be the permutation matrix on the points of $\mathcal{Q}$ corresponding to an automorphism a with order n so that $q_{ij} = 1$ if $P_i^a = P_j$, where $P_i, P_j \in \mathcal{P}$ and $q_{ij} = 0$ otherwise. Similarly, let $R = (r_{ij})$ be the permutation matrix belonging to a with respect to the action of a on the lines. Further, let D be the incidence matrix of $\mathcal{Q}$, with rows indexed by points and columns by lines, such that $M := DD^T = A + (t+1)I$ where A is the adjacency matrix of the point graph of $\mathcal{Q}$. The eigenvalues of M are given by $(s+1)(t+1), 0,$ and $(s+t)$ with appropriate multiplicities (see Lemma 3.1). Then $DR = QD$ and since Q and R are permutation matrices corresponding to a, $Q^T = Q^{-1}$ and $R^T = R^{-1}$. So $D = Q^{-1}DR$ and $D^T = R^T D^T (Q^{-1})^T$. Therefore,

$$QM = QDD^T = DRD^T = DRR^T D^T (Q^{-1})^T = DD^T Q = MQ,$$

and $QM = MQ$ with $(QM)^n = Q^n M^n = M^n$. Thus, the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate nth roots of unity. Since the rows of M have a constant sum, $(1+s)(1+t)$, then $MJ = (1+s)(1+t)J$, where J is the all 1's matrix, and $(QM)J = (1+s)(1+t)J$, so $(1+s)(1+t)$ is an eigenvalue of M

with multiplicity 1, it must be an eigenvalue of QM with multiplicity 1. Thus,

$$tr(QM) = (1 + s)(1 + t) + b_1\xi_1(s + t) + \cdots + b_n\xi_n(s + t),$$

where ξ_i is an nth root of unity and b_i is the multiplicity of $\xi_i(s + t)$. Now since conjugate eigenvalues appear with the same multiplicity, $\sum_i b_i\xi_i$ must be an integer. Thus $tr(QM) = (1 + s)(1 + t) + u_a(s + t)$. On the other hand, $tr(QM) = (t + 1)|\mathcal{P}_1(a)| + |\mathcal{P}_2(a)|$, since the entry in the ith row and the ith column is the number of lines incident with both P_i^a and P_i. Therefore,

$$(t + 1)|\mathcal{P}_1(a)| + |\mathcal{P}_2(a)| = tr(QM) = (1 + s)(1 + t) + u_a(s + t)$$

and $((t + 1)\mathcal{P}_1(a) + \mathcal{P}_2(a) - (1 + s)(1 + t))/(s + t)$ is an integer as desired. □

2.2 Yoshiara's Lemmas

Yoshiara [9] presents another important theorem about generalized quadrangles which is presented below. This theorem also serves as a model for our new results explained in Section 3.2. The first step of our project was to generalize the results in Yoshiara's paper. Throughout his paper, he considers a finite generalized quadrangle $\mathcal{Q} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of order (s, t) with $s, t \geq 2$ for which a group G of automorphisms acts regularly on the set $\mathcal{P}$ of points. Since this is a generalized quadrangle, we know that $|G| = |\mathcal{P}| = (s + 1)(st + 1)$ by Lemma 2.1. For two distinct points P and Q (respectively lines l and m), Yoshiara writes $P \sim Q$ if they are collinear (respectively $l \sim m$ if they are concurrent). Further, it is assumed that $P \sim P$. Fix a point O and let

$\Delta := \{g \in G | O^g \sim O\} \cup \{1\}.$

The symbol Δ^c is used to denote the complement of Δ in G. For each nontrivial automorphism $g \in G$, the point set is naturally divided into the following two parts, as g does not fix any point by the assumption:

$$\mathcal{P}_2(g) = \{P \in \mathcal{P} | P \neq P^g \sim P\}, \quad \mathcal{P}_3(g) = \{P \in \mathcal{P} | P^g \nsim P\}.$$

On the other hand, the set of lines is the disjoint union of the following three subsets:

$$\mathcal{L}_1(g) = \{l \in \mathcal{L} | l^g = l\}, \quad \mathcal{L}_2(g) = \{l \in \mathcal{L} | l \neq l^g \sim l\}, \quad \mathcal{L}_3(g) = \{l \in \mathcal{L} | l^g \nsim l\}.$$

We use the notation a^G to represent the conjugacy class of a in G. Therefore, for an element $a \in G$,

$a^G := \{a^g = g^{-1}ag | g \in G\}.$

Further, $C_G(a)$ represents the centralizer, so for an element $a \in G$,

$C_G(a) = \{g \in G | ga = ag\}.$

With this information, we state and prove Yoshiara's lemmas below. Note that the outline of Yoshiara's proof is used below, but with further explanations included by the author.

Lemma 2.3. *[9, Lemma 3] For a nontrivial automorphism $a \in G$ of a generalized quadrangle, we have*

$$\begin{aligned} |\mathcal{P}_2(a)| = (s+1)|\mathcal{L}_1(a)| + |\mathcal{L}_2(a)| &= |a^G \cap \Delta||C_G(a)| \\ &= (s+1)(t+1) + (s+t)u_a \end{aligned}$$

for some integer u_a. Furthermore, we have

$$|a^G \cap \Delta^c||C_G(a)| = t(s-1)(s+1) - (s+t)u_a.$$

Proof. There are $(s+1)$ points on a line and $|\mathcal{L}_1(a)|$ is the number of lines that remain fixed by automorphisms. Multiply these to find the number of points on the fixed lines. Now, by the axiom of generalized quadrangles (iii) of Definition 1.5, there is only one way for a line to remain concurrent; otherwise, the line l incident with both P and P^a is not fixed. But then $l \sim l^a$ and P^a is on both l and l^a, so it is concurrent. Therefore, add these remainder options onto the count so that $|\mathcal{P}_2(a)| = (s+1)|\mathcal{L}_1(a)| + |\mathcal{L}_2(a)|$.

Now we prove that $|\mathcal{P}_2(a)| = |a^G \cap \Delta||C_G(a)|$. Suppose that $O^x \in \mathcal{P}_2(a)$ for some $x \in G$. Then, O^x is sent through the automorphism $a \in G$ and becomes $(O^x)^a = O^{xa}$ and $O^x \sim O^{xa}$. Therefore $O^{xx^{-1}} = O \sim O^{xax^{-1}}$. Now since $a^G := \{g^{-1}ag | g \in G\}$, then for $x \in G$ we know that $(x^{-1})^{-1}ax^{-1} = xax^{-1} \in \Delta$ and since $xax^{-1} = a^{(x^{-1})} \in a^G$, then $xax^{-1} \in a^G \cap \Delta$. Now suppose that $xax^{-1} = yay^{-1}$ for $x, y \in G$. Then

$$xax^{-1}y = yay^{-1}y$$
$$x^{-1}xax^{-1}y = x^{-1}yay^{-1}y$$
$$ax^{-1}y = x^{-1}ya.$$

Therefore, $x^{-1}y \in C_G(a)$. Conversely, suppose $x^{-1}y \in C_G(a)$. Then

$$ax^{-1}y = x^{-1}ya$$
$$xax^{-1}y = xx^{-1}ya$$
$$xax^{-1}yy^{-1} = xx^{-1}yay^{-1}$$
$$xax^{-1} = yay^{-1}.$$

Therefore, $xax^{-1} = yay^{-1}$ if and only if $x^{-1}y \in C_G(a)$. Therefore, $y \in xC_G(a)$ and lies within the left coset of $C_G(a)$ and by Lagrange's Theorem, $|\mathcal{P}_2(a)| = |a^G \cap \Delta||C_G(a)|$.

We also must prove that $|\mathcal{P}_2(a)| = (s+1)(t+1) + (s+t)u_a$ for some integer u_a. From Benson's Lemma 2.2, we know that $(1+t)|\mathcal{P}_1(a)| + |\mathcal{P}_2(a)| \equiv (1+st)$

(mod $s+t$). Since we are working with regular automorphism groups, there are no fixed points, therefore we can simplify this to $|\mathcal{P}_2(a)| \equiv (1+st) \pmod{s+t}$ which can then be rewritten as $|\mathcal{P}_2(a)| = (s+1)(t+1)+(s+t)u_a$, for some $u_a \in \mathbb{Z}$.

Now we only have the last equality to verify. As $|a^G \cap \Delta^c| = |a^G| - |a^G \cap \Delta|$ and by the Orbit-Stabilizer Theorem, $|a^G||C_G(a)| = |G| = (s+1)(st+1)$, then

$$\begin{aligned} |a^G \cap \Delta^c||C_G(a)| &= (s+1)(st+1) - (s+1)(t+1) - (s+1)u_a \\ &= t(s-1)(s+1) - (s+t)u_a, \end{aligned}$$

as claimed. □

Lemma 2.4. *[9, Lemma 6] Let a be any nontrivial element of the automorphism group G, and let $d = \gcd(s,t)$ where (s,t) is the order of the generalized quadrangle. Then the following hold.*

(1) If $d > 1$, then $a^G \cap \Delta \neq \emptyset$

(2) $|a^G \cap \Delta^c|$ is a multiple of d (possibly equal to 0).

Proof. For (1), suppose $a^G \cap \Delta = \emptyset$. Then it follows from Lemma 2.3 that

$$0 = |a^G \cap \Delta||C_G(a)| = (s+1)(t+1) + (s+t)u_a$$

for some integer u_a. Thus $s+t$ divides $st+1$. In particular, for $d > 1$, d divides $st+1$. However, d is the greatest common divisor of s and t; therefore, $st+1 \equiv 1 \pmod{d}$ which is a contradiction. Therefore, $a^G \cap \Delta \neq \emptyset$.

Now consider statement (2). As d divides both t and $s+t$, it follows from Lemma 2.3 that $|a^G \cap \Delta^c||C_G(a)|$ is a multiple of d. On the other hand, $|G| = (s+1)(st+1) \equiv 1 \pmod{d}$. Thus d is prime to $|G|$, and hence to $|C_G(a)|$. Then d divides $|a^G \cap \Delta^c|$. □

Chapter 3

Strongly Regular Graphs

Strongly regular graphs served as a vital building block in this research. Before our results are stated, further background information must be presented. An adjacency matrix is a matrix with rows and columns labelled as the graph vertices with a 1 in position (n_i, n_j) if n_i and n_j are adjacent and a 0 if n_i and n_j are not adjacent. The eigenvalues also served as a key feature in our calculations, so we calculate these values first.

Lemma 3.1. *[4]. Let G be a $(v, k, \lambda, \mu) - SRG$ and let A be its adjacency matrix. Then the $v \times v$ matrix A has eigenvalues*

$$\nu_1 := k,$$

$$\nu_2 := \frac{1}{2}(\lambda - \mu + \sqrt{\Lambda}),$$

$$\nu_3 := \frac{1}{2}(\lambda - \mu - \sqrt{\Lambda}),$$

where $\Lambda = (\lambda - \mu)^2 + 4(k - \mu) = (\nu_2 - \nu_3)^2$.

Proof. Let A be the adjacency matrix for the $(v, k, \lambda, \mu) - SRG$ and J be the all ones matrix, so $J_{ij} = 1$ for all (i, j) entries. Then $AJ = JA = kJ$ by the Perron-Frobenius theorem, so k is an eigenvalue of the adjacency matrix with the $\vec{1}$ eigenvector. Additionally, the i, j-entry of A^2 represents the number of walks of length 2 from vertex i to vertex j. There are three cases to consider: $i = j, i$ and j are neighbors, and i and j are not neighbors. First consider if $i = j$. Then, there are k ways to accomplish this since there are k neighbors to go from i to a neighbor back to $i = j$. Now consider when i and j are neighbors. Then there are λ common neighbors, so there are λ possible ways to walk from i to j with length 2. If i and j are not neighbors, then there are μ points that are adjacent to each of them so that there are μ possibilities to walk from i to j. Therefore, we can write $A^2 = kI + \lambda A + (J - I - A)\mu$ where I is the identity matrix and J is the all ones matrix. We can simplify this to be $A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J$. Now suppose $\vec{v}$ is an eigenvector for A with eigenvalue $x \neq k$. Then $A\vec{v} = x\vec{v}$ and, since A is a real symmetric matrix, $\vec{v}$ must be orthogonal to $\vec{1}$ so that $\vec{v} \cdot \vec{1} = 0$. Therefore,

$$\begin{aligned}
(A^2 - (\lambda - \mu)A - (k - \mu)I)\vec{v} &= A^2\vec{v} - (\lambda - \mu)A\vec{v} - (k - \mu)\vec{v} \\
&= x^2\vec{v} - (\lambda - \mu)x\vec{v} - (k - \mu)\vec{v} \\
&= [x^2 - (\lambda - \mu)x - (k - \mu)]\vec{v} \\
&= \mu J\vec{v} \\
&= \vec{0}
\end{aligned}$$

Therefore, the other two eigenvalues are given by the roots of the quadratic $x^2 - (\lambda - \mu)x - (k - \mu) = 0$, thus proving the lemma. □

Note that we are working with strongly regular graphs which are not conference graphs; therefore, $2k + (v - 1)(\lambda - \mu) \neq 0$ and the eigenvalues are integers. To prove this statement, reference the two lemmas below.

Lemma 3.2. *Let G be a $(v, k, \lambda, \mu) - SRG$ and let A be its adjacency matrix. Then* *the $v \times v$ matrix A has eigenvalues $k, v_2,$ and v_3 with respective multiplicities*

$$m_k = 1,$$

$$m_2 = -\frac{(v-1)v_3 + k}{v_2 - v_3},$$

$$m_3 = \frac{(v-1)v_2 + k}{v_2 - v_3}.$$

Proof. Similar to the proof of Lemma 3.1, let A be the adjacency matrix for the (v, k, λ, μ)-SRG and J be the all ones matrix, so $J_{ij} = 1$ for all (i, j) entries. Then $AJ = JA = kJ$ by the Perron-Frobenius theorem, so k is an eigenvalue of the adjacency matrix with the $\vec{1}$ eigenvector and multiplicity 1. Additionally, since the sum of all the eigenvalues is equal to the trace of A (which is 0), then $k+m_2v_2+m_3v_3 = 0$ and

$$m_2v_2 + m_3v_3 = -k.$$

Also, we know that $m_k+m_2+m_3 = v$, so $m_2+m_3 = v-1$. By setting $m_3 = v-1-m_2$, we have

$$\begin{aligned} m_2v_2 + m_3v_3 &= m_2v_2 + (v-1-m_2)v_3 \\ &= m_2(v_2 - v_3) + (v-1)v_3 \\ &= -k. \end{aligned}$$

Therefore,

$$m_2 = -\frac{(v-1)v_3 + k}{v_2 - v_3}.$$

Similarly, we can set $m_2 = v - 1 - m_3$, so that

$$\begin{aligned} m_2 v_2 + m_3 v_3 &= (v - 1 - m_3)v_2 + m_3 v_3 \\ &= -m_3(v_2 - v_3) + (v - 1)v_2 \\ &= -k. \end{aligned}$$

Therefore,

$$m_3 = \frac{(v-1)v_2 + k}{v_2 - v_3},$$

and this completes the proof. □

Lemma 3.3. *Let G be a (v, k, λ, μ)-SRG which is not a conference graph. Then $2k + (v - 1)(\lambda - \mu) \neq 0$, and the eigenvalues are integers.*

Proof. From Lemma 3.2, we know that $m_2 = -\frac{(v-1)v_3+k}{v_2-v_3}$ and $m_3 = \frac{(v-1)v_2+k}{v_2-v_3}$. Then, from Lemma 3.1 we know $v_2 = \frac{1}{2}(\lambda - \mu + \sqrt{\Lambda})$ and $v_3 = \frac{1}{2}(\lambda - \mu - \sqrt{\Lambda})$. Therefore,

$$\begin{aligned} m_2 &= -\frac{(v-1)v_3 + k}{v_2 - v_3} \\ &= -\frac{(v-1)\frac{1}{2}(\lambda - \mu - \sqrt{\Lambda}) + k}{\frac{1}{2}(\lambda - \mu + \sqrt{\Lambda}) - \frac{1}{2}(\lambda - \mu - \sqrt{\Lambda})} \\ &= -\frac{1}{2}\frac{(v-1)(\lambda - \mu - \sqrt{\Lambda}) + 2k}{\sqrt{\Lambda}} \\ &= \frac{1}{2}\left((v-1) - \frac{(v-1)(\lambda-\mu) + 2k}{\sqrt{\Lambda}}\right). \end{aligned}$$

Similarly, we can say that

$$\begin{aligned} m_3 &= -\frac{(v-1)v_2 + k}{v_2 - v_3} \\ &= -\frac{(v-1)\frac{1}{2}(\lambda - \mu + \sqrt{\Lambda}) + k}{\frac{1}{2}(\lambda - \mu + \sqrt{\Lambda}) - \frac{1}{2}(\lambda - \mu - \sqrt{\Lambda})} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{(v-1)(\lambda-\mu+\sqrt{\Lambda})+2k}{\sqrt{\Lambda}} \\
&= \frac{1}{2}\left((v-1)+\frac{(v-1)(\lambda-\mu)+2k}{\sqrt{\Lambda}}\right).
\end{aligned}
$$

Now if we subtract the two multiplicities,

$$
\begin{aligned}
m_3 - m_2 &= \frac{1}{2}\left((v-1)+\frac{(v-1)(\lambda-\mu)+2k}{\sqrt{\Lambda}}\right) - \frac{1}{2}\left((v-1)-\frac{(v-1)(\lambda-\mu)+2k}{\sqrt{\Lambda}}\right) \\
&= \frac{(v-1)(\lambda-\mu)+2k}{\sqrt{\Lambda}}.
\end{aligned}
$$

If the numerator is not equal to 0, then since $m_3 - m_2$ is an integer, Λ must be a perfect square. From Lemma 3.1, $\sqrt{\Lambda} = v_2 - v_3$ so the eigenvalues are rational. From [6], we know that the roots of a monic quadratic with integer coefficients are algebraic integers. Thus the eigenvalues are integers, giving the final claim. □

Some of the techniques from combinatorics served as a helpful took in analyzing the properties of strongly regular graphs. Consider the proof of the Lemma below.

Lemma 3.4. *[4] If G is a strongly regular graph with parameters (v, k, λ, μ), then $k(k-\lambda-1) = (v-k-1)\mu$.*

Proof. To explain this relationship, consider any vertex u. Then u must have k neighbors and hence $v-k-1$ nonneighbors. We will count the total number of edges between the neighbors and nonneighbors of u in two ways. Each of the k neighbors of u is adjacent to u itself and to λ neighbors of u, therefore to $k-\lambda-1$ nonneighbors of u for a total of $k(k-\lambda-1)$ edges. On the other hand, each of the $v-k-1$ nonneighbors of u is adjacent to μ neighbors of u for a total of $(v-k-1)\mu$ edges. Therefore, we can say that $k(k-\lambda-1) = (v-k-1)\mu$. □

Lemma 3.5. *[1] If $\mu > 0$, then the parameters $v, k,$ and λ of a strongly regular graph* *can be expressed in terms of* ν_2, ν_3*, and* μ *as*

$$k = \mu - \nu_2\nu_3, \quad v = (k - \nu_2)(k - \nu_3)/\mu, \quad \lambda = \mu + \nu_2 + \nu_3$$

Proof. This can be explained by analyzing the quadratic equation $x^2 - (\lambda - \mu)x + (\mu - k) = 0$. Since eigenvalues are roots of this equation, then $(x - v_2)(x - v_3) = 0$ and therefore $x^2 - (v_2 + v_3)x + v_2v_3 = 0$. Then by equating the coefficients of the terms, we see that $\mu - k = v_2v_3$ and $\mu - \lambda = -v_2 - v_3$. Further, by referencing Lemma 3.4, we see that $k^2 - k\lambda - k = \mu v - \mu k - \mu$. Therefore, $\mu v = k^2 - (\lambda - \mu)k + (\mu - k) = (k - v_2)(k - v_3)$, thus proving the parameters stated above. □

3.1 De Winter, Kamischke, and Wang Theorem

De Winter, Kamischke, and Wang [11] generalize Benson's Lemma 2.2 for generalized quadrangles to strongly regular graphs. This paper later references some of these results, so we present the theorem and proof here.

Theorem 3.6. *[11, Theorem 1] Let* G *be a* $(v, k, \lambda, \mu) - SRG$ *whose adjacency matrix* A *has integer eigenvalues* k, v_2*, and* v_3*. Let* ϕ *be an automorphism of order* n *of* G*, and let* $\mu()$*denote the Möbius function. Then for every integer* r *and all positive divisors* d *of* n*, there are non-negative integers* a_d *and* b_d *(which are independent of* r*) such that*

$$k - r + \sum_{d|n} a_d\mu(d)(v_2 - r) + \sum_{d|n} b_d\mu(d)(v_3 - r) = -rf + g, \tag{3.1}$$

where f *is the number of fixed vertices of* ϕ *and* g *is the number of vertices that are adjacent to their image under* ϕ*. Furthermore,* $a_1 + b_1 = c - 1$*, where* c *is the number of cycles in the cycle decomposition of* ϕ*, and* $a_d + b_d = \sum_{d|l} c_l, d \neq 1$*, where* c_l *is the number of cycles of length* l *of* ϕ*. As a consequence the following equation and* *congruence hold:*

$$k - v_3 + \sum_{d|n} a_d \mu(d)(v_2 - v_3) = -v_3 f + g, \tag{3.2}$$

and

$$k - v_3 \equiv -v_3 f + g \pmod{v_2 - v_3}. \tag{3.3}$$

Proof. Let M be the matrix $M = A - rI$. Then M has integer eigevalues $k-r, v_2-r$, and $v_3 - r$. If P is the permutation matrix corresponding to ϕ, then $PM = MP$, and hence $(PM)^n = P^n M^n = M^n$. It follows that the eigenvalues of PM are the eigenvalues of M multiplied with appropriate nth roots of unity. Now let d be a positive divisor of n, and let ξ_d be the primitive dth root of unity. As the eigenvalues of M are integers, it follows that the multiplicity of the eigenvalue $\xi_d(v_2-r)$ only depends on d, and not on the specific primitive dth root of unity. Denote this multiplicity by a_d. Analogously the multiplicity of the eigenvalue $\xi_d(v_3 - r)$ will only depend on d. Denote this multiplicity by b_d. Since the sum of all primitive dth roots of unity is given by $\mu(d)$ where $\mu()$ is the Möbius function, then we obtain

$$tr(PM) = k - r + \sum_{d|n} a_d \mu(d)(v_2 - r) + \sum_{d|n} b_d \mu(d)(v_3 - r).$$

On the other hand, the trace of PM must equal $-rf + g$ for the same reasons as in the proof of Lemma 2.2, and hence

$$k - r + \sum_{d|n} a_d \mu(d)(v_2 - r) + \sum_{d|n} b_d \mu(d)(v_3 - r) = -rf + g. \tag{3.4}$$

Setting $r = v_3$ we obtain

$$k - v_3 + \sum_{d|n} a_d \mu(d)(v_2 - v_3) = -v_3 f + g \tag{3.5}$$

and

$$k - v_3 \equiv -v_3 f + g \pmod{v_2 - v_3}, \tag{3.6}$$

giving the final claim. □

3.2 New Results

Using Yoshiara's work [9] as an outline, we expanded his results to be applicable to strongly regular graphs. These new results are stated and proved below.

Lemma 3.7. *Let G be a group acting regularly on a point set of a non-conference $(v, k, \lambda, \mu) - SRG$. For a nontrivial automorphism $a \in G$, let $\mathcal{P}_2(a)$ be the set of points sent to adjacent points under the automorphism. Then we have*

$$\begin{aligned} |\mathcal{P}_2(a)| &= k - v_3 + u_a(v_2 - v_3) \\ &= \mu - v_3(v_2 + 1) + u_a(v_2 - v_3) \\ &= |a^G \cap \Delta||C_G(a)| \end{aligned}$$

for some integer u_a and eigenvalues v_2 and v_3. Furthermore, we have

$$|a^G \cap \Delta^c||C_G(a)| = \frac{v_2 v_3 (v_2 + 1)(v_3 + 1)}{\mu} - v_2(v_3 + 1) - u_a(v_2 - v_3).$$

Proof. As a fixes no point, it follows from Theorem 3.6 that we have $|\mathcal{P}_2(a)| =$

$k - v_3 + u_a(v_2 - v_3)$ for some integer u_a. Then since strongly regular graphs have the property that $k = \mu - v_2 v_3$, this is equivalent to $|\mathcal{P}_2(a)| = \mu - v_3(v_2+1) + u_a(v_2 - v_3)$. For $x \in G$, O^x lies in $\mathcal{P}_2(a)$ if and only if $O^x \sim O^{xa}$ (as $a \neq 1$) if and only if $O \sim O^{xax^{-1}}$, which is equivalent to $xax^{-1} \in a^G \cap \Delta$. (This is similar to the proof from Lemma 2.3). As $xax^{-1} = yay^{-1}$ for $x, y \in G$ if and only if $x^{-1}y \in C_G(a)$, we see that $|\mathcal{P}_2(a)| = |a^G \cap \Delta||C_G(a)|$. Thus we have verified all equalities except the last one.

As $|a^G \cap \Delta^c| = |a^G| - |a^G \cap \Delta|$ and

$$|a^G||C_G(a)| = |G| = \frac{(k - v_2)(k - v_3)}{\mu} = \frac{(\mu - v_2 v_3 - v_2)(\mu - v_2 v_3 - v_3)}{\mu},$$

then

$$\begin{aligned}
|a^G \cap \Delta^c||C_G(a)| &= \frac{(k - v_2)(k - v_3)}{\mu} - \mu + v_3(v_2+1) - u_a(v_2 - v_3) \\
&= \frac{(\mu - v_2 v_3 - v_2)(\mu - v_2 v_3 - v_3)}{\mu} - \mu + v_3(v_2+1) - u_a(v_2 - v_3) \\
&= \frac{v_2 v_3 (v_2+1)(v_3+1)}{\mu} - v_2(v_3+1) - u_a(v_2 - v_3),
\end{aligned}$$

as desired. □

Using the above lemma as well as the results from [11], we can confirm the following:

Lemma 3.8. *If* $(v_2 - v_3) \nmid (\mu - v_3(v_2+1))$*, then* $a^G \cap \Delta \neq \emptyset$*.*

Proof. Lemma 3.7 tells us that $|a^G \cap \Delta||C_G(a)| = \mu - v_3(v_2+1) + u_a(v_2 - v_3)$. If $0 = |a^G \cap \Delta|$, then $-\mu + v_3(v_2+1) = u_a(v_2 - v_3)$ so $(v_2 - v_3) | \mu - v_3(v_2+1)$. Therefore, if $(v_2 - v_3) \nmid \mu - v_3(v_2+1)$, then $|a^G \cap \Delta| \neq 0$, thus proving the statement. □

Lemma 3.9. *Let a be any nontrivial element of G, let $d = \gcd(v_2(v_3+1), v_3(v_2+1))$ with $d > 1$, and let μ be coprime with d. Then the following hold:*

(1) $a^G \cap \Delta \neq \emptyset$

(2)$|a^G \cap \Delta^c|$ *is a multiple of* d *(possibly* 0*).*

Proof. (1) Suppose $a^G \cap \Delta = \emptyset$. Then it follows from Lemma 3.7 that

$$0 = |a^G \cap \Delta||C_G(a)| = \mu - v_3(v_2+1) + u_a(v_2 - v_3)$$

for some integer u_a. Note that $v_2 - v_3 = v_2(v_3+1) - v_3(v_2+1)$. Thus $1 < d = gcd(v_2(v_3+1), v_3(v_2+1))$ divides μ. However, μ is coprime to d and therefore $a^G \cap \Delta \neq \emptyset$.

(2) Since $\gcd(d, \mu) = 1$, then d divides $\frac{v_2 v_3 (v_2+1)(v_3+1)}{\mu}$. As d divides both $v_2(v_3+1)$ and $v_2 - v_3$, then it follows from Lemma 3.7 that $|a^G \cap \Delta^c||C_G(a)|$ is a multiple of d. On the other hand,

$$\begin{aligned} |G| = v = \frac{(k-v_2)(k-v_3)}{\mu} &= \frac{(\mu - v_2v_3 - v_2)(\mu - v_2v_3 - v_3)}{\mu} \\ &= \mu - v_2(v_3+1) - v_3(v_2+1) + \frac{(v_2v_3 + v_3)(v_2v_3 + v_2)}{\mu}. \end{aligned}$$

Then, since d is coprime to μ but divides every other term, then d is coprime to $|G|$, and hence to $|C_G(a)|$. Then d divides $|a^G \cap \Delta^c|$ □

3.3 Application to Partial Geometries

In addition to strongly regular graphs, we also analyze Yoshiara's results with respect to partial geometries. The resulting lemmas are stated and proved below.

Lemma 3.10. *[12, Theorem 2.1] Let G be a group acting regularly on a point set of a (s, t, α) partial geometry. For a nontrivial automorphism $a \in G$, let $P_2(a)$ be the set of points sent to adjacent points under the automorphism. Then we have*

$$|\mathcal{P}_2(a)| = (1+t)(1+s) + u_a(s - \alpha + 1 + t) = |a^G \cap \Delta||C_G(a)|$$

for some integer u_a. Furthermore, we have

$$|a^G \cap \Delta^c||C_G(a)| = \frac{(1+s)t(s-\alpha)}{\alpha} - u_a(s-\alpha+1+t).$$

Proof. From [12], partial geometries have strongly regular point graphs with the parameters given by $(\frac{(s+1)(st+\alpha)}{\alpha}, s(t+1)s-1+t(\alpha-1), \alpha(t+1))$ and eigenvalues $v_2 = s-\alpha$ and $v_3 = -t-1$. Therefore, we can substitute these new values into Lemma 3.7 to get the desired result. □

Lemma 3.11. *Let a be any nontrivial element of G, let $d = \gcd(t(s-\alpha), s+t-(\alpha-1)))$, $d > 1$, and d be coprime with $\alpha(t+1)$. Then the following hold:*

(1) $a^G \cap \Delta \neq \emptyset$

(2)$|a^G \cap \Delta^c|$ *is a multiple of d (possibly 0).*

Proof. (1) Suppose $a^G \cap \Delta = \emptyset$. Then it follows from Lemma 3.7 with the appropriate substitutions that

$$0 = |a^G \cap \Delta||C_G(a)| = (1+t)(1+s) + u_a(s-\alpha+1+t)$$

for some integer u_a. Thus $1 < d = (t(s-\alpha), s+t-(\alpha-1)))$ divides $(s+1)$ or $(t+1)$. This contradiction implies $a^G \cap \Delta \neq 0$.

(2) We use Lemma 3.9 and plug in the appropriate values for partial geometries to obtain the desired results. □

Similarly, we can use Lemma 3.8 to say the following:

Lemma 3.12. *If $s+t-\alpha+1 \nmid (s+1)(t+1)$, then $a^G \cap \Delta \neq \emptyset$.*

Proof. This proof follows directly from the proof of Lemma 3.8 with the appropriate substitutions of parameters. □

Temmermans, Thas, and Van Maldeghem [3] proved a similar result for partial geometries. However, they used $d = (s, t, \alpha - 1)$ to be the greatest common divisor distinct from 1 of s, t, and $\alpha - 1$. Our conditions are slightly different and relate to the previous results for strongly regular graphs with the new parameter values.

Chapter 4

Constructing and Proving the Nonexistence of Partial Difference Sets in Nonabelian Groups

In order to construct the partial difference sets, we work with the equations for strongly regular graphs given in Lemma 3.7 and Lemma 3.9. Partial difference sets that are constructed from abelian groups are well known, and Ma gives an extensive list in his paper [10]. Nonabelian groups, however, have little to no results. This section of the paper is entirely devoted to the necessary conditions and results obtained for constructing partial difference sets in nonabelian groups.

A list of feasible parameters for strongly regular graphs was constructed by Brouwer, and from this we were able to test our conditions. Every known or feasible graph has a complement graph associated with it in which the adjacent points become nonadjacent points and the nonadjacent points become the adjacent points to any vertex. Therefore, we want to come up with equations for the complement as well.

Proposition 4.1. *If the parameters of a strongly regular graph are given by* (v, k, λ, μ) *with eigenvalues* v_2 *and* v_3*, then the parameters for its complement, which is also* *strongly regular, are given by* $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$ *with eigenvalues* $-v_3-1$ *and* $-v_2-1$.

Proof. Let the complement $\overline{\Gamma}$ have parameters given by $(\overline{v}, \overline{k}, \overline{\lambda}, \overline{\mu})$.

The number of vertices does not change depending upon whether we are considering the original graph, Γ, or its complement, $\overline{\Gamma}$. Therefore, $v = \overline{v}$.

Now consider $\overline{k}$. There are v points in total, but a vertex is not adjacent to itself. Nonneighbors become neighbors and vice versa in the complement, so $\overline{k} = v - k - 1$.

Let uw be an edge of $\overline{\Gamma}$. Then the number of vertices in Γ that are adjacent to u or w is the same as the number of vertices in $\overline{\Gamma}$ that are adjacent to neither u nor w. In Γ, uw is not an edge, and each of u and w has k neighbors of which μ vertices are common neighbors of u and w. Therefore, the number of vertices that are adjacent to at least one of u or w is $2k-\mu$. Then the number of vertices (other than u or w) that are adjacent to neither u nor w is given by $\overline{\lambda} = v - 2k + \mu - 2$.

Now suppose that u and w are non-adjacent vertices in $\overline{\Gamma}$. Then uw is an edge of Γ. Each of u and w has $k-1$ additional neighbors in Γ, and they have λ common neighbors. Therefore, there are $2k-2-\lambda$ vertices (other than u and w themselves) that are adjacent to at least one of u or w. That leaves $v-2-(2k-2-\lambda) = v-2k+\lambda$ vertices in Γ that are adjacent to neither u nor w. This is then the number of common neighbors of u and w in $\overline{\Gamma}$ and $\overline{\mu} = v - 2k + \lambda$.

To find the new eigenvalues, we can use the same equations given in Lemma 3.1 but using the new values for the complement. We know that $v_2 = \frac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})$.

Therefore,

$$\overline{v_2} = \frac{1}{2}(\overline{\lambda} - \overline{\mu} + \sqrt{(\overline{\lambda}-\overline{\mu})^2 + 4(\overline{k}-\overline{\mu})})$$

$$= \frac{1}{2}((v - 2k + \mu - 2) - (v - 2k + \lambda)$$

$$+ \sqrt{((v - 2k + \mu - 2) - (v - 2k + \lambda))^2 + 4((v - k - 1) - (v - 2k + \lambda))})$$

$$= \frac{1}{2}((\mu - \lambda) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1$$

$$= -\frac{1}{2}((\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1$$

$$= -v_3 - 1$$

A similar calculation can be made for $\overline{v_3}$ so that

$$\overline{v_3} = \frac{1}{2}(\overline{\lambda} - \overline{\mu} - \sqrt{(\overline{\lambda} - \overline{\mu})^2 + 4(\overline{k} - \overline{\mu})})$$

$$= \frac{1}{2}((v - 2k + \mu - 2) - (v - 2k + \lambda)$$

$$- \sqrt{((v - 2k + \mu - 2) - (v - 2k + \lambda))^2 + 4((v - k - 1) - (v - 2k + \lambda))})$$

$$= \frac{1}{2}((\mu - \lambda) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1$$

$$= -\frac{1}{2}((\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1$$

$$= -v_2 - 1,$$

which completes the proof. □

Now since we have the correlation between the original graph and its complement, we can focus on the conditions necessary to construct a partial difference set. Keep the same notation as in Proposition 4.1.

Proposition 4.2. *If* $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$ *and* $(v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)$, *then* $|a^G \cap \Delta| \neq \emptyset$ *and* $|a^G \cap \Delta^c| \neq \emptyset$.

Proof. By referencing Lemma 3.8, we already know that if $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$, then $|a^G \cap \Delta| \neq \emptyset$. By substituting the parameters of the complement into this relation, it follows that if $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$, then $|a^G \cap \Delta^c| \neq \emptyset$. $\square$

Corollary 4.3. *If G is a group of order v with $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$ and $(v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)$, and the center of G is nontrivial, then there cannot exist a (v, k, λ, μ)-PDS.*

Proof. Let $x \in Z(G)$ such that $x \neq 1$ and D is a (v, k, λ, μ)-PDS. Then $|x^G \cap D| \neq \emptyset$ and $|x^G \cap (D \cup \{1\})^c| = |x^G \cap G - (D \cup \{1\})| \neq \emptyset$. This is not possible because either $x^G = \{x\}$ lies in D or it lies in a set of points excluding D; it cannot lie in both. Therefore D cannot exist. $\square$

Similarly, all p-groups have non-trivial centers, so we can say:

Corollary 4.4. *If G is a p-group and (v, k, λ, μ), v_2 and v_3 satisfy the conditions that $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$ and $(v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)$, then there cannot exist a (v, k, λ, μ)-PDS in G.*

Therefore, we have our necessary conditions for ruling out the possibility of a partial difference set being constructed from a strongly regular graph.

4.1 Ruling Out Parameters

Brouwer [2] gives an extensive list of the parameters of strongly regular graphs along with their complements. We work through this list to find all the parameter sets through size 300 that satisfy the necessary conditions of $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$ and $(v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)$. In some cases, however, $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$ while $(v_2 - v_3) | v - 2k + \lambda - v_3(v_2 + 1)$. For these cases, we created a code in GAP [13] that tested the parameter sets and determined whether a possible partial difference set could exist. From there, we read the code into Gurobi [8] to solve and determine definitively whether or not a partial difference set exists. The first set of code is used when you are assuming that automorphisms exist other than the G automorphism acting regularly. (This is helpful in shortening the running time for large groups). The second code, however, is more accurate and does not necessarily assume that automorphisms exist. Both GAP codes are presented below.

4.1.1 Assuming Automorphisms

```
GurobifyAutPDS := function(mat, w1, w2, size, j1, j2, inversepairs, TorF,
                  conj, filename)
 local numrows, numcols, i, j, output, positions, total, numclasses;
 total := 2*(size-1);
 numrows := Length(mat);
 numcols := Length(mat[1]);
 numclasses:= Length(conj);
 output := OutputTextFile( filename, false );;
 SetPrintFormattingStatus(output, false);
 AppendTo(output,"Maximize\n");
 for i in [1..numrows] do
     AppendTo(output, Concatenation( " + 0 r", String(i)));
 od;
 AppendTo(output,"\n Subject To\n");
 for i in [1..numcols/2] do
     AppendTo(output, Concatenation( " + ", String(j1[i])," s", String(i)));
 od;
 AppendTo(output, Concatenation( " = ", String(size), "\n" ));
 for i in [1..numcols/2] do
     positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
     for j in [1..Size(positions)] do
         AppendTo(output, Concatenation( " + ", String(mat[positions[j]][i]),
```

```
                " r", String(positions[j])));
        od;
        AppendTo(output, Concatenation(" - ", String(w1-w2), " s", String(i),
               " = ", String(w2), "\n"));
  od;
  for i in [1..numrows] do
AppendTo(output, Concatenation(" + ", String(j2[i])," r", String(i)));
  od;
  AppendTo(output, Concatenation(" = ", String(size*(size - 1)), "\n"));
  ## put incidences in. e.g., r1 =1 => s1=s2=1.
  AppendTo(output, "\\ Incidences\n");
  for i in [1..numrows] do
      positions := Filtered([1..numcols/2], j -> IsOne(mat[i][j+numcols/2]));
      for j in positions do
          AppendTo(output, Concatenation("r", String(i)," - s", String(j),
                 " < 0\n"));
      od;
  od;
  AppendTo(output, "\\ Inverse Pairs\n");
  for i in inversepairs do
      AppendTo(output, Concatenation( "s", String(i[1]), " - s", String(i[2]),
             " = 0\n"));
  od;
  if TorF then
    AppendTo(output, "\\ Each Nonidentity Conjugacy Class Meets Connection
           Set\n");
    for i in [1..numclasses] do
```

```
      positions := Filtered([1..numcols/2], j -> IsOne(conj[i][j]));
      for j in positions do
          AppendTo(output, Concatenation(" + 1 s", String(j)));
      od;
      AppendTo(output, " > 1\n");
    od;
  fi;
  AppendTo(output, "\\ Variables\n");
  AppendTo(output,"Binary\n");
  for i in [1..numrows] do
      AppendTo(output, Concatenation( "r", String(i), "\n"));
  od;
  for i in [1..numcols/2] do
      AppendTo(output, Concatenation( "s", String(i), "\n"));
  od;
  AppendTo(output,"End\n");
  CloseStream(output);
  return;
end;

CreateAutGurobi:= function(g, aut, size, w1, w2, filename)

local orbs, count, notone, t, j1, j2, pairs, mat, i ,j, x, y, inversepairs,
       d, v2, v3, conj, conjclasses, TorF;

orbs:= Orbits(aut, g);;
notone := Filtered(AsList(orbs), t -> not IsOne(Random(t)));;
```

```
pairs := Tuples(notone, 2);;
j1:= List(notone, t -> Size(t));;
j2:= [];;
for t in pairs do
count:= 0;
for i in t[1] do
for j in t[2] do
if not IsOne(i*j^-1) then
count:= count + 1;
fi;
od;
od;
Add(j2, count);
od;
mat := NullMat(Size(pairs), 2*Size(notone), Rationals);;
for i in [1..Size(pairs)] do
  for j in [1..Size(notone)] do
t:= Random(notone[j]);
count:= 0;
for x in pairs[i][1] do
for y in pairs[i][2] do
   if  x*y^-1 = t then
        count:= count + 1;
fi;
od;
od;
mat[i][j]:= count;
```

```
    if notone[j] in pairs[i] then
       mat[i][j+Size(notone)] := 1;
    fi;
  od;
  if i mod 1000 = 0 then Print(i,".\c"); fi;
od;
inversepairs := Filtered(Combinations([1..Size(notone)],2),
       x-> Random(notone[x[1]])^-1 in notone[x[2]]);;
v2:= (w1 - w2 + Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
v3:= (w1 - w2 - Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
conjclasses:= Orbits(aut, List(Filtered(ConjugacyClasses(g),
       i -> not IsOne(Random(i))), j -> AsSet(j)), OnSets) ;
conjclasses:= Filtered(ConjugacyClasses(g), i -> not IsOne(Random(i)));
if (w2 - v3*(v2 + 1) mod (v2 - v3) = 0) then
    TorF:= false;
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
else
    TorF:= true;
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
    for i in [1..Size(conjclasses)] do
        for j in [1..Size(notone)] do
            if Size(Intersection(notone[j],conjclasses[i])) > 0 then
                conj[i][j] := 1;
            fi;
        od;
    od;
fi;
```

```
GurobifyAutPDS(mat, w1, w2, size, j1, j2, inversepairs, TorF, conj, filename);
end;

AutGurList:= function(G, list, size, w1, w2)
local x, name;
for x in [1..Length(list)] do
name:= Concatenation(String(Order(G)), "_", String(IdGroup(G)[2]), "_",
       String(x),".lp");

CreateAutGurobi(G, list[x], size, w1, w2, name);
od;
end;

FindOuts:= function(G)
local aut, inn, conjauts, x, i, list, nonid, reps;
aut:= AutomorphismGroup(G);
inn:= InnerAutomorphismsAutomorphismGroup(aut);
conjauts:= ConjugacyClassesSubgroups(aut);
reps:= List(conjauts, i -> Representative(i));
list:= Filtered(reps, i -> Size(Intersection(i, inn))=1);
nonid:= Filtered(list, i -> Order(i)>1);
return nonid;
end;

AutGur:= function(G, size, w1, w2)
local l, x;
```

```
l:= FindOuts(G);
AutGurList(G, l, size, w1, w2);
return l;
end;
```

4.1.2 Not Assuming Automorphisms

```
GurobifyPDS := function(mat, w1, w2, size, inversepairs, TorF, conj, filename)
  local numrows, numcols, i, j, output, positions, total, numclasses;
  total := 2*(size-1);
  numrows := Length(mat);
  numcols := Length(mat[1]);
  numclasses:= Length(conj);
  output := OutputTextFile( filename, false );;
  SetPrintFormattingStatus(output, false);
  AppendTo(output,"Maximize\n");
  for i in [1..numrows] do
      AppendTo(output, Concatenation( " + 0 r", String(i)));
  od;
  AppendTo(output,"\n Subject To\n");
  for i in [1..numcols/2] do
      AppendTo(output, Concatenation( " + 1 s", String(i)));
  od;
  AppendTo(output, Concatenation( " = ", String(size), "\n" ));
  for i in [1..numcols/2] do
      positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
      for j in [1..Size(positions)] do
```

```
        AppendTo(output, Concatenation( " + ", String(mat[positions[j]][i]),
                " r", String(positions[j])));
     od;
    AppendTo(output, Concatenation(" - ", String(w1-w2), " s", String(i),
            " = ", String(w2), "\n"));
od;
for i in [(numcols/2)+1..numcols] do
    positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
    for j in [1..Size(positions)] do
        AppendTo(output, Concatenation( " + ", String(mat[positions[j]][i]),
                " r", String(positions[j])));
     od;
    AppendTo(output, Concatenation(" - ", String(total), " s",
            String(i - numcols/2), " = 0 \n"));
od;
## put incidences in. e.g., r1 =1 => s1=s2=1.
AppendTo(output, "\\ Incidences\n");
for i in [1..numrows] do
    positions := Filtered([1..numcols/2], j -> IsOne( mat[i][j+numcols/2] ) );
    for j in positions do
        AppendTo(output, Concatenation("r", String(i)," - s", String(j),
                " < 0\n"));
    od;
od;
AppendTo(output, "\\ Inverse Pairs\n");
for i in inversepairs do
    AppendTo(output, Concatenation( "s", String(i[1]), " - s", String(i[2]),
```

```
            " = 0\n"));
  od;
  if TorF then
    AppendTo(output, "\\ Each Nonidentity Conjugacy Class Meets Connection
          Set\n");
    for i in [1..numclasses] do
        positions := Filtered([1..numcols/2], j -> IsOne(conj[i][j]));
        for j in positions do
            AppendTo(output, Concatenation(" + 1 s", String(j)));
        od;
        AppendTo(output, " > 1\n");
    od;
  fi;
  AppendTo(output, "\\ Variables\n");
  AppendTo(output,"Binary\n");
  for i in [1..numrows] do
      AppendTo(output, Concatenation( "r", String(i), "\n"));
  od;
  for i in [1..numcols/2] do
      AppendTo(output, Concatenation( "s", String(i), "\n"));
  od;
  AppendTo(output,"End\n");
  CloseStream(output);
  return;
end;


CreateGurobi:= function(g, size, w1, w2, filename)
```

```
local notone, t, pairs, mat, i ,j, inversepairs, v2, v3, conj, conjclasses,
         TorF;
notone := Filtered(AsList(g), t -> not IsOne(t));;
pairs := Filtered(Tuples(notone, 2), t -> not IsOne(t[1]*t[2]^-1));;
mat := NullMat(Size(pairs), 2*Size(notone), Rationals);;
for i in [1..Size(pairs)] do
  for j in [1..Size(notone)] do
    if pairs[i][1] * pairs[i][2]^-1 = notone[j] then
       mat[i][j] := 1;
    fi;
    if notone[j] in pairs[i] then
       mat[i][j+Size(notone)] := 1;
    fi;
  od;
  if i mod 1000 = 0 then Print(i,".\c"); fi;
od;

inversepairs := Filtered(Combinations([1..Size(notone)],2),x-> notone[x[1]]=
       notone[x[2]]^-1);;
v2:= (w1 - w2 + Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
v3:= (w1 - w2 - Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
conjclasses:= Filtered(ConjugacyClasses(g), i -> not IsOne(Random(i)));
if (w2 - v3*(v2 + 1) mod (v2 - v3) = 0) then
    TorF:= false;
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
else
    TorF:= true;
```

```
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
    for i in [1..Size(conjclasses)] do
        for j in [1..Size(notone)] do
            if notone[j] in conjclasses[i] then
                conj[i][j] := 1;
            fi;
        od;
    od;
fi;

GurobifyPDS(mat, w1, w2, size, inversepairs, TorF , conj, filename);

end;
```

4.2 Table of Nonexistence

Using the conditions outlined in Corollary 4.3 as well as the GAP code above, we worked through Brouwer's [2] list of parameters (up through $v = 300$ and some larger p-groups) to rule out the parameter sets for which partial difference sets cannot be constructed. The list below shows our results.

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2+1)$	method to rule out
28	12	6	4	4	-2	6	14	Corollary 4.3
	15	6	10	1	-5	6	20	
36	14	7	4	5	-2	7	16	Corollary 4.3
	21	10	15	1	-6	7	27	
40	12	2	4	2	-4	6	16	GAP code
	27	18	18	3	-3	6	30	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2+1)$	method to rule out
50	21	8	9	3	-4	7	25	Corollary 4.3
	28	15	16	3	-4	7	32	
56	10	0	2	2	-4	6	14	GAP
	45	36	36	3	-3	6	48	
63	30	13	15	3	-5	8	35	Corollary 4.3
	32	16	16	4	-4	8	36	
66	20	10	4	8	-2	10	22	Corollary 4.3
	45	28	36	1	-9	10	54	
70	27	12	9	6	-3	9	30	Corollary 4.3
	42	23	28	2	-7	9	49	
78	22	11	4	9	-2	11	24	Corollary 4.3
	55	36	45	1	-10	11	65	
88	27	6	9	3	-6	9	33	Corollary 4.3
	60	41	40	5	-4	9	64	
96	35	10	14	3	-7	10	42	Corollary 4.3
	60	38	36	6	-4	10	64	
100	33	14	9	8	-3	11	36	Corollary 4.3
	66	41	48	2	-9	11	75	
105	26	13	4	11	-2	13	28	Corollary 4.3
	78	55	66	1	-12	13	90	
105	32	4	12	2	-10	12	42	Corollary 4.3
	72	51	45	9	-3	12	75	
105	52	21	30	2	-11	13	63	Corollary 4.3
	52	29	22	10	-3	13	55	
112	30	2	10	2	-10	12	40	GAP code
	81	60	54	9	-3	12	84	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2+1)$	method to rule out
112	36	10	12	4	-6	10	42	GAP code
	75	50	50	5	-5	10	80	
117	36	15	9	9	-3	12	39	Corollary 4.3
	80	52	60	2	-10	12	90	
120	28	14	4	12	-2	14	30	Corollary 4.3
	91	66	78	1	-13	14	104	
120	42	8	18	2	-12	14	54	Corollary 4.3
	77	52	44	11	-3	14	80	
126	25	8	4	7	-3	10	28	Corollary 4.3
	100	78	84	2	-8	10	108	
126	50	13	24	2	-13	15	63	Corollary 4.3
	75	48	39	12	-3	15	78	
126	60	33	24	12	-3	15	63	Corollary 4.3
	65	28	39	2	-13	15	78	
130	48	20	16	8	-4	12	52	Corollary 4.3
	81	48	54	3	-9	12	90	
136	30	8	6	6	-4	10	34	Corollary 4.3
	105	80	84	3	-7	10	112	
136	30	15	4	13	-2	15	32	Corollary 4.3
	105	78	91	1	-14	15	119	
136	60	24	28	4	-8	12	68	Corollary 4.3
	75	42	40	7	-5	12	80	
136	63	30	28	7	-5	12	68	Corollary 4.3
	72	36	40	4	-8	12	80	
148	63	22	30	3	-11	14	74	Corollary 4.3
	84	50	44	10	-4	14	88	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2 + 1)$	method to rule out
148	70	36	30	10	-4	14	74	Corollary 4.3
	77	36	44	3	-11	14	88	
154	48	12	16	4	-8	12	56	Corollary 4.3
	105	72	70	7	-5	12	110	
154	72	26	40	2	-16	18	88	Corollary 4.3
	81	48	36	15	-3	18	84	
170	78	35	36	6	-7	13	85	Corollary 4.3
	91	48	49	6	-7	13	98	
171	34	17	4	15	-2	17	36	Corollary 4.3
	136	105	120	1	-16	17	152	
171	50	13	15	5	-7	12	57	Corollary 4.3
	120	84	84	6	-6	12	126	
171	60	15	24	3	-12	15	72	Corollary 4.3
	110	73	66	11	-4	15	114	
176	25	0	4	3	-7	10	32	Corollary 4.3
	150	128	126	6	-4	10	154	
176	45	18	9	12	-3	15	48	Corollary 4.3
	130	93	104	2	-13	15	143	
176	70	18	34	2	-18	20	88	Corollary 4.3
	105	68	54	17	-3	20	108	
176	70	24	30	4	-10	14	80	Corollary 4.3
	105	64	60	9	-5	14	110	
176	85	48	34	17	-3	20	88	Corollary 4.3
	90	38	54	2	-18	20	108	
189	48	12	12	6	-6	12	54	Corollary 4.3
	140	103	105	5	-7	12	147	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2 + 1)$	method to rule out
190	36	18	4	16	-2	18	38	Corollary 4.3
	153	120	136	1	-17	18	170	
190	45	12	10	7	-5	12	50	Corollary 4.3
	144	108	112	4	-8	12	152	
190	84	33	40	4	-11	15	95	Corollary 4.3
	105	60	55	10	-5	15	110	
190	84	38	36	8	-6	14	90	Corollary 4.3
	105	56	60	5	-9	14	114	
190	90	45	40	10	-5	15	95	Corollary 4.3
	99	48	55	4	-11	15	110	
195	96	46	48	6	-8	14	104	Corollary 4.3
	98	49	49	7	-7	14	105	
196	39	2	9	3	-10	13	49	Corollary 4.3
	156	125	120	9	-4	13	160	
196	60	23	16	11	-4	15	64	Corollary 4.3
	135	90	99	3	-12	15	147	
204	63	22	18	9	-5	14	68	Corollary 4.3
	140	94	100	4	-10	14	150	
208	75	30	25	10	-5	15	80	Corollary 4.3
	132	81	88	4	-11	15	143	
208	81	24	36	3	-15	18	88	Corollary 4.3
	126	80	70	14	-4	18	130	
210	38	19	4	17	-2	19	40	Corollary 4.3
	171	136	153	1	-18	19	189	
220	84	38	28	14	-4	18	88	Corollary 4.3
	135	78	90	3	-15	18	150	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2 + 1)$	method to rule out
222	51	20	9	14	-3	17	54	Corollary 4.3
	170	127	140	2	-15	17	185	
225	96	51	33	21	-3	24	99	Corollary 4.3
	128	64	84	2	-22	24	150	
231	30	9	3	9	-3	12	33	Corollary 4.3
	200	172	180	2	-10	12	210	
231	40	20	4	18	-2	20	42	Corollary 4.3
	190	153	171	1	-19	20	209	
231	90	33	36	6	-9	15	99	Corollary 4.3
	140	85	84	8	-7	15	147	
232	33	2	5	4	-7	11	40	Corollary 4.3
	198	169	168	6	-5	11	203	
232	63	14	18	5	-9	14	72	Corollary 4.3
	168	122	120	8	-6	14	174	
232	7	36	20	19	-3	22	80	Corollary 4.3
	154	96	114	2	-20	22	174	
232	81	30	27	9	-6	15	87	Corollary 4.3
	150	95	100	5	-10	15	160	
236	55	18	11	11	-4	15	59	Corollary 4.3
	180	135	144	3	-12	15	192	
238	75	20	25	5	-10	15	85	Corollary 4.3
	162	111	108	9	-6	15	168	
244	108	42	52	4	-14	18	121	Corollary 4.3
	135	78	70	13	-5	18	140	
244	117	60	52	13	-5	18	122	Corollary 4.3
	126	60	70	4	-14	18	140	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2+1)$	method to rule out
246	85	20	34	3	-17	20	102	Corollary 4.3
	160	108	96	16	-4	20	164	
246	105	36	51	3	-18	21	123	Corollary 4.3
	140	85	72	17	-4	21	144	
246	119	64	51	17	-4	21	123	Corollary 4.3
	126	57	72	3	-18	21	144	
266	45	0	9	3	-12	15	57	Corollary 4.3
	220	183	176	11	-4	15	224	
273	80	19	25	5	-11	16	91	Corollary 4.3
	192	136	132	10	-6	16	198	
273	102	41	36	11	-6	17	108	Corollary 4.3
	170	103	110	5	-12	17	182	
273	136	65	70	6	-11	17	147	Corollary 4.3
	136	69	66	10	-7	17	143	
275	112	30	56	2	-28	30	140	Corollary 4.3
	162	105	81	27	-3	30	165	
276	44	22	4	20	-2	22	46	Corollary 4.3
	231	190	210	1	-21	22	252	
276	75	10	24	3	-17	20	92	Corollary 4.3
	200	148	136	16	-4	20	204	
276	75	18	21	6	-9	15	84	Corollary 4.3
	200	145	144	8	-7	15	207	
276	110	52	38	18	-4	22	114	Corollary 4.3
	165	92	108	3	-19	22	184	
276	135	78	54	27	-3	30	138	Corollary 4.3
	140	58	84	2	-28	30	168	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2 + 1)$	method to rule out
279	128	52	64	4	-16	20	144	Corollary 4.3
	150	85	75	15	-5	20	155	
280	117	44	52	5	-13	18	130	Corollary 4.3
	162	96	90	12	-6	18	168	
285	64	8	16	4	-12	16	76	Corollary 4.3
	220	171	165	11	-5	16	225	
286	95	24	33	4	-15	19	108	Corollary 4.3
	190	129	120	14	-5	19	195	
286	125	60	50	15	-5	20	130	Corollary 4.3
	160	84	96	4	-16	20	176	
288	105	52	30	25	-3	28	108	Corollary 4.3
	182	106	130	2	-26	28	208	
290	136	63	64	8	-9	17	145	Corollary 4.3
	153	80	81	8	-9	17	161	
297	128	64	48	20	-4	24	132	Corollary 4.3
	168	87	105	3	-21	24	189	
300	46	23	4	21	-2	23	48	Corollary 4.3
	253	210	231	1	-22	23	275	
343	102	21	34	4	-17	21	119	Corollary 4.4
	240	171	160	16	-5	21	245	
343	114	45	34	16	-5	21	119	Corollary 4.4
	228	147	160	4	-17	21	245	
625	246	119	82	41	-4	45	250	Corollary 4.4
	378	213	252	3	-42	45	420	
729	208	37	68	4	-35	39	243	Corollary 4.4
	520	379	350	34	-5	39	525	

v	k	λ	μ	v_2	v_3	$(v_2 - v_3)$	$\mu - v_3(v_2+1)$	method to rule out
729	248	67	93	5	-31	36	279	Corollary 4.4
	480	324	300	30	-6	36	486	
729	280	127	95	37	-5	42	285	Corollary 4.4
	448	262	296	4	-38	42	486	

4.3 Existence of Partial Difference Sets

Our results show that partial difference sets are actually rather uncommon to find in nonabelian groups; however, GAP allowed us to find some examples for which partial difference sets do exist.

$(57, 32, 16, 20)$

$(57, 24, 11, 9)$

$(55, 18, 9, 4)$

$(55, 36, 21, 28)$

Although not many partial difference sets were found, we were easily able to rule out many parameter sets, which is a new result compared to past work.

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