Partial Difference Sets in Nonabelian Groups and Strongly Regular Cayley Graphs

Gabrielle Tauscheck

William & Mary

Follow this and additional works at: https://scholarworks.wm.edu/honorstheses

Part of the Algebra Commons

Recommended Citation
https://scholarworks.wm.edu/honorstheses/1322

This Honors Thesis is brought to you for free and open access by the Theses, Dissertations, & Master Projects at W&M ScholarWorks. It has been accepted for inclusion in Undergraduate Honors Theses by an authorized administrator of W&M ScholarWorks. For more information, please contact scholarworks@wm.edu.
Partial Difference Sets in Nonabelian Groups and Strongly Regular Cayley Graphs

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Mathematics from
The College of William & Mary

by

Gabrielle Tauscheck

Accepted for _____________________________
(Honors, High Honors, Highest Honors)

Eric Swartz, Advisor

Ryan Vinroot

Peter McHenry

Williamsburg, VA
April 26, 2019
Partial Difference Sets in Nonabelian Groups and Strongly Regular Cayley Graphs

Gabrielle Tauscheck

April 7, 2019
Abstract

A regular graph $\Gamma$ with $v$ vertices and valency $k$ is said to be a $(v, k, \lambda, \mu)$-strongly regular graph if any two adjacent vertices are both joined to exactly $\lambda$ other vertices and two nonadjacent vertices are both joined to exactly $\mu$ other vertices. Let $G$ be a group of order $v$ and $D$ a $k$-element subset of $G$. Then $D$ is called a $(v, k, \lambda, \mu)$-partial difference set if for every nonidentity element $g$ of $D$, the equation $d_1d_2^{-1} = g$ has exactly $\lambda$ solutions $(d_1, d_2) \in D \times D$; and for every nonidentity element $g'$ of $G$ not in $D$, the equation $d_1d_2^{-1} = g'$ has exactly $\mu$ solutions. It is known that a subset $D$ of $G$ with $e \notin D$ and $\{d^{-1} | d \in D\} = D$ is a partial difference set if and only if the Cayley graph generated by $D$ is strongly regular. Yoshiara [9] has given two lemmas that describe the conditions needed for an automorphism group to act regularly on a finite generalized quadrangle. De Winter, Kamischke, and Wang [11] build upon the work of Benson to construct partial difference sets in abelian groups. In this work, we confirm Yoshiara’s results, and use De Winter, Kamischke, and Wang’s result in place of Benson’s to generalize Yoshiara’s results to nonabelian groups. In the process, we are able to rule out the existence of many partial difference sets in nonabelian groups.
Acknowledgment

I would like to thank Professor Swartz, Professor Vinroot, and Professor McHenry for serving on my committee. In addition, thank you to my friends and family, especially my parents, for supporting me and encouraging me throughout my college experience. I would especially like to thank Professor Swartz for his patience, time, and assistance throughout this past year; it was a rewarding experience, and I greatly appreciate his willingness to guide me through this process.
# Contents

1. **Introduction** 4

2. **Generalized Quadrangles** 10
   - 2.1 Benson's Lemma 11
   - 2.2 Yoshiara's Lemmas 12

3. **Strongly Regular Graphs** 16
   - 3.1 De Winter, Kamischke, and Wang Theorem 21
   - 3.2 New Results 23
   - 3.3 Application to Partial Geometries 25

4. **Constructing and Proving the Nonexistence of Partial Difference Sets in Nonabelian Groups** 28
   - 4.1 Ruling Out Parameters 31
     - 4.1.1 Assuming Automorphisms 32
     - 4.1.2 Not Assuming Automorphisms 38
   - 4.2 Table of Nonexistence 42
   - 4.3 Existence of Partial Difference Sets 50

Bibliography 51
Chapter 1

Introduction

Yoshiara studies the regular automorphism groups acting on generalized quadrangles. This is the basic model from which we will build to study different structures throughout this paper. In this section, we will present the background and definitions of some of these structures, which will then be further analyzed later in the paper.

Definition 1.1. An automorphism is a structure-preserving map that sends a structure to itself. For a graph, it is a permutation of the vertex set which preserves the adjacency and nonadjacency of vertices.

Definition 1.2. The set of all automorphisms of an object forms the automorphism group.

Definition 1.3. An automorphism group is considered regular if it is transitive and no nonidentity elements of the group fix any elements of the set being permuted. Therefore, for all $i, j$ in the vertex set of a graph $G$, $\phi(i) = j$ where $i \neq j$ and $\phi$ represents the nonidentity automorphism.
Definition 1.4. An incidence structure has two distinct types of objects that are connected by a single relationship.

Definition 1.5. A generalized quadrangle (GQ) is an incidence structure described as $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint, nonempty sets of objects called points and lines, respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$), and two distinct points are incident with at most one line;

(ii) Each line is incident with $1 + s$ points ($s \geq 1$), and two distinct lines are incident with at most one point;

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

Figure 1.1: Generalized Quadrangle

The integers $s$ and $t$ are the parameters of the generalized quadrangle, and $S$ is said to have order $(s, t)$; if $s = t$, then $S$ is said to have order $s$.

Definition 1.6. The valency of a graph is the number of neighbors of any vertex. If all vertices in a graph have the same valency $k$, then the graph is said to be regular with valency $k$. (Note the distinction between “regular group” and “regular graph.”)
Definition 1.7. A strongly regular graph with parameters \((v, k, \lambda, \mu)\) (denoted by \((v, k, \lambda, \mu) - SRG\)) is an undirected graph, without loops or multiple edges, on \(v\) vertices which is regular with valency \(k\), and which has the following two properties:

(i) for each pair \((x, y)\) of adjacent vertices there are exactly \(\lambda\) vertices mutually adjacent to \(x\) and to \(y\), and

(ii) for each pair \((x, y)\) of nonadjacent vertices there are exactly \(\mu\) vertices mutually adjacent to \(x\) and to \(y\).

A strongly regular graph, therefore, has the property that the number of common neighbors of two distinct vertices depends only on whether they are adjacent or nonadjacent.

Definition 1.8. An adjacency matrix is a matrix with the rows and columns labelled as the graph vertices with a 1 in position \((n_i, n_j)\) if \(n_i\) and \(n_j\) are adjacent and a 0 if \(n_i\) and \(n_j\) are not adjacent.

Definition 1.9. A conference graph is a strongly regular graph with parameters \(v, k = \frac{v-1}{2}, \lambda = \frac{v-5}{4}, \text{ and } \mu = \frac{v-1}{4}\). A conference graph is unique in that the eigenvalues of its adjacency matrix need not be integers. In fact, if \(G\) is a conference graph and \(2k + (v-1)(\lambda - \mu) = 0\), then the eigenvalues are not integers.
Definition 1.10. Let $G$ be a group, and $D \subseteq G$ such that $D^{-1} = D$ and $1 \notin G$. The Cayley graph $\text{Cay}(G, D)$ is defined to be the undirected graph with vertex set $G$, which has no loops, and $g, h \in G$ are adjacent if and only if $gh^{-1} \in D$.

Example: Let $G = \langle x \rangle = C_5 = \{1, x, x^2, x^3, x^4\}$ and $D = \{x, x^4\}$. Then $x^2, x^3 \in G$ are adjacent since $x^3x^{-2} = x \in D$. However, $x, x^3 \in G$ are not adjacent since $x^3x^{-1} = x^2 \notin D$. For the reverse direction, since $x^4x^{-3} = x \in D$, then $x^3, x^4 \in G$ are adjacent.

\begin{center}
\begin{tikzpicture}
\node at (0,0) (x) {$x$};
\node at (2,2) (x3) {$x^3$};
\node at (4,0) (x4) {$x^4$};
\node at (-2,2) (x2) {$x^2$};
\node at (0,4) (1) {1};
\draw (x) -- (x3);
\draw (x3) -- (x4);
\draw (x4) -- (1);
\draw (1) -- (x2);
\draw (x2) -- (x);
\end{tikzpicture}
\end{center}

Definition 1.11. We say that $S$ is a $(v, k, \lambda, \mu)$-PDS (partial difference set) of a group $G$ if $|G| = v, |S| = k$, and each nonidentity element $g \in G$ can be written either $\lambda$ or $\mu$ different ways as $g = ab^{-1}$, where $a, b \in S$, depending on whether or not $g$ is in $S$.

Partial difference sets serve as a tool to construct strongly regular Cayley graphs. We prove this connection in the proposition below.

Proposition 1.12. A $(v, k, \lambda, \mu)$ partial difference set $D$, with $1 \notin D$ and $D^{-1} = D$, is equivalent to a $(v, k, \lambda, \mu)$ strongly regular Cayley graph, $\text{Cay}(G, D)$ arising from $G$.

Proof. Let $\Gamma$ be a strongly regular Cayley graph. Then by Definition 1.10, $\Gamma$ is an undirected graph with $D \subseteq G$ where $D = D^{-1}, 1 \notin D$, and $a, b \in G$ are adjacent if and only if $ab^{-1} \in D$. Let $d \in D$. Since $d$ is adjacent to $1$, there exist $\lambda$ elements
of $D$ that are adjacent to 1 and $d$. Let $a$ be one of these $\lambda$ elements. Then there is $c \in D$ such that $ac = d$. Since $D = D^{-1}$, $c = b^{-1}$ for some $b \in D$, so $d$ can be written as $ab^{-1}$ for $a, b \in D$ in $\lambda$ different ways. Now let $g \notin D$, so $g$ is not adjacent to 1. Then there exist $\mu$ elements of $D$ that are adjacent to 1 and $g$. Let $d$ be one of these $\mu$ elements. Then there is an $a \in D$ such that $gd^{-1} = a$ so $g = ad$. Since $D = D^{-1}$, $d = b^{-1}$ for some $b \in D$, so $g$ can be written as $ab^{-1}$ for $a, b \in D$ in $\mu$ different ways. This satisfies the requirements for a partial difference set given by Definition 1.11.

Now assume $D$ is a $(v, k, \lambda, \mu)$ PDS with $D^{-1} = D$ and $1 \notin D$. Let $d \in D$. Since $d$ can be written in $\lambda$ ways as $ab^{-1}$, with $a, b \in D$, then $d(b^{-1})^{-1} = a \in D$, and $d$ has exactly $\lambda$ neighbors in $D$. Now let $g \notin D$. Since $g$ can be written in $\mu$ ways as $ab^{-1}$, with $a, b \in D$, then $g(b^{-1})^{-1} = a \in D$, and $g$ has exactly $\mu$ neighbors in $D$.

**Definition 1.13.** A partial geometry is an incidence geometry said to be of order $(s, t, \alpha)$ and described as $S = (P, B, \mathcal{I})$ in which $P$ and $B$ are disjoint, nonempty sets of objects called points and lines, respectively. $\mathcal{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$), and two distinct points are incident with at most one line;

(ii) Each line is incident with $1 + s$ points ($s \geq 1$), and two distinct lines are incident with at most one point;

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there are $\alpha$ pairs \((y, M) \in P \times B\) for which $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

**Definition 1.14.** Given a partial geometry $G$ where two points determine at most one line, a collinearity graph, or point graph, of $G$ is a graph whose vertices are
the points of $G$. The vertices are considered to be adjacent if and only if they are collinear in $G$.

Partial geometries relate to our other definitions because their points graphs are strongly regular graphs with the parameters $((s + 1)(st + α)/α), s(t + 1), s - 1 + t(α - 1), α(t + 1)) - SRG$. Also, partial geometries for which $α = 1$ are known as generalized quadrangles. This is because of axiom $(iii)$ of Definition 1.5 stating that there is a unique pair satisfying the incidence relation. This unique pair means that $α = 1$. 
Chapter 2

Generalized Quadrangles

A generalized quadrangle gives rise to a specific type of strongly regular graph; therefore, it is necessary to better understand their properties.

Lemma 2.1. The point graph associated with a generalized quadrangle of order $(s, t)$ has parameters $((s+1)(st+1), s(t+1), s-1, t+1) - SRG$.

Proof. Consider a line $l$ in the generalized quadrangle. Each point not on $l$ is collinear with a unique point on $l$. Therefore, there are $st$ points off of $l$ which are collinear with a fixed point of $l$. This gives $st(s + 1)$ points off of $l$, and $(s + 1) + st(s + 1) = (s + 1)(st + 1)$ points in total. Now, each point of the generalized quadrangle lies on $t + 1$ lines of size $s + 1$, and none of these lines can have another point in common. Therefore, the point graph is regular of degree $s(t + 1)$. Now, by (ii) of Definition 1.5, two distinct lines are incident with at most one point. So if two points are collinear, no points off of the line can be collinear with both. Therefore, the only points in the graph adjacent to both are the other points on the line, so $\lambda = s - 1$. If two points, $P$ and $Q$ are not collinear, then $Q$ must be collinear with exactly one point on each line through $P$, so $\mu = t + 1$. \qed
2.1 Benson’s Lemma

De Winter, Kamischke, and Wang [11] generalize a theorem of Benson for generalized quadrangles. We present the original theorem of Benson below.

Lemma 2.2. [5, Benson’s Lemma]: Let $P_1(a)$ be the number of fixed points of an automorphism $a$ of a generalized quadrangle $Q$. Let $P_2(a)$ be the number of points $P$ such that $P$ is collinear with its image, $P^a$, under $a$. Then

$$
((t + 1)|P_1(a)| + |P_2(a)| - (1 + s)(1 + t))/(s + t)
$$

is an integer.

Proof. Let $Q$ be a generalized quadrangle, and index the points of $Q$. Define $Q$ to be the permutation matrix on the points of $Q$ corresponding to an automorphism $a$ with order $n$ so that $q_{ij} = 1$ if $P_i^a = P_j$, where $P_i, P_j \in P$ and $q_{ij} = 0$ otherwise. Similarly, let $R = (r_{ij})$ be the permutation matrix belonging to $a$ with respect to the action of $a$ on the lines. Further, let $D$ be the incidence matrix of $Q$, with rows indexed by points and columns by lines, such that $M := DD^T = A + (t + 1)I$ where $A$ is the adjacency matrix of the point graph of $Q$. The eigenvalues of $M$ are given by $(s + 1)(t + 1), 0,$ and $(s + t)$ with appropriate multiplicities (see Lemma 3.1). Then $DR = QD$ and since $Q$ and $R$ are permutation matrices corresponding to $a$, $Q^T = Q^{-1}$ and $R^T = R^{-1}$. So $D = Q^{-1}DR$ and $D^T = R^T D^T (Q^{-1})^T$. Therefore,

$$QM = QDD^T = DRD^T = DRR^T D^T (Q^{-1})^T = DD^T Q = MQ,$$

and $QM = MQ$ with $(QM)^n = Q^n M^n = M^n$. Thus, the eigenvalues of $QM$ are the eigenvalues of $M$ multiplied by the appropriate $n$th roots of unity. Since the rows of $M$ have a constant sum, $(1 + s)(1 + t)$, then $MJ = (1 + s)(1 + t)J$, where $J$ is the all 1’s matrix, and $(QM)J = (1 + s)(1 + t)J$, so $(1 + s)(1 + t)$ is an eigenvalue of $M.$
with multiplicity 1, it must be an eigenvalue of $QM$ with multiplicity 1. Thus,

$$tr(QM) = (1 + s)(1 + t) + b_1 \xi_1(s + t) + \cdots + b_n \xi_n(s + t),$$

where $\xi_i$ is an $n$th root of unity and $b_i$ is the multiplicity of $\xi_i(s + t)$. Now since conjugate eigenvalues appear with the same multiplicity, $\sum_i b_i \xi_i$ must be an integer. Thus $tr(QM) = (1 + s)(1 + t) + u_a(s + t)$. On the other hand, $tr(QM) = (t + 1)|P_1(a)| + |P_2(a)|$, since the entry in the $i$th row and the $i$th column is the number of lines incident with both $P_i$ and $P$. Therefore,

$$(t + 1)|P_1(a)| + |P_2(a)| = tr(QM) = (1 + s)(1 + t) + u_a(s + t)$$

and $((t + 1)P_1(a) + P_2(a) - (1 + s)(1 + t))/(s + t)$ is an integer as desired. 

\[2.2 \quad \text{Yoshiara’s Lemmas}\]

Yoshiara [9] presents another important theorem about generalized quadrangles which is presented below. This theorem also serves as a model for our new results explained in Section 3.2. The first step of our project was to generalize the results in Yoshiara’s paper. Throughout his paper, he considers a finite generalized quadrangle $Q = (P, B, I)$ of order $(s, t)$ with $s, t \geq 2$ for which a group $G$ of automorphisms acts regularly on the set $P$ of points. Since this is a generalized quadrangle, we know that $|G| = |P| = (s + 1)(st + 1)$ by Lemma 2.1. For two distinct points $P$ and $Q$ (respectively lines $l$ and $m$), Yoshiara writes $P \sim Q$ if they are collinear (respectively $l \sim m$ if they are concurrent). Further, it is assumed that $P \sim P$. Fix a point $O$ and let

$$\Delta := \{g \in G|Og \sim O\} \cup \{1\}.$$ 

The symbol $\Delta^c$ is used to denote the complement of $\Delta$ in $G$. For each nontrivial
automorphism $g \in G$, the point set is naturally divided into the following two parts, as $g$ does not fix any point by the assumption:

$$\mathcal{P}_2(g) = \{P \in \mathcal{P} | P \neq P^g \sim P\}, \quad \mathcal{P}_3(g) = \{P \in \mathcal{P} | P^g \sim P\}.$$ 

On the other hand, the set of lines is the disjoint union of the following three subsets:

$$\mathcal{L}_1(g) = \{l \in \mathcal{L} | l^g = l\}, \quad \mathcal{L}_2(g) = \{l \in \mathcal{L} | l \neq l^g \sim l\}, \quad \mathcal{L}_3(g) = \{l \in \mathcal{L} | l^g \sim l\}.$$ 

We use the notation $a^G$ to represent the conjugacy class of $a$ in $G$. Therefore, for an element $a \in G$,

$$a^G := \{a^g = g^{-1}ag | g \in G\}.$$ 

Further, $C_G(a)$ represents the centralizer, so for an element $a \in G$,

$$C_G(a) = \{g \in G | ga = ag\}.$$ 

With this information, we state and prove Yoshiara’s lemmas below. Note that the outline of Yoshiara’s proof is used below, but with further explanations included by the author.

**Lemma 2.3.** [Lemma 3] For a nontrivial automorphism $a \in G$ of a generalized quadrangle, we have

$$|\mathcal{P}_2(a)| = (s + 1) |\mathcal{L}_1(a)| + |\mathcal{L}_2(a)| = |a^G \cap \Delta| |C_G(a)|$$

$$= (s + 1)(t + 1) + (s + t)u_a$$

for some integer $u_a$. Furthermore, we have

$$|a^G \cap \Delta^c| |C_G(a)| = t(s - 1)(s + 1) - (s + t)u_a.$$ 

**Proof.** There are $(s + 1)$ points on a line and $|\mathcal{L}_1(a)|$ is the number of lines that
remain fixed by automorphisms. Multiply these to find the number of points on
the fixed lines. Now, by the axiom of generalized quadrangles (iii) of Definition 1.5,
there is only one way for a line to remain concurrent; otherwise, the line \( l \) incident
with both \( P \) and \( P^a \) is not fixed. But then \( l \sim l^a \) and \( P^a \) is on both \( l \) and \( l^a \),
so it is concurrent. Therefore, add these remainder options onto the count so that
\[
|P_2(a)| = (s + 1)|L_1(a)| + |L_2(a)|.
\]

Now we prove that \( |P_2(a)| = |a^G \cap \Delta||C_G(a)| \). Suppose that \( O^x \in P_2(a) \) for some
\( x \in G \). Then, \( O^x \) is sent through the automorphism \( a \in G \) and becomes \( (O^x)^a = O^{xa} \)
and \( O^x \sim O^{xa} \). Therefore \( O^{x^{-1}} = O \sim O^{xa^{-1}} \). Now since \( a^G := \{g^{-1}ag | g \in G \} \), then
for \( x \in G \) we know that \( (x^{-1})^{-1}ax^{-1} = xax^{-1} \in \Delta \) and since \( xax^{-1} = a^{(x^{-1})} \in a^G \),
then \( xax^{-1} \in a^G \cap \Delta \). Now suppose that \( xax^{-1} = yay^{-1} \) for \( x, y \in G \). Then

\[
\begin{align*}
xax^{-1}y &= yay^{-1} \\
x^{-1}xax^{-1}y &= x^{-1}yay^{-1} \\
ax^{-1}y &= x^{-1}ya.
\end{align*}
\]

Therefore, \( x^{-1}y \in C_G(a) \). Conversely, suppose \( x^{-1}y \in C_G(a) \). Then

\[
\begin{align*}
ax^{-1}y &= x^{-1}ya \\
xax^{-1}y &= xx^{-1}ya \\
xax^{-1}yy^{-1} &= xx^{-1}yay^{-1} \\
xax^{-1} &= yay^{-1}.
\end{align*}
\]

Therefore, \( xax^{-1} = yay^{-1} \) if and only if \( x^{-1}y \in C_G(a) \). Therefore, \( y \in xC_G(a) \) and lies
within the left coset of \( C_G(a) \) and by Lagrange’s Theorem, \( |P_2(a)| = |a^G \cap \Delta||C_G(a)| \).

We also must prove that \( |P_2(a)| = (s + 1)(t + 1) + (s + t)u_a \) for some integer
\( u_a \). From Benson’s Lemma 2.2, we know that \( (1 + t)|P_1(a)| + |P_2(a)| \equiv (1 + st)\)
(mod \(s + t\)). Since we are working with regular automorphism groups, there are no fixed points, therefore we can simplify this to \(|P_2(a)| \equiv (1 + st) \pmod{s + t}\) which can then be rewritten as \(|P_2(a)| = (s + 1)(t + 1) + (s + t)u_a\), for some \(u_a \in \mathbb{Z}\).

Now we only have the last equality to verify. As \(|a_G \cap \Delta_c| = |a_G| - |a_G \cap \Delta|\) and by the Orbit-Stabilizer Theorem, \(|a_G||C_G(a)| = |G| = (s + 1)(st + 1)\), then

\[
|a_G \cap \Delta_c||C_G(a)| = (s + 1)(st + 1) - (s + 1)(t + 1) - (s + 1)u_a
\]

\[
= t(s - 1)(s + 1) - (s + t)u_a,
\]

as claimed. \(\Box\)

**Lemma 2.4.** [9, Lemma 6] Let \(a\) be any nontrivial element of the automorphism group \(G\), and let \(d = \gcd(s, t)\) where \((s, t)\) is the order of the generalized quadrangle. Then the following hold.

1. If \(d > 1\), then \(a_G \cap \Delta \neq \emptyset\)
2. \(|a_G \cap \Delta_c|\) is a multiple of \(d\) (possibly equal to 0).

**Proof.** For (1), suppose \(a_G \cap \Delta = \emptyset\). Then it follows from Lemma 2.3 that

\[
0 = |a_G \cap \Delta||C_G(a)| = (s + 1)(t + 1) + (s + t)u_a
\]

for some integer \(u_a\). Thus \(s + t\) divides \(st + 1\). In particular, for \(d > 1\), \(d\) divides \(st + 1\). However, \(d\) is the greatest common divisor of \(s\) and \(t\); therefore, \(st + 1 \equiv 1 \pmod{d}\) which is a contradiction. Therefore, \(a_G \cap \Delta \neq \emptyset\).

Now consider statement (2). As \(d\) divides both \(t\) and \(s + t\), it follows from Lemma 2.3 that \(|a_G \cap \Delta_c||C_G(a)|\) is a multiple of \(d\). On the other hand, \(|G| = (s + 1)(st + 1) \equiv 1 \pmod{d}\). Thus \(d\) is prime to \(|G|\), and hence to \(|C_G(a)|\). Then \(d\) divides \(|a_G \cap \Delta_c|\). \(\Box\)
Chapter 3

Strongly Regular Graphs

Strongly regular graphs served as a vital building block in this research. Before our results are stated, further background information must be presented. An adjacency matrix is a matrix with rows and columns labelled as the graph vertices with a 1 in position \((n_i, n_j)\) if \(n_i\) and \(n_j\) are adjacent and a 0 if \(n_i\) and \(n_j\) are not adjacent. The eigenvalues also served as a key feature in our calculations, so we calculate these values first.

**Lemma 3.1.** Let \(G\) be a \((v, k, \lambda, \mu) - SRG\) and let \(A\) be its adjacency matrix. Then the \(v \times v\) matrix \(A\) has eigenvalues

\[
\nu_1 := k,
\nu_2 := \frac{1}{2}(\lambda - \mu + \sqrt{\Lambda}),
\nu_3 := \frac{1}{2}(\lambda - \mu - \sqrt{\Lambda}),
\]

where \(\Lambda = (\lambda - \mu)^2 + 4(k - \mu) = (\nu_2 - \nu_3)^2\).

**Proof.** Let \(A\) be the adjacency matrix for the \((v, k, \lambda, \mu) - SRG\) and \(J\) be the all ones matrix, so \(J_{ij} = 1\) for all \((i, j)\) entries. Then \(AJ = JA = kJ\) by the Perron-Frobenius
theorem, so \( k \) is an eigenvalue of the adjacency matrix with the \( \vec{1} \) eigenvector. Additionally, the \( i, j \)-entry of \( A^2 \) represents the number of walks of length 2 from vertex \( i \) to vertex \( j \). There are three cases to consider: \( i = j \), \( i \) and \( j \) are neighbors, and \( i \) and \( j \) are not neighbors. First consider if \( i = j \). Then, there are \( k \) ways to accomplish this since there are \( k \) neighbors to go from \( i \) to a neighbor back to \( i = j \). Now consider when \( i \) and \( j \) are neighbors. Then there are \( \lambda \) common neighbors, so there are \( \lambda \) possible ways to walk from \( i \) to \( j \) with length 2. If \( i \) and \( j \) are not neighbors, then there are \( \mu \) points that are adjacent to each of them so that there are \( \mu \) possibilities to walk from \( i \) to \( j \). Therefore, we can write \( A^2 = kI + \lambda A + (J - I - A)\mu \) where \( I \) is the identity matrix and \( J \) is the all ones matrix. We can simplify this to be
\[
A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J.
\]
Now suppose \( \vec{v} \) is an eigenvector for \( A \) with eigenvalue \( x \neq k \). Then \( A\vec{v} = x\vec{v} \) and, since \( A \) is a real symmetric matrix, \( \vec{v} \) must be orthogonal to \( \vec{1} \) so that \( \vec{v} \cdot \vec{1} = 0 \). Therefore,
\[
(A^2 - (\lambda - \mu)A - (k - \mu)I)\vec{v} = A^2\vec{v} - (\lambda - \mu)A\vec{v} - (k - \mu)\vec{v}
\]
\[
= x^2\vec{v} - (\lambda - \mu)x\vec{v} - (k - \mu)\vec{v}
\]
\[
= [x^2 - (\lambda - \mu)x - (k - \mu)]\vec{v}
\]
\[
= \mu J\vec{v}
\]
\[
= \vec{0}
\]

Therefore, the other two eigenvalues are given by the roots of the quadratic \( x^2 - (\lambda - \mu)x - (k - \mu) = 0 \), thus proving the lemma.

Note that we are working with strongly regular graphs which are not conference graphs; therefore, \( 2k + (v - 1)(\lambda - \mu) \neq 0 \) and the eigenvalues are integers. To prove this statement, reference the two lemmas below.

**Lemma 3.2.** Let \( G \) be a \((v, k, \lambda, \mu) - SRG\) and let \( A \) be its adjacency matrix. Then
the $v \times v$ matrix $A$ has eigenvalues $k, v_2$, and $v_3$ with respective multiplicities

\[
m_k = 1,
\]
\[
m_2 = -\frac{(v - 1)v_3 + k}{v_2 - v_3},
\]
\[
m_3 = \frac{(v - 1)v_2 + k}{v_2 - v_3}.
\]

**Proof.** Similar to the proof of Lemma 3.1, let $A$ be the adjacency matrix for the $(v, k, \lambda, \mu)$-SRG and $J$ be the all ones matrix, so $J_{ij} = 1$ for all $(i, j)$ entries. Then $AJ = JA = kJ$ by the Perron-Frobenius theorem, so $k$ is an eigenvalue of the adjacency matrix with the $\tilde{1}$ eigenvector and multiplicity 1. Additionally, since the sum of all the eigenvalues is equal to the trace of $A$ (which is 0), then $k + m_2v_2 + m_3v_3 = 0$ and

\[
m_2v_2 + m_3v_3 = -k.
\]

Also, we know that $m_k + m_2 + m_3 = v$, so $m_2 + m_3 = v - 1$. By setting $m_3 = v - 1 - m_2$, we have

\[
m_2v_2 + m_3v_3 = m_2v_2 + (v - 1 - m_2)v_3
\]
\[
= m_2(v_2 - v_3) + (v - 1)v_3
\]
\[
= -k.
\]

Therefore,

\[
m_2 = -\frac{(v - 1)v_3 + k}{v_2 - v_3}.
\]
Similarly, we can set \( m_2 = v - 1 - m_3 \), so that

\[
m_2 v_2 + m_3 v_3 = (v - 1 - m_3) v_2 + m_3 v_3
\]
\[
= -m_3 (v_2 - v_3) + (v - 1) v_2
\]
\[
= -k.
\]

Therefore,

\[
m_3 = \frac{(v - 1) v_2 + k}{v_2 - v_3},
\]

and this completes the proof. \( \square \)

**Lemma 3.3.** Let \( G \) be a \((v, k, \lambda, \mu)\)-SRG which is not a conference graph. Then \( 2k + (v - 1)(\lambda - \mu) \neq 0 \), and the eigenvalues are integers.

**Proof.** From Lemma 3.2 we know that \( m_2 = \frac{(v - 1)v_3 + k}{v_2 - v_3} \) and \( m_3 = \frac{(v - 1)v_2 + k}{v_2 - v_3} \). Then, from Lemma 3.1 we know \( v_2 = \frac{1}{2}(\lambda - \mu + \sqrt{\lambda}) \) and \( v_3 = \frac{1}{2}(\lambda - \mu - \sqrt{\lambda}) \). Therefore,

\[
m_2 = -\frac{(v - 1)v_3 + k}{v_2 - v_3}
\]
\[
= -\frac{(v - 1)\frac{1}{2}(\lambda - \mu - \sqrt{\lambda}) + k}{\frac{1}{2}(\lambda - \mu + \sqrt{\lambda}) - \frac{1}{2}(\lambda - \mu - \sqrt{\lambda})}
\]
\[
= -\frac{1}{2} \frac{(v - 1)(\lambda - \mu - \sqrt{\lambda}) + 2k}{\sqrt{\lambda}}
\]
\[
= \frac{1}{2} \left( (v - 1) - \frac{(v - 1)(\lambda - \mu) + 2k}{\sqrt{\lambda}} \right).
\]

Similarly, we can say that

\[
m_3 = -\frac{(v - 1)v_2 + k}{v_2 - v_3}
\]
\[
= -\frac{(v - 1)\frac{1}{2}(\lambda - \mu + \sqrt{\lambda}) + k}{\frac{1}{2}(\lambda - \mu + \sqrt{\lambda}) - \frac{1}{2}(\lambda - \mu - \sqrt{\lambda})}
\]
Now if we subtract the two multiplicities,

\[ m_3 - m_2 = \frac{1}{2} \left( (v - 1) + \frac{(v - 1)(\lambda - \mu) + 2k}{\sqrt{\Lambda}} \right) - \frac{1}{2} \left( (v - 1) - \frac{(v - 1)(\lambda - \mu) + 2k}{\sqrt{\Lambda}} \right) = \frac{(v - 1)(\lambda - \mu) + 2k}{\sqrt{\Lambda}}. \]

If the numerator is not equal to 0, then since \( m_3 - m_2 \) is an integer, \( \Lambda \) must be a perfect square. From Lemma 3.1, \( \sqrt{\Lambda} = v_2 - v_3 \) so the eigenvalues are rational. From [6], we know that the roots of a monic quadratic with integer coefficients are algebraic integers. Thus the eigenvalues are integers, giving the final claim.

Some of the techniques from combinatorics served as a helpful tool in analyzing the properties of strongly regular graphs. Consider the proof of the Lemma below.

**Lemma 3.4.** [4] If \( G \) is a strongly regular graph with parameters \( (v, k, \lambda, \mu) \), then \( k(k - \lambda - 1) = (v - k - 1)\mu \).

**Proof.** To explain this relationship, consider any vertex \( u \). Then \( u \) must have \( k \) neighbors and hence \( v - k - 1 \) nonneighbors. We will count the total number of edges between the neighbors and nonneighbors of \( u \) in two ways. Each of the \( k \) neighbors of \( u \) is adjacent to \( u \) itself and to \( \lambda \) neighbors of \( u \), therefore to \( k - \lambda - 1 \) nonneighbors of \( u \) for a total of \( k(k - \lambda - 1) \) edges. On the other hand, each of the \( v - k - 1 \) nonneighbors of \( u \) is adjacent to \( \mu \) neighbors of \( u \) for a total of \( (v - k - 1)\mu \) edges. Therefore, we can say that \( k(k - \lambda - 1) = (v - k - 1)\mu \). □

**Lemma 3.5.** [1] If \( \mu > 0 \), then the parameters \( v, k, \) and \( \lambda \) of a strongly regular graph
can be expressed in terms of $\nu_2, \nu_3,$ and $\mu$ as

$$k = \mu - \nu_2 \nu_3, \quad v = (k - \nu_2)(k - \nu_3)/\mu, \quad \lambda = \mu + \nu_2 + \nu_3$$

**Proof.** This can be explained by analyzing the quadratic equation $x^2 - (\lambda - \mu)x + (\mu - k) = 0$. Since eigenvalues are roots of this equation, then $(x - \nu_2)(x - \nu_3) = 0$ and therefore $x^2 - (\nu_2 + \nu_3)x + \nu_2 \nu_3 = 0$. Then by equating the coefficients of the terms, we see that $\mu - k = \nu_2 \nu_3$ and $\mu - \lambda = -\nu_2 - \nu_3$. Further, by referencing Lemma 3.4, we see that $k^2 - k \lambda - k = \mu v - \mu k - \mu$. Therefore, $\mu v = k^2 - (\lambda - \mu)k + (\mu - k) = (k - \nu_2)(k - \nu_3)$, thus proving the parameters stated above. \qed

### 3.1 De Winter, Kamischke, and Wang Theorem

De Winter, Kamischke, and Wang [11] generalize Benson’s Lemma 2.2 for generalized quadrangles to strongly regular graphs. This paper later references some of these results, so we present the theorem and proof here.

**Theorem 3.6.** [11, Theorem 1] Let $G$ be a $(v, k, \lambda, \mu)$-SRG whose adjacency matrix $A$ has integer eigenvalues $k, \nu_2,$ and $\nu_3$. Let $\phi$ be an automorphism of order $n$ of $G$, and let $\mu(\cdot)$ denote the Möbius function. Then for every integer $r$ and all positive divisors $d$ of $n$, there are non-negative integers $a_d$ and $b_d$ (which are independent of $r$) such that

$$k - r + \sum_{d|n} a_d \mu(d)(\nu_2 - r) + \sum_{d|n} b_d \mu(d)(\nu_3 - r) = -rf + g, \quad (3.1)$$

where $f$ is the number of fixed vertices of $\phi$ and $g$ is the number of vertices that are adjacent to their image under $\phi$. Furthermore, $a_1 + b_1 = c - 1$, where $c$ is the number of cycles in the cycle decomposition of $\phi$, and $a_d + b_d = \sum_{d|l} c_l, d \neq 1$, where $c_l$ is the number of cycles of length $l$ of $\phi$. As a consequence the following equation and
congruence hold:

\[ k - v_3 + \sum_{d|n} a_d\mu(d)(v_2 - v_3) = -v_3f + g, \quad (3.2) \]

and

\[ k - v_3 \equiv -v_3f + g \pmod{v_2 - v_3}. \quad (3.3) \]

**Proof.** Let \( M \) be the matrix \( M = A - rI \). Then \( M \) has integer eigenvalues \( k - r, v_2 - r, \) and \( v_3 - r \). If \( P \) is the permutation matrix corresponding to \( \phi \), then \( PM = MP \), and hence \((PM)^n = P^nM^n = M^n\). It follows that the eigenvalues of \( PM \) are the eigenvalues of \( M \) multiplied with appropriate \( n \)th roots of unity. Now let \( d \) be a positive divisor of \( n \), and let \( \xi_d \) be the primitive \( d \)th root of unity. As the eigenvalues of \( M \) are integers, it follows that the multiplicity of the eigenvalue \( \xi_d(v_2 - r) \) only depends on \( d \), and not on the specific primitive \( d \)th root of unity. Denote this multiplicity by \( a_d \). Analogously the multiplicity of the eigenvalue \( \xi_d(v_3 - r) \) will only depend on \( d \). Denote this multiplicity by \( b_d \). Since the sum of all primitive \( d \)th roots of unity is given by \( \mu(d) \) where \( \mu() \) is the Möbius function, then we obtain

\[ \text{tr}(PM) = k - r + \sum_{d|n} a_d\mu(d)(v_2 - r) + \sum_{d|n} b_d\mu(d)(v_3 - r). \]

On the other hand, the trace of \( PM \) must equal \(-rf + g\) for the same reasons as in the proof of Lemma 2.2 and hence

\[ k - r + \sum_{d|n} a_d\mu(d)(v_2 - r) + \sum_{d|n} b_d\mu(d)(v_3 - r) = -rf + g. \quad (3.4) \]
Setting $r = v_3$ we obtain

$$k - v_3 + \sum_{d|n} a_d \mu(d)(v_2 - v_3) = -v_3 f + g$$

(3.5)

and

$$k - v_3 \equiv -v_3 f + g \pmod{v_2 - v_3},$$

(3.6)

giving the final claim. \qed

### 3.2 New Results

Using Yoshiara’s work [9] as an outline, we expanded his results to be applicable to strongly regular graphs. These new results are stated and proved below.

**Lemma 3.7.** Let $G$ be a group acting regularly on a point set of a non-conference $(v, k, \lambda, \mu) - \text{SRG}$. For a nontrivial automorphism $a \in G$, let $\mathcal{P}_2(a)$ be the set of points sent to adjacent points under the automorphism. Then we have

$$|\mathcal{P}_2(a)| = k - v_3 + u_a(v_2 - v_3)$$

$$= \mu - v_3(v_2 + 1) + u_a(v_2 - v_3)$$

$$= |a^G \cap \Delta||C_G(a)|$$

for some integer $u_a$ and eigenvalues $v_2$ and $v_3$. Furthermore, we have

$$|a^G \cap \Delta^e||C_G(a)| = \frac{v_2 v_3 (v_2 + 1)(v_3 + 1)}{\mu} - v_2(v_3 + 1) - u_a(v_2 - v_3).$$

**Proof.** As $a$ fixes no point, it follows from Theorem 3.6 that we have $|\mathcal{P}_2(a)| =$
$k - v_3 + u_a(v_2 - v_3)$ for some integer $u_a$. Then since strongly regular graphs have the property that $k = \mu - v_2v_3$, this is equivalent to $|P_2(a)| = \mu - v_3(v_2 + 1) + u_a(v_2 - v_3)$. For $x \in G$, $O^x$ lies in $P_2(a)$ if and only if $O^x \sim O^{xa}$ (as $a \neq 1$) if and only if $O \sim O^{xax^{-1}}$, which is equivalent to $xax^{-1} \in a^G \cap \Delta$. (This is similar to the proof from Lemma 2.3). As $xax^{-1} = yay^{-1}$ for $x, y \in G$ if and only if $x^{-1}y \in C_G(a)$, we see that $|P_2(a)| = |a^G \cap \Delta||C_G(a)|$. Thus we have verified all equalities except the last one.

As $|a^G \cap \Delta^c| = |a^G| - |a^G \cap \Delta|$ and

$$ |a^G||C_G(a)| = |G| = \frac{(k - v_2)(k - v_3)}{\mu} = \frac{\mu - v_2v_3 - v_2(\mu - v_2v_3 - v_3)}{\mu}, $$

then

$$ |a^G \cap \Delta^c||C_G(a)| = \frac{(k - v_2)(k - v_3)}{\mu} - \mu + v_3(v_2 + 1) - u_a(v_2 - v_3) $$

$$ = \frac{(\mu - v_2v_3 - v_2)(\mu - v_2v_3 - v_3)}{\mu} - \mu + v_3(v_2 + 1) - u_a(v_2 - v_3) $$

$$ = \frac{v_2v_3(v_2 + 1)(v_3 + 1)}{\mu} - v_2(v_3 + 1) - u_a(v_2 - v_3), $$

as desired.  

Using the above lemma as well as the results from [11], we can confirm the following:

**Lemma 3.8.** If $(v_2 - v_3) \nmid (\mu - v_3(v_2 + 1))$, then $a^G \cap \Delta \neq \emptyset$.

**Proof.** Lemma 3.7 tells us that $|a^G \cap \Delta||C_G(a)| = \mu - v_3(v_2 + 1) + u_a(v_2 - v_3)$. If $0 = |a^G \cap \Delta|$, then $-\mu + v_3(v_2 + 1) = u_a(v_2 - v_3)$ so $(v_2 - v_3)|\mu - v_3(v_2 + 1)$. Therefore, if $(v_2 - v_3) \nmid \mu - v_3(v_2 + 1)$, then $|a^G \cap \Delta| \neq 0$, thus proving the statement.  

**Lemma 3.9.** Let $a$ be any nontrivial element of $G$, let $d = \gcd(v_2(v_3 + 1), v_3(v_2 + 1))$ with $d > 1$, and let $\mu$ be coprime with $d$. Then the following hold:
(1) \( a^G \cap \Delta \neq \emptyset \)

(2) \(|a^G \cap \Delta^c|\) is a multiple of \( d \) (possibly \( 0 \)).

Proof. (1) Suppose \( a^G \cap \Delta = \emptyset \). Then it follows from Lemma 3.7 that

\[ 0 = |a^G \cap \Delta||C_G(a)| = \mu - v_3(v_2 + 1) + u_a(v_2 - v_3) \]

for some integer \( u_a \). Note that \( v_2 - v_3 = v_2(v_3 + 1) - v_3(v_2 + 1) \). Thus \( 1 < d = \gcd(v_2(v_3 + 1), v_3(v_2 + 1)) \) divides \( \mu \). However, \( \mu \) is coprime to \( d \) and therefore \( a^G \cap \Delta \neq \emptyset \).

(2) Since \( \gcd(d, \mu) = 1 \), then \( d \) divides \( \frac{v_2 v_3(v_2 + 1)(v_3 + 1)}{\mu} \). As \( d \) divides both \( v_2(v_3 + 1) \) and \( v_2 - v_3 \), then it follows from Lemma 3.7 that \(|a^G \cap \Delta^c||C_G(a)|\) is a multiple of \( d \).

On the other hand,

\[ |G| = \frac{(k - v_2)(k - v_3)}{\mu} = \frac{(\mu - v_2 v_3 - v_2)(\mu - v_2 v_3 - v_3)}{\mu} = \mu - v_2(v_3 + 1) - v_3(v_2 + 1) + \frac{(v_2 v_3 + v_3)(v_2 v_3 + v_2)}{\mu}. \]

Then, since \( d \) is coprime to \( \mu \) but divides every other term, then \( d \) is coprime to \(|G|\), and hence to \(|C_G(a)|\). Then \( d \) divides \(|a^G \cap \Delta^c|\) \( \square \)

3.3 Application to Partial Geometries

In addition to strongly regular graphs, we also analyze Yoshiara’s results with respect to partial geometries. The resulting lemmas are stated and proved below.

Lemma 3.10. [12, Theorem 2.1] Let \( G \) be a group acting regularly on a point set of a \((s, t, \alpha)\) partial geometry. For a nontrivial automorphism \( a \in G \), let \( P_2(a) \) be the set of points sent to adjacent points under the automorphism. Then we have

\[ |P_2(a)| = (1 + t)(1 + s) + u_a(s - \alpha + 1 + t) = |a^G \cap \Delta||C_G(a)| \]
for some integer $u_a$. Furthermore, we have

$$|a^G \cap \Delta^c||C_G(a)| = \frac{(1 + s)t(s - \alpha)}{\alpha} - u_a(s - \alpha + 1 + t).$$

Proof. From [12], partial geometries have strongly regular point graphs with the parameters given by $(\frac{(s+1)(st+\alpha)}{\alpha}, s(t+1)s-1+t(\alpha-1), \alpha(t+1))$ and eigenvalues $v_2 = s - \alpha$ and $v_3 = -t - 1$. Therefore, we can substitute these new values into Lemma 3.7 to get the desired result.

**Lemma 3.11.** Let $a$ be any nontrivial element of $G$, let $d = \gcd(t(s - \alpha), s + t - (\alpha - 1)))$, $d > 1$, and $d$ be coprime with $\alpha(t + 1)$. Then the following hold:

1. $a^G \cap \Delta \neq \emptyset$
2. $|a^G \cap \Delta^c|$ is a multiple of $d$ (possibly 0).

Proof. (1) Suppose $a^G \cap \Delta = \emptyset$. Then it follows from Lemma 3.7 with the appropriate substitutions that

$$0 = |a^G \cap \Delta||C_G(a)| = (1 + t)(1 + s) + u_a(s - \alpha + 1 + t)$$

for some integer $u_a$. Thus $1 < d = (t(s - \alpha), s + t - (\alpha - 1)))$ divides $(s+1)$ or $(t+1)$. This contradiction implies $a^G \cap \Delta \neq 0$.

(2) We use Lemma 3.9 and plug in the appropriate values for partial geometries to obtain the desired results.

Similarly, we can use Lemma 3.8 to say the following:

**Lemma 3.12.** If $s + t - \alpha + 1 \nmid (s + 1)(t + 1)$, then $a^G \cap \Delta \neq \emptyset$.

Proof. This proof follows directly from the proof of Lemma 3.8 with the appropriate substitutions of parameters.
Temmermans, Thas, and Van Maldeghem [3] proved a similar result for partial geometries. However, they used \( d = (s, t, \alpha - 1) \) to be the greatest common divisor distinct from 1 of \( s, t, \) and \( \alpha - 1 \). Our conditions are slightly different and relate to the previous results for strongly regular graphs with the new parameter values.
Chapter 4

Constructing and Proving the Nonexistence of Partial Difference Sets in Nonabelian Groups

In order to construct the partial difference sets, we work with the equations for strongly regular graphs given in Lemma 3.7 and Lemma 3.9. Partial difference sets that are constructed from abelian groups are well known, and Ma gives an extensive list in his paper [10]. Nonabelian groups, however, have little to no results. This section of the paper is entirely devoted to the necessary conditions and results obtained for constructing partial difference sets in nonabelian groups.

A list of feasible parameters for strongly regular graphs was constructed by Brouwer, and from this we were able to test our conditions. Every known or feasible graph has a complement graph associated with it in which the adjacent points become nonadjacent points and the nonadjacent points become the adjacent points to any vertex. Therefore, we want to come up with equations for the complement as well.

**Proposition 4.1.** If the parameters of a strongly regular graph are given by \((v, k, \lambda, \mu)\) with eigenvalues \(v_2\) and \(v_3\), then the parameters for its complement, which is also
strongly regular, are given by \((v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)\) with eigenvalues \(-v_3 - 1\) and \(-v_2 - 1\).

Proof. Let the complement \(\overline{\Gamma}\) have parameters given by \((\overline{v}, \overline{k}, \overline{\lambda}, \overline{\mu})\).

The number of vertices does not change depending upon whether we are considering the original graph, \(\Gamma\), or its complement, \(\overline{\Gamma}\). Therefore, \(v = \overline{v}\).

Now consider \(\overline{k}\). There are \(v\) points in total, but a vertex is not adjacent to itself. Nonneighbors become neighbors and vice versa in the complement, so \(\overline{k} = v - k - 1\).

Let \(uw\) be an edge of \(\Gamma\). Then the number of vertices in \(\Gamma\) that are adjacent to \(u\) or \(w\) is the same as the number of vertices in \(\overline{\Gamma}\) that are adjacent to neither \(u\) nor \(w\). In \(\Gamma\), \(uw\) is not an edge, and each of \(u\) and \(w\) has \(k\) neighbors of which \(\mu\) vertices are common neighbors of \(u\) and \(w\). Therefore, the number of vertices that are adjacent to at least one of \(u\) or \(w\) is \(2k - \mu\). Then the number of vertices (other than \(u\) or \(w\)) that are adjacent to neither \(u\) nor \(w\) is given by \(\overline{\lambda} = v - 2k + \mu - 2\).

Now suppose that \(u\) and \(w\) are non-adjacent vertices in \(\overline{\Gamma}\). Then \(uw\) is an edge of \(\Gamma\). Each of \(u\) and \(w\) has \(k - 1\) additional neighbors in \(\Gamma\), and they have \(\lambda\) common neighbors. Therefore, there are \(2k - 2 - \lambda\) vertices (other than \(u\) and \(w\) themselves) that are adjacent to at least one of \(u\) or \(w\). That leaves \(v - 2 - (2k - 2 - \lambda) = v - 2k + \lambda\) vertices in \(\Gamma\) that are adjacent to neither \(u\) nor \(w\). This is then the number of common neighbors of \(u\) and \(w\) in \(\Gamma\) and \(\overline{\mu} = v - 2k + \lambda\).

To find the new eigenvalues, we can use the same equations given in Lemma 3.1 but using the new values for the complement. We know that \(v_2 = \frac{1}{2}(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})\).

Therefore,

\[
v_2 = \frac{1}{2}(\overline{\lambda} - \overline{\mu} + \sqrt{(\overline{\lambda} - \overline{\mu})^2 + 4(\overline{k} - \overline{\mu})})
\]
\[
\begin{align*}
\frac{1}{2}((v - 2k + \mu - 2) - (v - 2k + \lambda) \\
+ \sqrt{((v - 2k + \mu - 2) - (v - 2k + \lambda))^2 + 4((v - k - 1) - (v - 2k + \lambda))}
\end{align*}
\]

\[
= \frac{1}{2}((\mu - \lambda) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1
\]

\[
= -\frac{1}{2}((\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1
\]

\[
= -v_3 - 1
\]

A similar calculation can be made for \( v_3 \) so that

\[
\begin{align*}
\bar{v}_3 &= \frac{1}{2}(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) \\
&= \frac{1}{2}((v - 2k + \mu - 2) - (v - 2k + \lambda) \\
- \sqrt{((v - 2k + \mu - 2) - (v - 2k + \lambda))^2 + 4((v - k - 1) - (v - 2k + \lambda))}
\end{align*}
\]

\[
= \frac{1}{2}((\mu - \lambda) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1
\]

\[
= -\frac{1}{2}((\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) - 1
\]

\[
= -v_2 - 1,
\]

which completes the proof.

Now since we have the correlation between the original graph and its complement, we can focus on the conditions necessary to construct a partial difference set. Keep the same notation as in Proposition 4.1.

**Proposition 4.2.** If \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\) and \((v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)\), then \(|a^G \cap \Delta| \neq \emptyset\) and \(|a^G \cap \Delta^c| \neq \emptyset\).

**Proof.** By referencing Lemma 3.8, we already know that if \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\), then \(|a^G \cap \Delta| \neq \emptyset\). By substituting the parameters of the complement into this
relation, it follows that if \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\), then \(|a^G \cap \Delta^c| \neq \emptyset\).

**Corollary 4.3.** If \(G\) is a group of order \(v\) with \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\) and \((v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)\), and the center of \(G\) is nontrivial, then there cannot exist a \((v, k, \lambda, \mu)\)-PDS.

**Proof.** Let \(x \in Z(G)\) such that \(x \neq 1\) and \(D\) is a \((v, k, \lambda, \mu)\)-PDS. Then \(|x^G \cap D| \neq \emptyset\) and \(|x^G \cap (D \cup \{1\})^c| = |x^G \cap G - (D \cup \{1\})| \neq \emptyset\). This is not possible because either \(x^G = \{x\}\) lies in \(D\) or it lies in a set of points excluding \(D\); it cannot lie in both. Therefore \(D\) cannot exist.

Similarly, all \(p\)-groups have non-trivial centers, so we can say:

**Corollary 4.4.** If \(G\) is a \(p\)-group and \((v, k, \lambda, \mu)\), \(v_2\) and \(v_3\) satisfy the conditions that \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\) and \((v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)\), then there cannot exist a \((v, k, \lambda, \mu)\)-PDS in \(G\).

Therefore, we have our necessary conditions for ruling out the possibility of a partial difference set being constructed from a strongly regular graph.

### 4.1 Ruling Out Parameters

Brouwer [2] gives an extensive list of the parameters of strongly regular graphs along with their complements. We work through this list to find all the parameter sets through size 300 that satisfy the necessary conditions of \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\) and \((v_2 - v_3) \nmid v - 2k + \lambda - v_3(v_2 + 1)\). In some cases, however, \((v_2 - v_3) \nmid \mu - v_3(v_2 + 1)\) while \((v_2 - v_3)|v - 2k + \lambda - v_3(v_2 + 1)\). For these cases, we created a code in GAP [13] that tested the parameter sets and determined whether a possible partial difference set could exist. From there, we read the code into Gurobi [8] to solve and determine definitively whether or not a partial difference set exists. The first set of code is used when you are assuming that automorphisms exist other than the \(G\) automorphism.
acting regularly. (This is helpful in shortening the running time for large groups).
The second code, however, is more accurate and does not necessarily assume that
automorphisms exist. Both GAP codes are presented below.

4.1.1 Assuming Automorphisms

GurobifyAutPDS := function(mat, w1, w2, size, j1, j2, inversepairs, TorF,
    conj, filename)
    local numrows, numcols, i, j, output, positions, total, numclasses;
    total := 2*(size-1);
    numrows := Length(mat);
    numcols := Length(mat[1]);
    numclasses:= Length(conj);
    output := OutputTextFile( filename, false );;
    SetPrintFormattingStatus(output, false);
    AppendTo(output,"Maximize\n");
    for i in [1..numrows] do
        AppendTo(output, Concatenation( " + 0 r", String(i)));
    od;
    AppendTo(output,"\n Subject To\n");
    for i in [1..numcols/2] do
        AppendTo(output, Concatenation( " + ", String(j1[i])," s", String(i)));
    od;
    AppendTo(output, Concatenation( " = ", String(size), "\n" ));
    for i in [1..numcols/2] do
        positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
        for j in [1..Size(positions)] do
            AppendTo(output, Concatenation( " + ", String(mat[positions[j]][i]),
        od;
    od;
end;
" r", String(positions[j]));

od;

AppendTo(output, Concatenation(" - ", String(w1-w2), " s", String(i), " = ", String(w2), "\n"));

od;
for i in [1..numrows] do
  AppendTo(output, Concatenation(" + ", String(j2[i])," r", String(i)));
od;
AppendTo(output, Concatenation(" = ", String(size*(size - 1)), "\n"));
## put incidences in. e.g., r1 =1 => s1=s2=1.
AppendTo(output, "\\ Incidences\n");
for i in [1..numrows] do
  positions := Filtered([1..numcols/2], j -> IsOne(mat[i][j+numcols/2]));
  for j in positions do
    AppendTo(output, Concatenation("r", String(i)," - s", String(j), " < 0\n"));
  od;
od;
AppendTo(output, "\\ Inverse Pairs\n");
for i in inversepairs do
  AppendTo(output, Concatenation("s", String(i[1]), " - s", String(i[2]), " = 0\n"));
od;
if TorF then
  AppendTo(output, "\\ Each Nonidentity Conjugacy Class Meets Connection Set\n");
  for i in [1..numclasses] do
positions := Filtered([1..numcols/2], j -> IsOne(conj[i][j]));
for j in positions do
    AppendTo(output, Concatenation(" + 1 s", String(j))); od;
AppendTo(output, " > 1
"); od;
fi;
AppendTo(output, "\\ Variables\n");
AppendTo(output,"Binary\n");
for i in [1..numrows] do
    AppendTo(output, Concatenation( "r", String(i), "\n")); od;
for i in [1..numcols/2] do
    AppendTo(output, Concatenation( "s", String(i), "\n")); od;
AppendTo(output,"End\n");
CloseStream(output);
return;
end;

CreateAutGurobi:= function(g, aut, size, w1, w2, filename)

local orbs, count, notone, t, j1, j2, pairs, mat, i ,j, x, y, inversepairs, d, v2, v3, conj, conjclasses, TorF;

orbs:= Orbits(aut, g);;
notone := Filtered(AsList(orbs), t -> not IsOne(Random(t)));;

34
pairs := Tuples(notone, 2);

j1:= List(notone, t -> Size(t));

j2:= [];

for t in pairs do
    count:= 0;
    for i in t[1] do
        for j in t[2] do
            if not IsOne(i*j^-1) then
                count:= count + 1;
            fi;
        od;
    od;
    Add(j2, count);
od;

mat := NullMat(Size(pairs), 2*Size(notone), Rationals);

for i in [1..Size(pairs)] do
    for j in [1..Size(notone)] do
        t:= Random(notone[j]);
        count:= 0;
        for x in pairs[i][1] do
            for y in pairs[i][2] do
                if x*y^-1 = t then
                    count:= count + 1;
                fi;
            od;
        od;
        mat[i][j]:= count;
if notone[j] in pairs[i] then
    mat[i][j+Size(notone)] := 1;
fi;
od;
if i mod 1000 = 0 then Print(i,".\c"); fi;
od;
inversepairs := Filtered(Combinations([1..Size(notone)],2),
    x-> Random(notone[x[1]])^-1 in notone[x[2]]);
v2:= (w1 - w2 + Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
v3:= (w1 - w2 - Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
conjclasses:= Orbits(aut, List(Filtered(ConjugacyClasses(g),
    i -> not IsOne(Random(i))), j -> AsSet(j)), OnSets) ;
inversepairs := Filtered(ConjugacyClasses(g), i -> not IsOne(Random(i)));
if (w2 - v3*(v2 + 1) mod (v2 - v3) = 0) then
    TorF:= false;
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
else
    TorF:= true;
    conj:= NullMat(Size(conjclasses), Size(notone), Rationals);
for i in [1..Size(conjclasses)] do
    for j in [1..Size(notone)] do
        if Size(Intersection(notone[j],conjclasses[i])) > 0 then
            conj[i][j] := 1;
        fi;
    od;
od;
fi;
GurobifyAutPDS(mat, w1, w2, size, j1, j2, inversepairs, TorF, conj, filename);
end;

AutGurList:= function(G, list, size, w1, w2)
local x, name;
for x in [1..Length(list)] do
name:= Concatenation(String(Order(G)), "_", String(IdGroup(G)[2]), "_",
    String(x),".lp");
CreateAutGurobi(G, list[x], size, w1, w2, name);
od;
end;

FindOuts:= function(G)
local aut, inn, conjauts, x, i, list, nonid, reps;
aut:= AutomorphismGroup(G);
inn:= InnerAutomorphismsAutomorphismGroup(aut);
conjauts:= ConjugacyClassesSubgroups(aut);
reps:= List(conjauts, i -> Representative(i));
list:= Filtered(reps, i -> Size(Intersection(i, inn))=1);
nonid:= Filtered(list, i -> Order(i)>1);
return nonid;
end;

AutGur:= function(G, size, w1, w2)
local l, x;
l := FindOuts(G);
AutGurList(G, l, size, w1, w2);
return l;
end;

4.1.2 Not Assuming Automorphisms

GurobifyPDS := function(mat, w1, w2, size, inversepairs, TorF, conj, filename)
    local numrows, numcols, i, j, output, positions, total, numclasses;
    total := 2*(size-1);
    numrows := Length(mat);
    numcols := Length(mat[1]);
    numclasses := Length(conj);
    output := OutputTextFile( filename, false );;
    SetPrintFormattingStatus(output, false);
    AppendTo(output,"Maximize\n");
    for i in [1..numrows] do
        AppendTo(output, Concatenation( " + 0 r", String(i)));
    od;
    AppendTo(output,"\n Subject To\n");
    for i in [1..numcols/2] do
        AppendTo(output, Concatenation( " + 1 s", String(i)));
    od;
    AppendTo(output, Concatenation( " = ", String(size), "\n" ));
    for i in [1..numcols/2] do
        positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
        for j in [1..Size(positions)] do
            
38
AppendTo(output, Concatenation(" + ", String(mat[positions[j]][i]), " r", String(positions[j])));

od;
AppendTo(output, Concatenation(" - ", String(w1-w2), " s", String(i), " = ", String(w2), "\n"));

od;
for i in [(numcols/2)+1..numcols] do
  positions := Filtered([1..numrows], j -> not IsZero( mat[j][i] ) );
  for j in [1..Size(positions)] do
    AppendTo(output, Concatenation(" + ", String(mat[positions[j]][i]), " r", String(positions[j])));
  od;
  AppendTo(output, Concatenation(" - ", String(total), " s", String(i - numcols/2), " = 0 \n"));
od;
## put incidences in. e.g., r1 =1 => s1=s2=1.
AppendTo(output, "\\ Incidences\n"); for i in [1..numrows] do
  positions := Filtered([1..numcols/2], j -> IsOne( mat[i][j+numcols/2] ) );
  for j in positions do
    AppendTo(output, Concatenation("r", String(i)," - s", String(j), " < 0\n"));
  od;
od;
AppendTo(output, "\\ Inverse Pairs\n"); for i in inversepairs do
  AppendTo(output, Concatenation("s", String(i[1]), " - s", String(i[2]), " < 0\n"));
od;
if TorF then
    AppendTo(output, "\\ Each Nonidentity Conjugacy Class Meets Connection Set\\n");
    for i in [1..numclasses] do
        positions := Filtered([1..numcols/2], j -> IsOne(conj[i][j]));
        for j in positions do
            AppendTo(output, Concatenation(" + 1 s", String(j)));
        od;
        AppendTo(output, " > 1\\n");
    od;
fi;
AppendTo(output, "\\ Variables\\n");
AppendTo(output,"Binary\\n");
for i in [1..numrows] do
    AppendTo(output, Concatenation("r", String(i), "\\n"));
od;
for i in [1..numcols/2] do
    AppendTo(output, Concatenation("s", String(i), "\\n"));
od;
AppendTo(output,"End\\n");
CloseStream(output);
return;
end;

CreateGurobi:= function(g, size, w1, w2, filename)
local notone, t, pairs, mat, i, j, inversepairs, v2, v3, conj, conjclasses, TorF;

notone := Filtered(AsList(g), t -> not IsOne(t));;
pairs := Filtered(Tuples(notone, 2), t -> not IsOne(t[1]*t[2]^-1));;
mat := NullMat(Size(pairs), 2*Size(notone), Rationals);
for i in [1..Size(pairs)] do
    for j in [1..Size(notone)] do
        if pairs[i][1] * pairs[i][2]^-1 = notone[j] then
            mat[i][j] := 1;
        fi;

        if notone[j] in pairs[i] then
            mat[i][j+Size(notone)] := 1;
        fi;
    od;
    if i mod 1000 = 0 then Print(i,"\.c"); fi;
od;

inversepairs := Filtered(Combinations([1..Size(notone)], 2), x -> notone[x[1]] = notone[x[2]]^-1);;

v2 := (w1 - w2 + Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
v3 := (w1 - w2 - Sqrt((w1 - w2)^2 + 4*(size - w2)))/2;
conjclasses := Filtered(ConjugacyClasses(g), i -> not IsOne(Random(i)));
if (w2 - v3*(v2 + 1) mod (v2 - v3) = 0) then
    TorF := false;
    conj := NullMat(Size(conjclasses), Size(notone), Rationals);
else
    TorF := true;
end if;
\begin{verbatim}
conj := NullMat(Size(conjclasses), Size(notone), Rationals);
for i in [1..Size(conjclasses)] do
    for j in [1..Size(notone)] do
        if notone[j] in conjclasses[i] then
            conj[i][j] := 1;
        fi;
    od;
od;
fi;

GurobifyPDS(mat, w1, w2, size, inversepairs, TorF , conj, filename);
end;
\end{verbatim}

\section*{4.2 Table of Nonexistence}

Using the conditions outlined in Corollary 4.3 as well as the GAP code above, we worked through Brouwer's list of parameters (up through $v = 300$ and some larger $p$-groups) to rule out the parameter sets for which partial difference sets cannot be constructed. The list below shows our results.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$v$ & $k$ & $\lambda$ & $\mu$ & $v_2$ & $v_3$ & $(v_2 - v_3)$ & $\mu - v_3(v_2 + 1)$ & method to rule out \\
\hline
28 & 12 & 6 & 4 & 4 & -2 & 6 & 14 & Corollary 4.3 \\
\hline
15 & 6 & 10 & 1 & -5 & 6 & 20 & & \\
\hline
36 & 14 & 7 & 4 & 5 & -2 & 7 & 16 & Corollary 4.3 \\
\hline
21 & 10 & 15 & 1 & -6 & 7 & 27 & & \\
\hline
40 & 12 & 2 & 4 & 2 & -4 & 6 & 16 & GAP code \\
\hline
27 & 18 & 18 & 3 & -3 & 6 & 30 & & \\
\hline
\end{tabular}
\end{center}
<table>
<thead>
<tr>
<th>$v$</th>
<th>$k$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$(v_2 - v_3)$</th>
<th>$\mu - v_3(v_2 + 1)$</th>
<th>method to rule out</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>21</td>
<td>8</td>
<td>9</td>
<td>3</td>
<td>-4</td>
<td>7</td>
<td>25</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>15</td>
<td>16</td>
<td>3</td>
<td>-4</td>
<td>7</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>10</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>-4</td>
<td>6</td>
<td>14</td>
<td>GAP</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>36</td>
<td>36</td>
<td>3</td>
<td>-3</td>
<td>6</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>30</td>
<td>13</td>
<td>15</td>
<td>3</td>
<td>-5</td>
<td>8</td>
<td>35</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>16</td>
<td>16</td>
<td>4</td>
<td>-4</td>
<td>8</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>66</td>
<td>20</td>
<td>10</td>
<td>4</td>
<td>8</td>
<td>-2</td>
<td>10</td>
<td>22</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>28</td>
<td>36</td>
<td>1</td>
<td>-9</td>
<td>10</td>
<td>54</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>27</td>
<td>12</td>
<td>9</td>
<td>6</td>
<td>-3</td>
<td>9</td>
<td>30</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>42</td>
<td>23</td>
<td>28</td>
<td>2</td>
<td>-7</td>
<td>9</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>78</td>
<td>22</td>
<td>11</td>
<td>4</td>
<td>9</td>
<td>-2</td>
<td>11</td>
<td>24</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>55</td>
<td>36</td>
<td>45</td>
<td>1</td>
<td>-10</td>
<td>11</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>88</td>
<td>27</td>
<td>6</td>
<td>9</td>
<td>3</td>
<td>-6</td>
<td>9</td>
<td>33</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>41</td>
<td>40</td>
<td>5</td>
<td>-4</td>
<td>9</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>35</td>
<td>10</td>
<td>14</td>
<td>3</td>
<td>-7</td>
<td>10</td>
<td>42</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>38</td>
<td>36</td>
<td>6</td>
<td>-4</td>
<td>10</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>33</td>
<td>14</td>
<td>9</td>
<td>8</td>
<td>-3</td>
<td>11</td>
<td>36</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>66</td>
<td>41</td>
<td>48</td>
<td>2</td>
<td>-9</td>
<td>11</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>105</td>
<td>26</td>
<td>13</td>
<td>4</td>
<td>11</td>
<td>-2</td>
<td>13</td>
<td>28</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>78</td>
<td>55</td>
<td>66</td>
<td>1</td>
<td>-12</td>
<td>13</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>105</td>
<td>32</td>
<td>4</td>
<td>12</td>
<td>2</td>
<td>-10</td>
<td>12</td>
<td>42</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>72</td>
<td>51</td>
<td>45</td>
<td>9</td>
<td>-3</td>
<td>12</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>105</td>
<td>52</td>
<td>21</td>
<td>30</td>
<td>2</td>
<td>-11</td>
<td>13</td>
<td>63</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>52</td>
<td>29</td>
<td>22</td>
<td>10</td>
<td>-3</td>
<td>13</td>
<td>55</td>
<td></td>
</tr>
<tr>
<td>112</td>
<td>30</td>
<td>2</td>
<td>10</td>
<td>2</td>
<td>-10</td>
<td>12</td>
<td>40</td>
<td>GAP code</td>
</tr>
<tr>
<td></td>
<td>81</td>
<td>60</td>
<td>54</td>
<td>9</td>
<td>-3</td>
<td>12</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>--------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>-------------</td>
<td>------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>112</td>
<td>36</td>
<td>10</td>
<td>12</td>
<td>4</td>
<td>-6</td>
<td>10</td>
<td>42</td>
<td>GAP code</td>
</tr>
<tr>
<td>75</td>
<td>50</td>
<td>50</td>
<td>5</td>
<td>-5</td>
<td>10</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>117</td>
<td>36</td>
<td>15</td>
<td>9</td>
<td>9</td>
<td>-3</td>
<td>12</td>
<td>39</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>80</td>
<td>52</td>
<td>60</td>
<td>2</td>
<td>-10</td>
<td>12</td>
<td>90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>28</td>
<td>14</td>
<td>4</td>
<td>12</td>
<td>-2</td>
<td>14</td>
<td>30</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>91</td>
<td>66</td>
<td>78</td>
<td>1</td>
<td>-13</td>
<td>14</td>
<td>104</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>42</td>
<td>8</td>
<td>18</td>
<td>2</td>
<td>-12</td>
<td>14</td>
<td>54</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>77</td>
<td>52</td>
<td>44</td>
<td>11</td>
<td>-3</td>
<td>14</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>126</td>
<td>25</td>
<td>8</td>
<td>4</td>
<td>7</td>
<td>-3</td>
<td>10</td>
<td>28</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>100</td>
<td>78</td>
<td>84</td>
<td>2</td>
<td>-8</td>
<td>10</td>
<td>108</td>
<td></td>
<td></td>
</tr>
<tr>
<td>126</td>
<td>50</td>
<td>13</td>
<td>24</td>
<td>2</td>
<td>-13</td>
<td>15</td>
<td>63</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>75</td>
<td>48</td>
<td>39</td>
<td>12</td>
<td>-3</td>
<td>15</td>
<td>78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>126</td>
<td>60</td>
<td>33</td>
<td>24</td>
<td>12</td>
<td>-3</td>
<td>15</td>
<td>63</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>65</td>
<td>28</td>
<td>39</td>
<td>2</td>
<td>-13</td>
<td>15</td>
<td>78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>130</td>
<td>48</td>
<td>20</td>
<td>16</td>
<td>8</td>
<td>-4</td>
<td>12</td>
<td>52</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>81</td>
<td>48</td>
<td>54</td>
<td>3</td>
<td>-9</td>
<td>12</td>
<td>90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>136</td>
<td>30</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>-4</td>
<td>10</td>
<td>34</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>105</td>
<td>80</td>
<td>84</td>
<td>3</td>
<td>-7</td>
<td>10</td>
<td>112</td>
<td></td>
<td></td>
</tr>
<tr>
<td>136</td>
<td>30</td>
<td>15</td>
<td>4</td>
<td>13</td>
<td>-2</td>
<td>15</td>
<td>32</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>105</td>
<td>78</td>
<td>91</td>
<td>1</td>
<td>-14</td>
<td>15</td>
<td>119</td>
<td></td>
<td></td>
</tr>
<tr>
<td>136</td>
<td>60</td>
<td>24</td>
<td>28</td>
<td>4</td>
<td>-8</td>
<td>12</td>
<td>68</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>75</td>
<td>42</td>
<td>40</td>
<td>7</td>
<td>-5</td>
<td>12</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>136</td>
<td>63</td>
<td>30</td>
<td>28</td>
<td>7</td>
<td>-5</td>
<td>12</td>
<td>68</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>72</td>
<td>36</td>
<td>40</td>
<td>4</td>
<td>-8</td>
<td>12</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>148</td>
<td>63</td>
<td>22</td>
<td>30</td>
<td>3</td>
<td>-11</td>
<td>14</td>
<td>74</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>84</td>
<td>50</td>
<td>44</td>
<td>10</td>
<td>-4</td>
<td>14</td>
<td>88</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>---------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------------</td>
<td>----------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>148</td>
<td>70</td>
<td>36</td>
<td>30</td>
<td>10</td>
<td>-4</td>
<td>14</td>
<td>74</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>77</td>
<td>36</td>
<td>44</td>
<td>3</td>
<td>-11</td>
<td>14</td>
<td>88</td>
<td></td>
</tr>
<tr>
<td>154</td>
<td>48</td>
<td>12</td>
<td>16</td>
<td>4</td>
<td>-8</td>
<td>12</td>
<td>56</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>72</td>
<td>70</td>
<td>7</td>
<td>-5</td>
<td>12</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>154</td>
<td>72</td>
<td>26</td>
<td>40</td>
<td>2</td>
<td>-16</td>
<td>18</td>
<td>88</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>81</td>
<td>48</td>
<td>36</td>
<td>15</td>
<td>-3</td>
<td>18</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>78</td>
<td>35</td>
<td>36</td>
<td>6</td>
<td>-7</td>
<td>13</td>
<td>85</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>91</td>
<td>48</td>
<td>49</td>
<td>6</td>
<td>-7</td>
<td>13</td>
<td>98</td>
<td></td>
</tr>
<tr>
<td>171</td>
<td>34</td>
<td>17</td>
<td>4</td>
<td>15</td>
<td>-2</td>
<td>17</td>
<td>36</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>136</td>
<td>105</td>
<td>120</td>
<td>1</td>
<td>-16</td>
<td>17</td>
<td>152</td>
<td></td>
</tr>
<tr>
<td>171</td>
<td>50</td>
<td>13</td>
<td>15</td>
<td>5</td>
<td>-7</td>
<td>12</td>
<td>57</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>84</td>
<td>84</td>
<td>6</td>
<td>-6</td>
<td>12</td>
<td>126</td>
<td></td>
</tr>
<tr>
<td>171</td>
<td>60</td>
<td>15</td>
<td>24</td>
<td>3</td>
<td>-12</td>
<td>15</td>
<td>72</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>73</td>
<td>66</td>
<td>11</td>
<td>-4</td>
<td>15</td>
<td>114</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>25</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-7</td>
<td>10</td>
<td>32</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>128</td>
<td>126</td>
<td>6</td>
<td>-4</td>
<td>10</td>
<td>154</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>45</td>
<td>18</td>
<td>9</td>
<td>12</td>
<td>-3</td>
<td>15</td>
<td>48</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>130</td>
<td>93</td>
<td>104</td>
<td>2</td>
<td>-13</td>
<td>15</td>
<td>143</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>70</td>
<td>18</td>
<td>34</td>
<td>2</td>
<td>-18</td>
<td>20</td>
<td>88</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>68</td>
<td>54</td>
<td>17</td>
<td>-3</td>
<td>20</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>70</td>
<td>24</td>
<td>30</td>
<td>4</td>
<td>-10</td>
<td>14</td>
<td>80</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>64</td>
<td>60</td>
<td>9</td>
<td>-5</td>
<td>14</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>85</td>
<td>48</td>
<td>34</td>
<td>17</td>
<td>-3</td>
<td>20</td>
<td>88</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>38</td>
<td>54</td>
<td>2</td>
<td>-18</td>
<td>20</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>189</td>
<td>48</td>
<td>12</td>
<td>12</td>
<td>6</td>
<td>-6</td>
<td>12</td>
<td>54</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td></td>
<td>140</td>
<td>103</td>
<td>105</td>
<td>5</td>
<td>-7</td>
<td>12</td>
<td>147</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>----------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>----------------</td>
<td>----------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>190</td>
<td>36</td>
<td>18</td>
<td>4</td>
<td>16</td>
<td>-2</td>
<td>18</td>
<td>38</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>153</td>
<td>120</td>
<td>136</td>
<td>1</td>
<td>-17</td>
<td>18</td>
<td>170</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190</td>
<td>45</td>
<td>12</td>
<td>10</td>
<td>7</td>
<td>-5</td>
<td>12</td>
<td>50</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>144</td>
<td>108</td>
<td>112</td>
<td>4</td>
<td>-8</td>
<td>12</td>
<td>152</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190</td>
<td>84</td>
<td>33</td>
<td>40</td>
<td>4</td>
<td>-11</td>
<td>15</td>
<td>95</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>105</td>
<td>60</td>
<td>55</td>
<td>10</td>
<td>-5</td>
<td>15</td>
<td>110</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190</td>
<td>84</td>
<td>36</td>
<td>8</td>
<td>-6</td>
<td>14</td>
<td>90</td>
<td></td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>105</td>
<td>56</td>
<td>60</td>
<td>5</td>
<td>-9</td>
<td>14</td>
<td>114</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190</td>
<td>90</td>
<td>45</td>
<td>40</td>
<td>10</td>
<td>-5</td>
<td>15</td>
<td>95</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>99</td>
<td>48</td>
<td>55</td>
<td>4</td>
<td>-11</td>
<td>15</td>
<td>110</td>
<td></td>
<td></td>
</tr>
<tr>
<td>195</td>
<td>96</td>
<td>46</td>
<td>48</td>
<td>6</td>
<td>-8</td>
<td>14</td>
<td>104</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>98</td>
<td>49</td>
<td>49</td>
<td>7</td>
<td>-7</td>
<td>14</td>
<td>105</td>
<td></td>
<td></td>
</tr>
<tr>
<td>196</td>
<td>39</td>
<td>2</td>
<td>9</td>
<td>3</td>
<td>-10</td>
<td>13</td>
<td>49</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>156</td>
<td>125</td>
<td>120</td>
<td>9</td>
<td>-4</td>
<td>13</td>
<td>160</td>
<td></td>
<td></td>
</tr>
<tr>
<td>196</td>
<td>60</td>
<td>23</td>
<td>16</td>
<td>11</td>
<td>-4</td>
<td>15</td>
<td>64</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>135</td>
<td>90</td>
<td>99</td>
<td>3</td>
<td>-12</td>
<td>15</td>
<td>147</td>
<td></td>
<td></td>
</tr>
<tr>
<td>204</td>
<td>63</td>
<td>22</td>
<td>18</td>
<td>9</td>
<td>-5</td>
<td>14</td>
<td>68</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>140</td>
<td>94</td>
<td>100</td>
<td>4</td>
<td>-10</td>
<td>14</td>
<td>150</td>
<td></td>
<td></td>
</tr>
<tr>
<td>208</td>
<td>75</td>
<td>30</td>
<td>25</td>
<td>10</td>
<td>-5</td>
<td>15</td>
<td>80</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>132</td>
<td>81</td>
<td>88</td>
<td>4</td>
<td>-11</td>
<td>15</td>
<td>143</td>
<td></td>
<td></td>
</tr>
<tr>
<td>208</td>
<td>81</td>
<td>24</td>
<td>36</td>
<td>3</td>
<td>-15</td>
<td>18</td>
<td>88</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>126</td>
<td>80</td>
<td>70</td>
<td>14</td>
<td>-4</td>
<td>18</td>
<td>130</td>
<td></td>
<td></td>
</tr>
<tr>
<td>210</td>
<td>38</td>
<td>19</td>
<td>4</td>
<td>17</td>
<td>-2</td>
<td>19</td>
<td>40</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>171</td>
<td>136</td>
<td>153</td>
<td>1</td>
<td>-18</td>
<td>19</td>
<td>189</td>
<td></td>
<td></td>
</tr>
<tr>
<td>220</td>
<td>84</td>
<td>38</td>
<td>28</td>
<td>14</td>
<td>-4</td>
<td>18</td>
<td>88</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>135</td>
<td>78</td>
<td>90</td>
<td>3</td>
<td>-15</td>
<td>18</td>
<td>150</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>--------------</td>
<td>-------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>222</td>
<td>51</td>
<td>20</td>
<td>9</td>
<td>14</td>
<td>-3</td>
<td>17</td>
<td>54</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>170</td>
<td>127</td>
<td>140</td>
<td>2</td>
<td>-15</td>
<td>17</td>
<td>185</td>
<td></td>
<td></td>
</tr>
<tr>
<td>225</td>
<td>96</td>
<td>51</td>
<td>33</td>
<td>21</td>
<td>-3</td>
<td>24</td>
<td>99</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>128</td>
<td>64</td>
<td>84</td>
<td>2</td>
<td>-22</td>
<td>24</td>
<td>150</td>
<td></td>
<td></td>
</tr>
<tr>
<td>231</td>
<td>30</td>
<td>9</td>
<td>3</td>
<td>9</td>
<td>-3</td>
<td>12</td>
<td>33</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>200</td>
<td>172</td>
<td>180</td>
<td>2</td>
<td>-10</td>
<td>12</td>
<td>210</td>
<td></td>
<td></td>
</tr>
<tr>
<td>231</td>
<td>40</td>
<td>20</td>
<td>4</td>
<td>18</td>
<td>-2</td>
<td>20</td>
<td>42</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>190</td>
<td>153</td>
<td>171</td>
<td>1</td>
<td>-19</td>
<td>20</td>
<td>209</td>
<td></td>
<td></td>
</tr>
<tr>
<td>231</td>
<td>90</td>
<td>33</td>
<td>36</td>
<td>6</td>
<td>-9</td>
<td>15</td>
<td>99</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>140</td>
<td>85</td>
<td>84</td>
<td>8</td>
<td>-7</td>
<td>15</td>
<td>147</td>
<td></td>
<td></td>
</tr>
<tr>
<td>232</td>
<td>33</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>-7</td>
<td>11</td>
<td>40</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>198</td>
<td>169</td>
<td>168</td>
<td>6</td>
<td>-5</td>
<td>11</td>
<td>203</td>
<td></td>
<td></td>
</tr>
<tr>
<td>232</td>
<td>63</td>
<td>14</td>
<td>18</td>
<td>5</td>
<td>-9</td>
<td>14</td>
<td>72</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>168</td>
<td>122</td>
<td>120</td>
<td>8</td>
<td>-6</td>
<td>14</td>
<td>174</td>
<td></td>
<td></td>
</tr>
<tr>
<td>232</td>
<td>7</td>
<td>36</td>
<td>20</td>
<td>19</td>
<td>-3</td>
<td>22</td>
<td>80</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>154</td>
<td>96</td>
<td>114</td>
<td>2</td>
<td>-20</td>
<td>22</td>
<td>174</td>
<td></td>
<td></td>
</tr>
<tr>
<td>232</td>
<td>81</td>
<td>30</td>
<td>27</td>
<td>9</td>
<td>-6</td>
<td>15</td>
<td>87</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>150</td>
<td>95</td>
<td>100</td>
<td>5</td>
<td>-10</td>
<td>15</td>
<td>160</td>
<td></td>
<td></td>
</tr>
<tr>
<td>236</td>
<td>55</td>
<td>18</td>
<td>11</td>
<td>11</td>
<td>-4</td>
<td>15</td>
<td>59</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>180</td>
<td>135</td>
<td>144</td>
<td>3</td>
<td>-12</td>
<td>15</td>
<td>192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>238</td>
<td>75</td>
<td>20</td>
<td>25</td>
<td>5</td>
<td>-10</td>
<td>15</td>
<td>85</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>162</td>
<td>111</td>
<td>108</td>
<td>9</td>
<td>-6</td>
<td>15</td>
<td>168</td>
<td></td>
<td></td>
</tr>
<tr>
<td>244</td>
<td>108</td>
<td>42</td>
<td>52</td>
<td>4</td>
<td>-14</td>
<td>18</td>
<td>121</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>135</td>
<td>78</td>
<td>70</td>
<td>13</td>
<td>-5</td>
<td>18</td>
<td>140</td>
<td></td>
<td></td>
</tr>
<tr>
<td>244</td>
<td>117</td>
<td>60</td>
<td>52</td>
<td>13</td>
<td>-5</td>
<td>18</td>
<td>122</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>126</td>
<td>60</td>
<td>70</td>
<td>4</td>
<td>-14</td>
<td>18</td>
<td>140</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>--------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>-------------</td>
<td>-----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>246</td>
<td>85</td>
<td>20</td>
<td>34</td>
<td>3</td>
<td>-17</td>
<td>20</td>
<td>102</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>160</td>
<td>108</td>
<td>96</td>
<td>16</td>
<td>-4</td>
<td>20</td>
<td>102</td>
<td>164</td>
<td></td>
</tr>
<tr>
<td>246</td>
<td>105</td>
<td>36</td>
<td>51</td>
<td>3</td>
<td>-18</td>
<td>21</td>
<td>123</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>140</td>
<td>85</td>
<td>72</td>
<td>17</td>
<td>-4</td>
<td>21</td>
<td>123</td>
<td>144</td>
<td></td>
</tr>
<tr>
<td>246</td>
<td>119</td>
<td>64</td>
<td>51</td>
<td>17</td>
<td>-4</td>
<td>21</td>
<td>123</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>126</td>
<td>57</td>
<td>72</td>
<td>3</td>
<td>-18</td>
<td>21</td>
<td>123</td>
<td>144</td>
<td></td>
</tr>
<tr>
<td>266</td>
<td>45</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>-12</td>
<td>15</td>
<td>57</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>220</td>
<td>183</td>
<td>176</td>
<td>11</td>
<td>-4</td>
<td>15</td>
<td>224</td>
<td></td>
<td></td>
</tr>
<tr>
<td>273</td>
<td>80</td>
<td>19</td>
<td>25</td>
<td>5</td>
<td>-11</td>
<td>16</td>
<td>91</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>192</td>
<td>136</td>
<td>132</td>
<td>10</td>
<td>-6</td>
<td>16</td>
<td>198</td>
<td></td>
<td></td>
</tr>
<tr>
<td>273</td>
<td>102</td>
<td>41</td>
<td>36</td>
<td>11</td>
<td>-6</td>
<td>17</td>
<td>108</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>170</td>
<td>103</td>
<td>110</td>
<td>5</td>
<td>-12</td>
<td>17</td>
<td>182</td>
<td></td>
<td></td>
</tr>
<tr>
<td>273</td>
<td>136</td>
<td>65</td>
<td>70</td>
<td>6</td>
<td>-11</td>
<td>17</td>
<td>147</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>136</td>
<td>69</td>
<td>66</td>
<td>10</td>
<td>-7</td>
<td>17</td>
<td>143</td>
<td></td>
<td></td>
</tr>
<tr>
<td>275</td>
<td>112</td>
<td>30</td>
<td>56</td>
<td>2</td>
<td>-28</td>
<td>30</td>
<td>140</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>162</td>
<td>105</td>
<td>81</td>
<td>27</td>
<td>-3</td>
<td>30</td>
<td>165</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>44</td>
<td>22</td>
<td>4</td>
<td>20</td>
<td>-2</td>
<td>22</td>
<td>46</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>231</td>
<td>190</td>
<td>210</td>
<td>1</td>
<td>-21</td>
<td>22</td>
<td>252</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>75</td>
<td>10</td>
<td>24</td>
<td>3</td>
<td>-17</td>
<td>20</td>
<td>92</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>200</td>
<td>148</td>
<td>136</td>
<td>16</td>
<td>-4</td>
<td>20</td>
<td>204</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>75</td>
<td>18</td>
<td>21</td>
<td>6</td>
<td>-9</td>
<td>15</td>
<td>84</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>200</td>
<td>145</td>
<td>144</td>
<td>8</td>
<td>-7</td>
<td>15</td>
<td>207</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>110</td>
<td>52</td>
<td>38</td>
<td>18</td>
<td>-4</td>
<td>22</td>
<td>114</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>165</td>
<td>92</td>
<td>108</td>
<td>3</td>
<td>-19</td>
<td>22</td>
<td>184</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>135</td>
<td>78</td>
<td>54</td>
<td>27</td>
<td>-3</td>
<td>30</td>
<td>138</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>140</td>
<td>58</td>
<td>84</td>
<td>2</td>
<td>-28</td>
<td>30</td>
<td>168</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$(v_2 - v_3)$</td>
<td>$\mu - v_3(v_2 + 1)$</td>
<td>method to rule out</td>
</tr>
<tr>
<td>------</td>
<td>------</td>
<td>------------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>--------------</td>
<td>---------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>279</td>
<td>128</td>
<td>52</td>
<td>64</td>
<td>4</td>
<td>-16</td>
<td>20</td>
<td>144</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>150</td>
<td>85</td>
<td>75</td>
<td>15</td>
<td>-5</td>
<td>20</td>
<td>155</td>
<td></td>
<td></td>
</tr>
<tr>
<td>280</td>
<td>117</td>
<td>44</td>
<td>52</td>
<td>5</td>
<td>-13</td>
<td>18</td>
<td>130</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>162</td>
<td>96</td>
<td>90</td>
<td>12</td>
<td>-6</td>
<td>18</td>
<td>168</td>
<td></td>
<td></td>
</tr>
<tr>
<td>285</td>
<td>64</td>
<td>8</td>
<td>16</td>
<td>4</td>
<td>-12</td>
<td>16</td>
<td>76</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>220</td>
<td>171</td>
<td>165</td>
<td>11</td>
<td>-5</td>
<td>16</td>
<td>225</td>
<td></td>
<td></td>
</tr>
<tr>
<td>286</td>
<td>95</td>
<td>24</td>
<td>33</td>
<td>4</td>
<td>-15</td>
<td>19</td>
<td>108</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>190</td>
<td>129</td>
<td>120</td>
<td>14</td>
<td>-5</td>
<td>19</td>
<td>195</td>
<td></td>
<td></td>
</tr>
<tr>
<td>286</td>
<td>125</td>
<td>60</td>
<td>50</td>
<td>15</td>
<td>-5</td>
<td>20</td>
<td>130</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>160</td>
<td>84</td>
<td>96</td>
<td>4</td>
<td>-16</td>
<td>20</td>
<td>176</td>
<td></td>
<td></td>
</tr>
<tr>
<td>288</td>
<td>105</td>
<td>52</td>
<td>30</td>
<td>25</td>
<td>-3</td>
<td>28</td>
<td>108</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>182</td>
<td>106</td>
<td>130</td>
<td>2</td>
<td>-26</td>
<td>28</td>
<td>208</td>
<td></td>
<td></td>
</tr>
<tr>
<td>290</td>
<td>136</td>
<td>63</td>
<td>64</td>
<td>8</td>
<td>-9</td>
<td>17</td>
<td>145</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>153</td>
<td>80</td>
<td>81</td>
<td>8</td>
<td>-9</td>
<td>17</td>
<td>161</td>
<td></td>
<td></td>
</tr>
<tr>
<td>297</td>
<td>128</td>
<td>64</td>
<td>48</td>
<td>20</td>
<td>-4</td>
<td>24</td>
<td>132</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>168</td>
<td>87</td>
<td>105</td>
<td>3</td>
<td>-21</td>
<td>24</td>
<td>189</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>46</td>
<td>23</td>
<td>4</td>
<td>21</td>
<td>-2</td>
<td>23</td>
<td>48</td>
<td>Corollary 4.3</td>
</tr>
<tr>
<td>253</td>
<td>210</td>
<td>231</td>
<td>1</td>
<td>-22</td>
<td>23</td>
<td>275</td>
<td></td>
<td></td>
</tr>
<tr>
<td>343</td>
<td>102</td>
<td>21</td>
<td>34</td>
<td>4</td>
<td>-17</td>
<td>21</td>
<td>119</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>240</td>
<td>171</td>
<td>160</td>
<td>16</td>
<td>-5</td>
<td>21</td>
<td>245</td>
<td></td>
<td></td>
</tr>
<tr>
<td>343</td>
<td>114</td>
<td>45</td>
<td>34</td>
<td>16</td>
<td>-5</td>
<td>21</td>
<td>119</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>228</td>
<td>147</td>
<td>160</td>
<td>4</td>
<td>-17</td>
<td>21</td>
<td>245</td>
<td></td>
<td></td>
</tr>
<tr>
<td>625</td>
<td>246</td>
<td>119</td>
<td>82</td>
<td>41</td>
<td>-4</td>
<td>45</td>
<td>250</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>378</td>
<td>213</td>
<td>252</td>
<td>3</td>
<td>-42</td>
<td>45</td>
<td>420</td>
<td></td>
<td></td>
</tr>
<tr>
<td>729</td>
<td>208</td>
<td>37</td>
<td>68</td>
<td>4</td>
<td>-35</td>
<td>39</td>
<td>243</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>520</td>
<td>379</td>
<td>350</td>
<td>34</td>
<td>-5</td>
<td>39</td>
<td>525</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 4.3 Existence of Partial Difference Sets

Our results show that partial difference sets are actually rather uncommon to find in nonabelian groups; however, GAP allowed us to find some examples for which partial difference sets do exist.

- $(57, 32, 16, 20)$
- $(57, 24, 11, 9)$
- $(55, 18, 9, 4)$
- $(55, 36, 21, 28)$

Although not many partial difference sets were found, we were easily able to rule out many parameter sets, which is a new result compared to past work.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$k$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$(v_2 - v_3)$</th>
<th>$\mu - v_3(v_2 + 1)$</th>
<th>Method to Rule Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>729</td>
<td>248</td>
<td>67</td>
<td>93</td>
<td>5</td>
<td>-31</td>
<td>36</td>
<td>279</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>480</td>
<td>324</td>
<td>300</td>
<td>30</td>
<td>-6</td>
<td>36</td>
<td>486</td>
<td></td>
<td></td>
</tr>
<tr>
<td>729</td>
<td>280</td>
<td>127</td>
<td>95</td>
<td>37</td>
<td>-5</td>
<td>42</td>
<td>285</td>
<td>Corollary 4.4</td>
</tr>
<tr>
<td>448</td>
<td>262</td>
<td>296</td>
<td>4</td>
<td>-38</td>
<td>42</td>
<td>486</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Bibliography


52