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Rankin-Cohen Brackets and Fusion Rules for Discrete Series Representations of SL(2,R)

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Rankin-Cohen Brackets and Fusion Rules for Discrete Series Representations of $\mathrm{SL}_2(\mathbb{R})$

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Abstract

In this paper, we discuss the holomorphic discrete series representations of $\mathrm{SL}_2(\mathbb{R})$. We give an overview of general representation theory, from the perspective of both groups and Lie algebras. We then consider tensor products of representations, specifically investigating tensor products of the holomorphic discrete series and their associated algebraic objects, called $(\mathfrak{g}, K)$-modules. We then use algebraic techniques to study the fusion rules of the discrete series. We conclude by giving explicit intertwiners, recovering the formula of number-theoretic objects, called Rankin-Cohen brackets.

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1 Introduction

Representation theory, and more specifically, the study of unitary representations of Lie groups, is a classical field of mathematics with many applications, including physics and number theory. Representations are vector spaces that represent the action of a group or an algebra by linear transformations. When these vector spaces have added structure, like Hilbert spaces, we can also use analytic and geometric techniques to understand the action of the group and therefore, the group itself.

The simplest representations are called *irreducible representations*, and they are used as building blocks in the theory. Thus, one may ask, given an arbitrary representation, how can we decompose it into irreducible representations? Can we decompose it as a direct sum of irreducible representations, or do we need a more complicated decomposition, like direct integrals? This is precisely the motivation of *branching problems*.

In general, a given *irreducible unitary representation* of a group G is not irreducible when restricted to a subgroup H. Branching problems are the question of whether this restriction can be decomposed into irreducible representations. If so, how often does a given irreducible representation occur in the decomposition? Specifically, does every irreducible occur only finitely many times? In [Har54], Harish-Chandra famously proved that if K is a maximal compact subgroup of G, then the restriction of a representation of G to K can, in fact, be decomposed in such a way. However, even in the case of the locally compact group $\mathbb{R}$ represented on the vector space of $\mathrm{L}^2(\mathbb{R})$, the theory of Fourier transform shows that we must use direct integrals to decompose representations into irreducible representations.

A modern approach to branching problems of unitary representations is described by Kobayashi [Kob02]. The decomposition of tensor product representations, or *fusion rules*, can be rephrased as branching problems in the following way: Fusion rules consider the tensor product of two representations of a group G. For two representations (ρ_1, V) and (ρ_2, W) of G, we can consider the action of the group $G \times G$ on $V \otimes W$ given by

$$\rho_1(g_1) \otimes \rho_2(g_2)$$

for $(g_1, g_2) \in G \times G$. Then, we can consider the *diagonal action of* G, or the action of $(g, g) \in G \times G$. Then, the diagonal subgroup of $G \times G$, denoted $\Delta(G)$, is isomorphic to G. Hence, understanding the branching problem of the restriction from $G \times G$ to $\Delta(G)$ is equivalent to understanding how the representation

$$\rho_1(g) \otimes \rho_2(g)$$

of G decomposes. For representations of Hilbert spaces, we are specifically interested in whether they can be *discretely decomposed*, or decomposed as a Hilbert direct sum.

In this paper, we focus on the fusion rules of the *holomorphic discrete series of* $\mathrm{SL}_2(\mathbb{R})$. As we shall prove, each discrete series representation is both irreducible and unitary, and the tensor product of two discrete series representations is discretely decomposable. In fact, this is an instance of a recent result of Kobayashi [Kob08], that for a non-compact simple Lie group of Hermitian type, the tensor product of two holomorphic discrete series representations does in fact decompose as a Hilbert direct sum. Determining whether this holds in larger generality is currently an active area of research. While analyzing the case of $\mathrm{SL}_2(\mathbb{R})$, we will work at the *infinitesimal level*, using Lie theory to determine the fusion rules of the discrete series. Known classifications for representations of $\mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, will allow us to give explicit calculations of intertwiners, giving us the Hilbert direct sum decomposition.

The case of discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ is also related to number-theoretic objects called modular forms. A *holomorphic modular form* of weight $k \in \mathbb{N}$ is a holomorphic function on the upper half plane such in that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-k} f(z)$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix in an arithmetic subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$. The space of modular forms of weight k is denoted $M^k(\Gamma)$. Further, in the study of these modular forms, *Rankin-Cohen brackets* are classical functions which bring modular forms to new modular forms, as discussed in Zagier's paper [Zag94]. Rankin-Cohen brackets are given by the formula

$$\mathrm{RC}_n(f_1, f_2) = \sum_{j=0}^{n} (-1)^j \binom{k_1+n-1}{n-j} \binom{k_2+n-1}{j} f_1^{(j)} f_2^{(n-j)}$$

for $f_1 \in M^{k_1}(\Gamma)$ and $f_2 \in M^{k_2}(\Gamma)$ where $f^{(k)}$ is the k^{th} derivative of f. Thus, we see Rankin-Cohen brackets map modular forms from $M^{k_1}(\Gamma)$ and $M^{k_1}(\Gamma)$ to $M^{k_1+k_2+2n}(\Gamma)$. Surprisingly, we recover the exact formula of Rankin-Cohen brackets as *intertwiners* in the decomposition of tensors of the discrete series $V_{m_1} \otimes V_{m_2}$ to $V_{m_1+m_2+2n}$, as we will show in the conclusion of this paper. This phenomenon has been noted in other papers, prompting the investigation of more general objects, called *Rankin-Cohen operators*, as noted in Kobayashi and Pevzner's paper [KP16].

We begin this paper with an exposition of representation theory and Lie theory in Section 2, which will allow us to understand the holomorphic discrete series of $\mathrm{SL}_2(\mathbb{R})$ and their infinitesimal counterparts, called *underlying* $(\mathfrak{g}, K)$*-modules*, in Section 3. Then, we use a classification discussed in Pevner's paper [Pev12], and Howe and Tan's book [HT92]. This classification allows us to recognize a decomposition of tensors of the discrete series, in which we perform explicit calculations in Section 4, recovering the formula of Rankin-Cohen brackets.

2 Preliminary Material

In this section, we lay the groundwork for understanding the discrete series. We first discuss group-theoretic representations and their tensor products. We then describe a type of groups, called *Lie groups*, their associated algebras, and representations of these algebras. Finally, we describe $(\mathfrak{g}, K)$-*modules* of a Lie group G, which carry compatible representations of both the Lie algebra $\mathfrak{g}$ and the maximal compact subgroup K.

2.1 Basic Representation Theory

We begin by letting G be a locally compact, Hausdorff topological group, $\mathbb{F}$ a field, and V a vector space over $\mathbb{F}$. Additionally, we define

$$\mathrm{GL}(V) := \{T : V \to V \mid T \text{ is linear and invertible}\},$$

the *general linear group* of V, or the group of all linear automorphisms on V.

Definition 2.1. A **representation** (ρ, V) of G is a group homomorphism

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto \rho(g) \end{aligned}$$

Additionally, V, the vector space associated with the representation, is often called the representation space or carrying space.

Remark 2.1.1. We may also refer to a representation (ρ, V) simply by its carrying space V, occasionally denoted V_ρ for clarity, or by the homomorphism ρ.

Definition 2.2. Given a representation (ρ, V) of G, the vector space V is also a G**-module**, by defining

$$g \cdot v := \rho(g)v$$

for $g \in G$, $v \in V$. Then, we see that

- For $g \in G$ and $u, v \in V$,

  $$g \cdot (u + v) = g \cdot u + g \cdot v$$

  since

  $$\rho(g)(u + v) = \rho(g)u + \rho(g)v$$

- For $g \in G$ and $v \in V$, $\lambda \in \mathbb{F}$,

  $$g \cdot (\lambda v) = \lambda g \cdot v$$

  since

  $$\rho(g)(\lambda v) = \lambda \rho(g)v$$

Definition 2.3. A **continuous representation** (ρ, V) of G is normed vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$ such that

$$G \times V \to V : (g, v) \longmapsto \rho(g)v$$

is continuous for all $v \in V$, $g \in G$.

Definition 2.4. A **unitary representation** is a group homomorphism

$$\rho : G \to U(V),$$

where $U(V) \subseteq \mathrm{GL}(V)$ is the space of unitary operators on the Hilbert space V. In other words, (ρ, V) is a continuous representation and

$$\langle \rho(g)v_1, \rho(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V,$$

where $\langle \cdot, \cdot \rangle_V$ is the inner product on V.

To illustrate these definitions, we will discuss a representation called the *regular representation*, but first, we must describe the *Haar measure* that is associated with any locally compact topological group:

Theorem 2.1. *Let G be a locally compact group. Then, there exists a Borel measure μ, unique up to positive scalars, such that*

- *$\mu(S) = \mu(gS)$ for all $g \in G$ and every measurable set $S \subseteq G$*
- *$\mu(U) > 0$ for all nonempty open sets $U \subseteq G$*
- *$\mu(K) < \infty$ for all compact sets $K \subseteq G$*

This measure is called the (left) ***Haar measure****.*

Notably, this measure allows us to integrate over G in the form

$$\int_G f(g) d\mu(g)$$

and maintain left invariance in that, for arbitrary $h \in G$, we have

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g)$$

A full proof of this theorem and detailed explanation of the resulting integral can be found in Kowalski's text [Kow14, Section 5.2].

Now, we define

$$\mathcal{L}^2(G) := \{f : G \to \mathbb{C} : \int_G |f(g)|^2 d\mu(g) < \infty\}.$$

Then, define an equivalence relation by identifying functions if and only if they differ only on a set of measure zero. When we quotient $\mathcal{L}^2(G)$ by this equivalence relation, we call the resulting space

$$\mathrm{L}^2(G).$$

Then, $\mathrm{L}^2(G)$ is a Hilbert space with an inner product defined by

$$\langle f_1, f_2\rangle := \int_G \overline{f_1(g)} f_2(g) d\mu(g).$$

Example 2.2. The *(left) regular representation* λ is a representation of G on the Hilbert space $\mathrm{L}^2(G)$. The regular representation has the form

$$\lambda(g) : f \longmapsto f(g^{-1}\cdot)$$

for all $g \in G$. Below, we show that the left regular representation is unitary. For $g, h \in G$, $f_1, f_2 \in \mathrm{L}^2(G)$, we have

$$\begin{aligned}\langle \rho(g)f_1, \rho(g)f_2\rangle &= \int_G \overline{\rho(g)f_1(h)}\rho(g)f_2(h)d\mu(h)\\ &= \int_G \overline{f_1(g^{-1}h)} f_2(g^{-1}h)d\mu(h)\\ &= \int_G \overline{f_1(h)} f_2(h)d\mu(h)\\ &= \langle f_1, f_2\rangle\end{aligned}$$

by the left-translation invariance of the Haar measure.

Now, we shift our attention and discuss which representations may be considered the most simplified, or *irreducible.*

Let U_1 be a closed subspace of the representation space V_ρ. The subspace U_1 is *invariant* if

$$\rho(g)v \in U_1$$

for all $g \in G$ and all $v \in U_1$. If $U_1 \neq \{0\}$ is invariant, then

$$\rho_1 := \rho\restriction_{U_1} : U_1 \to \mathrm{GL}(U_1)$$

defines a representation of G on U_1, called a *subrepresentation.*

Definition 2.5. A representation (ρ, V) is **irreducible** if the only invariant subspaces of V are $\{0\}$ and V itself, thus having no nontrivial subrepresentations.

We define the direct sum of representations

$$\rho_1 \oplus \rho_2 : G \to \mathrm{GL}(U_1 \oplus U_2)$$

by the formula

$$(\rho_1 \oplus \rho_2)(g)(u_1, u_2) := \rho_1(g)u_1 + \rho_2(g)u_2$$

Definition 2.6. A representation (ρ, V) is **completely reducible** if

$$(\rho, V) = \bigoplus_i (\rho_i, U_i)$$

where each (ρ_i, U_i) is an irreducible representation. Note that any (ρ_i, U_i) may have infinite multiplicity in the sum while satisfying this definition.

Definition 2.7. A **Hilbert space direct sum** of Hilbert spaces H_i with associated inner products $\langle \cdot, \cdot \rangle_i$ is the completion

$$H = \widehat{\bigoplus}_i H_i$$

under the inner product given by

$$\langle u, v \rangle = \sum_i \langle u_i, v_i \rangle_i$$

for $u_i, v_i \in H_i$ and $u, v \in H$ where $u = \oplus_i u_i$ and $v = \oplus_i v_i$.

Theorem 2.3 (Peter-Weyl)**.** *Let G be a compact topological group with left Haar measure μ. Then the regular representation of G on the space $\mathrm{L}^2(G, \mu)$ decomposes as a Hilbert space direct sum*

$$\mathrm{L}^2(G, \mu) = \widehat{\bigoplus}_{\rho \in \widehat{G}} \dim(\rho) V_\rho$$

where $\widehat{G}$ is the set of finite-dimensional irreducible unitary representations of G.

A proof of Peter-Weyl theorem can be found Kowalski's text [Kow14, Section 5.2].

Example 2.4. Recall the (left) regular representation,

$$\lambda(g) : f \longmapsto f(g^{-1} \cdot)$$

for all $g \in G$ on the Hilbert space $\mathrm{L}^2(G)$. We see by Peter-Weyl theorem that if G is compact, then the regular representation is completely reducible.

However, this is not the case for general locally compact G. In fact, this is not the case when G is the locally compact group $\mathbb{R}$. In order to decompose the regular representation of $\mathbb{R}$ into irreducible representations, the theory of Fourier transform shows that we would need to consider *direct integrals* of representations, in addition to direct sums. This is fully illustrated in Kowalski's text [Kow14, Section 7.3].

2.2 Tensor Products

Now we define (algebraic) tensor products, tensor products of Hilbert spaces, and finally, tensor products of representations.

Consider two vector spaces V and W, both over the field $\mathbb{F}$. We first consider the Cartesian product

$$V \times W = \{(v, w) : v \in V, w \in W\}.$$

Then, to create a vector space, we consider the *free vector space*,

$$F(V \times W).$$

To construct $F(V \times W)$, we consider $V \times W$ as the basis that generates our vector space, thus including scalar multiples and finite sums of the ordered pairs. Rather than write (v, w), we write $v \otimes w$, so $F(V \times W)$ is a vector space with elements of the form

$$\sum_{i=0}^{n} \lambda(v_i \otimes w_i),$$

where $v_i \in V$ and $w_i \in W$ are arbitrary vectors and $\lambda \in \mathbb{F}$ is an arbitrary scalar.

To construct the (algebraic) tensor product $V \otimes W$, we define a subspace of $F(V \times W)$. Let $v, v' \in V$ and $w, w' \in W$ and $\lambda \in \mathbb{F}$, and let F_0 be the subspace generated by

$$(v + v') \otimes w - v \otimes w - v' \otimes w,$$

$$v \otimes (w + w') - v \otimes w - v \otimes w',$$

$$(\lambda v) \otimes w - \lambda(v \otimes w),$$

and

$$v \otimes (\lambda w) - \lambda(v \otimes w).$$

Then, we define the *(algebraic) tensor product* of V and W to be

$$V \otimes W := F(V \times W)/F_0,$$

which gives us that

$$(v + v') \otimes w = v \otimes w + v' \otimes w,$$

$$v \otimes (w + w') = v \otimes w + v \otimes w',$$

$$(\lambda v) \otimes w = \lambda(v \otimes w),$$

and

$$v \otimes (\lambda w) = \lambda(v \otimes w).$$

In this space, each $v \otimes w$ is called a *pure tensor*. Thus, an arbitrary vector in $V \otimes W$ is a linear combination of pure tensors, as shown by

$$\sum_{i=0}^{n} \lambda_i(v_i \otimes w_i)$$

where $v_i \in V$ and $w_i \in W$ are arbitrary vectors and $\lambda \in \mathbb{F}$ is an arbitrary scalar.

Further, let $\{\beta_i\}_{i\in\mathbb{N}}$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ be bases for V and W, respectively. Then, $\{\beta_i \otimes \gamma_j\}_{i,j\in\mathbb{N}}$ serves as a basis for $V \otimes W$. Because $\{\beta_i\}_{i\in\mathbb{N}}$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ are bases, we see $\{\beta_i \otimes \gamma_j\}_{i,j\in\mathbb{N}}$ are linearly independent. So, we show that $\{\beta_i \otimes \gamma_j\}_{i,j\in\mathbb{N}}$ span $V \otimes W$. Consider a pure tensor $v \otimes w$ where $v \in V$ and $w \in W$ are arbitrary. Then,

$$\begin{aligned} v \otimes w &= \left(\sum_{i=0}^{m} \beta_i\right) \otimes w \\ &= \sum_{i=0}^{m} (\beta_i \otimes w) \\ &= \sum_{i=0}^{m} \left(\beta_i \otimes \left(\sum_{j=0}^{n} \gamma_j\right)\right) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} (\beta_i \otimes \gamma_j)\,. \end{aligned}$$

Then, since arbitrary vectors in $V \otimes W$ are linear combinations of pure tensors, we see that any tensor can be written as a linear combination of $\{\beta_i \otimes \gamma_j\}_{i,j\in\mathbb{N}}$.

Now we define the *tensor product of Hilbert spaces* V and W with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Then, we can define an inner product on pure tensors of V and W by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle_V \langle w_1, w_2 \rangle_W$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Then, since inner products are sesquilinear, we can extend this inner product for all tensors of V and W. Finally, when V and W are both Hilbert spaces, we define the tensor product of Hilbert spaces V and W to be $V \mathbin{\widehat{\otimes}} W$, *or the completion of* $V \otimes W$ *under this inner product.* In this paper, however, we will simply write $V \otimes W$, where it is understood that it is meant as the completion under the inner product when V and W are Hilbert spaces.

Definition 2.8. Given representations (ρ, V) and (φ, W) of G, the **tensor product of representations** is the vector space $V \otimes W$, on which G acts by

$$(\rho \otimes \varphi)(g)(v \otimes w) := \rho(g)v \otimes \varphi(g)w$$

for all $v \otimes w \in V \otimes W$. Note that because $\rho(g)$ and $\varphi(g)$ are linear maps, we only need to define $(\rho \otimes \varphi)(g)$ on pure tensors to understand the function on all of $V \otimes W$.

Remark 2.8.1. Let (ρ, V) and (φ, W) be unitary representations of G, where V and W are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Then, for $g \in G$,

$v_1, v_2 \in V$, and $w_1, w_2 \in W$, we see that $\rho \otimes \varphi$ is also unitary by computing

$$\begin{aligned}
\langle \rho \otimes \varphi(g)(v_1 \otimes w_1), \rho \otimes \varphi(g)(v_2 \otimes w_2)\rangle &= \langle \rho(g)v_1 \otimes \varphi(g)w_1, \rho(g)v_2 \otimes \varphi(g)w_2\rangle \\
&= \langle \rho(g)v_1, \rho(g)v_2\rangle_V \langle \varphi(g)w_1, \varphi(g)w_2\rangle_W \\
&= \langle v_1, v_2\rangle_V \langle w_1, w_2\rangle_W \\
&= \langle v_1 \otimes v_2, w_1 \otimes w_2\rangle.
\end{aligned}$$

Note that because $\rho(g) \in \mathrm{GL}(V)$ and $\varphi(g) \in \mathrm{GL}(W)$ are linear for all $g \in G$ and because pure tensors are dense in $V \widehat{\otimes} W$, we only must consider the pure tensors of $V \otimes W$. Thus, we see that tensor products preserve unitarity.

2.3 Lie Groups and Lie Algebras

A *Lie group* is a group G that is also a smooth manifold on which the group operations are differentiable. Every Lie group has an associated *Lie algebra*. We can gain information about representations of Lie groups by studying the representations at the level of Lie algebras, as we will do when studying fusion rules in following sections.

Definition 2.9. A **reductive Lie group** is a closed connected subgroup of $\mathrm{GL}_n\mathbb{C}$ that is stable under conjugate transposition

$$M \longmapsto \overline{M}^T$$

for a matrix $M \in \mathrm{GL}_n\mathbb{C}$.

Definition 2.10. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ along with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies

- Bilinearity:
$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$
for all scalars $a, b \in \mathbb{F}$ and all $X, Y, Z \in \mathfrak{g}$;
- Anti-symmetry: $[X, X] = 0$ for all $X \in \mathfrak{g}$;
- The Jacobi Identity:
$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$
for all $X, Y, Z \in \mathfrak{g}$.

Example 2.5. For a vector space V over $\mathbb{F}$, we can define

$$\mathfrak{gl}(V) = \mathrm{End}(V),$$

the space of all linear endomorphisms on V. We can equip $\mathfrak{gl}(V)$ with the bracket given by the commutator,

$$[X, Y] = XY - YX$$

giving us that $\mathfrak{gl}(V)$ is a Lie algebra, also known as the *general linear Lie algebra*.

As mentioned, every Lie group has an associated Lie algebra. More specifically, because Lie groups are also smooth manifolds, we can consider tangent spaces, and the tangent space at the identity of a Lie group naturally carries the structure of a Lie algebra, defining the Lie algebra associated with the Lie group.

Reductive Lie groups, like $\mathrm{SL}_2(\mathbb{R})$, have Lie algebras, satisfying Definition 2.10, defined to be

$$\mathfrak{g} := \{X \in M_n(\mathbb{C}) : e^{tX} \in G \text{ for all } t \in \mathbb{R}\}$$

where $M_n(\mathbb{C})$ is the ring of n by n matrices with complex entries and the matrix exponential is defined as

$$e^{tX} = \exp(tX) = \sum_{n\geq 0} \frac{1}{n!}(tX)^n$$

We note that this series converges since the series is absolutely convergent in $M_n(\mathbb{C})$, a Banach space.

2.4 Lie-Theoretic Representation Theory

Just as we defined representations of groups, we can define representations of Lie algebras. Studying Lie group representations at the level of their associated Lie algebraic representations allows us to use additional Lie algebraic tools to understand the original group representations.

Definition 2.11. A **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras such that, for all $X, Y \in \mathfrak{g}$,

$$\phi([X, Y]) = [\phi(X), \phi(Y)].$$

Definition 2.12. A **representation** (ρ, V) of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

So, ρ must be a linear map that satisfies

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

for all $X, Y \in \mathfrak{g}$.

Definition 2.13. Similar to G-modules, we can consider V as a **$\mathfrak{g}$-module**. We define

$$X \cdot v := \rho(X)v$$

for $X \in \mathfrak{g}$, $v \in V$, which satisfies

- $X \cdot (u+v) = \rho(X)(u+v) = \rho(X)u + \rho(X)v = X \cdot u + X \cdot v$ for $X \in \mathfrak{g}$ and $u, v \in V$
- $X \cdot (\lambda v) = \rho(X)(\lambda v) = \lambda(\rho(X)v) = \lambda(g \cdot v)$ for $X \in \mathfrak{g}$, $v \in V$, and $\lambda \in \mathbb{F}$

In the following examples, we discuss two Lie algebra representations that are associated with representations of the Lie group. Both examples will be used in future calculations.

Example 2.6. Let G be a reductive Lie group, $\mathfrak{g}$ its associated Lie algebra, and $\mathfrak{gl}(\mathfrak{g})$ the general linear Lie algebra over $\mathfrak{g}$. Then, we can define the *adjoint action*. The adjoint action is the Lie algebra representation

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined for $X, Y \in \mathfrak{g}$ by

$$\mathrm{ad}(X)Y = [X, Y].$$

Then, ad is a Lie algebra representation, as shown by the Jacobi identity:

$$\begin{aligned} \mathrm{ad}([X,Y])(Z) &= [[X,Y],Z] \\ &= [X,[Y,Z]] + [Y,[Z,X]] \\ &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= (\mathrm{ad}(X)\,\mathrm{ad}(Y))(Z) - (\mathrm{ad}(Y)\,\mathrm{ad}(X))(Z) \\ &= [\mathrm{ad}(X), \mathrm{ad}(Y)](Z) \end{aligned}$$

Additionally, we can define the *adjoint representation* by

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

since $X \in \mathfrak{g} \subset M_n(\mathbb{C})$ and $G \subseteq \mathrm{GL}_n(\mathbb{C})$. Then, the adjoint representation is a group representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

as shown by

$$\begin{aligned} \mathrm{Ad}(gh)(X) &= (gh)X(gh)^{-1} \\ &= g(hXh^{-1})g^{-1} \\ &= g(\mathrm{Ad}(h)(X))g^{-1} \\ &= \mathrm{Ad}(g)(\mathrm{Ad}(h)(X)) \end{aligned}$$

Example 2.7. Let (ρ, V) be a representation of the reductive Lie group G. Then, we can define the *derived representation* of the associated Lie algebra $\mathfrak{g}$. First, we define the carrying space of the derived representation to be the space of smooth vectors,

$$V^\infty := \{v \in V : g \mapsto \rho(g)v \text{ is smooth}\}.$$

Then, we define the Lie algebra homomorphism. Recall that $\exp(tX) \in G$ for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. The derived representation $d\rho$ is defined on V^∞ by

$$d\rho(X)v = \frac{d}{dt}\rho(\exp(tX))v \restriction_{t=0} := \lim_{t\to 0} \frac{1}{t}(\rho(\exp(tX))v - v). \tag{1}$$

Then, $d\rho$ preserves the Lie brackets from $\mathfrak{g}$ to $\mathfrak{gl}(V^\infty)$ by

$$d\rho([X,Y]) = d\rho(X)d\rho(Y) - d\rho(Y)d\rho(X),$$

as proven in Lang's book [Lan75, Chapter VI]. Thus, we have defined V^∞, the representation space of the Lie algebra homomorphism

$$d\rho : \mathfrak{g} \to \mathfrak{gl}(V^\infty),$$

and so we see $(d\rho, V^\infty)$ is a Lie algebra representation.

Note that the adjoint action is the derived representation of the adjoint representation, since, by equation (1),

$$d\,\mathrm{Ad}(X)Y = \frac{d}{dt}\,\mathrm{Ad}(\exp(tX))Y \restriction_{t=0} = \lim_{t\to 0}\frac{1}{t}(\mathrm{Ad}(\exp(tX))Y - Y) = [X,Y] = \mathrm{ad}(X)Y.$$

Having introduced Lie algebra representations, we now define their tensor products, as we will use them later when studying tensor products of the discrete series.

Definition 2.14. Given representations (ρ_1, V) and (ρ_2, W) of $\mathfrak{g}$, the **tensor product of Lie algebra representations** is the vector space $V \otimes W$ along with the Lie algebra homomorphism

$$(\rho_1 \otimes \rho_2) : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$$

given by

$$(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_2(X)$$

where $X \in \mathfrak{g}$ and Id is the identity map. Again, since $\rho_1(X)$, $\rho_2(X)$, and Id are linear maps, we can extend this definition on pure tensors to arbitrary tensors in $V \otimes W$.

To see this definition, we consider the derived representation.

Example 2.8. Let (ρ, V) and (φ, W) be representations of a reductive Lie group G. Then, we know we can consider $(\rho\otimes\varphi, V\otimes W)$, the representation of G. Now, consider the derived representation

$$(d(\rho\otimes\varphi), (V\otimes W)^\infty).$$

Then, by definition of $\rho\otimes\varphi$, we see

$$\begin{aligned}
d(\rho\otimes\varphi)(X)(v\otimes w) &= \frac{d}{dt}\left[(\rho\otimes\varphi)(\exp(tX))(v\otimes w)\right]_{t=0} \\
&= \frac{d}{dt}\left[(\rho(\exp(tX))(v)\otimes\varphi(\exp(tX))(w))\right]_{t=0} \\
&= \Bigg[\left(\frac{d}{dt}[\rho(\exp(tX))(v)]\otimes\varphi(\exp(tX))(w)\right) \\
&\qquad + \left(\rho(\exp(tX))(v)\otimes\frac{d}{dt}[\varphi(\exp(tX))(w)]\right)\Bigg]_{t=0} \\
&= (d\rho(X)(v)\otimes\varphi(I)(w)) + (\rho(I)(v)\otimes d\varphi(X)(w)) \\
&= d\rho(X)(v)\otimes w + v\otimes d\varphi(X)(w).
\end{aligned}$$

Thus, we have

$$d(\rho\otimes\varphi) = d\rho\otimes\mathrm{Id} + \mathrm{Id}\otimes d\varphi = d\rho\otimes d\varphi,$$

verifying our definition of tensor products of Lie algebra representations.

Definition 2.15. An **intertwiner** between representations ρ_1 and ρ_2 of a Lie algebra $\mathfrak{g}$, over field $\mathbb{F}$, acting on vector spaces V and W, is an $\mathbb{F}$-linear map

$$\phi : V \to W$$

such that

$$\phi(\rho_1(X)v) = \rho_2(X)(\phi(v)) \in W$$

for any $X\in\mathfrak{g}$ and $v\in V$.

This gives us the following commutative diagram for all $X\in\mathfrak{g}$:

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "\rho_1(X)"] & W \arrow[d, "\rho_2(X)"] \\
V \arrow[r, "\phi"] & W
\end{tikzcd}$$

2.5 Admissible Representations

In the following two sections, we let G be a reductive Lie group, K a maximal compact subgroup of G, and $\mathfrak{g}$ its Lie algebra. Here, we define terms and types of representations that will be used in the following section to define $(\mathfrak{g}, K)$-modules.

Let (ρ, V) be a unitary representation of G. It follows from Peter-Weyl Theorem that (ρ, V) restricted to K decomposes into the direct sum of irreducible unitary representations of K. That is, (ρ, V) restricted to K is

$$(\rho, V) \restriction_K \simeq \bigoplus_{\sigma \in \widehat{K}} m_\sigma V_\sigma,$$

where $\widehat{K}$ is the set of irreducible unitary representations of K and m_σ is the possibly infinite multiplicity of the representation V_σ.

Definition 2.16. An irreducible unitary K-representation is a K**-type** of (ρ, V) if it is one of the (σ, V_σ) such that

$$(\rho, V) \restriction_K \simeq \bigoplus_{\sigma \in \widehat{K}} m_\sigma V_\sigma$$

However, $(\rho \restriction_K, V)$ may have many subspaces that are isomorphic as K-representations to a given irreducible representation of K. In fact, m_σ may be infinite.

Definition 2.17. For a given K-type σ of (ρ, V), the σ**-isotypic component** of a representation is the direct sum of all subspaces of V that are K-isomorphic to V_σ.

Definition 2.18. A continuous representation of G is **admissible** if the multiplicity of each σ-isotypic component is finite.

Remark 2.18.1. All irreducible unitary representations of G are known to be admissible, as given in [Har54].

Definition 2.19. A vector v in a representation V_ρ of K is a K**-finite vector** if

$$\{\rho(k)v : k \in K\}$$

is a finite dimensional subspace of V_ρ.

2.6 $(\mathfrak{g}, K)$-modules

In this section, we discuss $(\mathfrak{g}, K)$-modules, as the $(\mathfrak{g}, K)$-modules of $\mathrm{SL}_2(\mathbb{R})$ will allow us to use additional classifications and techniques to decompose tensor products of holomorphic discrete series.

Definition 2.20. A $(\mathfrak{g}, K)$**-module** is a vector space V such that:

- V is a $\mathfrak{g}$-module and a K-module, in that V carries a group representation ρ of K and a Lie algebra representation ρ' of $\mathfrak{g}$.
- V decomposes into an algebraic direct sum of finite-dimensional K-invariant subspaces.
- For $k \in K$, $X \in \mathfrak{g}$, and $v \in V$,

$$\rho(k)\rho'(X)\rho(k^{-1})v = \rho(\mathrm{Ad}(k)(X))v$$

Remark 2.20.1. When given a representation of K, the *derived representation* from Example 2.8 satisfies the compatibility condition of $(\mathfrak{g}, K)$-modules as we now check. Let $\mathfrak{k}$ be the associated Lie algebra of K. Then, for a representation ρ of K, $v \in V^\infty$, $k \in K$, and $X \in \mathfrak{k}$, we can consider

$$\begin{aligned}\rho(k)d\rho(X)\rho(k^{-1})v &= \rho(k)\frac{d}{dt}\rho(\exp(tX))\rho(k^{-1})v \restriction_{t=0}\\ &= \frac{d}{dt}\rho(k\exp(tX)k^{-1})v \restriction_{t=0}\\ &= \rho(\mathrm{Ad}(k)(X))v\end{aligned}$$

The purpose of defining $(\mathfrak{g}, K)$-modules in this paper is to use their classifications to understand $\mathrm{SL}_2(\mathbb{R})$ representations, the discrete series. The following theorem suggests how we will associate $(\mathfrak{g}, K)$-modules to irreducible, unitary representations, including discrete series representations.

Theorem 2.9. *Given any irreducible, unitary representation (ρ, V) of a reductive Lie group G, the space of all K-finite vectors, denoted $V_{(K)}$, is a $(\mathfrak{g}, K)$-module.*

Proof. Because we are considering K-finite vectors, $V_{(K)}$ is exactly an algebraic direct sum of finite-dimensional K-invariant subspaces. As mentioned in Remark 2.20.1, the compatibility requirements are satisfied by the definition of $d\rho$. Finally, $V_{(K)}$ must be both a K-module and a $\mathfrak{g}$-module. If V is a G-module, $V_{(K)}$ is certainly a K-module. Now, the rest of the proof will show that $V_{(K)}$ is a $\mathfrak{g}$-module.

We will show that $V_{(K)}$ is a $\mathfrak{g}$-module first by showing that $V_{(K)} \subseteq V^\infty$ and then that $V_{(K)}$ is invariant under the action of $\mathfrak{g}$ by the derived representation.

In this proof, we follow Bump's proof [Bum97, Proposition 2.4.5], in which he shows that $V_{(K)}$ is dense in V using matrix coefficients. We have previously defined

$$V^\infty = \{v \in V : g \mapsto \rho(g)v \ \text{ is smooth}\},$$

so we see that V^∞ is G-invariant because if

$$g \mapsto \rho(g)v$$

is smooth for all $g \in G$, then

$$g \mapsto \rho(g)(\rho(g')v) = \rho(gg')v$$

must also be smooth, since $gg' \in G$. Thus, V^∞ is also K-invariant, so we see that

$$V_0 := V_{(K)} \cap V^\infty$$

is K-invariant. Then, let σ be an admissible irreducible representation of K, and let $V_0(\sigma)$ be the σ-isotypic component of V_0. We know that

$$V_{(K)} = \bigoplus_{\sigma \in \widehat{K}} V(\sigma),$$

where $\widehat{K}$ is the set of irreducible unitary representations of K. We know for each $\sigma \in \widehat{K}$ that $V_0(\sigma) \subseteq V(\sigma)$, so we show $V_0(\sigma) \supseteq V(\sigma)$. Suppose this is not the case. Then, there must exist some nonzero

$$u \in V(\sigma) \cap V_0(\sigma)^\perp$$

by properties of Hilbert spaces. From the full statement of Peter-Weyl Theorem, it can be shown that the carrying spaces of irreducible representations on a Hilbert space are orthogonal by again considering matrix coefficients. Hence, we must have that u is also orthogonal to $V_0(\tau)$ for any $\tau \neq \sigma$. Thus, u is orthogonal to all of V_0. However, V_0 is dense in V, so we have reached a contradiction.

Finally, we prove that $V_{(K)}$ is invariant under the action of $\mathfrak{g}$. Let $v \in V_{(K)}$. Additionally, let $W \subset V$ be finite dimensional and stable under the action of $\mathfrak{k}$. Then, for $X \in \mathfrak{k}$, $Y \in \mathfrak{g}$, and $w \in W$, we have

$$d\rho(X)(d\rho(Y)w) = d\rho([X,Y])w + d\rho(Y)(d\rho(X)w).$$

Both $d\rho([X,Y])w$ and $d\rho(Y)(d\rho(X)w)$ are in

$$W' = \operatorname{span}\{d\rho(Y)(w) : w \in W, Y \in \mathfrak{g}\}.$$

Then, for arbitrary $Y \in \mathfrak{g}$, $d\rho(Y)v \in W'$. Yet, ρ is a unitary irreducible representation of G and so is admissible by Remark 2.18.1. Thus, W' is a finite dimensional subspace that is stable under the action of $\mathfrak{k}$. Thus, $d\rho(Y)v \in V_{(K)}$, as desired. □

Definition 2.21. We call $V_{(K)}$ the **underlying** $(\mathfrak{g}, K)$**-module** of (ρ, V).

Now, we see that all unitary representations have underlying $(\mathfrak{g}, K)$-modules. But in fact, many $(\mathfrak{g}, K)$-modules can be related back to unitary representations. Thus, when we analyze the discrete series at the Lie algebraic level, it will be directly relevant in the understanding of the representations at the level of $\mathrm{SL}_2(\mathbb{R})$.

Definition 2.22. A $(\mathfrak{g}, K)$-module is **admissible** if each irreducible representation of K appears only finitely many times in the vector space of the module.

Proposition 2.10. *Every admissible $(\mathfrak{g}, K)$-module can be realized as the space of K-finite vectors of an admissible representation of G.*

This is a result proven by Casselman and Wallach, as discussed in Casselman's paper [Cas89]. Additionally, they show that a $(\mathfrak{g}, K)$-module may be realized as the K-finite vectors of at most one unique *unitary* group representation.

3 Representations of $\mathrm{SL}_2(\mathbb{R})$

Having introduced preliminary representation theory and Lie theory, we now consider these concepts in the context of $\mathrm{SL}_2(\mathbb{R})$, the group of two by two real-valued matrices of determinant 1:

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

As previously noted, $\mathrm{SL}_2(\mathbb{R})$ is a reductive Lie group, as a subgroup of $\mathrm{GL}_2(\mathbb{C})$. Its associated Lie algebra is the linear space of two by two real-valued matrices with trace 0:

$$\mathfrak{sl}_2(\mathbb{R}) = \{X \in M_2(\mathbb{R}) : \mathrm{tr}(X) = 0\}.$$

Additionally, $\mathrm{SL}_2(\mathbb{R})$ has a maximal compact subgroup, the special orthogonal group:

$$\mathrm{SO}_2 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Like $\mathbb{R}$, we know $\mathrm{SL}_2(\mathbb{R})$ is locally compact, but not compact, and the regular representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{L}^2(\mathrm{SL}_2(\mathbb{R}))$ is not completely reducible. Instead, we will consider representations of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{L}^2(\mathbb{H})$ on the upper half plane,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

equipped with the measure

$$\frac{dxdy}{y^2}.$$

We see that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations, defined by

$$\frac{az+b}{cz+d}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \mathrm{SL}_2(\mathbb{R}).$$

We show that this action of $\mathrm{SL}_2(\mathbb{R})$ is transitive on $\mathbb{H}$ by showing that the orbit of i under $\mathrm{SL}_2(\mathbb{R})$ is all of $\mathbb{H}$. Let $z = a + ib \in \mathbb{H}$. Then, we can write

$$z = a + ib = \frac{bi + a}{0 \cdot i + 1} = \frac{bi + a}{0 \cdot i + 1} \cdot \frac{\frac{1}{\sqrt{b}}}{\frac{1}{\sqrt{b}}} = \frac{\frac{b}{\sqrt{b}} i + \frac{a}{\sqrt{b}}}{0 \cdot i + \frac{1}{\sqrt{b}}}$$

Thus, we see for

$$g = \begin{pmatrix} \frac{b}{\sqrt{b}} & \frac{a}{\sqrt{b}} \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

we have $g \cdot i = z$ for arbitrary $z = a + ib \in H$, so the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

Additionally, note that

$$\mathrm{stab}_G(i) = \mathrm{SO}_2.$$

Since the action is transitive, we have $\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$, as $\mathrm{SL}_2(\mathbb{R})$-modules, or spaces carrying an action of $\mathrm{SL}_2(\mathbb{R})$.

3.1 Holomorphic Discrete Series Representations

Here, we describe the holomorphic discrete series representations of $\mathrm{SL}_2(\mathbb{R})$. In essence, the discrete series representations are subrepresentations of the regular representation of $\mathrm{SL}_2(\mathbb{R})$. However, rather than acting upon $\mathrm{L}^2(\mathrm{SL}_2(\mathbb{R}))$, we consider $\mathrm{SL}_2(\mathbb{R})$ acting on a subspace of $\mathrm{L}^2(\mathbb{H})$, as described below.

For a fixed $m \in \mathbb{N}$ and $m \geq 2$, we can define a measure on $\mathbb{H}$ by

$$d\mu_m = y^m \frac{dxdy}{y^2}$$

where $x + iy = z \in \mathbb{H}$. This measure then allows us to perform integration, and thus, define the space $\mathrm{L}^2(\mathbb{H}, \mu_m)$, the L^2 functions on the measure space $(\mathbb{H}, \mu_m)$. We then define the space $\mathcal{H}_m = \mathrm{Hol}(\mathbb{H}) \cap \mathrm{L}^2(\mathbb{H}, \mu_m)$, where $\mathrm{Hol}(\mathbb{H})$ is the set of holomorphic functions on $\mathbb{H}$.

Now, we define representations of $G = \mathrm{SL}_2(\mathbb{R})$ acting on $\mathcal{H}_m$. For $g \in \mathrm{SL}_2(\mathbb{R})$, let

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

For $f \in \mathcal{H}_m$ and $g \in \mathrm{SL}_2(\mathbb{R})$, we define $\pi_m(g)$ to be

$$\begin{aligned} \pi_m(g) \colon \mathcal{H}_m &\longrightarrow \mathcal{H}_m \\ f &\longmapsto f(g^{-1}z)(cz+d)^{-m} = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-m} \end{aligned}$$

Then, $(\pi_m, \mathcal{H}_m)$ are the **holomorphic discrete series representations** of $\mathrm{SL}_2(\mathbb{R})$. As mentioned in the introduction, these representations are important in various fields of mathematics, including the study of modular forms in number theory. In the representation-theoretic context, their importance stems from the fact that all $(\pi_m, \mathcal{H}_m)$ are unitary and irreducible representations.

Theorem 3.1. *Each $(\pi_m, \mathcal{H}_m)$ representation is unitary.*

Proof. First, we show that π_m is a continuous representation. It suffices to show that π_m is continuous on a dense subset of $\mathrm{L}^2(\mathbb{H})$, since $\mathcal{H}_m \subseteq \mathrm{L}^2(\mathbb{H})$. Hence, we consider an arbitrary function $f \in \mathrm{L}^2(\mathbb{H})$ that is continuous with compact support. Then, $|f|^2 \leq M$ for some $M \in \mathbb{R}$. Let $g \in \mathrm{SL}_2(\mathbb{R})$. Then, let $\{g_i\}_{i\in\mathbb{N}}$ be a sequence of matrices in $\mathrm{SL}_2(\mathbb{R})$ converging to I_2, the identity matrix in $\mathrm{SL}_2(\mathbb{R})$. To show continuity of π_m, we only need that $\pi_m(g_i)f$ converges to f under the L^2 norm. By definition of π_m, we know that each $\pi_m(g_i)f$ is a square-integrable function in $\mathcal{H}_m$. Since $\{g_i\}_{i\in\mathbb{N}}$ converges to I_2, we also know that $\pi_m(g_i)f$ converges to f pointwise.

Now, fix $\varepsilon > 0$ and let

$$g_i^{-1} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Since f is continuous and compactly supported, we know that $\pi(g)f$ is also continuous and compactly supported. Hence, for $||g_i - \mathrm{I}_2|| < \varepsilon$, we have

$$\mathrm{supp}(\pi(g)f) \subseteq X := \mathrm{supp}(f) + B(0, \varepsilon),$$

which is a bounded set. On this set, $(c_i z + d_i)^{-m}$ converges uniformly to 1 as i approaches infinity. Thus, for $||g_i - \mathrm{I}_2|| < \varepsilon$, we see that

$$|\pi(g_i)f| \leq M(1 + \varepsilon) \cdot 1_X$$

where 1_X is the indicator function for the set X.

Then, we can apply Lebesgue's Dominated Convergence Theorem to see that

$$\int_{\mathbb{H}} |f|^2 d\mu = \lim_{n\to\infty} \int_{\mathbb{H}} |\pi_m(g_i)f|^2 d\mu,$$

giving us L^2 convergence and continuity of π_m.

Now, we prove that π_m preserves the inner product on $\mathcal{H}_m$. Let $g \in \mathrm{SL}_2(\mathbb{R})$, where

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, for $z = x + iy \in \mathbb{H}$, define $g^{-1}z = w = x_1 + iy_1 \in \mathbb{H}$. We note that

$$\begin{aligned}
y_1 &= \operatorname{Im}(g^{-1}z) \\
&= \operatorname{Im}\left(\frac{az+b}{cz+d}\right) \\
&= \operatorname{Im}\left(\frac{(az+b)\overline{cz+d}}{|cz+d|^2}\right) \\
&= \operatorname{Im}\left(\frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}\right) \\
&= \operatorname{Im}\left(\frac{adz + bc\overline{z}}{|cz+d|^2}\right) \\
&= \frac{\det(g)}{|cz+d|^2}\operatorname{Im}(z) \\
&= \frac{y}{|cz+d|^2}.
\end{aligned}$$

Then, using a change of variables from z to w, we complete the calculation for $f(z), f'(z) \in \mathcal{H}_m$,

$$\begin{aligned}
\langle \pi_m(g)(f), \pi_m(g)(f') \rangle &= \int_{\mathbb{H}} \overline{\pi_m(g)(f(z))}\pi_m(g)(f'(z))d\mu \\
&= \int_0^\infty \int_{-\infty}^\infty \overline{f(g^{-1}z)}(cz+d)^{-m} f(g^{-1}z)(cz+d)^{-m} y^m \frac{dxdy}{y^2} \\
&= \int_0^\infty \int_{-\infty}^\infty \overline{f(g^{-1}z)} f'(g^{-1}z)\left(\frac{y}{(cz+d)^2}\right)^m \frac{dxdy}{y^2} \\
&= \int_0^\infty \int_{-\infty}^\infty \overline{f(w)} f'(w) y_1^m \frac{dx_1 dy_1}{y_1^2} \\
&= \langle f, f' \rangle.
\end{aligned}$$

Thus, we see that $(\pi_m, \mathcal{H}_m)$ is unitary for each m. □

Theorem 3.2. *Each $(\pi_m, \mathcal{H}_m)$ representation is irreducible.*

First, we will consider an isomorphism between $\mathcal{H}_m$ and square-integrable, holomorphic functions on the unit disk in the complex plane. This will allow us to more easily analyze the vectors of $\mathcal{H}_m$ in an effort to show irreducibility.

To move from $\mathbb{H}$, the upper half plane to the unit disk in the complex plane, denoted $\mathbb{D}$, we consider the *Cayley transform*, an analytic isomorphism from $\mathbb{H}$ to $\mathbb{D}$, defined by

$$z \mapsto w = \frac{z-i}{z+i}$$

for $z \in \mathbb{H}$ and $w \in \mathbb{D}$, and we note that the inverse is

$$z = -i\frac{w+1}{w-1}$$

We use this to define our isometry in the following.

Lemma 3.2.1. *Let $(\mathbb{D}, v_m)$ be the unit disk in the complex plane with the measure*

$$dv_m := \frac{4}{4^m}(1-|w|^2)^m \frac{dudv}{(1-|w|^2)^2}$$

for $w = u + iv \in \mathbb{D}$. Define

$$\mathcal{D}_m = \mathrm{Hol}(\mathbb{D}) \cap \mathrm{L}^2(\mathbb{D}, v_m)$$

Then, there is an isometry

$$\begin{aligned} T_m : \mathcal{H}_m &\longrightarrow \mathcal{D}_m \\ f(z) &\longmapsto f\left(-i\frac{w+1}{w-1}\right)\left(\frac{-2i}{w-1}\right)^m \end{aligned}$$

Proof. We see that T_m is a linear map, sending functions on $\mathbb{H}$ to functions on $\mathbb{D}$, so we prove that T_m preserves inner product on $\mathcal{H}_m$ to that of $\mathcal{D}_m$. This is shown by the following calculation for $w = u + iv \in \mathbb{D}$, $z = x + iy \in \mathbb{H}$, and $f, g \in \mathcal{H}_m$:

$$\begin{aligned} &\langle T_m f, T_m g\rangle_{\mathbb{D}_m} \\ &= \int_0^1 \int_0^1 \overline{f\left(-i\frac{w+1}{w-1}\right)\left(\frac{-2i}{w-1}\right)^m} g\left(-i\frac{w+1}{w-1}\right)\left(\frac{-2i}{w-1}\right)^m \frac{4}{4^m}(1-|w|^2)^m \frac{dudv}{(1-|w|^2)^2} \\ &= \int_0^1 \int_0^1 \overline{f\left(-i\frac{w+1}{w-1}\right)} g\left(-i\frac{w+1}{w-1}\right)\left|\frac{1}{w-1}\right|^m (1-|w|^2)^m \frac{4dudv}{(1-|w|^2)^2} \\ &= \int_0^\infty \int_{-\infty}^\infty \overline{f\left(z\right)} g\left(z\right) y^m \frac{dxdy}{y^2} \\ &= \langle f, g\rangle_{\mathcal{H}_m} \end{aligned}$$

$\square$

Thus, we can consider our discrete series representations on the disk. The proof of irreducibility is completed in [Kna86, Proposition 2.7], in which Knapp uses an integral calculation to show that any invariant subspace must include the entire representation.

Proposition 3.3. *Every holomorphic discrete representation $(\pi_m, \mathcal{H}_m)$ has an underlying $(\mathfrak{g}, K)$-module*

$$V_m := (\mathcal{H}_m)_{(\mathrm{SO}_2)}$$

where $(\mathcal{H}_m)_{(\mathrm{SO}_2)}$ is defined to be the SO_2-finite vectors of $(\pi_m, \mathcal{H}_m)$.

This is immediate from Theorems 2.9, 3.1, and 3.2.

3.2 Classification of $(\mathfrak{g}, K)$-modules of $\mathrm{SL}_2(\mathbb{R})$

In this section, we describe a classification of the $(\mathfrak{g}, K)$-modules of $\mathrm{SL}_2(\mathbb{R})$, as described in detail by Howe and Tan [HT92]. This classification is critical in our decomposition of the tensors of the holomorphic discrete series.

For $\mathrm{SL}_2(\mathbb{R})$, we know we have the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and the maximal compact subgroup $K = \mathrm{SO}_2$. Note that

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2(\mathbb{R})$ and give the following commutation relations:

$$[h, e^-] = -2e^-, [h, e^+] = 2e^+, [e^+, e^-] = h.$$

In the classification, we will consider these elements by their action in the derived representation of tensor products of the holomorphic discrete series.

To apply the classification, we must further specify properties of Lie algebra representations. Let (ρ, V) be a Lie algebra representation of $\mathfrak{sl}_2(\mathbb{R})$. In this section, we will denote $\rho(x)v$ by $x \cdot v$ for $x \in \mathfrak{sl}_2(\mathbb{R})$ and $v \in V$. Now, we define the *generalized h-eigenspace* of V for $\lambda \in \mathbb{C}$ as

$$V_\lambda = \{v \in V : (h - \lambda Id)^n \cdot v = 0\}.$$

These generalized eigenvalues of h,

$$\{\lambda \in \mathbb{C} : V_\lambda \neq \{0\}\}$$

are called *weights*. Then, the representation V is **h-admissible** if

$$V = \sum_{\lambda \in \mathbb{C}} V_\lambda$$

and

$$\dim V_\lambda < \infty$$

for all $\lambda \in \mathbb{C}$. If ρ is h-admissible and also each V_λ is a genuine eigenspace, in that

$$V_\lambda = \{v \in V : hv = \lambda v\}$$

for every $\lambda \in \mathbb{C}$, then ρ is called **h-semisimple**. Finally, to define a *quasi-simple* representation (ρ, V), we must define the *Casimir element* to be

$$c = h^2 + 2(e^+e^- + e^-e^+) = h^2 + 2h + 4e^-e^+.$$

The Casimir element is, in fact, the generator of the center of the universal enveloping algebra, $\mathrm{U}(\mathfrak{sl}_2(\mathbb{R}))$, but we will not use this fact here. Then, a **quasi-simple** representation (ρ, V) is a representation such that the Casimir element c acts by a multiple of identity on V. Then, for all h-admissible, h-semisimple, quasi-simple $(\mathfrak{g}, K)$-modules of $\mathrm{SL}_2(\mathbb{R})$, we can classify each module into one of three categories, described in the following definitions.

Definition 3.1. A **lowest weight module**, V_λ, for $\lambda \in \mathbb{C}$, is an $\mathfrak{sl}_2(\mathbb{R})$-module that has a basis of h-eigenvectors $\{v_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned} h \cdot v_j &= (\lambda + 2j)v_j \text{ for } j \in \mathbb{N} \\ e^+ \cdot v_j &= v_{j+1} \text{ for } j \in \mathbb{N} \\ e^- \cdot v_j &= -j(\lambda + j - 1)v_{j-1} \text{ for } j \in \mathbb{N} \setminus \{0\} \\ \text{and } e^- \cdot v_0 &= 0. \end{aligned}$$

Here, the basis element v_0 is called the **lowest weight vector** and λ is the *lowest weight* of the module V_λ.

Definition 3.2. A **highest weight module**, $\overline{V}_\lambda$, for $\lambda \in \mathbb{C}$, is an $\mathfrak{sl}_2(\mathbb{R})$-module that has a basis of h-eigenvectors $\{\overline{v}_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned} h \cdot \overline{v}_j &= (\lambda - 2j)v_j \text{ for } j \in \mathbb{N} \\ e^- \cdot \overline{v}_j &= \overline{v}_{j+1} \text{ for } j \in \mathbb{N} \\ e^+ \cdot \overline{v}_j &= j(\lambda - j - 1)\overline{v}_{j-1} \text{ for } j \in \mathbb{N} \setminus \{0\} \\ \text{and } e^+ \cdot \overline{v}_0 &= 0. \end{aligned}$$

In this case, the basis element v_0 is called the **highest weight vector** and λ is the *highest weight* of the module V_λ.

Definition 3.3. A **module**, $W(\mu, \lambda)$, for $\mu, \lambda \in \mathbb{C}$, is an $\mathfrak{sl}_2(\mathbb{R})$-module that has a basis of h-eigenvectors $\{\overline{v}_j, j \in \mathbb{Z}\}$ such that

$$\begin{aligned} h \cdot v_j &= (m + 2j)v_j \text{ for } j \in \mathbb{N} \\ e^+ \cdot v_j &= v_{j+1} \text{ for } j \in \mathbb{N} \\ \text{and } e^- \cdot v_j &= \frac{1}{4}(\mu - (\lambda + 2j - 1)^2 + 1)v_{j-1} \text{ for } j \in \mathbb{N}. \end{aligned}$$

Theorem 3.4. *An h-semisimple, quasi-simple $\mathfrak{sl}_2(\mathbb{R})$-module V is either a lowest weight module, highest weight module, or module, as defined above.*

This theorem is discussed in Pevzner's paper [Pev12], and a full proof of this theorem can be found in Howe and Tan's book [HT92].

As we have noted in Proposition 3.3, the holomorphic discrete series of representations have underlying $(\mathfrak{g}, K)$-modules, and these modules are h-admissible, h-semisimple, and quasi-simple. Thus, we can apply Theorem 3.4 to the underlying $(\mathfrak{g}, K)$-modules of the holomorphic discrete series. In fact, all underlying $(\mathfrak{g}, K)$-modules of the holomorphic discrete series, $(\pi_m, \mathcal{H}_m)$, are lowest weight modules. In order to show this, we will give some preliminary calculations using the basis elements e^-, e^+, h. Recall that underlying $(\mathfrak{g}, K)$-modules, V_m, of the discrete series have the associated action of the *derived representation*, as defined in Example 2.8. Rather than writing $d\pi_m(x)v$ for $x \in \mathfrak{sl}_2(\mathbb{R})$ and $v \in V_m$, we write $x \cdot v$. Additionally, vectors in V_m are functions $v = f(z)$ on $z \in \mathbb{H}$, so in the following calculations, we write $x \cdot f(z)$ for $x \cdot v$.

Now, for a fixed $m \geq 2$, we compute the action of e^-, e^+, and h on an arbitrary element of V_m, as follows:

We first compute the action of e^-,

$$\begin{aligned}
e^- \cdot f(z) &= \frac{d}{dt}\pi_m\left(e^{t\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\sum_{n\geq 0}\frac{1}{n!}\begin{pmatrix}0&0\\t&0\end{pmatrix}^n\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\t&0\end{pmatrix}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\begin{pmatrix}1&0\\t&1\end{pmatrix} f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\left[(-tz+1)^{-m} f\left(\frac{z}{-tz+1}\right)\right]_{t=0} \\
&= (-m)(-tz+1)^{-m-1}(-z) f\left(\frac{z}{-tz+1}\right) \\
&\qquad + (-tz+1)^{-m} f'\left(\tfrac{z}{-tz+1}\right) \cdot \left(\frac{0 - z(-z)}{(-tz+1)^2}\right) \restriction_{t=0} \\
&= mzf(z) + f'(z)(z^2).
\end{aligned}$$

We then compute the action of e^+,

$$
\begin{aligned}
e^+ \cdot f(z) &= \frac{d}{dt}\pi_m\left(e^{t\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\sum_{n\geq 0}\frac{1}{n!}\begin{pmatrix}0&t\\0&0\end{pmatrix}^n\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t\\0&0\end{pmatrix}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\begin{pmatrix}1&t\\0&1\end{pmatrix} f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\left[f(z-t)\right]_{t=0} \\
&= \left[f'(z-t)\cdot(-1)\right]_{t=0} \\
&= -f'(z).
\end{aligned}
$$

And finally, we compute the action of h,

$$
\begin{aligned}
h \cdot f(z) &= \frac{d}{dt}\pi_m\left(e^{t\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\sum_{n\geq 0}\frac{1}{n!}\begin{pmatrix}t&0\\0&-t\end{pmatrix}^n\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\left(\sum_{n\geq 0}\begin{pmatrix}\frac{t^n}{n!}&0\\0&\frac{(-t)^n}{n!}\end{pmatrix}\right) f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\pi_m\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix} f(z) \restriction_{t=0} \\
&= \frac{d}{dt}\left[(e^t)^{-m} f\left(\frac{e^{-t}z}{e^t}\right)\right]_{t=0} \\
&= \frac{d}{dt}\left[e^{-mt} f\left(e^{-2t}z\right)\right]_{t=0} \\
&= \left[-me^{-mt} f\left(e^{-2t}z\right) + e^{-mt} f'\left(e^{-2t}z\right)\left(-2e^{-2t}z\right)\right]_{t=0} \\
&= -mf(z) - 2zf'(z).
\end{aligned}
$$

These calculations allow us to prove the following theorem, which will give us an understanding of the holomorphic discrete series by using the classification of Theorem 3.4. After showing that the underlying $(\mathfrak{g}, K)$-modules of the discrete series are lowest weight modules, we will use this fact to decompose tensors of the discrete series, as we can identify lowest weight modules by the action of e^-.

Theorem 3.5. *The underlying $(\mathfrak{g}, K)$-modules of $(\pi_m, \mathcal{H}_m)$, denoted V_m, can be identified as lowest weight modules, using the basis*

$$v_j := \frac{(m+j-1)!}{(m-1)!} z^{-m-j}.$$

Proof. Fix $m \geq 2$. Recall from the preliminary calculations that

$$e^- \cdot f(z) = mzf(z) + f'(z)(z^2)$$

$$e^+ \cdot f(z) = -f'(z)$$

and

$$h \cdot f(z) = -mf(z) - 2zf'(z).$$

Now, we calculate the action of e^-, e^+, and h on our basis to prove that V_m is a lowest weight module.

For $j \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned}
e^- \cdot v_j &= mzv_j + (v_j)'(z^2) \\
&= mz \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right) + \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right)' (z^2) \\
&= m \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j+1} \right) + (-m-j) \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j+1} \right) \\
&= -j \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j+1} \right) \\
&= -j(m+j-1) \left(\frac{(m+j-2)!}{(m-1)!} z^{-m-(j-1)} \right) \\
&= -j(m+j-1)v_{j-1}
\end{aligned}$$

For $j = 0$, we have

$$\begin{aligned}
e^- \cdot v_0 &= mzv_0 + (v_0')(z^2) \\
&= mz \left(\frac{(m-1)!}{(m-1)!} z^{-m} \right) + \left(\frac{(m-1)!}{(m-1)!} z^{-m} \right)' (z^2) \\
&= m \left(z^{-m+1} \right) - m \left(z^{-m+1} \right) \\
&= 0
\end{aligned}$$

For the final two calculations, we let $j \in \mathbb{N}$. Then we have

$$\begin{aligned}
e^+ \cdot v_j &= -f'(z) \\
&= -\left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right)' \\
&= -(-m-j) \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j-1} \right) \\
&= (m+j) \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-(j+1)} \right) \\
&= \left(\frac{(m+j)!}{(m-1)!} z^{-m-(j+1)} \right) \\
&= v_{j+1},
\end{aligned}$$

and

$$\begin{aligned}
h \cdot v_j &= -mf(z) - 2zf'(z) \\
&= -m \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right) - 2z \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right)' \\
&= -m \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right) - 2(-m-j) \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right) \\
&= (-m - 2(-m-j)) \left(\frac{(m+j-1)!}{(m-1)!} z^{-m-j} \right) \\
&= (m+2j) v_j,
\end{aligned}$$

completing the proof. □

4 Fusion Rules and Rankin-Cohen Brackets

Having introduced the holomorphic discrete series, we now consider the fusion rules of its tensor products, $(\pi_{m_1} \otimes \pi_{m_2}, \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2})$. We will use the material discussed in all previous sections, including the classification of $(\mathfrak{g}, K)$-modules of $\mathrm{SL}_2(\mathbb{R})$. By Theorem 3.1 and 3.2, the representations $(\pi_{m_1}, \mathcal{H}_{m_1})$ and $(\pi_{m_2}, \mathcal{H}_{m_2})$ are unitary and irreducible, and by Remark 2.8.1 the tensor products are also unitary. Thus, for $(\pi_{m_1} \otimes \pi_{m_2}, \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2})$, we know the underlying $(\mathfrak{g}, K)$-module $V_{m_1} \otimes V_{m_2}$ exists. Additionally, we have shown that the $(\mathfrak{g}, K)$-modules V_{m_1} and V_{m_2} are lowest weight modules, by considering the derived representation. We now show that $V_{m_1} \otimes V_{m_2}$ decomposes into a sum of other lowest weight modules and recover the formula of *Rankin-Cohen brackets*, thus giving us a representation-theoretic perspective of a number-theoretic object.

Theorem 4.1. *For $m_1, m_2 \in \mathbb{N}$ and $m_1, m_2 \geq 2$, consider*

$$V_{m_1} \otimes V_{m_2}$$

which is the underlying $(\mathfrak{g}, K)$-module of $(\pi_{m_1} \otimes \pi_{m_2}, \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2})$. Then, the $(\mathfrak{g}, K)$-module $V_{m_1} \otimes V_{m_2}$ decomposes into lowest weight modules

$$\bigoplus_{n \in \mathbb{N}} V_{m_1+m_2+2n}$$

Proof. To show this decomposition, we want to find a nonzero tensor $T_n \in V_{m_1} \otimes V_{m_2}$ that is annihilated by the action of e^- for each $n \in \mathbb{N}$. This will show us that we are in a lowest weight module, by the classification given in Theorem 3.4. Then, once we have the lowest weight (tensor) vector, we know how the rest of the basis of $\mathfrak{sl}_2(\mathbb{R})$ must act on the other basis vectors, so it is possible to discern which other tensors in $V_{m_1} \otimes V_{m_2}$ belong to the subspace of $V_{m_1+m_2+2n}$.

Now, as discussed, we fix $n \in \mathbb{N}$, and we want to find $T_n \in V_{m_1} \otimes V_{m_2}$ that is annihilated by the action of e^-. So we let $\{v_i\}_{i \in \mathbb{N}}$ and $\{\widetilde{v}_j\}_{j \in \mathbb{N}}$ be bases of V_{m_1} and V_{m_2}, respectively. Then, we see that an arbitrary tensor in $V_{m_1} \otimes V_{m_2}$ is a linear combination of tensors of the form

$$\sum_{i=0}^{n} a_i (v_i \otimes \widetilde{v}_{n-i})$$

for some $n \in \mathbb{N}$. Now, fix $n \in \mathbb{N}$, and for

$$T_n = \sum_{i=0}^{n} a_i (v_i \otimes \widetilde{v}_{n-i})$$

we find $\{a_i\}_{i \in \mathbb{N}}$ such that $e^- \cdot T_n = 0$. So we calculate the action of e^- as follows

$$\begin{aligned} e^- \cdot T_n &= e^- \cdot \left(\sum_{i=0}^{n} a_i (v_i \otimes \widetilde{v}_{n-i}) \right) \\ &= \sum_{i=0}^{n} e^- \cdot (a_i (v_i \otimes \widetilde{v}_{n-i})). \end{aligned}$$

By our definition of tensors of representations, we next apply e^- to each pure tensor

$$\begin{aligned}
e^- \cdot T_n &= \sum_{i=0}^{n} a_i(e^- \cdot v_i \otimes \widetilde{v}_{n-i} + v_i \otimes e^- \cdot \widetilde{v}_{n-i}) \\
&= a_0(0 \otimes \widetilde{v}_n + v_0 \otimes (-n)(m_2 + n - 1)\widetilde{v}_{n-1}) \\
&\quad + \sum_{i=1}^{n-1} a_i((-i)(m_1 + i - 1)v_{i-1} \otimes \widetilde{v}_{n-i}) \\
&\quad + \sum_{i=1}^{n-1} a_i(v_i \otimes (-n + i)(m_2 + n - i - 1)\widetilde{v}_{n-i-1}) \\
&\quad + a_n((-n)(m_1 + n - 1)v_n \otimes \widetilde{v}_0 + v_n \otimes 0).
\end{aligned}$$

Then, we pull the constants to the outside of each pure tensor and again include the first and last elements in the summations

$$\begin{aligned}
e^- \cdot T_n &= a_0(-n)(m_2 + n - 1)(v_0 \otimes \widetilde{v}_{n-1}) \\
&\quad + \sum_{i=1}^{n-1} a_i(-i)(m_1 + i - 1)(v_{i-1} \otimes \widetilde{v}_{n-i}) \\
&\quad + \sum_{i=1}^{n-1} a_i(-n + i)(m_2 + n - i - 1)(v_i \otimes \widetilde{v}_{n-i-1}) \\
&\quad + a_n(-n)(m_1 + n - 1)(v_n \otimes \widetilde{v}_0) \\
&= \sum_{i=1}^{n} a_i(-i)(m_1 + i - 1)(v_{i-1} \otimes \widetilde{v}_{n-i}) \\
&\quad + \sum_{i=0}^{n-1} a_i(-n + i)(m_2 + n - i - 1)(v_i \otimes \widetilde{v}_{n-i-1}).
\end{aligned}$$

Finally, we reorder the indices to combine the two summations, giving us:

$$\begin{aligned}
e^- \cdot T_n = \sum_{i=0}^{n-1} & a_{i+1}(-i - 1)(m_1 + i)(v_i \otimes \widetilde{v}_{n-i-1}) \\
& + a_i(-n + i)(m_2 + n - i - 1)(v_i \otimes \widetilde{v}_{n-i-1}).
\end{aligned}$$

Thus, by setting this calculation of $e^- \cdot T_n = 0$, we see

$$a_{i+1}(i + 1)(m_1 + i) + a_i(n - i)(m_2 + n - i - 1) = 0,$$

which gives us the recurrence relation

$$a_{i+1} = -\left(\frac{(n - i)(m_2 + n - i - 1)}{(i + 1)(m_1 + i)}\right) a_i.$$

Then we see

$$\begin{aligned}
a_i &= (-1)^i \frac{\left(\frac{(m_1+n-1)!}{(m_1-1+i)!}\right)}{(n-i)!} \frac{\left(\frac{(m_2+n-1)!}{(m_2+n-1+i)!}\right)}{i!} \\
&= (-1)^i \left(\frac{(m_1+n-1)!}{(n-i)!(m_1+n-1-(n-i))!}\right)\left(\frac{(m_2+n-1)!}{i!(m_2+n-1+i)!}\right) \\
&= (-1)^i \binom{m_1+n-1}{n-i}\binom{m_2+n-1}{i}.
\end{aligned}$$

Thus, we have confirmed that the T_n is a lowest weight vector for some lowest weight module V_λ by finding its lowest weight vector. To find precisely the value of λ, we recall that we can consider the action of h on the lowest weight vector. Since T_n is the lowest weight vector, we will find $h \cdot T_n = \lambda T_n$. We compute λ below:

$$\begin{aligned}
h \cdot T_n &= h \cdot \left(\sum_{i=0}^{n} a_i (v_i \otimes \widetilde{v}_{n-i})\right) \\
&= \sum_{i=0}^{n} h \cdot (a_i(v_i \otimes \widetilde{v}_{n-i})).
\end{aligned}$$

Then, again by our definition of tensors of representations, we apply h to each pure tensor. Recall that

$$h \cdot v_i = (m_1 + 2i) v_i$$

and

$$h \cdot \widetilde{v}_{n-i} = (m_2 + 2(n-i)) \widetilde{v}_{n-i}$$

for basis elements $v_i \in V_{m_1}$ and $\widetilde{v}_{n-i} \in V_{m_2}$. So we see

$$\begin{aligned}
h \cdot T_n &= \sum_{i=0}^{n} a_i (h \cdot v_i \otimes \widetilde{v}_{n-i} + v_i \otimes h \cdot \widetilde{v}_{n-i}) \\
&= \sum_{i=0}^{n} a_i ((m_1+2i) v_i \otimes \widetilde{v}_{n-i} + v_i \otimes (m_1 + 2(n-i)) \widetilde{v}_{n-i}) \\
&= \sum_{i=0}^{n} a_i ((m_1+2i) v_i \otimes \widetilde{v}_{n-i} + v_i \otimes (m_2 + 2(n-i)) \widetilde{v}_{n-i}) \\
&= \sum_{i=0}^{n} ((m_1+2i) + (m_2 + 2(n-i))) a_i (v_i \otimes \widetilde{v}_{n-i}) \\
&= (m_1 + m_2 + 2n) \sum_{i=0}^{n} a_i (v_i \otimes \widetilde{v}_{n-i}) \\
&= (m_1 + m_2 + 2n) T_n.
\end{aligned}$$

Thus, for every fixed $n \in \mathbb{N}$, T_n can be recognized as the lowest weight vector for the lowest weight module $V_{m_1+m_2+2n}$, and so we have given an explicit intertwiner from $V_{m_1} \otimes V_{m_2}$ to $V_{m_1+m_2+2n}$ for all $n \in \mathbb{N}$. □

Finally, we note the analogous formula given by Rankin-Cohen brackets. We have shown that for every $n \in \mathbb{N}$, there is a tensor $T_n \in V_{m_1} \otimes V_{m_2}$ that is isomorphic to the lowest weight vector in $V_{m_1+m_2+2n}$. In finding this T_n, we have shown an intertwiner from $V_{m_1} \otimes V_{m_2}$ to $V_{m_1+m_2+2n}$ for a fixed $n \in \mathbb{N}$. Similarly, we can consider Rankin-Cohen brackets, which bring $M^{k_1}(\Gamma)$ and $M^{k_2}(\Gamma)$ to $M^{k_1+k_2+2n}(\Gamma)$ for a fixed $n \in \mathbb{N}$. Then, we can compare the explicit formulas, where the intertwiner from $V_{m_1} \otimes V_{m_2}$ to $V_{m_1+m_2+2n}$ is given by

$$T_n = \sum_{i=0}^{n} (-1)^i \binom{m_1+n-1}{n-i} \binom{m_2+n-1}{i} (v_i \otimes \widetilde{v}_{n-i})$$

for bases $\{v_i\}_{i \in \mathbb{N}}$ and $\{\widetilde{v}_j\}_{j \in \mathbb{N}}$ for V_{m_1} and V_{m_2}, respectively, and Rankin-Cohen brackets are given by

$$\mathrm{RC}_n(f_1, f_2) = \sum_{j=0}^{n} (-1)^j \binom{k_1+n-1}{n-j} \binom{k_2+n-1}{j} f_1^{(j)} f_2^{(n-j)}$$

for $f_1 \in M^{k_1}(\Gamma)$ and $f_2 \in M^{k_2}(\Gamma)$. We notice that these formulas are exactly the same. Hence, we have interpreted Rankin-Cohen brackets, which are number-theoretic objects, in the context of representation theory, as intertwiners. This phenomenon of Rankin-Cohen brackets as explicit intertwiners in the fusion rules of the holomorphic discrete series has led to further investigation of explicit intertwiners, as detailed in Kobayashi and Pevzner's paper [KP16].

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