4-2019

# A General Weil-Brezin Map and Some Applications

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A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science in Mathematics from The College of William and Mary

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Accepted for Honors

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April 23, 2019

# A General Weil-Brezin Map and Some Applications

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May 10, 2019

**Abstract**

We recall a theory generalizing the Heisenberg group on $\mathbb{R}$ to an analogous structure using a locally compact abelian group $G$. Then, using our new, general Heisenberg groups, we generalize the classical Weil-Brezin map, first introduced in [9] and [1], from an operator on $L^2(\mathbb{R})$ and develop a theory of that generalized Weil-Brezin map on $L^2(G)$ for some locally compact abelian group $G$. We then apply our generalized Weil-Brezin map to recover the Poisson Summation Formula as well as the Plancherel Theorem.

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# 1 Introduction and The Real Case

In the classical case, Fourier analysis is concerned with representing a real-valued function as a sum of trigonometric functions. Intuitively, this means that functions could be studied by breaking them down into combinations of simple periodic wave functions. Historically, this was an immensely powerful technique in the analysis of functions, as trigonometric functions were relatively well understood, and they have numerous special properties that make them amenable to study in detail. Furthermore, the relationship between integration theory and the analysis of real valued functions plays an incredibly important role, as it allows for the Fourier Transform, an integral transform that changes a function of time into a function of frequencies.

This serves as excellent motivation for this thesis, which will be concerned with Harmonic Analysis on groups. This theory generalizes Fourier analysis by easing the restriction to real functions. Now, our functions may be functions on abstract groups that are far more general than just the real numbers. In this case, we will also need to generalize the idea of trigonometric functions. In this paper, these will become the character functions of a group. Unfortunately, we will therefore be unable to use many of the pleasant properties of the real numbers. However, by restricting our attention to certain classes of groups, in particular, those upon which we can integrate, we will be able to use methods that are largely analogous to the methods used in the real case. In keeping with the spirit of Fourier analysis, we will break down the Fourier transform itself on these general groups into several constituent parts that we can study more easily, and from those constituent parts, we will recover several known facts about it.

In a recent paper by Munthe-Kaas, [7], a brief overview of the general Heisenberg group on locally compact abelian groups was given, and several known results were obtained as by-products of the generalization. However, many of those results relied on indirect methods of proof. In this paper, we will use direct analytic and group-theoretic methods to prove those results in greater detail without recourse to the indirect methods and homological algebra used in [7]. We will begin by providing a brief overview of the Heisenberg Group and the use of the classical Weil-Brezin map on $\mathbb{R}$. This will serve as motivation for the generalizations to come later. After that, we will move on to a discussion of the theory of topological groups, and, in particular, locally compact abelian groups. These will be the primary focus of this paper. Once we have developed that theory enough to proceed, we will construct an analogue to the real Heisenberg group on an arbitrary locally compact abelian group. Finally, we will use those groups to define a general Weil-Brezin map, from whose properties we will immediately derive the Poisson Summation Formula and the Plancherel Theorem.

**Definition 1.1.** The **Heisenberg Group of** $\mathbb{R}$ is the noncommutative group of three-by-three matrices of the form :

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. Note that this group is isomorphic to the semidirect product $\mathbb{R} \times \mathbb{R} \rtimes \mathbb{R}$, where for any $(a, b, c), (a', b', c') \in \mathbb{R} \times \mathbb{R} \rtimes \mathbb{R}$, we define the operation $(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + ab')$. We will denote this group $\mathscr{H}$

Now, we will consider the discrete subgroup $\Gamma$ of $\mathscr{H}$, where $\Gamma = \{(a, b, c) : a, b, c \in \mathbb{Z}\}$. Then, since $\Gamma$ will act on $\mathscr{H}$ from the left, we will consider the left quotient $\Gamma \backslash \mathscr{H}$, and, in particular, function spaces on $\Gamma \backslash \mathscr{H}$. It must be noted that the product of the Lebesgue measures on the copies of $\mathbb{R}$ in the Heisenberg group will induce a measure on $\Gamma \backslash \mathscr{H}$, and so we can define the space $L^2(\Gamma \backslash \mathscr{H})$. Finally, in the real case, we may decompose the space Hilbert space $L^2(\Gamma \backslash \mathscr{H})$ in the following way:

$$L^2(\Gamma \backslash \mathscr{H}) = \bigoplus_{n \in \mathbb{N}} \mathcal{V}_n$$

Where each $\mathcal{V}_n = \{f \in L^2(\Gamma \backslash \mathscr{H}) : f(\Gamma(a, b, c + t)) = e^{2\pi int} f(\Gamma(a, b, c))\}$. With all of these relevant spaces defined, we may define the classical Weil-Brezin Map, which we will denote $W_{\mathbb{R}}^{\mathbb{Z}}$ to distinguish it from the generalized Weil-Brezin map which we will encounter later.

**Definition 1.2.** The **Classical Weil-Brezin map** is a unitary operator $W_{\mathbb{R}}^{\mathbb{Z}} : \mathcal{S}(\mathbb{R}) \to \mathcal{V}_\infty \subset L^2(\Gamma \backslash \mathscr{H})$ (where $\mathcal{S}$ is the space of Schwartz functions) defined by

$$\mathcal{W}_{\mathbb{R}}^{\mathbb{Z}}(f)(a, b, c) = \sum_{n \in \mathbb{Z}} f(a + n) e^{2\pi inb} e^{2\pi ic}$$

It is a fact that the classical Weil-Brezin map is a unitary transformation. In this paper we shall omit the proof, as the result will follow from our proof of the unitarity of the general Weil-Brezin map. Nevertheless, the unitarity $W_{\mathbb{R}}^{\mathbb{Z}}$ and the density of the Schwartz functions in $L^2(\mathbb{R})$ [**?**] allows us to uniquely extend $W_{\mathbb{R}}^{\mathbb{Z}}$ to a map on $L^2(\mathbb{R})$ by a limit argument. From here, we may also define the inverse to the Classical Weil-Brezin map.

**Definition 1.3.** The inverse to $W_{\mathbb{R}}^{\mathbb{Z}}$ is the map $W_{\mathbb{R}}^{\mathbb{Z}^{-1}} : \mathcal{V}_1 \to L^2(\mathbb{R})$ defined by

$$W_{\mathbb{R}}^{\mathbb{Z}^{-1}}(\phi)(x) = \int_0^1 \phi(\Gamma(x, y, 0)) dy$$

Again, we shall omit the proof that this is indeed the inverse, as it will follow from more general results proven later in the paper. The Classical Weil-Brezin map is and its inverse can now be used to obtain several famous results in Fourier Analysis on $\mathbb{R}$, including the Plancherel Theorem and the Poisson Summation Formula, by factoring the Fourier operator into a product of Weil-Brezin operators. We will use an analogous method to prove a general Plancherel Theorem and Poisson Formula later in this paper.

The above definition generalizes relatively readily to higher dimensions, where $a$ and $b$ can be taken to be vectors in $\mathbb{R}^n$ and multiplication can be replaced by the standard inner product. That is, in this case the Heisenberg group is isomorphic to the group of matrices of the form

$$\begin{bmatrix} 1 & \mathbf{x} & z \\ 0 & 1 & \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix}$$

Where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$ and $\mathbf{y}$ is a row vector also in $\mathbb{R}^n$. Furthermore, in this case we can consider the semidirect product $\mathbb{R} \times \mathbb{R} \rtimes \mathbb{T}$, and define

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', cc' e^{2\pi iba'})$$

In the rest of this paper, we will generalize these results still further to abstract groups with certain topological structures, using the $n$-dimensional real case as a starting point. This will allow us to us the same methods as in the real case to obtain more general results regarding abstract harmonic analysis.

# 2 Topological Groups

As in the real case, we will be considering operators on Heisenberg groups. However, in place of considering a semidirect product with entries from $\mathbb{R}$ we will need to develop a theory of general groups that we can then use to construct our Heisenberg manifold. As we will be using largely analytic methods in this paper, we shall begin by considering groups with a topological structure, which will, in turn, allow for a measure-theoretic discussion of abstract groups. In many of the present examples, we will use $\mathbb{R}$, as it is probably the most intuitive topological group.

**Definition 2.1.** Let a group $G$ be a topological space with a topology $\mathscr{T}$. Then, $G$ is called a **topological group** if the following hold with respect to $\mathscr{T}$:

1. The mapping $(x, y) \to xy$ from $G \times G$ to $G$ is continuous
2. The mapping $x \to x^{-1}$ is continuous

We will need a few more propositions about topological groups before we can proceed.

**Proposition 2.2.** *Let $G$ be a topological group. Then, the the following are true:*

1. *The topology of $G$ is translation invariant, that is, if $U$ is an open set in $G$, then for all $g \in G$, $gU = \{gu : u \in U\}$ is open.*
2. *Every open subgroup of $G$ is closed*
3. *If $A, B \subset G$ are compact, then $AB = \{ab : a \in A, b \in B\}$ is compact.*

**Proposition 2.3.** *Let $G$ be a topological group, with $H \leq G$. Then, the following are true:*

1. *If $H$ is closed, then $G/H$ is Hausdorff in the quotient topology*
2. *If $G$ is locally compact, then so is $G/H$*
3. *If $H$ is a normal subgroup, then $G/H$ is a topological group.*

Definition 2.1 and Propositions 2.2 and 2.3 should come as no surprise, as these are all properties we would expect to be given based on an understanding of the properties of $\mathbb{R}$. In Definition 2.1, we require that our group operations be continuous, as otherwise, the group structure would be largely incompatible with the topological structure, meaning that one could consider $G$ as a topological space or a group, but would gain no extra information by examining the interactions between the imposed structures. Propositions 2.2 and 2.3 are immediately and intuitively apparent when one considers the Euclidean topology on $\mathbb{R}$. However, these are by no means obvious given nothing but the definition of a topological group. For full and complete proofs of the propositions, see [3].

It is not enough to only consider the topological structure of these groups. In order to generalize the ideas in Section 1, we will need to consider the analytic structure. Since we have been given information about the open sets of $G$, it is logical to consider the Borel $\sigma$-algebra on $G$ and, from that, to construct a measure space, and thereby perform our analysis. Fortunately, there is a well-developed theory of integration on groups, beginning with the theory of the Haar measure.

**Theorem 2.4.** *Consider an locally compact group $G$. Then, there exists a nonzero, (left or right) translation-invariant, Radon measure on $G$ that is unique up to scaling. This measure will be called the **Haar Measure**.*

Note here that this theorem restricts our use of the Haar measure to locally compact groups. As a result of this, it appears unlikely that there is any simple generalization of the results in this paper to infinite-dimensional spaces. Because of the importance of the Haar measure and its properties, we will spend some time examining those properties, as well as several examples of the Haar measure on elementary groups.

**Remark.** Since the Haar measure $\lambda$ is a well defined Radon measure, we may use the general definition of the Lebesgue integral to define an integral on any Borel function $f$ on a locally compact group $G$. This will be called the **Haar Integral**.

**Remark.** A full construction of the Haar measure is beyond the scope of this paper; for a complete proof of the existence and properties of the Haar measure, see [3]. The construction of the Haar measure on a group $G$ is accomplished by first constructing a positive linear functional on $C_C^+(G)$, the space of positive, continuous, compact-supported, functions on $G$, extending it to $C_C(G)$, the space of all continuous compact-supported functions on $G$, and then applying the Riesz Representation Theorem to gain a measure with the desired properties. It should be noted that the Riesz Theorem guarantees uniqueness of the measure, and by the properties of the construction of the relevant positive linear functional, we gain the following proposition.

**Proposition 2.5.** *The Haar measure of a group $G$ is finite if and only if $G$ is compact.*

This uniqueness of the Haar measure, combined with the uniqueness of the Lebesgue measure and the additive group properties of vectors in $\mathbb{R}^n$ gives us the following corollary:

**Corollary 2.6.** *The (properly scaled) Haar measure on $\mathbb{R}^n$ is identical to the Lebesgue measure.*

This corollary is particularly striking, because it then immediately follows from properties of the Lebesgue measure that when the Haar measure is scaled properly, the direct product of Haar measures on $\mathbb{R}$ is equivalent as a measure to the Haar measure on a direct product of $\mathbb{R}$. We will use similar notions later when we consider measures on abstract Heisenberg groups.

**Example.** It should be noted that the above equivalence between the Haar and Lebesgue measures holds when $\mathbb{R}$ is viewed as an additive group. If we consider the multiplicative group $\mathbb{R}\backslash\{0\}$, a possible Haar measure is $dx/|x|$ where $dx$ denotes the Lebesgue measure.

**Example.** If we consider $\mathbb{R}$ in the discrete topology, a possible Haar measure is simply the counting measure on $\mathbb{R}$. This is worth noting, as it shows that there are finite dimensional topological groups that are not $\sigma$-finite. However, those are largely sources of counterexamples, and have few applications outside of abstract measure theory, which is beyond the scope of this paper. In this paper, we will consider $\sigma$-finite measure spaces, and that will provide results in a level of generality that will include the vast majority of interesting or applicable cases.

**Example.** We have already shown that the Lebesgue measure on $\mathbb{R}$ is identical to a scaled Haar measure on $\mathbb{R}$. Another group of note in this paper is the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with operation multiplication. In order to define a Haar measure on this, consider the function $f : [0, 2\pi) \to \mathbb{T}$ defined by $f = (\cos(t), \sin(t))$. Then, one possible Haar measure $\lambda$ on this particular group is

$$\lambda(E) = \mu(f^{-1}(E))$$

where $\mu$ is the Lebesgue measure, and where this may be defined for any Borel set $E$. Note that is is quite consistent with our intuition, as we are using the auxiliary function $f$ to relate $\lambda$ and $\mu$, when we know well that $\mu$ is a Haar measure on $\mathbb{R}$. It is also worth noting that in this particular measure, $\lambda(\mathbb{T}) = 2\pi$. In many applications, including in this paper it may well be preferred to define a measure by $\frac{1}{2\pi}\mu(f^{-1}(E))$ so that the measure of the torus is one. This will dramatically simplify calculation. This is an ideal example of the way the scaling property of the Haar measure may be used to one's advantage.

**Example.** So far, our examples have been abelian groups, and indeed, these will be the focus of our paper. However, the Haar measure can be constructed on nonabelian groups. If we consider the general linear group $GL(n, \mathbb{R})$ and let $dT$ represent the Lebesgue measure on the space of all $n \times n$ matrices, then

$$|\det T|^{-n} dT$$

is a Haar measure on $GL(n, \mathbb{R})$.

As per the Theorem 2.4, and as shown in the previous example, the Haar measure can certainly be constructed on nonabelian groups. However, the following property of the Haar on abelian groups will cause us to restrict our considerations in this paper.

**Definition 2.7.** Let $\lambda$ be a left-translation invariant measure on a group $G$. Then, for all $x \in G$, define $\lambda_x(E) = \lambda(Ex)$ for a measurable set $E$. By uniqueness of the Haar measure there exists a number $\Delta(x)$ such that $\lambda x = \Delta(x)\lambda$. Moreover, $\Delta$ does not depend on the initial choice of $\lambda$ We may then define a function $\Delta : G \to \mathbb{R}_+$ is this way. The function $\Delta$ is called the **modular function** of $G$. For a proof that $\Delta$ is a continuous homomorphism, see [8]. If $\Delta$ is identically equal to 1, then $G$ is called **unimodular**.

In Definition 2.7, the modular function essentially indicates how close $\lambda$ is to being a right-invariant measure. If $G$ is unimodular, then it is both right and left-invariant.

**Corollary 2.8.** *All abelian groups are unimodular. The converse is untrue. There are many nonabelian groups that are unimodular, including all compact groups. For more on this, see [3].*

**Example.** The subgroup of $GL(2,\mathbb{R})$ defined as the set

$$\left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$

under matrix multiplication is not a unimodular group, when considered in the Haar measure we previously imposed in $GL(2,\mathbb{R})$. Note that the modular function is

$$\Delta(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}) = \frac{1}{|\alpha^2|}$$

Now, we have seen the structure of topological groups as well as the behavior of useful measures on groups. As a result of our understanding of the analytic properties or groups, it now makes sense to restrict the types of groups that will be examined in this paper. Henceforth, all groups considered in this paper are Hausdorff topological groups with $\sigma$-finite measures. Our paper will be concerned primarily with locally compact abelian groups, as we will need to use the Haar measure, and the right and left-invariance of that measure will be valuable in our examination of abstract groups. A locally compact abelian group will be referred to as an **LCA**. LCAs will be a natural setting for Fourier Analysis.

# 3 Locally Compact Abelian Groups

**Remark.** It is important to note that locally compact abelian groups can be largely understood by considering the **elementary locally compact abelian groups** $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}_p$ and their direct products. Other locally compact abelian groups can be well characterized using those groups, and so they will often suffice to provide surprising generality. For more information on the classification of locally compact abelian groups, see [5].

**Definition 3.1.** Let $G$ be an LCA. A homomorphism $\gamma : G \to \mathbb{T}$ is called a **character** of $G$. The set of all continuous characters of $G$ is called the **dual group** of $G$ and will be denoted $\hat{G}$.

**Proposition 3.2.** *Let $G$ be an LCA. Then $\hat{G}$ is also an LCA in the compact-open topology.*

**Theorem 3.3.** *If $G$ is discrete, then $\hat{G}$ is compact and if $G$ is compact, then $\hat{G}$ is discrete.*

For a proof of Theorems 3.2 and 3.3, see [8].

Now, we may use the concept of the dual group of an LCA to develop a theory of Fourier Transforms. Note that $L^2(G)$ denotes the space of all square Haar integrable functions on $G$. For any $x \in G, \xi \in \hat{G}$, we will denote $\xi(x)$ by $\langle \xi, x \rangle$.

**Definition 3.4.** The **Fourier Transform** on $G$ is a function $\mathcal{F} : L^2(G) \to L^2(\hat{G})$ defined by

$$\mathcal{F}f(k) = \int_G \langle -k, x \rangle f(x) dx$$

The notation $\hat{f}$ will often be used to denote $\mathcal{F}f$.

**Theorem 3.5** (Pontryagin Duality). *For any LCA $G$, there is a bicontinuous isomorphism $\alpha : G \to \hat{\hat{G}}$ where $\alpha$ is the evaluation map defined by $\alpha_g(\gamma) = \gamma(g)$.*

Pontryagin Duality will allow us to effectively learn about the properties of LCAs by using analysis on both our chosen group and its dual space. It is extremely useful in the development of the theory that we will continue to use, and it is another important reason that we restrict our perspective to LCAs, as there it is extremely difficult to obtain a general analogue in the nonabelian case. For more information on analogues on nonabelian groups, see [7]. There are several corollaries to the above theorems, some of which will be helpful moving forward.

**Corollary 3.6.** *Several important dualities to consider in the elementary LCAs are that $\mathbb{R}$ is isomorphic to the dual of $\mathbb{R}$, $\mathbb{T}$ is isomoprhic to the dual of $\mathbb{Z}$ (and vice versa) and $\mathbb{Z}_p$ is isomorphic to its own dual.*

**Example.** The following are the Fourier Transforms on some of the elementary LCAs:
If $G = \mathbb{R}$, then

$$\mathcal{F}_G f(y) = \int_{-\infty}^{\infty} e^{iyx} f(x) dx$$

If $G = \mathbb{T}$, then

$$\mathcal{F}_G f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} f(e^{i\theta}) d\theta$$

If $G = \mathbb{Z}$, then

$$\mathcal{F}_G f(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} f(n) e^{in\theta}$$

So far, we have generalized several aspects of the information given in section 1. In place of $\mathbb{R}$, we can now consider an LCA $G$. However, when we defined the classical Weil-Brezin map, we used a summation over $\mathbb{Z} \subset \mathbb{R}$, in effect, "sampling" a function on $\mathbb{R}$ at discrete points. In order to generalize this, we will need to define a subgroup of our general LCA over which we can perform a similar operation.

**Definition 3.7.** Let $H \leq G$. $H$ is called a **lattice subgroup** of $G$ if $H$ is discrete and $G/H$ is compact.

**Example.** $\mathbb{Z}$ is a lattice in $\mathbb{R}$, because $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ is compact, so our lattice definition already lines up well with our understanding of Fourier Analysis on $\mathbb{R}$. However, $\mathbb{Z}$ is far from the only lattice in $\mathbb{R}$: $2\mathbb{Z}$ is a lattice in $\mathbb{Z}$ because $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$ is compact. Indeed, in $\mathbb{R}$ we may consider any lattice of the form $k\mathbb{Z}$ for some $k \in \mathbb{R}$. Already, we have dramatically enlarged the amount of sets over which we can meaningfully sample. This generalizes readily to the $n$-dimensional case.

It should be noted that $\mathbb{R}^n$ is not the only LCA that can have lattice subgroups. However, it is the easiest to visualize in a meaningful way.

**Definition 3.8.** Given a subgroup $H$ in an LCA $G$, the **annihilator subgroup** denoted $H^\perp$ is the subgroup $H^\perp \leq \hat{G}$ of all elements $k \in \hat{G}$ such that $\langle k, h \rangle = 1$ for all $h \in H$.

**Lemma 3.9.** *That the short exact sequence of LCAs*

$$0 \longrightarrow H \xrightarrow{\phi_0} G \xrightarrow{\phi_1} K \longrightarrow 0$$

*implies that the short sequence*

$$0 \longrightarrow \hat{K} \xrightarrow{\hat{\phi}_1} \hat{G} \xrightarrow{\hat{\phi}_0} \hat{H} \longrightarrow 0$$

*is exact.*

*Proof.* Since the natural map $\hat{\phi}_1 : \hat{K} \to \hat{G}$ that sends an element of $k$ to the same element in $G$ is injective by definition and the map $\hat{\phi}_0 : \hat{G} \to \hat{H}$ similarly defined must be surjective, since by the properties of the dual space, $K, H \leq G$, we must show that $Im(\hat{\phi}_1) = \ker(\hat{\phi}_0)$.

If $x \in \ker(\hat{\phi}_0)$, then $\hat{\phi}_0(x) = 0$, so $\langle \hat{\phi}_0(x), h \rangle = 1$ for all $h \in H$. Since the first sequence is exact, we note that $\langle x, \phi_0(h) \rangle = 1$ implies that $\langle x, k \rangle = 1$ for all $k \in Im(\phi_1)$, so $x$ is in the image of $\phi_1$.

Conversely, if $x \in Im(\hat{\phi}_1)$, then $x = \hat{\phi}_1(k')$ for some $k' \in \hat{K}$. Then, consider $\langle x, h \rangle$ for any $h \in H < G$. $\langle x, h \rangle = \langle \hat{\phi}_1(k'), h \rangle$. But, $h$ is in the image of $\phi_0$, so it is in the kernel of $\phi_1$, so $\langle \hat{\phi}_1(k'), h \rangle = \langle \hat{\phi}_1(k'), \phi_0(h') \rangle = \langle \hat{\phi}_0 \circ \hat{\phi}_1(k'), h' \rangle = 1$, so $x \in \ker(\hat{\phi}_0)$, and the sequence is exact.

Moreover, in if $x \in \ker(\hat{\phi}_0)$, then $\langle \hat{\phi}_0(x), h \rangle = 1$ for all $h \in H$, so $\langle x, \phi_0(h) \rangle = 1$ for all $h \in H$. Then, $x \in H^\perp$. Similarly, if $x \in H^\perp$, then $\langle x, \phi_0(h) \rangle = 1$ for all $h \in H$, so $\langle \hat{\phi}_0(x), h \rangle = 1$ for all $h \in H$, so $\hat{\phi}_0(x) = 1$, and $x \in \ker(\phi)$. □

**Proposition 3.10.** *Let $G$ be an LCA. For any lattice $H \leq G$, the annihilator subgroup $H^\perp \leq \hat{G}$ is a lattice in $\hat{G}$. In this case, we will call $H^\perp$ the* ***reciprocal lattice*** *of $H$.*

This proposition will be proven in the proof of Theorem 3.11. We now have enough information to make several important identifications that will allow us to work with lattice subgroups more effectively.

**Theorem 3.11.** *Let $G$ be a group with a lattice $H < G$. We will denote the compact set $G/H$ as $K$. Then we see that $\hat{G}/\hat{K} = \hat{H}$ and moreover $\hat{K} = H^\perp < G$.*

*Proof.* Since we assume $H$ is a lattice in $G$, we know that

$$0 \longrightarrow H \xrightarrow{\phi_0} G \xrightarrow{\phi_1} K \longrightarrow 0$$

is a short exact sequence. Then, by Lemma 3.9, we have that

$$0 \longrightarrow \hat{K} \xrightarrow{\hat{\phi}_1} \hat{G} \xrightarrow{\hat{\phi}_0} \hat{H} \longrightarrow 0$$

is exact. So, $\hat{K} = \ker(\hat{\phi}_0)$ and then $\hat{G}/\hat{K} = \hat{G}/\ker(\hat{\phi}_0) \cong \hat{H}$ by the First Isomorphism Theorem. Moreover, for any $h \in H$ and $k \in K$, we note that

$$\langle \hat{\phi}_1(k), \phi_0(h) \rangle = \langle k, \phi_1\phi_0(h) \rangle = \langle k, 0 \rangle = 1$$

Then, if we consider $\xi \in \hat{G}$ such that $\langle \xi, h \rangle = 1$ for all $h \in H$, we have that $\xi = \hat{\phi}_1(k)$ for some $k$ since we are dealing with an exact sequence. Then, we note that $H^\perp = \hat{K} = \hat{G}/\hat{H}$ by the properties of the short exact sequence. □

These identifications will be useful in computing integrals and developing the Weil-Brezin map.

# 4 The Heisenberg Group on LCAs

Now that we have a handle on the nature of LCAs and we have a structure that we can work with in the form of the lattice, we can begin generalize Heisenberg group to arbitrary LCAs, and we can use this to analyze functions on LCAs in the same way we used the Heisenberg Group on $\mathbb{R}$. The semidirect product $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ can, with a few tweaks, be used to define a Heisenberg group on an general LCA $G$. We will begin our discussion of general Heisenberg Groups with a brief note on notation:

**Definition 4.1.** Let $G$ be an LCA. Then, we define the **Heisenberg Group on** $G$ by considering the set $\mathscr{H}_G = \hat{G} \times G \times \mathbb{T}$ and equip it with the operation

$$(\xi, x, z) \cdot (\xi', x', z') = (\xi + \xi', x + x', \langle \xi', x \rangle z)$$

Note that with this operation, the Heisenberg group is the semidirect product $\hat{G} \times G \rtimes \mathbb{T}$.

We should note before we continue that in this generalization, unlike the classical Heisenberg group, we are not dealing with a product of multiple copies of the same space ($\mathbb{R}$). Here, the fact that we are dealing with a product of three different LCAs will complicate things. However, the Pontryagin Duality between $G$ and $\hat{G}$ will allow use to perform the same analyses effectively.

Now, it is important that we note that we have been working with individual topological groups, and that we are now *not* working with a direct product of topological groups. Indeed, we are working with a semidirect product, and so we must check to be sure that $\mathscr{H}_G$ indeed has a topological structure compatible with the group operation. Otherwise, our previous theorems about analysis on topological groups will not help us.

**Lemma 4.2.** *Let $G$ be a topological group and let $H, N \subset G$ be closed subgroups of $G$ such that $H \subset N_G(N)$. Then, the inner semidirect product $HN = \{hn, h \in H, n \in N\}$ is homeomorphic to the direct product $H \times N$, with the homeomorphism induced by the natural map $(h, n) \mapsto hn$.*

*Proof.* We will show that $f : H \times N \to HN$ defined by $f(h, n) \to hn$ is a homeomorphism. Since $H, N$ are closed subgroups of $G$, and the mapping $\tilde{f} : G \times G \to G$ defined by $\tilde{f}(g_1, g_2) = g_1 g_2$ is continuous by definition of a topological group, $f$ must be continuous in the subspace topology, as it is the restriction of the continuous $\tilde{f}$ to a closed subspace.

Now, since the inversion map and the operation map are continuous in $G$, their composition must be continuous. So, the function $g : HN \to H$ defined by $g(hn) = hn^{-1} = h$ is continuous. Similarly, $h : HN \to N$ defined by $h(hn) = h^{-1}n = n$ is continuous. So, the function $f^{-1}(hn) = (h, n)$ is a finite Cartesian product of two continuous functions, so $f^{-1}$ is continuous. So, $f$ is a homeomorphism. $\square$

**Proposition 4.3.** *$\mathscr{H}_G$ is a topological group for any LCA $G$ and lattice $H < G$.*

*Proof.* $\mathscr{H}_G = \hat{G} \times G \rtimes \mathbb{T}$, but an external semidirect product can always be viewed as an internal semidirect product lying in a larger group [2]. So, $\hat{G} \times G \rtimes \mathbb{T} \cong \hat{G} \times G\mathbb{T}$. But, since $G\mathbb{T}$ is homeomorphic to $G \times \mathbb{T}$, we have that $\hat{G} \times G\mathbb{T}$ is homeomorphic to the direct product$\hat{G} \times G \times \mathbb{T}$. So, since the product group $\hat{G} \times G \times \mathbb{T}$ is a topological group, $\mathscr{H}_G$ must be a topological group. $\square$

From our definition of $\mathscr{H}_G$, we can define a left action $\mathscr{H}_G \times \mathscr{F}G \to \mathscr{F}G$, so long as $G$ has a norm, where $\mathscr{F}G$ denotes the set of all functions on $G$ such that $f \leq C(1+|x|)^{-n-\varepsilon}$ for some constant $C$, by $(\xi, x, z) \cdot f(t) = z \cdot \langle \xi, t \rangle \cdot f(t+x)$. We will largely consider $G = \mathbb{R}$, as the norm is well-understood. While in the future, we will tend to use generalized Schwartz functions on LCAs, this broader class of functions gives a much better intuition in the case of $\mathbb{R}^n$. Note that all Schwartz functions are in $\mathscr{F}G$

Now, we may introduce the Weil-Brezin map, which will later allow us to decompose the Fourier transform on $L^2(G)$.

**Definition 4.4.** Let $G$ be an LCA and let $H$ be a lattice in $G$ with $f \in \mathscr{F}G$ and $(\xi, x, z) \in \mathscr{H}_G$. The Weil-Brezin map $\mathcal{W}_G^H$ is defined by

$$\mathcal{W}_G^H f(\xi, x, z) = \sum_{h \in H} f(x+h) \langle \xi, h \rangle z$$

**Lemma 4.5.** *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a (continuous) function such that $f \leq C(1+|x|)^{-n-\varepsilon}$ for some constant $C$. Then, the series $\sum_{x \in H} f(x)$ converges for any lattice $H \leq \mathbb{R}$.*

*Proof.* Let $D_1$ be the unit disk, and define $D_r = \{x \in \mathbb{R}^n : r-1 \leq ||x|| < r\}$ for all $r \in \mathbb{N}$. Note that each $D_r$ is a compact set in $\mathbb{R}^n$. Thus, we can define $||f||_\infty^r$ to be the sup norm of $f$ on $D_r$, and we note $||f||_\infty^r$ is finite by compactness. Recall that the volume of an $n$-ball of radius $r$ is

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r^n$$

and therefore the volume of $D_r$ can be computed as

$$Vol(D_r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} (r^n - (r-1)^n)$$

Now, because $H$ is a lattice in $\mathbb{R}$, $H = \mathbb{Z}a_1 \oplus ... \oplus \mathbb{Z}a_n$ for some collection of scalars $\{a_1, ..., a_n\}$. Then letting $N_r$ be the number of points of $H$ contained in $D_r$, we note that

$$N_r \leq 2^n \frac{Vol(D_r)}{|det(a_1|...|a_n)|}$$

Let $S_r = \sum_{x \in D_r} f(x)$. Then, $S_r \leq ||f||_\infty^r N_r$. So, $\sum_{r \in \mathbb{Z}} S_r \leq \sum_{r \in \mathbb{Z}} ||f||_\infty^r N_r$. Then, because of our definition of $Vol(D)$, we see that $Vol(D) \leq Kr^n$ for some constant $K$ and that $N_r \leq Kr^n$. Then,

$$\sum_{r \in \mathbb{Z}} S_r \leq \sum_{r \in \mathbb{Z}} ||f||_\infty^r K r^n \leq \sum_{r \in \mathbb{Z}} ||C(1+|x|)^{-n-\epsilon}||_\infty^r K r^n < \infty$$

Because the partial sums of $\sum_{r \in \mathbb{Z}} S_r$ are partial sums of $\sum_{x \in H} f(x)$, we conclude that $\sum_{x \in H} f(x)$ converges. ∎

In order to be sure of the properties of the Weil-Brezin map, any real function $f$ will have the property that $|f| \leq C(1+|x|)^{-n-\epsilon}$ unless stated otherwise. Note that by the classification of the elementary LCAs, $\mathbb{R}$ and direct products of $\mathbb{R}$ are the only non-discrete, noncompact groups that need be considered. As such, the restriction of our functions to functions on $\mathbb{R}^n$ for now will still provide results in surprising generality.

# 5 Properties and Applications of the Weil-Brezin Map

Now, we have a well-defined generalization of the Weil-Brezin map, but we are still far from being able to apply it to gain any interesting results. First of all, while we have defined a domain space for the Weil-Brezin map, we do not yet have a well defined codomain. To do so, we will need to consider what sort of values $Wf$ can assume. In the real case, we considered our discrete subgroup $\Gamma$ acting from the left. However, since the classical Heisenberg group was just a product of $\mathbb{R}$ with itself, we could take $\Gamma$ to be a product of lattices in $\mathbb{R}$. Now, to define a more general analogue, we will need to do a bit more work.

**Lemma 5.1.** *Let $(h', h, 1) \in H^{\perp} \times H \times \mathbb{T}$ and let $z \in \mathbb{T}$. Then,*

$$(Wf)((h', h, 1) \cdot (\xi, x, s)) = (Wf)(\xi, x, s)$$

*and*

$$(Wf)(\xi, x, z) = z \cdot (Wf)(\xi, x, 1)$$

*Proof.* First, we see that

$$(Wf)((g', g, 1) \cdot (\xi, x, s)) = (Wf)(\xi+g', x+g, 1 \cdot s\langle \xi, g\rangle) = s\langle \xi, g\rangle \cdot \sum_{h\in H} f(x+g+h)\langle \xi+g', h\rangle$$

But, note that $\langle \xi + g', h\rangle = \langle \xi, h\rangle\langle g', h\rangle$, and since $g' \in H^{\perp}$, the reciprocal lattice of $H$, $\langle g', h\rangle = 1$, so $\langle \xi + g', h\rangle = \langle \xi, h\rangle\langle g', h\rangle = \langle \xi, h\rangle$. Moreover, since we are summing over all of $H$, $\sum_{h\in H} f((x + g) + h) = \sum_{h\in H} f(x + h)$ as long as $g \in H$. Finally, since $\xi$ is a character on $G$ and therefore only takes values in $\mathbb{T}$ and $g \in H$ is an element of the lattice $H < G$,

$$(Wf)((g', g, 1) \cdot (\xi, x, s)) = s\langle \xi, g\rangle \cdot \sum_{h\in H} f(x + g + h)\langle \xi + g', h\rangle =$$

$$s \cdot \sum_{h\in H} f(x + h)\langle \xi, h\rangle = (Wf)(\xi, x, s)$$

Proving the second half of the lemma is much simpler, as by linearity of the convergent series,

$$(Wf)(\xi, x, z) = \sum_{h\in H} f(x + h)\langle \xi, h\rangle z = z \cdot \sum_{h\in H} f(x + h)\langle \xi, h\rangle = z \cdot (Wf)(\xi, x, 1)$$

□

So, we see that $Wf$ is well-defined on cosets of $\Gamma = H^{\perp} \times H \times 1$. However, $H^{\perp} \times H \times 1$ is not necessarily a normal subgroup of $\mathscr{H}_G = \hat{G} \times G \rtimes \mathbb{T}$. In fact, because of the semidirect product in $\mathscr{H}_G$, $H^{\perp} \times H \times 1$ will never be a normal subgroup, so we cannot consider $\Gamma\backslash\mathscr{H}_G$ as a group. However, we will not need the group properties here, as we are primarily concerned with the topological, and thus analytic, structure.

**Lemma 5.2.** *As manifolds, $\Gamma\backslash\mathscr{H}_G$ is homeomorphic to $\hat{H} \times K \times \mathbb{T}$*

*Proof.* Since we are not concerned with the preservation of the operation, we will largely consider the individual quotients. By our earlier identifications, $\hat{G}/H^{\perp} = \hat{G}/\hat{K} = \hat{H}$, $G/H = K$, and $\mathbb{T}/1 = \mathbb{T}$. Since the topological products are well-behaved with respect to locally compact spaces [6], our identifications give us that $\Gamma\backslash\mathscr{H}_G = H^{\perp}\backslash\hat{G} \times h\backslash G \times 1\backslash\mathbb{T} = \hat{H} \times K \times \mathbb{T}$ as manifolds though, importantly, not as groups. □

Now that we are working on $\hat{H} \times K \times \mathbb{T}$, we need to ensure that we have a well-defined measure.

**Lemma 5.3.** *$\mathscr{H}_G$ and $\hat{H} \times K \times \mathbb{T}$ have left and right invariant measures defined as the direct products of the Haar measures on $\hat{G}$, $G$, and $\mathbb{T}$.*

*Proof.* This follows from a change of variables in the integral of an arbitrary function in Ł$^1(\mathscr{H}_G)$. □

Now, we can see the Weil-Brezin map must send functions on $G$ to functions on $\hat{H} \times K \times \mathbb{T}$, and since we have a well-defined measure on that space, we can define the $L^p$ spaces on $\hat{H} \times K \times \mathbb{T}$, and in particular, we may consider the Hilbert space $L^2(\hat{H} \times K \times \mathbb{T})$. Moreover, Lemma 5.1 allows us to consider a subspace of $L^2(\hat{H} \times K \times \mathbb{T})$ that will be the most relevant to our discussion.

**Definition 5.4.** By Fourier decomposition in the last entry, we may decompose $L^2(\hat{H} \times K \times \mathbb{T})$ into a direct sum of the orthogonal subspaces $V_k$, where

$$V_k = \{f \in L^2(\hat{H} \times K \times \mathbb{T}) : f(\xi, x, z) = z^k \cdot f(\xi, x, 1)\}$$

Now, by Lemma 5.1, we note that $Wf \in V_1$ for all $f$. This allows us to define a domain space for the inverse of the Weil-Brezin map. We will use the Weil-Brezin map in conjunction with its inverse to gain different proofs of several established results.

**Lemma 5.5.** *The operator* $: V_1 \subset L^2(\hat{H} \times K \times \mathbb{T}) \to L^2(G)$ *defined by*

$$T(f) = \frac{1}{\lambda(\hat{H})} \int_{\hat{H}} f(\xi, x, 1) d\xi$$

*is a (left) inverse of the Weil-Brezin map where $\lambda$ is a left Haar measure on $\hat{G}$. Not that $\lambda(\hat{H})$ is finite, since $\hat{H} \cong \hat{G}/\hat{K}$, and is therefore compact, since $\hat{K}$ is a lattice in $G$.*

*Proof.* Consider a Schwartz function $\phi$ on $G$. Then, the Weil-Brezin map applied to $\phi$ yields

$$W(\phi)(\xi, x, 1) = \sum_{h \in H} \phi(x + h)\langle \xi, h \rangle$$

We apply $T$ to see that

$$TW(\phi)(\xi, x, 1) = \frac{1}{\lambda(\hat{H})} \int_{\hat{H}} \sum_{h \in H} \phi(x + h)\langle \xi, h \rangle d\xi$$

Now, if we consider the expression

$$\sum_{h \in H} \int_{\hat{H}} \phi(x + h)\langle \xi, h \rangle d\xi$$

Since by the orthogonality of characters over a compact set $K$ [8] we have that for any $\chi_1, \chi_2 \in \hat{K}$

$$\int_K \chi_1(t)\overline{\chi_2(t)}dt = 0$$

We note that, as $\xi$ is in $\hat{H}$, the elements $h \in H$ are isomorphic to characters that can act elements of $\hat{H}$ by Pontryagin Duality. So, for any $h \in H, h \neq 0$, we have that:

$$\int_{\hat{H}} \langle \xi, h \rangle d\xi = \int_{\hat{H}} \langle \xi, h \rangle \overline{\langle \xi, 0 \rangle} d\xi = 0$$

So, for all $h \neq 0$

$$\int_{\hat{H}} \phi(x+h)\langle \xi, h \rangle d\xi = 0$$

So, finally, when we evaluate the composition of $W$ and $T$, we have that

$$TW(\phi) = \sum_{h\in H} \int_{\hat{H}} \phi(x+h)\langle \xi, h \rangle d\xi = \int_{\hat{H}} \phi(x+h)\langle \xi, 0 \rangle d\xi = \phi(x+h) \int_{\hat{H}} 1 d\xi = \lambda(\hat{H})\phi(x)$$

Since $\hat{H}$ is compact, $\lambda(\hat{H})\phi(x) < \infty$, so we conclude that

$$\lambda(\hat{H})\phi(x) = \sum_{h\in H} \int_{\hat{H}} \phi(x+h)\langle \xi, h \rangle d\xi = \int_{\hat{H}} \sum_{h\in H} \phi(x+h)\langle \xi, h \rangle d\xi$$

Then, $T = W^{-1}$. □

To prove the unitarity of the Weil-Brezin map, we need the following important theorem about integration on groups.

**Theorem 5.6.** *Let $G$ be an LCA with subgroup $H$. Then, there exists a measure $dgH$ such that:*

$$\int_G f(g)dg = \int_{G/H} \int_H f(gh) dh dgH$$

*for any $f$.*

For a proof of this theorem, see [4].

**Lemma 5.7.** *The Weil-Brezin Map is a unitary transformation on any normed group.*

*Proof.* Let $f, g \in \mathcal{F}G$. Now, let us consider the inner product

$$\langle W(f), W(g) \rangle_{L^2(\hat{H}\times K\times\mathbb{T})} = \int_{\hat{H}\times K\times\mathbb{T}} (z\sum_{h\in H} f(x+h)\langle \xi, h \rangle) \overline{(z \sum_{h'\in H} g(x+h)\langle \xi, h' \rangle)} dz d\xi dx$$

$$= \int_{\hat{H}\times K\times\mathbb{T}} |z|^2 (\sum_{h\in H} f(x+h)\langle \xi, h \rangle) \overline{(\sum_{h'\in H} g(x+h)\langle \xi, h' \rangle)} dz d\xi dx$$

Now, when we integrate with respect to $z$, since we have chosen the normalization of the Haar measure on $\mathbb{T}$ such that the measure of $\mathbb{T}$ is 1, since $|z|^2 = 1$ for all $z$, we have that $\int_{\mathbb{T}} |z|^2 dz = 1$. Then, the above equation is equal to

$$\int_{\hat{H}\times K}(\sum_{h\in H} f(x+h)\langle\xi,h\rangle)\overline{(\sum_{h'\in H} g(x+h)\langle\xi,h'\rangle)}d\xi dx$$

Now, since we are summing over all of $H$ in all cases, we can rewrite this as

$$\int_{\hat{H}\times K}(\sum_{h\in H} f(x+h)\langle\xi,h\rangle)\overline{(\sum_{h\in H} g(x+h)\langle\xi,h\rangle)}d\xi dx$$

And, when we multiply everything out, (convergence on a wider class of functions would be difficult, but since we are working with extremely convergent functions and most of the terms will integrate zero, this is a formal multiplication), we have that that expression becomes

$$\int_{\hat{H}\times K}\sum_{h\in H}\sum_{h'\in H} f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle}$$

Now, where $h=h'$, $f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle} = f(x+h)\overline{g(x+h)}\langle\xi,h\rangle\overline{\langle\xi,h\rangle}$. But, since the characters take values in $\mathbb{T}$, $\langle\xi,h\rangle\overline{\langle\xi,h\rangle}=1$. So, $f(x+h)\overline{g(x+h)}\langle\xi,h\rangle\overline{\langle\xi,h\rangle} = f(x+h)g(x+h)$. On the other hand, for elements where $h\neq h'$, by the orthogonality of characters, we have that

$$\int_K f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle}=0$$

So, since we are working with measurable functions in $\sigma$-finite spaces, we can pull out our summations, and

$$\int_{\hat{H}\times K}\sum_{h\in H}\sum_{h'\in H^{\perp}} f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle} =$$

$$\sum_{h\in H}\sum_{h'\in H^{\perp}}\int_{\hat{H}\times K} f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle}$$

But, since every time $h\neq h'$ the integral evaluates to zero,

$$\sum_{h\in H}\sum_{h'\in H}\int_{\hat{H}\times K} f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle} = \sum_{h\in H}\int_{\hat{H}\times K} f(x+h)\overline{g(x+h)}\langle\xi,h\rangle\overline{\langle\xi,h\rangle}$$

$$=\sum_{h\in H}\int_{\hat{H}\times K} f(x+h)\overline{g(x+h)}d\xi dx$$

So, finally, applying Fubini's Theorem again and using the averaging argument from Theorem 5.6, since the sum over $H$ can be considered an integral in the counting measure, and since $K=G/H$ by definition, we see that

$$\int_{\hat{H}\times K}\sum_{h\in H}\sum_{h'\in H^{\perp}} f(x+h)\overline{g(x+h')}\langle\xi,h\rangle\overline{\langle\xi,h'\rangle}d\xi dx = \int_{\hat{H}\times K}\sum_{h\in H} f(x+h)g(x+h)d\xi dx$$

$$= \int_K \sum_{h\in H} f(x+h)\overline{g(x+h)}dx = \int_G f(x)\overline{g(x)}dx = \langle f, g\rangle_{L^2(G)}$$

And so, we may conclude that the Weil-Brezin map is a unitary transformation. □

Now, we can state several important corollaries.

**Corollary 5.8.** *The Weil-Brezin map can be uniquely extended to a map on $L^2(G)$.*

*Proof.* We previously defined $W$ on the space of generalized Schwartz functions on $G$, which are dense in $L^2(G)$. Then, since the map is unitary in the $L^2$ norm, we may extend the map by a limiting argument to all of $L^2(G)$. □

Finally, we will relate the Weil-Brezin map to the Fourier transform on $G$ by writing the Fourier transform as a composition of Weil-Brezin operators. From this, we will derive the Poission Summation Formula, as well as the Plancherel Theorem.

**Proposition 5.9.** *The operator $J$ on $L^2(\hat{H} \times K \times \mathbb{T})$ induced by the transformation $J_0 : \hat{H} \times K \times \mathbb{T} \to K \times \hat{H} \times \mathbb{T}$ defined by $J_0(\xi, x, z) = (x, -\xi, z\langle\xi, x\rangle)$ is a unitary transformation.*

*Proof.* This is a change of variables argument:

$$\int_{\hat{H}\times K\times\mathbb{T}} |J(f)(\xi, x, z)|^2 dzdxd\xi = \int_{\hat{H}\times K\times\mathbb{T}} |f(x, -\xi, z\langle\xi, x\rangle)|^2 dzdxd\xi$$

$$\int_{\hat{H}\times K\times\mathbb{T}} |f(\xi', x', z')|^2 - d(-\xi')dx'd(z\langle\xi, x\rangle) = \int_{\hat{H}\times K\times\mathbb{T}} |f(\xi', x', z')|^2 d(\xi')dx'dz'$$

Where the fact that we integrate over all of $\mathbb{T}$ allows us leeway in changing that variable. □

As we are largely concerned with Fourier Analysis on the Heisenberg group, we ought to examine how the Fourier Transform interacts with the the group operation and with the induced operations.

**Theorem 5.10.** *Let $G$ be an LCA and let $H < G$ be a lattice. Then, the Fourier transform on $L^2(G)$ factorizes as below, with $C = \lambda(K)$:*

$$\mathcal{F}_G = CW_{\hat{G}}^{-1} \circ J \circ W_G$$

*Proof.*

$$CW_{\hat{G}}^{-1} \circ J \circ W_G(f) = (W_{\hat{G}}^{-1} \circ J)(\sum_{h\in H} f(x+h)\langle\xi, h\rangle z)$$

$$= \int_K (\sum_{h\in H} f(x+h)\langle -\xi, x+h\rangle)dx$$

Which, by Theorem 5.6, is equal to

$$\int_G f(g)\langle -\xi, g\rangle dg = \hat{f}(\xi)$$

which is precisely the Fourier transform of $f$. □

Now, from Theorem 5.10, we can immediately derive several major results in Fourier analysis on LCAs. In this way, it becomes clear that the use of the Weil-Brezin map here has allowed us to avoid the classical proofs and encounter the results from a different angle.

**Corollary 5.11** (The Plancherel Theorem)**.** *The Haar measures on $G$ and $\hat{G}$ can be renormalized in such a way that the Fourier transform on $G$ is a unitary map.*

**Corollary 5.12** (The Poisson Summation Formula)**.** *Let $G$ be an LCA and let $\lambda$ be the Haar measure on $\hat{G}$. Then, given any lattice $H < G$ and reciprocal lattice $H^{\perp} < \hat{G}$, for any Schwartz function $f$, for $C = \lambda(K)$ such that*

$$\sum_{h \in H} f(h) = \frac{1}{C} \sum_{h' \in H^{\perp}} \hat{f}(h')$$

*Proof.* Note that

$$\sum_{h \in H} f(h) = W_G(f)(0,0,1) = (JW_{\hat{G}})(f)(0,0,1) = \frac{1}{C} W_{\hat{G}}(\hat{f})(0,0,1) = \frac{1}{C} \sum_{h' \in H^{\perp}} \hat{f}(h')$$

□

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