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# The change in multiplicity of an eigenvalue of a Hermitian matrix associated with the removal of an edge from its graph

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ABSTRACT

When an edge is removed from an undirected graph, there is a limited change that can occur in the multiplicity of an eigenvalue of a Hermitian matrix with that graph. Primarily for trees, we identify the changes that can occur and characterize the circumstances under which they occur. This extends known results for the removal of vertices. A catalog of examples is given to illustrate the possibilities that can occur and to contrast the case of trees with that of general graphs.



## 1. Introduction

For an undirected graph $G$, we denote by $\mathcal{H}(G)$ the set of all Hermitian matrices whose graph is $G$; no restriction is placed by $G$ on diagonal entries. For any matrix $A \in \mathcal{H}(G)$, we denote by $A(i)$ the principal submatrix of $A$ created by deleting the $i$th row and column. Similarly, we mean by $G(i)$ the subgraph of $G$ resulting from deletion of the $i$th vertex, or, equivalently, the graph of $A(i)$. For $G = (V, E)$, if $(i, j) \in E$, we know that $a_{ij}, a_{ji} \neq 0$; we denote by $A(e_{i,j})$ the matrix $A$ altered only so that $a_{ij} = a_{ji} = 0$. Similarly, we denote by $G(e_{i,j})$ the graph resulting from removing the edge $(i, j)$ from $G$. For the purpose of some of our arguments, the order in which we remove vertices or edges will be relevant (although, of course, the end result is independent of the order); we thus apply the convention of removing edges and vertices in the order they are listed. For example, $G(e_{1,2}, 4)$ is the subgraph of $G$ in which we first delete the edge $(1, 2)$ and then the 4th vertex.

For any $A \in \mathcal{H}(G)$ and $\lambda \in \mathbb{R}$, we denote the multiplicity of $\lambda$ as $m_A(\lambda)$. Recall that, by the classical interlacing inequalities for Hermitian matrices, $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$. Since deleting a row and a column with the same index from $A$ is analogous to deleting a vertex from $G$, it is useful to classify vertices in the following way: if $m_A(\lambda) - m_{A(i)}(\lambda) = -1$, then the vertex $i$ is called a *Parter vertex* for $\lambda$ in $G$; if $m_A(\lambda) - m_{A(i)}(\lambda) = 0$, then $i$ is a *neutral vertex*; and if $m_A(\lambda) - m_{A(i)}(\lambda) = 1$, then $i$ is a *downer vertex*. Previous papers, such as [2] have explored the theory of such vertices extensively. Our interest here is the possible effects upon multiplicities of eigenvalues of matrices in $\mathcal{H}(T)$ when an edge is removed from a tree $T$. However, some of our observations are more general, and we try to indicate what is special about trees.

## 2. Preliminaries

As we mentioned before, although we consider the order in which we remove vertices or edges from a graph, the end result is the same. More specifically, we see that $G(e_{i,j}, i) = G(i, e_{i,j}) = G(i)$, since removal of a vertex $i$ from $G$ also results in the removal of all edges incident to $i$. Similarly, $A(e_{i,j}, i) = A(i)$. This leads us to a simple but important observation:

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**Remark 2.1.** For $A \in \mathcal{H}(G)$, $m_{A(e_{i,j},i)}(\lambda) = m_{A(i)}(\lambda)$.

This leads us to the following lemma:

**Lemma 2.2.** *Let $G = (V, E)$ be any undirected graph, and $A \in \mathcal{H}(G)$. For $(i, j) \in E$, $\lambda \in \mathbb{R}$,*

$$m_A(\lambda) - 2 \leq m_{A(e_{i,j})}(\lambda) \leq m_A(\lambda) + 2.$$

**Proof.** Assume, to the contrary, that $|m_{A(e_{i,j})}(\lambda) - m_A(\lambda)| \geq 3$. Then, by the interlacing inequalities, $|m_{A(e_{i,j},i)}(\lambda) - m_A(\lambda)| \geq 2$. So, by Remark 2.1, $|m_{A(i)}(\lambda) - m_A(\lambda)| \geq 2$, which cannot be the case, since interlacing dictates that the multiplicity of an eigenvalue can increase or decrease by at most 1 from $A$ to $A(i)$. □

Thus we can define edges as Parter, neutral, and downer in the same way as we defined vertices as such, with the obvious additional classifications of 2-Parter and 2-downer edges. It turns out, however, that although 2-downer edges can exist in a general graph (see Appendix of Examples (1)), they do not exist in trees, which we will show below using the following definition and lemma.

**Definition 2.3.** Let $T = (V, E)$ be a tree, and $(i, j) \in E$. If $j$ is a downer vertex for $\lambda$ in the forest $T(i)$, then we call the connected subgraph of $T(i)$ containing $j$ a *downer branch at i* for $\lambda$ in $T$.

Identifying downer branches can be very useful, thanks to the following lemma [1]:

**Lemma 2.4.** *For a tree $T$, a vertex $i$ is Parter for $\lambda$ in $T$ if and only if there is a downer branch at $i$ for $\lambda$ in $T$.*

We are now ready to prove that 2-downer edges do not exist in trees.

**Theorem 2.5.** *Let $T = (V, E)$ be a tree, $A \in \mathcal{H}(T)$, $(i, j) \in E$, and $\lambda \in \mathbb{R}$. Then*

$$m_A(\lambda) - 1 \leq m_{A(e_{i,j})}(\lambda) \leq m_A(\lambda) + 2.$$

**Proof.** From Lemma 2.2, we need only show that $m_{A(e_{i,j})}(\lambda)$ cannot be $m_A(\lambda) - 2$. Assume, to the contrary, that $m_{A(e_{i,j})}(\lambda) = m_A(\lambda) - 2$. By Remark 2.1 and interlacing, we have

$$m_A(\lambda) - 1 \geq m_{A(e_{i,j},i)}(\lambda) = m_{A(i)}(\lambda) \geq m_A(\lambda) - 1.$$

Thus $m_{A(e_{i,j},i)}(\lambda) = m_A(\lambda) - 1 = m_{A(i)}(\lambda)$, so the vertex $i$ must be downer for $\lambda$ in $T$ and Parter for $\lambda$ in $T(e_{i,j})$. But, since $i$ is downer in $T$, then by Lemma 2.4 there is no downer branch at $i$, which means that there cannot be a downer branch at $i$ in $T(e_{i,j})$ since $T(i)$ and $T(e_{i,j}, i)$ are the same. So $i$ cannot be Parter in $T(e_{i,j})$, and we have a contradiction. □

Since 2-Parter, Parter, downer, and neutral edges do exist in trees (see Appendix of examples (2), (3), (4), and (5), respectively), we devote the rest of this paper to determining how they occur and certain properties about them.

## 3. Locating edges using vertex classification

We have already discussed Parter vertices, but, for our purposes, we segregate them into two different categories:

**Definition 3.1.** Let $T = (V, E)$ be a tree with $i \in V$ a Parter vertex for $\lambda$. If there is exactly one downer branch at $i$ for $\lambda$, we call $i$ *singly Parter*; if there is more than one downer branch at $i$, we call $i$ *multiply Parter*.

**Lemma 3.2.** *Let $A \in \mathcal{H}(T)$ for some tree $T$, and let $i$ be a vertex in $T$ such that, with respect to an eigenvalue $\lambda$ of $A$, $i$ is multiply Parter. Then, for any vertex $j \neq i$ in $T$, $i$ is Parter (of some sort) in $T(j)$. Moreover, the classification of $j$ (as Parter, downer, or neutral with respect to $\lambda$) is the same in $T(i)$ as in $T$.*

**Proof.** Since $i$ is multiply Parter in $T$, there is a downer branch at $i$ in $T$ that does not contain $j$, and thus there is a downer branch at $i$ in $T(j)$. Therefore, by Lemma 2.4, $i$ is Parter in $T(j)$. Now assume that the classification of $j$ changes with the deletion of $i$ from $T$. Then we have that $m_{A(i,j)}(\lambda) \neq m_{A(j,i)}(\lambda)$, since $i$ is Parter in both $T$ and $T(j)$. Since $A(i, j) = A(j, i)$, this is a contradiction. □

We can thus make the following remark, using the same reasoning as we did in the proof of Lemma 3.2.

**Remark 3.3.** Let $i$ be multiply Parter in $T$, and let $(j, k)$ be any edge in $T$. Then $i$ is Parter in $T(e_{j,k})$, and $(j, k)$ has the same classification in $T(i)$ as in $T$, assuming $(j, k)$ is not incident to $i$.

From this point onwards, to avoid confusion between edges and vertices, we will abbreviate Parter as $P$, singly Parter as $P_s$, multiply Parter as $P_m$, neutral as $N$, and downer as $D$ whenever we talk about vertices.

**Table 1**
Possible change in multiplicity of $\lambda$ with the removal of adjacent vertices $i$ and $j$.

| $i$ | $j$ | $m_A(\lambda) - m_{A(i,j)}(\lambda)$ |
|---|---|---|
| $P$ | $P$ | $-2, 0$ |
| $P$ | $N$ | $-1, 0$ |
| $P$ | $D$ | $0$ |
| $N$ | $N$ | $0$ |
| $N$ | $D$ | Not possible |
| $D$ | $D$ | $1$ |

**Remark 3.4.** Note that $|m_{A(e_{i,j})}(\lambda) - m_{A(i)}(\lambda)| \leq 1$; this follows from Remark 2.1 and interlacing.

The results above in Table 1, will also prove helpful. The vertices $i$ and $j$ are adjacent in $T$, and their given classifications are with respect to the matrix $A \in \mathcal{H}(T)$.
So, for example, if $i$ and $j$ are adjacent vertices that are both $D$, we know that $j$ must be $N$ in $T(i)$.

Now we can prove some necessary conditions for the existence of Parter and 2-Parter edges.

**Theorem 3.5.** *Let $T = (V, E)$ be a tree, $A \in \mathcal{H}(T)$, and $(i, j) \in E$ be 2-Parter for $\lambda \in \mathbb{R}$. Then $i$ and $j$ are both $P_s$ for $\lambda$ in $T$.*

**Proof.** By Remark 3.4 and interlacing, we see that $i$ must be $P$ for $\lambda$ in $T$, so that $m_A(\lambda) + 1 = m_{A(i)}(\lambda) = m_{A(e_{i,j},i)}(\lambda)$, the last equality justified by Remark 2.1. Therefore, since $(i, j)$ is 2-Parter for $\lambda$ in $T$, $i$ must be $D$ for $\lambda$ in $T(e_{i,j})$. Thus, by Remark 3.3, $i$ must be $P_s$ for $\lambda$ in $T$; the same argument holds for $j$. $\square$

**Theorem 3.6.** *Let $T = (V, E)$ be a tree, $A \in \mathcal{H}(T)$, and $(i, j) \in E$ be Parter for $\lambda \in \mathbb{R}$. Then $i$ is $P_s$ and $j$ is $N$ for $\lambda$ in $T$, or vice versa.*

**Proof.** By Remark 3.4 and interlacing, $i$ cannot be $D$. So we consider 2 cases:

(a) Let $i$ be $N$. Thus, by Remark 2.1, $m_A(\lambda) = m_{A(i)}(\lambda) = m_{A(e_{i,j},i)}(\lambda)$, so $i$ must be $D$ in $T(e_{i,j})$, and therefore is also $D$ in $T(j)$. Thus there is a downer branch at $j$, so $j$ must be $P$. Since $i$ is $N$ in $T$ and $D$ in $T(j)$, by Lemma 3.2, $j$ is $P_s$.

(b) Let $i$ be $P$. By Remark 2.1, $m_A(\lambda) + 1 = m_{A(i)}(\lambda) = m_{A(e_{i,j},i)}(\lambda)$, so $i$ is $N$ in $T(e_{i,j})$. Thus $i$ is also $N$ in $T(j)$, and, by Lemma 3.2, $i$ is $P_s$. By the first and third lines of Table 1, $j$ cannot be $P$ or $D$; otherwise, $m_A(\lambda) - m_{A(j,i)}(\lambda)$ would be $-1$ or $1$, respectively. Thus $j$ is $N$. $\square$

For downer edges, however, we can make a stronger claim:

**Theorem 3.7.** *Let $T = (V, E)$ be a tree and $A \in \mathcal{H}(T)$. Then $(i, j) \in E$ is downer for $\lambda \in \sigma(A)$ if and only if $i$ and $j$ are both $D$ for $\lambda$ in $T$.*

**Proof.** ($\Rightarrow$) Let $(i, j) \in E$ be downer. By Remark 3.4 and interlacing, $i$ cannot be $P$ in $T$; we assume, for a contradiction, that $i$ is $N$. Thus, by Remark 2.1, $m_A(\lambda) = m_{A(i)}(\lambda) = m_{A(e_{i,j},i)}(\lambda)$, so $i$ must be $P$ in $T(e_{i,j})$. So, according to Lemma 2.4, there is a downer branch at $i$ for $\lambda$ in $T(e_{i,j})$ but no downer branch at $i$ in $T$, which is clearly a contradiction. Thus $i$ must be $D$, and the same argument holds for $j$.

($\Leftarrow$) Let $(i, j) \in E$, where $i$ and $j$ are both $D$. By Remark 3.4 and interlacing, $(i, j)$ is either downer or neutral; we assume, for a contradiction, that it is neutral. By Table 1, $i$ is $N$ in $T(j)$, and $j$ is $N$ in $T(i)$; thus both are $N$ in $T(e_{i,j})$. We therefore have

$$m_{A(e_{i,j},i)}(\lambda) = m_A(\lambda)$$
$$m_{A(i)}(\lambda) = m_A(\lambda) - 1.$$

By Remark 2.1, however, these multiplicities should be equal, so we have a contradiction. $\square$

We notice that the only type of edge for whose existence in a tree we do not have a necessary condition is a neutral edge. Thus, from these theorems, we obtain the following corollary:

**Corollary 3.8.** *Let $T = (V, E)$ be a tree and $(i, j) \in E$. If $i$ and $j$ are both $N$, then $(i, j)$ is neutral. If, instead, at least one of $i$ and $j$ is $P_m$, then $(i, j)$ is also neutral.*

We summarize our results in Table 2:

**Remark 3.9.** If $i$ is $P_s$ and $j$ is $D$, $i$ and $j$ cannot be adjacent. If they were, by Table 1, $m_{A(i,j)}(\lambda) = m_A(\lambda)$, so $j$ remains $D$ with the removal of $i$. Thus, there is a downer branch for $\lambda$ at $i$ that contains $j$. But that equality also implies that $i$ must remain $P$ with the removal of $j$, so there is also a downer branch at $i$ that does not contain $j$ by Lemma 2.4. Since $i$ is $P_s$, this is a contradiction.

We conclude with the following corollary.

**Corollary 3.10.** *Let $T = (V, E)$ be a tree and $(i, j) \in E$ be neutral. Then the classifications of $i$ and $j$ (as $P$, $N$, or $D$) are the same in $T(e_{i,j})$ as in $T$.*

**Table 2**
Possible classification for $(i, j)$ given classifications of $i$ and $j$.

| $i$ | $j$ | Possible classifications for $(i, j)$ |
|---|---|---|
| $P_m$ | $P_m$ | Neutral |
| $P_m$ | $P_s$ | Neutral |
| $P_m$ | $N$ | Neutral |
| $P_m$ | $D$ | Neutral |
| $P_s$ | $P_s$ | 2-Parter, neutral* |
| $P_s$ | $N$ | Parter, neutral** |
| $P_s$ | $D$ | $i, j$ cannot be adjacent*** |
| $N$ | $N$ | Neutral |
| $N$ | $D$ | $i, j$ cannot be adjacent**** |
| $D$ | $D$ | Downer |

* See Appendix of examples (6) for an example.
** See Appendix of examples (5) for an example.
*** See Remark 3.9.
**** See Table 1.

**Proof.** Let $(i, j) \in E$ be neutral. Assume, to the contrary, that the classification of $i$ changes when we remove $(i, j)$. By Remark 2.1,

$$m_{A(e_{i,j},i)}(\lambda) = m_{A(i)}(\lambda). \tag{3.1}$$

Since $(i, j)$ is neutral,

$$m_{A(e_{i,j})}(\lambda) = m_A(\lambda). \tag{3.2}$$

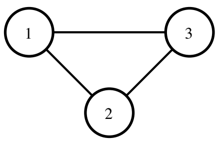
But the classification of $i$ is different in $T(e_{i,j})$ from $T$, which makes it impossible that both (3.1) and (3.2) are satisfied, so we have a contradiction. The same argument holds for $j$. $\square$

## 4. Appendix of examples

For $\alpha \subset \{1, \ldots, n\}$, $A[\alpha]$ denotes the principal submatrix of $A$ that lies in the rows and columns indexed by $\alpha$. A blank entry in a matrix denotes 0.

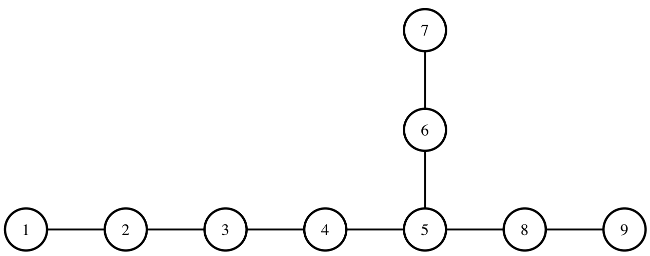
**Examples.**

(1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$



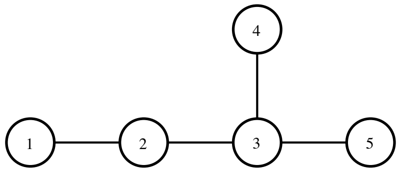

Note that $A$ has rank 1, and thus $m_A(0) = 2$. If you set $a_{13} = a_{31} = 0$, however, the resulting matrix $A(e_{1,3})$ has full rank, so $m_{A(e_{1,3})}(0) = 0$. Thus $(1, 3)$ is a 2-downer edge for the eigenvalue 0, and we have an example of a 2-downer edge in a non-tree.

(2) $A = \begin{bmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & \\ & & 1 & 0 & 1 & & & & \\ & & & 1 & 0 & 1 & & 1 & \\ & & & & 1 & 0 & 1 & & \\ & & & & & 1 & 0 & & \\ & & & & 1 & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{bmatrix}$



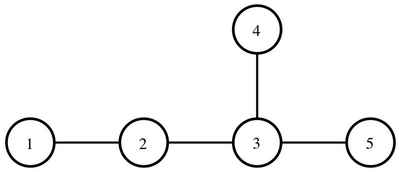
Here, $m_A(-1) = 1$ and $m_{A(e_{2,3})}(-1) = 3$, so $(2, 3)$ is 2-Parter for the eigenvalue $-1$. Note that vertices 2 and 3 are both $P_s$.

(3) $A = \begin{bmatrix} 2 & 1 & & & \\ 1 & 8 & 1 & & \\ & 1 & 2 & 1 & 1 \\ & & 1 & 8 & \\ & & 1 & & 8 \end{bmatrix}$



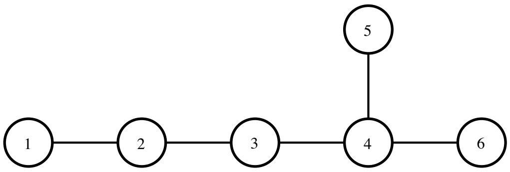

Here, $m_A(8) = 1$ and $m_{A(e_{1,2})}(8) = 2$, so $(1, 2)$ is Parter for the eigenvalue 8. Note that vertex 1 is $P_s$ and vertex 2 is $N$.

(4) $A = \begin{bmatrix} 9 & 1 & & & \\ 1 & 9 & 1 & & \\ & 1 & 2 & 1 & 1 \\ & & 1 & 8 & \\ & & 1 & & 8 \end{bmatrix}$



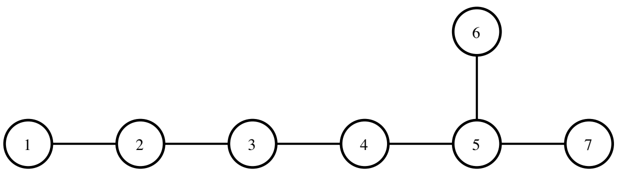

Here, $m_A(8) = 2$ and $m_{A(e_{1,2})}(8) = 1$, so $(1, 2)$ is downer for the eigenvalue 8. Note that vertices 1 and 2 are both $D$.

(5) $A = \begin{bmatrix} 8 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & 1 \\ & & & 1 & 8 & \\ & & & 1 & & 8 \end{bmatrix}$




Here, $m_A(8) = 1 = m_{A(e_{2,3})}(8)$, so $(2, 3)$ is neutral for the eigenvalue 8. Note that vertex 2 is $P_s$ and vertex 3 is $N$.

(6) $A = \begin{bmatrix} 8 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 3 & 1 & & & \\ & & 1 & 8 & 1 & & \\ & & & 1 & 2 & 1 & 1 \\ & & & & 1 & 8 & \\ & & & & 1 & & 8 \end{bmatrix}$




Here, $m_A(8) = 1 = m_{A(e_{2,3})}(8)$, so $(2, 3)$ is neutral for the eigenvalue 8. Note that vertices 2 and 3 are both $P_s$.

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